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Stieltjes Integral Inequalities of Gronwall Type and Applications (*).

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Summary. – *We obtain estimates for solutions of integral inequalities of Gronwall type involving Stieltjes integrals and their inverse inequalities. From these we obtain some new results for integral inequalities for Riemann integrals and functional integral inequalities. Extensions are also given to Bihari type integral inequalities.*

1. – Introduction.

In the study of existence, uniqueness, stability, boundedness, and certain other aspects of the qualitative behavior of solutions of differential and integral equations, integral inequalities often play a fundamental role. Moreover, since integral inequalities of Stieltjes type may include, as special cases, Riemann as well as functional integral inequalities, it seems reasonable that such results would be of use. However, there seem to be relatively few results dealing with integral inequalities of Stieltjes type. In [4] the authors obtain a result for modified Stieltjes integrals which however does not hold in general and in [5] some results are given which are somewhat incomplete since the integrals involved may not exist (cf. Theorem 3.1 and 3.2—if $u(t)$ has a discontinuity from the right (left) then so also does $\int_s^t K(\sigma)\,du(\sigma)$). There appears to be a similar problem in [2] with Lemma 5).

In this paper, by modifying the conditions in [5], we discuss a more general kind of integral inequalities of Gronwall type involving Stieltjes integrals and their inverse inequalities, from which we derive some results on integral inequalities for Riemann integrals and functional integral inequalities of the same type. As an application several properties of solutions of a second order retarded differential equation are obtained by using the given inequalities. We also indicate extensions to more general Bihari-type integral inequalities of Stieltjes type.

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2. - Basic results.

For convenience in the proofs, we first introduce two properties of Stieltjes integrals.

PROPERTY 1. - Let $\alpha(t), \varphi(t) \in C[a, b]$ and suppose $u(t)$ is of bounded variation on $[a, b]$. Let $\beta(t) = \beta_0 + \int_0^t \varphi(s)\, du(s)$. Then

$$\int_a^b \alpha(t)\, d\beta(t) = \int_a^b \alpha(t) \varphi(t)\, du(t) .$$

PROPERTY 2. - Let $\alpha(t), u(t) \in C[a, b]$ and suppose $u(t)$ is of bounded variation on $[a, b]$ and $\beta(t)$ is of bounded variation on the range of $u(t)$ and has a continuous derivative. Then

$$\int_a^b \alpha(t) \beta'(u(t))\, du(t) = \int_a^b \alpha(t)\, d\beta(u(t)) .$$

We now give three basic results of this paper.

THEOREM 1. - Suppose

i) $\tau(t)$ is continuous and strictly increasing on $[0, \infty)$ and $\tau(t) \leqslant t$,

ii) $u(t, s)$ is defined on $[0, \infty) \times [\tau(0), \infty)$, and continuous and nondecreasing in s,

iii) $f(t)$ and $g(t) \geqslant 0 \in C[0, \infty)$, $x(t) \geqslant 0 \in C[\tau(0), \infty)$,

iv) $x(t) \equiv \varphi(t)$ for $t \in [\tau(0), 0]$, where $\varphi(t)$ is a given continuous function.

If

$$x(t) \leqslant f(t) + g(t) \int_{\tau(0)}^{\tau(t)} x(s)\, d_s u(k, s) \tag{1}$$

for $0 \leqslant t \leqslant k < \infty$, then

$$\begin{aligned} x(t) \leqslant f(t) + g(t) \int_0^{\tau(t)} f(s) \exp\left[\int_s^{\tau(t)} g(\sigma)\, d_\sigma(u(k, \sigma))\right] d_s u(k, s) + \\ + g(t) R(\tau^{-1}(0); k, \varphi) \exp\left[\int_0^{\tau(t)} g(s)\, d_s u(k, s)\right] \end{aligned} \tag{2}$$

for $0 \leqslant t \leqslant k < \infty$, where $f(t) \equiv g(t) \equiv 0$ when $t \in [\tau(0), 0]$,

$$R(t;\, k,\, x) = \int_{\tau(0)}^{\tau(t)} x(s)\, d_s u(k,\, s)\,.$$

PROOF. – From (1) we have

$$x(t) \leqslant f(t) + g(t) R(t;\, k,\, x)\,. \tag{3}$$

When $0 \leqslant s \leqslant k < \infty$,

$$x(s) \exp\left[-\int_0^s g(\sigma)\, d_\sigma u(k,\, \sigma)\right] \leqslant [f(s) + g(s) R(s;\, k,\, x)] \exp\left[\int_0^s -g(\sigma)\, d_\sigma u(k,\, \sigma)\right]. \tag{4}$$

So when $\tau^{-1}(0) \leqslant t \leqslant k < \infty$, noting that $R(t;\, k,\, x)$ is nondecreasing in t, and by Properties 1 and 2, we get

$$\begin{aligned}
&\int_0^{\tau(t)} g(s) R(s;\, k,\, x) \exp\left[-\int_0^s g(\sigma)\, d_\sigma u(k,\, \sigma)\right] d_s u(k,\, s) \\
&\qquad = -\int_0^{\tau(t)} R(s;\, k,\, x) \exp\left[-\int_0^s g(\sigma)\, d_\sigma u(k,\, \sigma)\right] d_s\left(-\int_0^s g(\sigma)\, d_\sigma u(k,\, \sigma)\right) = \\
&\qquad = -\int_0^{\tau(t)} R(s;\, k,\, x)\, d_s \exp\left[-\int_0^s g(\sigma)\, d_\sigma u(k,\, \sigma)\right] \leqslant \\
&\qquad \leqslant -\int_0^{\tau(t)} R(\tau^{-1}(s);\, k,\, x)\, d_s \exp\left[-\int_0^s g(\sigma)\, d_\sigma u(k,\, \sigma)\right] = \\
&\qquad = -R(t;\, k,\, x) \exp\left[-\int_0^{\tau(t)} g(\sigma)\, d_\sigma u(k,\, \sigma)\right] + R(\tau^{-1}(0);\, k,\, \varphi) + \\
&\qquad + \int_0^{\tau(t)} \exp\left[-\int_0^s g(\sigma)\, d_\sigma u(k,\, \sigma)\right] d_s R(\tau^{-1}(s);\, k,\, x) = \\
&\qquad = -R(t;\, k,\, x) \exp\left[-\int_0^{\tau(t)} g(\sigma)\, d_\sigma u(k,\, \sigma)\right] + R(\tau^{-1}(0);\, k,\, \varphi) + \\
&\qquad\qquad + \int_0^{\tau(t)} x(s) \exp\left[-\int_0^s g(\sigma)\, d_\sigma u(k,\, \sigma)\right] d_s u(k,\, s)\,.
\end{aligned} \tag{5}$$

Integrating both sides of (4) with respect to $u(k,\cdot)$ from 0 to $\tau(t)$, and combining with (5), we can see when $\tau^{-1}(0)\leqslant t\leqslant k<\infty$,

$$R(t;k,x)\leqslant\int_0^{\tau(t)}f(s)\exp\left[\int_s^{\tau(t)}g(\sigma)\,d_\sigma u(k,\sigma)\right]d_s u(k,s)+$$
$$+R(\tau^{-1}(0);k,\varphi)\exp\left[\int_0^{\tau(t)}g(\sigma)\,d_\sigma u(k,\sigma)\right].$$

Substituting the above expression into (3), we obtain the conclusion for $\tau^{-1}(0)\leqslant$ $\leqslant t\leqslant k<\infty$.

When $0\leqslant t<\tau^{-1}(0)$ and $t\leqslant k$ from (1) we have

$$x(t)\leqslant f(t)+g(t)\int_{\tau(0)}^{\tau(t)}\varphi(s)\,d_s u(k,s)\leqslant f(t)+g(t)\int_{\tau(0)}^{0}\varphi(s)\,d_s u(k,s)=f(t)+g(t)R(\tau^{-1}(0);k,\varphi)\,.$$

Noting that $f(t)\equiv g(t)\equiv 0$ when $t\in[\tau(0),0]$, we see (2) is also true for $0\leqslant t<\tau^{-1}(0)$ and $t\leqslant k$.

THEOREM 2. - Suppose

i) $\tau(t)$ is continuous and strictly increasing on $[0,\infty)$, and $\tau(t)\geqslant\tau$,

ii) $u(t,s)$ is defined on $[0,\infty)\times[\tau(0),\infty)$, and continuous and nondecreasing in s,

iii) $f(t),g(t)\geqslant 0\in C[0,\infty)$.

If

$$x(t)\geqslant f(t)+g(t)\int_{\tau(0)}^{\tau(t)}x(s)\,d_s u(k,s)$$

for $0\leqslant t\leqslant k<\infty$, then

$$x(t)\geqslant f(t)+g(t)\int_{\tau(0)}^{\tau(t)}f(s)\exp\left[\int_s^{\tau(t)}g(\sigma)\,d_\sigma u(k,s)\right]d_s u(k,s)$$

for $0\leqslant t\leqslant k<\infty$.

The proof of Theorem 2 is similar to that of Theorem 1 so we shall omit it here.

THEOREM 3. - Suppose

i) $\tau_i(t)$ $(i=1,2,\ldots,n)$ is continuous and strictly increasing, $\tau_i(t)\leqslant t$, $\tau_*(t)=$ $=\min\limits_{1\leqslant i\leqslant n}\{\tau_i(t)\}$,

ii) $u_i(t, s)$ $(i = 1, 2, \ldots, n)$ are defined on $[0, \infty) \times [\tau_i(0), \infty)$ and continuous and nondecreasing in s,

iii) $f(t)$ and $g_i(t) \geq 0$ $(i = 1, 2, \ldots, n) \in C[0, \infty)$, and $x(t) \geq 0 \in C[\tau_*(0), \infty]$,

iv) $x(t) \equiv \varphi(t)$ for $t \in [\tau_*(0), 0]$, where $\varphi(t)$ is a given continuous function.

If

$$x(t) \leq f(t) + \sum_{i=1}^{n} g_i(t) \int_{\tau_i(0)}^{\tau_i(t)} x(s)\, d_s u_i(k, s) \tag{6}$$

for $0 \leq t \leq k < \infty$, then

$$x(t) \leq A_{n+1}(f) + \sum_{i=1}^{n} A_{n-i+1}(H_i) \tag{7}$$

for $0 \leq t \leq k < \infty$, where $A_i(v)$ $(i = 1, 2, \ldots, n+1)$ are defined as follows:

$$A_1(v) = v(t),$$

$$A_{i+1}(v) = A_i(v) + A_i(g_i) \int_0^{\tau_i(t)} A_i(v(s)) \exp\left[\int_0^{\tau_i(t)} A_i(g_i)\, d_\sigma u_i(k, \sigma)\right] d_s u_i(k, s)$$

for $0 < t < \infty$, and

$$A_i(v(t)) = 0 \qquad \text{for } t \in [\tau_i(0), 0], \tag{8}$$

and

$$H_i(t) = A_i(g_i(t))\, R_i(\tau_i^{-1}(0); k, \varphi) \exp\left[\int_0^{\tau_i(t)} A_i(g_i)(\sigma)\right] d_\sigma u_i(k, \sigma,$$

$$R_i(\tau_i^{-1}(0); k, \varphi) = \int_{\tau_i(0)}^{0} \varphi(s)\, d_s u_i(k, s).$$

PROOF. - We can easily see by induction that the functions $A_i(v)$ $(i = 1, 2, \ldots$ $\ldots, n+1)$ defined by (8) satisfy that for $v_1(t) \geq 0$, $v_2(t) \geq 0$,

(9) i) $A_i(v_1 + v_2) = A_i(v_1) + A_i(v_2)$, and

ii) $A_i(v_1 v_2)(t) \leq (A_1(v_1) v_2)(t)$ if $v_2(t)$ is nondecreasing.

Clearly, (7) holds for $n = 1$ by Theorem 1. Assume that (7) holds for $n = r - 1$ $(1 < r \leq n + 1)$. Then for $n = r$, from (6) we have

$$x \leq \left(f + g_r \int_{\tau_r(0)}^{\tau_r(t)} x d_s u_r(k, s)\right) + \sum_{i=1}^{r-1} g_i \int_{\tau_i(0)}^{\tau_i(t)} x d_s u_i(k, s) .$$

According to the assumption and (9) we get

$$x \leq A_r\left(f + g_r \int_{\tau_r(0)}^{\tau_r(t)} x d_s u_r(k, s)\right) + \sum_{i=1}^{r-1} A_{r-i}(H_i) \leq A_r(f) + \sum_{i=1}^{r-1} A_{r-i}(H_i) + A_r(g_r) \int_{\tau_r(0)}^{\tau_r(t)} x d_s u_r(k, s) .$$

Using Theorem 1, we get

$$x \leq A_r(f) + \sum_{i=1}^{r-1} A_{r-i}(H_i) + H_r +$$
$$+ A_r(g_r) \int_0^{\tau_r(t)} \left[A_r(f) + \sum_{i=1}^{r-1} A_{r-i}(H_i)\right] \exp\left[\int_s^{\tau_r(t)} A_r(g_r)\, d_\sigma u(k, \sigma)\right] d_s u(k, s) =$$
$$= A_{r+1}(f) + \sum_{i=1}^{r} A_{r-i+1}(H_i) .$$

So (7) holds for $n = r$. This completes the proof.

COROLLARY 1. – Suppose

i) $u(t, s)$ is defined on $[0, \infty) \times [0, \infty)$, and continuous and nondecreasing in s,

ii) $f(t), g(t) \geq 0$ and $x(t) \geq 0 \in C[0, \infty)$.

i) If

$$x(t) \leq f(t) + g(t) \int_0^t x(s)\, d_s u(k, s)$$

for $0 \leq t \leq k < \infty$, then

$$x(t) \leq f(t) + g(t) \int_0^t f(s) \exp\left[\int_s^t g(\sigma)\, d_\sigma u(k, \sigma)\right] d_s u(k, s)$$

for $0 \leq t \leq k < \infty$;

ii) If

$$x(t) \geq f(t) + g(t) \int_0^t x(s)\, d_s u(k, s)$$

for $0 \leqslant t \leqslant k < \infty$, then

$$x(t) \geqslant f(t) + g(t) \int_0^t f(s) \exp\left[\int_s^t g(\sigma)\, d_\sigma u(k, \sigma)\right] d_s u(k, s)$$

for $0 \leqslant t \leqslant k < \infty$.

COROLLARY 2. – Suppose

i) $u_i(t, s)$ $(i = 1, 2, \dots, n)$ are defined on $[0, \infty) \times [0, \infty)$ and continuous and nondecreasing in s,

ii) $f(t)$, $g_i(t) \geqslant 0$ $(i = 1, 2, \dots, n)$ and $x(t) \geqslant 0 \in C[0, \infty)$.

If

$$x(t) \leqslant f(t) + \sum_{i=1}^{n} g_i(t) \int_0^t x(s)\, d_s u_i(k, s)$$

for $0 \leqslant t \leqslant k < \infty$, then

$$x(t) \leqslant A_{n+1}(f)$$

for $0 \leqslant t \leqslant k < \infty$ where $A_i(v)$ $(i = 1, 2, \dots, n+1)$ are defined as follows:

$$A_1(v) = v\,,$$

$$A_{i+1}(v) = A_i(v) + A_i(g_i) \int_0^t A_i(v) \exp\left[\int_s^t A_i(g_i)\, d_\sigma u_i(k, \sigma)\right] d_s u_i(k, s)\,,$$

REMARK. – Corollary 1, i) includes Th. 3.2 in [5] (for modified conditions) as a special case. In fact, letting $g(t) \equiv 1$, $u(k, s) = \int_0^s K(\sigma)\, d\sigma$ in Corollary 1, i), we obtain Theorem 3.2 in [5]. Furthermore, we also obtain results on the inverse inequality and the inequality with n linear terms.

3. – Special cases of the above Theorems.

1. *On functional integral inequalities.*

For the purpose of proofs we need the following lemma.

LEMMA. – For functions $g_i(t) > 0$ $(i = 1, 2, \dots, n) \in C^1[a, b]$, there exist functions $q_i(t)$ $(i = 1, 2) \in C^1[a, b]$, nondecreasing functions $\alpha_i(t)$ $(i = 1, 2, \dots, n) \in C^1[a, b]$ and nonincreasing functions $\beta_i(t)$ $(i = 1, 2, \dots, n) \in C^1[a, b]$, such that $g_i(t) = q_1(t)\alpha_i(t) = q_2(t)\beta_i(t)$, $(i = 1, , \dots, n)$.

PROOF. – Denote $w_1(t) = \min_{1\le i\le n} \{g_i'(t)/g_i(t)\}$, $q_1(t) = \exp\Big[\int_a^t w_1(s)\,ds\Big]$, and $\alpha_i(t) = g_i(t)/q_1(t)$ $(i = 1, 2, \dots, n)$ for $t \in [a, b]$. Obviously, $q_1(t) > 0$ $\alpha_i(t) > 0 \in C^1[a, b]$, and

$$\alpha_i'(t) = [g_i(t)/q_1(t)]' = \frac{g_i'(t)q_1(t) - g_i(t)q_1(t)w_1(t)}{q_1^2(t)} \geqslant \frac{g_i'(t) - g_i(t)(g_i'(t)/g_i(t))}{q_1(t)} = 0\,.$$

So $\alpha_i(t)$ $(k = 1, 2, \dots, n)$ are nondecreasing.

Similarly, if we denote

$$w_2(t) = \max_{1\le i\le n} \{g_i'(t)/g_i(t)\}\,, \qquad q_2(t) = \exp\Big[\int_a^t w_2(s)\Big], \qquad \beta_i(t) = g_i(t)/q_2(t)$$
$$(i = 1, 2, \dots, n)\,, \qquad t \in [a, b]$$

then $q_2(t) > 0$, $\beta_i(t) > 0 \in C^1[a, b]$ and we can prove that $\beta_i'(t) \leqslant 0$, therefore $\beta_i(t)$ $(i = 1, 2, \dots, n)$ are nonincreasing.

In the sequel, we assume that $\tau_i(t)$ $(i = 1, 2, \dots, n) \in C^1[a, \infty)$, and $\tau_i'(t) > 0$. Denote $\tau_*(t) = \min_{1\le i\le n} \{\tau_i(t)\}$, $\tau^*(t) = \max_{1\le i\le n} \{\tau_i(t)\}$.

THEOREM 4. – Suppose

i) $f(t) \in C[0, \infty)$, $g_i(t) > 0$ $(i = 1, 2, \dots, n) \in C^1[0, \infty)$, $x(t) \geqslant 0 \in C[\tau_*(0), \infty)$, $h_i(t, s) \geqslant 0$ $(i = 1, 2, \dots, n)$ are nondecreasing in t on $[0, \infty)$, and continuous in s on $[0, \infty)$,

ii) $\tau_i(t) \leqslant t$ and $\tau_1(0) = \tau_2(0) = \dots = \tau_n(0)$.

If

$$x(t) \leqslant f(t) + \sum_{i=1}^n g_i(t)\int_0^t h_i(t, s)x(\tau_i(s))\,ds\,, \qquad t \geqslant 0 \tag{10}$$
$$x(t) = \varphi(t)\,, \qquad t \in [\tau_*(0), 0]$$

then

$$x(t) \leqslant f(t) + \sum_{i=1}^n g_i(t)\int_{\tau_i^{-1}(0)}^{\tau_i^{-1}(\tau^*(t))} h_i(t, s)f(\tau_i(s)) \exp\left[\sum_{j=1}^n \alpha_j(t)\int_{\tau_j^{-1}(\tau_i(s))}^{\tau_j^{-1}(\tau^*(t))} q_1(\tau(\sigma))h_j(t, \sigma)\,d\sigma\right] ds +$$
$$+ \left(\sum_{i=1}^n g_i(t)\int_0^{\tau_i^{-1}(0)} h_i(t, s)\varphi(\tau_i(s))\,ds\right) \exp\left[\sum_{j=1}^n \alpha_j(t)\int_{\tau_j^{-1}(0)}^{\tau_j^{-1}(\tau^*(t))} q_1(\tau_j(s))h_j(t, s)\,ds\right], \qquad t \geqslant 0,$$

where $q_1(t)$ and $\alpha_i(t)$ $(i=1,2,\dots,n)$ are defined as in the Lemma, and $q_1(t)\equiv 0$ when $t<0$.

PROOF. – From (10) we have

$$x(t)\leqslant f(t)+\sum_{i=1}^{n} g_i(t)\int_{\tau_i(0)}^{\tau_i(t)} h_i(t,\tau_i^{-1}(\sigma))(\tau_i^{-1}(\sigma))'x(\sigma)\,d\sigma\,.$$

Let

$$u(t,s)=\sum_{i=1}^{n}\alpha_i(t)\int_0^s h_i(t,\tau_i^{-1}(\sigma))(\tau_i^{-1}(\sigma))'\,d\sigma\,.$$

According to the Lemma, noting that $\alpha_i(t)$ and $h_i(t,s)$ $(i=1,2,\dots,n)$ are nondecreasing in t, we get

$$x(t)\leqslant f(t)+q_1(t)\int_{\tau^*(0)}^{\tau^*(t)} x(s)\,d_s u(k,s)$$

for $0\leqslant t\leqslant k<\infty$. Using Theorem 1 and letting $k=t$, we obtain

$$x(t)\leqslant f(t)+q_1(t)\int_0^{\tau^*(t)} f(s)\exp\left[\int_s^{\tau^*(t)} q_1(s)\,d_\sigma u(t,\sigma)\right]d_s u(t,s)+ \tag{11}$$

$$+\,q_1(t)R(\tau^{*-1}(0);t,\varphi)\exp\left[\int_0^{\tau^*(t)} q_1(t)\,d_s u(t,s)\right]=$$

$$=f(t)+q_1(t)\int_0^{\tau^*(t)} f(s)\exp\left[\sum_{j=1}^{n}\alpha_j(t)\int_s^{\tau^*(t)} q_1(s)h_j(t,\tau_j^{-1}(\sigma))(\tau_j^{-1}(\sigma))'\,d\sigma\right]\cdot$$

$$\cdot\Big(\sum_{i=1}^{n}\alpha_i(t)h_i(t,\tau_i^{-1}(s))(t_i^{-1}(s))'\Big)ds+\left(q_1(t)\int_{\tau^*(0)}^{0}\varphi(s)\Big[\sum_{i=1}^{n}\alpha_i(t)h_i(t,\tau_i^{-1}(s))(\tau_i^{-1}(s))'\Big]ds\right)\cdot$$

$$\cdot\exp\left[\sum_{j=1}^{n}\alpha_j(t)\int_0^{\tau^*(t)} q_1(s)h_j(t,\tau_j^{-1}(s))(\tau_j^{-1}(s))'\,ds\right]=$$

$$=f(t)+\sum_{i=1}^{n} g_i(t)\int_{\tau_i^{-1}(0)}^{\tau_i^{-1}(\tau^*(t))} h_i(t,s)f(\tau_i(s))\exp\left[\sum_{j=1}^{n}\alpha_j(t)\int_{\tau_j^{-1}(\tau_i(s))}^{\tau_j^{-1}(\tau^*(t))} q_1(\tau_j(s))h_j(t,\sigma)\,d\sigma\right]ds+$$

$$+\left(\sum_{i=1}^{n} g_i(t)\int_0^{\tau_i^{-1}(0)} h_i(t,s)\varphi(\tau_i(s))\,ds\right)\exp\left[\sum_{j=1}^{n}\alpha_j(t)\int_{\tau_j^{-1}(0)}^{\tau_j^{-1}(\tau^*(t))} q_1(\tau_j(s))h_j(t,s)\,ds\right].$$

THEOREM 5. – Suppose

i) $f(t)$ and $x(t)\geqslant 0 \in C[0, \infty)$, $g_i(t) > 0$ $(i = 1, 2, \dots, n) \in C^1[0, \infty)$, $h_i(t, s) \geqslant 0$ $(i = 1, 2, \dots, n)$ are nonincreasing in t on $[0, \infty)$ and continuous in s on $[0, \infty)$.

ii) $\tau_i(t) \geqslant t$ $(i = 1, 2, \dots, n)$, and $\tau_1(0) = \tau_2(0) = \dots = \tau_n(0)$.

If:

$$x(t) \geqslant f(t) + \sum_{i=1}^{n} g_i(t) \int_0^t h_i(t, s) x(\tau_i(s))\, ds\,, \qquad x \geqslant 0\,, \tag{12}$$

then

$$x(t) \geqslant f(t) + \sum_{i=1}^{n} g_i(t) \int_0^{\tau_i^{-1}(\tau_*(t))} h_i(t, s) f(\tau_i(s)) \exp\left[\sum_{j=1}^{n} \beta_j(t) \int_{\tau_j^{-1}(\tau_i(s))}^{\tau_j^{-1}(\tau_*(t))} q_2(s) h_j(t, \sigma)\, d\sigma\right] ds\,, \qquad t \geqslant 0\,,$$

where $q_2(t)$ and $\beta_i(t)$ $(i = 1, 2, \dots, n)$ are defined as in the Lemma.

COROLLARY 3. – i) In addition to the conditions of Theorem 4, if $g_i(t) \equiv 1$ $(i = 1, 2, \dots, n)$, $f(t)$ is nondecreasing, then the solutions of (10) satisfy

$$x(t) \leqslant \left[f(t) + \sum_{i=1}^{n} \int_0^{\tau_i^{-1}(0)} h_i(t, s) \varphi(\tau_i(s))\, ds\right] \exp\left[\sum_{j=1}^{n} \int_{\tau_j^{-1}(0)}^{\tau_j^{-1}(\tau^*(t))} \delta(\tau_j(\sigma)) h_j(t, \sigma)\, d\sigma\right], \qquad t \geqslant 0\,,$$

where

$$\delta(t) = \begin{cases} 1 & t \geqslant 0 \\ 0 & t < 0; \end{cases}$$

ii) in addition to the conditions of Theorem 5, if $g_i(t) \equiv 1$ $(i = 1, 2, \dots, n)$ $f(t)$ is nonincreasing, then the solution of (12) satisfy

$$x(t) \geqslant f(\tau^*(t)) \exp\left[\sum_{i=1}^{n} \int_0^{\tau_i^{-1}(\tau_*(t))} h_i(t, s)\, ds\right], \qquad t \geqslant 0\,.$$

PROOF. – i) The solutions of (10) satisfy (11). Since $g_i(t) \equiv 1$, $f(t)$ is nondecreasing, and $\tau^*(t) \leqslant t$, in view of $q_1(t) \equiv 0$ when $t < 0$, we see

$$x(t) \leqslant f(t)\left[1 + \int_0^{\tau^*(t)} \exp\left[\int_s^{\tau^*(t)} \delta(\sigma)\, d_\sigma u(t, \sigma)\right] d_s u(t, s)\right] +$$

$$+ R(\tau^{*-1}(0), t, \varphi)\exp\left[\int_0^{\tau^*(t)}\delta(s)\,d_s u(t,s)\right] = f(t)\left[\!\left[1 - \int_0^{\tau^*(t)} d_s \exp\left[\int_s^{\tau^*(t)}\delta(s)\,d_\sigma u(t,\sigma)\right]\right]\!\right] +$$

$$+ R(\tau^{*-1}(0); t, \varphi)\exp\left[\int_0^{\tau^*(t)}\delta(s)\,d_s u(t,s)\right] =$$

$$= f(t)\exp\left[\int_0^{\tau^*(t)}\delta(s)\,d_\sigma u(t,\sigma)\right] + R(\tau^{*-1}(0); t, \varphi)\exp\left[\int_0^{\tau^*(t)}\delta(s)\,d_s u(t,s)\right] =$$

$$= \left[f(t) + \sum_{i=1}^{n}\int_0^{\tau_i^{-1}(0)} h_i(t,s)\varphi(\tau_i(s))\,ds\right]\exp\left[\sum_{i=1}^{n}\int_{\tau_i^{-1}(0)}^{\tau_i^{-1}(\tau^*(t))} h_i(t,s)\,ds\right].$$

The proof of ii) is similar.

Theorems 4 and 5 and Corollary 3 do not apply to the case when $\tau_1(0) = \tau_2(0) = \dots = \tau_n(0)$ is not satisfied. In that case we have an inductory estimate expression for the solutions of inequality (10).

THEOREM 6. - Suppose

i) $f(t)$ is continuous and nondecreasing in $[0, \infty)$, $x(t) \geq 0 \in C[\tau_*(0), \infty)$, $f_i(t, s) \geq 0$ $(i = 1, 2, \dots, n)$ are nondecreasing in t on $[0, \infty)$, and continuous in s on $[0, \infty)$,

ii) $\tau_i(t) \leq t$ $(i = 1, 2, \dots, n)$.

If

$$x(t) \leq f(t) + \sum_{i=1}^{n}\int_0^t h_i(t,s)x(\tau_i(s))\,ds, \qquad t \geq 0,$$

$$x(t) = \varphi(t), \qquad t \in [\tau_*(0), 0],$$

then the solutions of (10) satisfy

$$x(t) \leq B_{n+1}(f) + \sum_{i=1}^{n} B_{n-i+1}(H_i)$$

where $B_i(v)$ $(i = 1, 2, \dots, n+1)$ are defined as follows:

$$B_1(v) = v(t),$$

$$B_{i+1}(v) = v(t)B_i(1)\exp\left[\int_0^{\tau_i(t)} B_i(1)h_i(t, \tau_i^{-1}(s))(\tau_i^{-1}(s))'\,ds\right] \qquad (i = 1, 2, \dots, n)$$

for $0 \leq t < \infty$, and $B_i(1) \equiv 0$ $(i = 1, 2, \dots, n)$ for $t \in [\tau_i(0), 0]$,

$$H_i(t) = \Big[\int_0^{\tau_i^{-1}(0)} h_i(t, s)\varphi(\tau_i(s))\, ds\Big] \exp\Big[\int_{\tau_i^{-1}(0)}^{t} \delta(\tau_i(s))\, h_i(t, s)\, ds\Big].$$

$\delta(t)$ is defined as in Corollary 3.

PROOF. - From (10) we have

$$x(t) \leq f(t) + \sum_{i=1}^{n} \int_{\tau_i(0)}^{\tau_i(t)} h_i(t, \tau_i^{-1}(\sigma))(\tau_i^{-1}(\sigma))' x(\sigma)\, d\sigma .$$

Denote

$$u_i(t, s) = \int_0^s h_i(t, \tau_i^{-1}(\sigma))(\tau_i^{-1}(\sigma))'\, d\sigma .$$

Noting that $h_i(t, s)$ $(i = 1, 2, \dots, n)$ are nondecreasing in t, we get

$$x(t) \leq f(t) + \sum_{i=1}^{u} \int_{\tau_i(0)}^{\tau_i(t)} x(s)\, d_s u_i(k, s)$$

for $0 \leq t \leq k < \infty$. According to Theorem 3,

$$x(t) \leq A_{n+1}(f) + \sum_{i=1}^{n} a_{n-i+1}(H_i)$$

for $0 \leq t \leq k < \infty$, where

$$A_1(v) = v(t),$$

$$A_{i+1}(v) = A_i(v) + A_i(1)\int_0^{\tau_i(t)} A_i(v) \exp\Big[\int_s^{\tau_i(t)} A_i(1)\, d_\sigma u_i(k, \sigma)\Big] d_s u_i(k, s) \qquad (i = 1, 2, \dots, n)$$

for $0 \leq t < \infty$, and $A_i(v) \equiv 0$ for $t \in [\tau_i(0), 0]$, and

$$H_i(t) \equiv \Big[\int_{\tau_i(0)}^{0} \varphi(s)\, d_s u(k, s)\Big] \exp\Big[\int_0^{\tau_i(t)} \delta(s)\, d_\sigma u(k, \sigma)\Big] =$$

$$= \Big(\int_0^{\tau_i^{-1}(0)} h_i(t, s)\varphi(\tau_i(s))\, ds\Big) \exp\Big[\int_{\tau_i^{-1}(0)}^{t} \delta(\tau_i(s))\, h_i(t, s)\, ds\Big].$$

When $k = t$, we can easily prove that $A_r(v) \leqslant B_r(v)$ $(t = 1, 2, \ldots, n+1)$ if $v(t)$ is nondecreasing. In fact, $A_1(v) = B_1(v) = v(t)$. Assume that $A_i(v) \leqslant B_i(v)$ for $1 \leqslant i \leqslant n$. Then from (9) we have

$$A_{i+1}(v) \leqslant v(t)A_i(1)\Big[1 - \int_0^{\tau_i(t)} \exp\Big[\int_s^{\tau_i(t)} A_i(1)\, d_\sigma u_i(k,\sigma)\Big]\, d_s\Big(\int_s^{\tau_i(t)} A_i(1)\, d_\sigma u_i(k,\sigma)\Big)\Big] =$$

$$= v(t)A_i(1)\exp\Big[\int_0^{\tau_i(t)} A_1(1) h_i(t, \tau_i^{-1}(s))(\tau_i^{-1}(s))'\, ds\Big] \leqslant$$

$$\leqslant v(t)B_i(1)\exp\Big[\int_0^{\tau_i(t)} B_i(1) h_i(t, \tau_i^{-1}(s))(\tau_i^{-1}(s))'\, ds\Big] = B_{i+1}(v)\,.$$

Therefore $A_{n+1}(v) \leqslant B_{n+1}(v)$ if $v(t)$ is nondecreasing. Since $f(t)$ and $H(t)$ are nondecreasing, the conclusion follows.

2. *On Gronwall's inequalities for Riemann integrals.*

Letting $u(t, s) = \int_0^s h(s)\, ds$ in Corollary 1, we obtain the well-known Gronwall's inequality and its inverse inequality; letting $u_i(t, s) = \int_0^s h_i(s)\, ds$ $(i = 1, 2, \ldots, n)$ in Corollary 2, we obrain Theorem 1 in [3]; letting $\tau_i(t) \equiv t$ $(i = 1, 2, \ldots, n)$ in Theorem 6, we obtain a better result than Theorem 1 in [1] and [6] under the same condition. We next derive new estimate expressions for the solutions of Gronwall's inequality with n linear terms and its inverse inequality from Theorems 4 and 5 and Corollary 3.

THEOREM 7. - Suppose $f(t)$ and $x(t) \geqslant 0 \in C[0, \infty)$, $g_i(t) > 0$ $(i = 1, 2, \ldots, n) \in C^1[0, \infty)$, $h_i(t, s) \geqslant 0$ $(i = 1, 2, \ldots, n)$ are nondecreasing in t on $[0, \infty)$ and continuous in s on $[0, \infty)$. If

$$x(t) \leqslant f(t) + \sum_{i=1}^n g_i(t) \int_0^t h_i(t, s) x(s)\, ds\,, \qquad x > 0\,, \tag{13}$$

then

$$x(t) \leqslant f(t) + \sum_{i=1}^n g_i(t) \int_0^t h_i(t, s) f(s) \exp\Big[\sum_{j=1}^n \alpha_j(t) \int_s^t h_j(t, \sigma) q_1(\sigma)\, d\sigma\Big] ds\,, \qquad x \geqslant 0\,,$$

where $q_1(t)$, $\alpha_i(t)$ $(i = 1, 2, \ldots, n)$ are defined as in the Lemma.

THEOREM 8. - Suppose $f(t)$ and $x(t) \geqslant 0 \in C[0, \infty)$, $g_i(t) > 0$ $(i = 1, 2, \ldots, n) \in C^1[0, \infty)$, $h_i(t, s) \geqslant 0$ $(i = 1, 2, \ldots, n)$ are nonincreasing in t on $[0, \infty)$ and continuous in s on $[0, \infty)$. If

$$x(t) \geq f(t) + \sum_{i=1}^{n} g_i(t) \int_0^t h_i(t, s) x(s)\, ds\,, \quad x > 0\,, \tag{14}$$

then

$$x(t) \geq f(t) + \sum_{i=1}^{n} g_i(t) \int_0^t h_i(t, s) f(s) \exp\left[\sum_{j=1}^{n} \beta_j(t) \int_s^t h_j(t, \sigma) q_2(\sigma)\, d\sigma\right] ds\,, \quad x \geq 0\,,$$

where $q_2(t)$, $\beta_i(t)$ $(i = 1, 2, \ldots, n)$ are defined as in the Lemma.

COROLLARY 4. – i) In addition to the conditions of Theorem 7, if $g_i(t) \equiv 1$ $(i = 1, 2, \ldots, n)$, $f(t)$ is nondecreasing, then the solutions of (13) satisfy

$$x(t) \leq f(t) \exp\left[\sum_{i=1}^{n} \int_0^t h_i(t, s) x(s)\, ds\right], \quad x \geq 0;$$

ii) in addition to the conditions of Theorem 8, if $g_i(t) \equiv 1$ $(i = 1, \ldots, n)$, $f(t)$ is nonincreasing, then the solutions of (14) satisfy

$$x(t) \geq f(t) \exp\left[\sum_{i=1}^{n} h_i(t, s) x(s)\, ds\right], \quad x \geq 0\,.$$

The importance of Theorems 7 and 8 and Corollary 4 lies in the fact that they offer explicit expressions, rather than inductory expressions as given by [1, 3, 6], to estimate the solutions of Gronwall's inequality with n linear terms and its inverse inequality. Therefore it is more convenient to use them. And in many cases the estimates are quite sharp.

EXAMPLE. – Consider the inequality

$$x(t) \leq 1 + (1 + t) \int_0^t x(s)\, ds + 2 \int_0^t (1 + s) x(s)\, ds\,.$$

With simple computation we see, from Corollary 4 i),

$$x(t) \leq \exp\,[3t + 2t^2];$$

and from Theorem 1 in [5],

$$x(t) \leq 2(1 + t) \exp\left[t + \frac{t^2}{2} + 2 \int_0^t (1 + s)^2 \exp\left[s + \frac{s^2}{2}\right] ds\right],$$

and the latter is greater than

$$2(1+t)\exp\left[3t+\frac{5}{2}t^2+\frac{2}{3}t^3\right].$$

The difference between the two results is quite large.

4. – Bihari's type of integral inequalities.

DEFINITION. – A function $g(u)$ is said to belong to the class $\mathcal{F}$ if $g(u)>0$ is nondecreasing and continuous on $(-\infty,+\infty)$ and $g(u)\leqslant vg(u/v)$ for $u\geqslant 0$ and $v\geqslant 1$. Clearly, if $g\in\mathcal{F}$ then $\int^{\infty}(1/g(s))\,ds=+\infty$.

THEOREM 9. – Suppose

i) $\tau(t)$ is continuous and strictly increasing on $[0,\infty)$ and $\tau(t)\leqslant t$,

ii) $u(t,s)$ is defined on $[0,\infty)\times[\tau(0),\infty)$, and nondecreasing in s,

iii) $f(t)>0$ is nondecreasing on $[0,\infty)$, $x(t)\in C[\tau(0),\infty)$,

iv) $w(u)\in\mathcal{F}$.

If

$$x(t)\leqslant f(t)+\int_{\tau(0)}^{\tau(t)} w(x(s))\,d_s u(k,s) \tag{15}$$

for $0\leqslant t\leqslant k<\infty$, and $x(t)\equiv\varphi(t)$ for $t\in[\tau(0),0]$, where $\varphi(t)$ is a given continuous function. Then

$$x(t)\leqslant\frac{f(t)}{f(0)}\,G^{-1}\left[G\left(f(0)+\int_{\tau(0)}^{0} w(\varphi(s))\,d_s u(k,s)\right)+\int_{0}^{\tau(t)}\delta(s)\,d_s u(k,s)\right] \tag{16}$$

for $0\leqslant t\leqslant k<\infty$, where

$$\delta(t)=\begin{cases}1 & t\geqslant 0\\ 0 & t<0,\end{cases}$$

$$G(u)=\int_{u_0}^{u}\frac{ds}{w(s)}\quad\text{for } u_0>,\ u>0.$$

PROOF. – i) We prove that (16) holds for $\tau^{-1}(0)\leqslant t\leqslant k<\infty$ first. The proof is similar to Theorem 1. Denote $f(t)\equiv f(0)$ where $t\in[\tau(0),0]$. Then $f(t)$ is nondecreasing on $[\tau(0), \infty)$. From (15), when $0 \leqslant t \leqslant k < \infty$

$$\frac{f(0)}{f(t)}\, x(t) \leqslant f(0) + \int_{\tau(0)}^{\tau(t)} \frac{f(0)}{f(s)}\, w(x(s))\, d_s u(k, s)\, .$$

Since $f(0)/f(t) \leqslant 1$ and $w(u) \in \mathcal{F}$, we have

$$\frac{f(0)}{f(t)}\, x(t) \leqslant f(0) + \int_{\tau(0)}^{\tau(t)} w\left(\frac{f(0)}{f(s)}\, x(s)\right) d_s u(k, s) = R(\tau(t); R, x)\, .$$

Hence

$$w\left(\frac{f(0)}{f(s)}\, x(s)\right) \leqslant w\big(R(\tau(t); k, x) \leqslant w(R(t; k, x))\big)\, ,$$

i.e.,

$$\frac{w\big((f(0)/f(t))\, x(t)\big)}{w(R(t; k, x))} \leqslant 1$$

for $0 \leqslant t \leqslant k < \infty$. Integrating from 0 to $\tau(t)$ with respect to $u(k, \cdot)$, we get for $\tau^{-1}(0) \leqslant t \leqslant k < \infty$

$$\int_0^{\tau(t)} \frac{w\big((f(0)/f(s))\, x(s)\big)}{w(R(s; k, x))}\, d_s u(k, s) = \int_0^{\tau(t)} \frac{d_s(R(s; k, x))}{w(R(s; k, x))} \leqslant \int_0^{\tau(t)} d_s u(k, s)\, .$$

Hence

$$G\big(R(\tau(t); k, x)\big) - G(R(0; k, x)) \leqslant \int_0^{\tau(t)} d_s u(k, s)\, ,$$

$$R(\tau(t); k, x) \leqslant G^{-1}\left[G(R(0; k, x)) + \int_0^{\tau(t)} d_s u(k, s)\right] =$$

$$= G^{-1}\left[G\left(f(0) + \int_{\tau(0)}^{0} w(\varphi(s))\, d_s u(k, s)\right) + \int_0^{\tau(t)} d_s u(k, s)\right].$$

Therefore the conclusion is true for $\tau^{-1}(0) \leqslant t \leqslant k < \infty$.

ii) When $0 \leqslant t < \tau^{-1}(0)$ and $t \leqslant k$, from (15) we have

$$x(t) \leqslant f(t) + \int_{\tau(0)}^{\tau(t)} w(\varphi(s))\, d_s u(k, s) \leqslant$$

$$\leqslant \frac{f(t)}{f(0)}\left[f(0) + \int_{\tau(0)}^{0} w(\varphi(s))\, d_s u(k, s)\right] = \frac{f(t)}{f(0)}\, G^{-1}\left[G\left(f(0) + \int_{\tau(0)}^{0} w(\varphi(s))\, d_s u(k, s)\right)\right].$$

Noting that $\delta(t) \equiv 0$ for $t \in [\tau(0), 0]$, we see (2) is true for $0 \leqslant t < \tau^{-1}(0)$ and $t \leqslant k$.

THEOREM 10. – Suppose

i) $\tau_i(t)$ $(i = 1, 2, \ldots, n)$ are continuous and increasing, $\tau_i(t) \leqslant t$, $\tau_*(t) = \min_{1\leqslant i\leqslant n} \{\tau_i(t)\}$,

ii) $u_i(t, s)$ $(i = 1, 2, \ldots, n)$ are defined on $[0, \infty) \times [\tau(0), \infty)$ and nondecreasing in s,

iii) $f(t) > 0$ is continuous and nondecreasing on $[0, \infty)$, $x(t) \in C[\tau_*(0), \infty)$,

iv) $w_i(u) \in \mathcal{F}$ $(i = 1, 2, \ldots, n)$.

If

$$x(t) \leqslant f(t) + \sum_{i=1}^{n} \int_{\tau_i(0)}^{\tau_i(t)} w_i(x(s))\, d_s u_i(k, s) \tag{17}$$

for $0 \leqslant t \leqslant k < \infty$, and $x(t) \equiv \varphi(t)$ for $t \in [\tau_*(0), 0]$, where $\varphi(t)$ is a given continuous function. Then

$$x(t) \leqslant E_{t,k}^{(n+1)}(f) \tag{18}$$

for $0 \leqslant t \leqslant k < \infty$, where $E_{t,k}^{(n+1)}(f)$ is defined by induction as follows:

$$\begin{aligned} E_{t,k}^{(1)}(u) &= u(t, k)\,, \\ E_{t,k}^{(i+1)}(u) &= \frac{E_{t,k}^{(i)}(u)}{E_{0,k}^{(i)}(f)}\, G_i^{-1}\Bigg[G_i\Bigg(E_{0,k}^{(i)}(f) + E_{k,k}^{(i)}(1)\int_{\tau_i(0)}^{0} w_i(\varphi(s))\, d_s u_i(k, s)\Bigg) + \\ &\qquad + E_{k,k}^{(i)}(1)\int_{0}^{\tau_i(t)} \delta(s)\, d_s u_i(k, s)\Bigg] \end{aligned} \tag{19}$$

where $\delta(t)$ is given by Theorem 9, and

$$G_i(u) = \int_{u_0}^{u} \frac{ds}{w_i(s)} \quad \text{for } u_0 > 0\,,\ u > 0\,.$$

PROOF. – $E_{t,k}^{(i)}(u)$ $(i = 1, 2, \ldots, n+1)$ obviously satisfy that for $u, v \in R$

$$E_{t,k}^{(i)}(u + v) = E_{t,k}^{(i)}(u) + E_{t,k}^{(i)}(v)\,,$$
$$E_{t,k}^{(i)}(uv) = [E_{t,k}^{(i)}(u)]v\,,$$

and $E_{t,k}^{(i)}(1)$ are nondecreasing in t. By Theorem 9 we know the conclusion is true for $n = 1$. Assume the conclusion is true for $n = r - 1$ $(1 < r \leqslant n)$. Then for $n = r$

$$x(\ ) \leqslant \Bigg[f(t) + \int_{\tau_r(0)}^{\tau_i(t)} w_r(x(s))\, d_s u_r(k, s)\Bigg] + \sum_{i=1}^{r-1} \int_{\tau_i(0)}^{\tau_i(t)} w_i(x(s))\, d_s u_i(k, s)\,.$$

According to the assumption we have

$$x(t)\leqslant E^{(r)}_{t,k}\Big[f(t)+\int\limits_{\tau_r(0)}^{\tau_r(t)} w_r\big((s)\big)\,d_s u_r(k,s)\Big]=E^{(r)}_{t,k}(f)+E^{(r)}_{t,k}(1)\int\limits_{\tau_r(0)}^{\tau_r(t)} w_r\big(x(s)\big)\,d_s u_r(k,s)\leqslant$$

$$\leqslant E^{(r)}_{t,k}(f)+\int\limits_{\tau_r(0)}^{\tau_r(t)} w_r\big(x(s)\big)\,d_s[E^{(i)}_{k,k}(1)\,u_r(k,s)]\,.$$

By Theorem 9 we get for $0\leqslant t\leqslant k<\infty$

$$x(t)\leqslant\frac{E^{(r)}_{t,k}(f)}{E^{(r)}_{0,k}(f)}\,G_r^{-1}\left[G_r\left(E^{(r)}_{0,k}(f)+\int\limits_{\tau_r(0)}^{0} w(\varphi(s))\,d_s[E^{(r)}_{k,k}(1)\,u_r(k,s)]+\int\limits_{0}^{\tau_r(t)}\delta(s)\,d_s[E^{(r)}_{k,k}(1)\,u_r(k,s)]\right)\right]\leqslant$$

$$\leqslant\frac{E^{(r)}_{t,k}(f)}{E^{(r)}_{0,k}(f)}\,G_r^{-1}\left[G_r\left(E^{(r)}_{0,k}(f)+E^{(r)}_{k,k}(1)\int\limits_{\tau_r(0)}^{0} w(\varphi(s))\,d_s u_r(k,s)\right)+E^{(r)}_{k,k}(1)\int\limits_{0}^{\tau_r(t)}\delta(s)\,d_s u_r(k,s)\right]=$$

$$=E^{(r+1)}_{t,k}(f)\,.$$

This completes the proof.

5. – Examples of applications.

Using the above inequalities, we can not only solve problems about the existence, uniqueness, continuous dependence on the initial values (functions), but also solve problems concerning boundedness and stability of the solutions of differential equations. Here we indicate only the latter case.

EXAMPLE 1. – Consider equation

$$\left(\frac{x(t)}{a(t)}\right)^{(m)}=f\big(t,\,x(\tau(t))\big) \tag{20}$$

where $f(t,u)\in C[[0,\infty)\times(-\infty,\infty)]$,

$$|f\big(t,\,x(\tau(t))\big)|\leqslant b(t)|x(\tau(t))|\,,$$

and $a(t)>0\in C^m[0,\infty)$, $b(t)\geqslant 0\in C[0,\infty)$ and $\tau(t)\in C^1[0,\infty)$, $\tau'(t)>0$, $\tau(t)\leqslant t$.

RESULT 1. – If there is a $M>0$ such that

$$t^{m-1}a(t)\leqslant M \quad\text{for } t\geqslant 0 \qquad\text{and}\qquad \lim_{t\to\infty}\int\limits_{\tau^{-1}(0)}^{t}\frac{(t-\sigma)^{m-1}}{(m-1)!}\,a\big(\tau(s)\big)\,b(s)\,ds\leqslant M\,, \tag{21}$$

then all solutions of eq. (20) are bounded, and the zero solution is stable.

PROOF. – Assume that $x(t)$ is a solution of eq. (15) which satisfies that $x(t) = \varphi(t)$ for $t \in [\tau(0), 0]$ and

$$\left(\frac{x(t)}{a(t)}\right)^{(i)}\Bigg|_{t=0} = C_i \quad (i = 1, 2, \ldots, m-1).$$

Then from (15) we have

$$x(t) = a(t)\sum_{i=1}^{m} \frac{C_{i-1}}{(i-1)!}\, t^{i-1} + a(t)\int_0^t \frac{(t-s)^{m-1}}{(m-1)!} f\big(s, x(\tau(s))\big)\, ds .$$

Hence

$$|x(t)| \leqslant a(t)\sum_{i=1}^{m} \frac{|C_{i-1}|}{(i-1)!}\, t^{i-1} + a(t)\int_0^t \frac{t-s)^{m-1}}{(m-1)!}\, b(s)|x(\tau(s))|\, ds .$$

Using Theorem 4, we get

$$|x(t)| \leqslant \left[a(t)\sum_{i=1}^{m} \frac{|C_{i-1}|}{(i-1)!}\, t^{i-1} + a(t)\int_0^{\tau^{-1}(0)} \frac{(t-s)^{m-1}}{(m-1)!}\, b(s)|\varphi(\tau(s))|\, ds\right]\cdot$$
$$\cdot \exp\left[\int_{\tau^{-1}(0)}^{t} \frac{(t-\sigma)^{m-1}}{(m-1)!}\, b(\sigma)a(\tau(\sigma))\, d\sigma\right].$$

Therefore, when (21) holds, $x(t)$ is obviously bounded. Furthermore for any $\varepsilon > 0$, if we let $|\varphi(t)| < \varepsilon$ for $t \in [\tau(0), 0]$, and $|C_i| < \varepsilon$ $(i = 1, 2, \ldots, n)$, then there is a $N > 0$ such that

$$|x(t)| \leqslant N\varepsilon ,$$

therefore the zero solution is table.

EXAMPLE 2. – Consider equation

$$x'' + \alpha(t)x' = f\big(t, x(\tau_1(t)), \ldots, x(\tau_n(t))\big) , \tag{22}$$

where $f(t, u_1, \ldots, u_n)$ is continuous on $R_+ \times R^n$ and

$$\big|f\big(t, x(\tau_1(t)), \ldots, x(\tau_n(t))\big)\big| \leqslant \beta(t) + \sum_{i=1}^{n} \gamma_i(t)x(\tau_i(t)) , \tag{23}$$

$\tau_i(t)$ $(i = 1, 2, \ldots, n) \in C^1[0, \infty)$, $\tau_i(t) \leqslant t$ and $\tau_1'(t) > 0$. Denote $\tau_*(t) = \min\limits_{1\leqslant i\leqslant n} \{\tau_i(l)\}$, $\tau^*(t) = \max\limits_{1\leqslant i\leqslant n} \{\tau_i(t)\}$. Assume that $\alpha(t)$, $\beta(t)$ and $\gamma_i(t)$ $(i = 1, 2, \ldots, n)$ are continuous on $[0, \infty)$.

RESULT 2. - If

$$\int_0^\infty \exp\left[-\int_0^s \alpha(r)\,dr\right] ds = M_1 < \infty\,,$$

$$\int_0^\infty ds \int_0^s |\beta(\sigma)| \exp\left[-\int_\sigma^s \alpha(r)\,dr\right] d\sigma = M_2 < \infty\,, \tag{24}$$

$$\lim_{t\to\infty} \int_0^t \left(|\gamma_i(s)| \int_s^t \exp\left[-\int_s^\sigma \alpha(r)\,dr\right] d\sigma\right) ds \leqslant M_3 < \infty\,,$$

then all solutions of eq. (22) are bounded, and if $\beta(t) \equiv 0$, then the zero solution is stable.

PROOF. - Equation (22) is equivalent to

$$\left(x' \exp\left[\int_0^t \alpha(s)\,ds\right]\right)' = \exp\left[\int_0^t \alpha(s)\,ds\right] f\left(t, x(\tau_1(t)), \ldots, x(\tau_n(t))\right). \tag{25}$$

Assume that $x(t)$ is a solution of eq. (17) which satisfies that $x(t) = \varphi(t)$ for $t \in [\tau_*(0), 0]$, and $x'(0) = c$. Then integrating both sides of (25) from 0 to t, we have

$$x' = c \exp\left[-\int_0^t \alpha(s)\,ds\right] + \int_0^t \exp\left[-\int_s^t \alpha(\sigma)\,d\sigma\right] f\left(s, x(\tau_1(s)), \ldots, x(\tau_n(s))\right) ds\,.$$

Integrating both sides from 0 to t once more, we get

$$x(t) = \varphi_0 + c\int_0^t \exp\left[-\int_0^s \alpha(r)\,dr\right] ds + \int_0^t ds \int_0^s \exp\left[-\int_\sigma^s \alpha(r)\,dr\right] f\left(\sigma, x(\tau_1(\sigma)), \ldots, x(\tau_n(\sigma))\right) d\sigma\,.$$

Let

$$f(t) = |\varphi_0| + |c|\int_0^t \exp\left[-\int_0^s \alpha(r)\,dr\right] ds + \int_0^t ds \int_0^s |\beta(\sigma)| \exp\left[-\int_\sigma^s \alpha(r)\,dr\right] d\sigma$$

$$h_i(t, s) = |\gamma_i(s)| \int_s^t \exp\left[-\int_s^\sigma \alpha(r)\,dr\right] d\sigma\,.$$

According to (23),

$$|x(t)| \leqslant f(t) + \sum_{i=1}^n \int_0^t h_i(t, s)\,|x(\tau_i(s))|\,ds\,.$$

If (24) holds, we can prove $B_i(1)$ $(i = 1, 2, \dots, n)$ defined as in Theorem 6 are bounded. In fact, $B_1(1) \leqslant 1$ is bounded. Suppose $|B_i(1)| \leqslant N_i$ for $1 \leqslant i \leqslant n$. Then

$$|B_{i+1}(1)| \leqslant |B_i(1)| \exp\left[\int_0^{\tau_i(t)} |B_i(1)| h_i(t, \tau_i^{-1}(s))(\tau_i^{-1}(s))'\, ds\right] \leqslant$$

$$\leqslant N_i \exp\left[N_i \int_{\tau_i^{-1}(0)}^{t} h_i(t, s)\, ds\right] \leqslant N_i \exp(N_i M_3) = N_{i+1} < \infty .$$

Because $f(t)$ and $H_i(t)$ $(i = 1, 2, \dots, n)$ are nondecreasing and bounded, according to Theorem 6, we get

$$|x(t)| \leqslant B_{n+1}(f) + \sum_{i=1}^{n} B_{n-i+1}(H_i) \leqslant B_{n+1}(1) f + \sum_{i=1}^{n} B_{n-i+1}(1) H_i ,$$

therefore $x(t)$ is bounded.

If $\beta(t) \equiv 0$, since $\int_0^{\tau_i^{-1}(0)} h_i(t, s)\, ds \leqslant M_3$. Then for any $\varepsilon > 0$, if we let $|\varphi(t)| < \varepsilon$ for $t \in [\tau_*(0), 0]$ and $|c| < \varepsilon$, then

$$|x(t)| \leqslant B_{n+1}(1) f + \sum_{i=1}^{n} B_{n-i+1}(1) H_i \leqslant N_{n+1}(1 + M_1)\varepsilon + \sum_{i=1}^{n} N_{n-i+1} M_3 \exp(M_3)\varepsilon .$$

Therefore, the zero solution is stable.

It appears difficult to obtain the results 1 and 2 without the integral inequalities in this paper.

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