Tangency and Differentiation: Some Applications of Convergence Theory (*).

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Summary. – *We present a unified approach based on convergence theory to approximating cones and generalized derivatives.*

A lot of research has been carried in the realm of tangency and differentiation. I neither intend or am capable to give a complete bibliograhical account of that work, but I rely on some existent accounts ([16], [22], [24], [25], [29], [35], [36]) limiting myself either to most recent or to most remote contributions.

Probably Severi was the first to consider derivatives of functions defined on non open sets [33]. He « differentiated functions along » approximating cones (contingents, paratingents) of the sets of definition. In [33] one also finds the idea (attributed to Guareschi) of defining derivatives of functions via approximating cones to the graphs of those functions. The theory had been developed contemporarily by Bouligand [1]. One finds related ideas e.g., in [25] and [35] and in their references.

One-sided derivatives (Dini derivatives and their generalizations) turned out to be adequate tools for optimization problems. Much attention over last few years was attracted by the generalized directional derivative of Clarke [5]. A conceptual turnover was to approach this derivative via appropriate approximating cones of the epigraphs of the considered functions (Hiriart - Urruty [16] [17], Rockafellar [28], [29], [30]).

Graphs and epigraphs are examples of relations (subsets of product spaces). Whether we consider a function as its graph or as its epigraph depends on the nature of the confronted problem that involves that function (Sometimes the zeal of one—sidedness goes so far as to attempts of the use of one-sided concepts to intrinsically double-sided problems).

Therefore one may look at differentiation and one-sided differentiation as at instances of tangency theory (theory of approximating cones to subsets of topological vector spaces). However this viewpoint does not exhaust all the aspects of differentiation. There are classical notions that make essential use of the fact that they

(*) Entrata in Redazione il 24 ottobre 1981.

concern relations (say in $X \times Y$): they admit non symmetric interplay between X and Y. Our ***Hadamard approximating cone*** embraces these notions.

We observe that the ***Hadamard derivative*** in first countable spaces [24] (see also [6] and the *Ursescu derivative* [35]) may be interpreted as the Hadamard approximating cone; also the *Severi derivatives* [33] carry a presentiment of this idea.

Our aim in this paper is to present tangency theory based on the study of the relation we call ***homothety***, which plays a role analogous to that of difference quotient in differentiation.

In particular, the homothety of the epigraph is the epigraph of the difference quotient, that of the graph is the graph of the difference quotient. This is the clue to unifying tangency and differentiation.

The concept of homothety has been used before to similar ends (without being named) by ROCKAFELLAR [28] [29] [30], PENOT [25] [26], HIRIART-URRUTY [16], but we attempt to exploit in its respect (in a more systematic way) the consequences of convergence theory. Much work can be avoided by choosing appropriate topologies and filters to work with. For instance the radial variants of approximating cones result from the choice of the discrete topology. This attitude allows us to argue at a higher level of abstraction of several reasonigs. There is no need of convincing about conceptual gain of treating the theories of convergence and tangency simultaneously. We point out that CHOQUET developed convergence theory [4] with, in mind, applications to tangency theory. But already BOULIGAND interpreted his contingents and paratingents as upper limits (ensembles d'accumulation) [1].

One of the principal problems of convergence theory is that of finding conditions under which a limit of intersection includes the intersection of limits. This question may be formulated in terms of convergence of functions [9] [30] [3]. We provide a sufficient condition for lower limits (akin to the one used for another purpose in [11]) and apply it to homotheties thus obtaining, among otherr esults, a ROCKAFELLAR's condition for $\mathcal{C}_{C \cap D} \supset \mathcal{C}_C(x) \cap \mathcal{C}_D(x)$ to hold [30] ($\mathcal{C}_C(x)$ being, in our terminology the hypertangent of C at x).

We somewhat refine a theory of equi-semicontinuity, recently developed by DOLECKI, SALINETTI and WETS [14] and apply it to give a sufficient condition for the directional derivatives of Clarke and Rockafellar to be equal. Another such condition was proposed by ROCKAFELLAR [29]. We indicate a class of functions that satisfy our condition but not that of [29]. We also extend the theory of [14] to relations and to general *Γ-limits of De Giorgi.*

There are known connections between Γ- and G-limits and the classical *Kuratowski limits* ([7] [9] [8] [2]; see also [37] [20] [23] [14]). We introduce hyperlimits, show their equivalence to some *G-limits* and show that they constitute the type of convergence of homotheties (difference quotients) that gives rise to the Hadamard derivative.

We do not explore much uniform convergences which for homotheties correspond to Fréchet differentiability and do not stress the uniform character (uniform on compact sets) of Hadamard differentiability.

CONVERGENCE

1. – Limits of families of sets.

We recollect some basic facts about convergences and add some others that are either new or not widely known. The exposition is intended to be in the vein of the fundamental paper of CHOQUET [4].

We are primarily concerned with the concepts of upper and lower limits (said *of Kuratowski*).

Let $\mathcal{F}$ be a filter in a set I and let $\boldsymbol{A} = \{A_i\}_{i\in I}$ be a family of subsets of a set X. Equip X with a topology τ. The *upper limit* of $\{A_i\}_{i\in I}$ (filtered by $\mathcal{F}$, with respect to τ) is defined by

$$\mathrm{Ls}\,\boldsymbol{A} = \mathrm{Ls}^{\tau}_{\mathcal{F}} A_i = \bigcap_{F\in\mathcal{F}} \mathrm{Cl}_\tau \bigcup_{i\in F} A_i\,. \tag{1.1}$$

The upper limit is equal to the set of cluster point of the filter $\boldsymbol{A}_{\mathcal{F}}$ generated by $\left\{A_F = \bigcup_{i\in F} A_i,\ F\in\mathcal{F}\right\}$. If the upper limit is a subset of a set A, we say that $\{A_i\}_{i\in I}$ *subconverges* to A.

Recall that the grill $\ddot{\mathcal{F}}$ of a family $\mathcal{F}$ of subsets of I consists of all these subsets of I which meet every member of $\mathcal{F}$. The *lower limit* (of $\{A_i\}_{i\in I}$) is, by definition,

$$\mathrm{Li}\,\boldsymbol{A} = \mathrm{Li}^{\tau}_{\mathcal{F}} A_i = \bigcap_{F\in\ddot{\mathcal{F}}} \mathrm{Cl}_\tau \bigcup_{i\in F} A_i\,. \tag{1.2}$$

When all the sets A_i are singletons, then the lower limit is equal to the limit of the filter $\boldsymbol{A}_{\mathcal{F}}$ (see below). If A is a subset of the lower limit, we say that $\{A_i\}_{i\in I}$ *superconverges* to A.

We shall also write $\mathrm{Li}_{i\to i_0}$ $(\mathrm{Ls}_{i\to i_0})$, when $\mathcal{F}$ will be a neighborhood filter of i_0.

Obviously, both limits are τ-closed and if we consider different topologies on X, then the weaker the topology, the larger is the limit. Since $\mathcal{F}$ is a subfamily of $\ddot{\mathcal{F}}$, the lower limit is a subset of the upper limit. Finally the coarser the filter the larger is the upper limit, but the smaller the lower limit.

It follows from the definitions that $x\in \mathrm{Ls}\,A_i$, iff for every $Q\in\mathcal{N}(x)$ every $F\in\mathcal{F}$ there is $i\in F$ such that $Q\cap A_i\neq\emptyset$; due to the duality of filters and their grills, if for every $Q\in\mathcal{N}(x)$ there is $H\in\ddot{\mathcal{F}}$ such that $Q\cap A_i\neq\emptyset$ for each $i\in H$.

An $x\in\mathrm{Li}\,A_i$, iff for every $Q\in\mathcal{N}(x)$ and every $H\in\ddot{\mathcal{F}}$ there is $i\in H$ such that $Q\cap A_i\neq\emptyset$. Dually, if for every $Q\in\mathcal{N}(x)$ there is $F\in\mathcal{F}$ such that for each $i\in F$ $A_i\cap Q\neq\emptyset$.

PROPOSITION 1.1. – (KURATOWSKI [20] for sequences, DENKOWSKI [10] for nets)

$$\mathrm{Li}_{\mathcal{F}} A_i = \bigcap_{H\in\ddot{\mathcal{F}}} \mathrm{Ls}_{H\cap\mathcal{F}} A_i\,.$$

PROOF. – An x belongs to the right-hand side, whenever for each H in $\ddot{\mathcal{F}}$ every F from $\mathcal{F}$ and every Q in $\mathcal{N}(x)$, $A_{H \cap F} \cap Q \neq \emptyset$. Thus $A_H \cap Q \neq \emptyset$ and $x \in \mathrm{Li}_{\mathcal{F}} A_i$. On the other hand, $H \cap F$ is an element of $\ddot{\mathcal{F}}$, hence the opposite inclusion is valid.

Of course, a family $\{A_i\}_{i \in I}$ constitutes a relation in $I \times X$ and by standard convention we have that $A^{-1}Q = \{i: A_i \cap Q \neq \emptyset\}$. On using this language, we obtain that $x \in \mathrm{Li}\, A_i$, if and only if for every $Q \in \mathcal{N}(x)$ there is $F \in \mathcal{F}$ such that $A^{-1}Q \supset F$.

PROPOSITION 1.2.

$$\mathrm{Ls}_{\mathcal{F}} A_i = \bigcup_{\mathcal{G} \supset \mathcal{F}} \mathrm{Li}_{\mathcal{G}} A_i .$$

PROOF. – Always $\mathrm{Li}_{\mathcal{G}} A_i \subset \mathrm{Ls}_{\mathcal{G}} A_i \subset \mathrm{Ls}_{\mathcal{F}} A_i$.

Let $x \in \mathrm{Ls}_{\mathcal{F}} A_i$: for every $Q \in \mathcal{N}(x)$ and every $F \in \mathcal{F}$, $Q \cap A_F \neq \emptyset$, that is $A^{-1}Q \cap \cap F \neq \emptyset$. As a result there is a filter $\mathcal{G}$ finer than both $A^{-1}\mathcal{N}(x)$ and $\mathcal{F}$. Consequently for every $Q \in \mathcal{N}(x)$ there is $G \in \mathcal{G}$ such that $Q \cap A_i \neq \emptyset$ for $i \in G$. This means that $x \in \mathrm{Li}_{\mathcal{G}} A_i$. If we consider sequences in metric spaces we may also put ([20])

$$\mathrm{Ls}_{\mathcal{F}} A_i = \bigcup_{H \in \ddot{\mathcal{F}}} \mathrm{Li}_{\mathcal{F} \cap H} A_i .$$

PROPOSITION 1.3. – The upper limit with respect to the infimum of topologies θ and τ is equal to the union of the upper limits with respect to θ and to τ.

Analogously

PROPOSITION 1.4. – The upper limit of a family filtered by the infimum of filters $\mathcal{F}$ and $\mathcal{H}$ is equal to the union of the upper limits filtered by $\mathcal{F}$ and $\mathcal{H}$.

But

PROPOSITION 1.5. – The lower limit filtered by the infimum of $\mathcal{F}$ and $\mathcal{H}$ is equal to the intersection of the lower limits corresponding to $\mathcal{F}$ and $\mathcal{H}$.

Consider now the particular case when a family of subsets of X is indexed by the product $I \times Z$. Let $\mathcal{F}$ be a filter in I and $\mathcal{H}$ be a filter of subsets of Z containing ζ_0. Then

$$\mathrm{Ls}_{\mathcal{F} \times \mathcal{H}} A_{i\zeta} \supset \mathrm{Ls}_{\mathcal{F}} A_{i\zeta_0}, \qquad \mathrm{Li}_{\mathcal{F} \times \mathcal{H}} A_{i\zeta} \subset \mathrm{Li}_{\mathcal{F}} A_{i\zeta_0}, \tag{1.3}$$

as the right-hand-side limits are, in fact, filtered by $\mathcal{F} \times \mathcal{N}_\iota(\zeta_0)$—the discrete neighborhood filter of ζ_0.

When X is equipped with the discrete topology the discussed limits become set-theoretical

$$\mathrm{Ls}^\iota_{\mathcal{F}} A_i = \bigcap_{F \in \mathcal{F}} \bigcup_{i \in F} A_i, \qquad \mathrm{Li}^\iota_{\mathcal{F}} A_i = \bigcap_{H \in \ddot{\mathcal{F}}} \bigcup A_i . \tag{1.4}$$

The duality of filters and grills yields a more familiar representation for the lower limit

$$\mathrm{Li}^{\tau}_{\mathcal{F}} A_i = \bigcup_{F\in\mathcal{F}} \bigcap_{i\in F} A_i .$$

If a topology τ is first countable and a filter $\mathcal{F}$ in I has a countable base, then an x is in $\mathrm{Ls}^{\tau}_{\mathcal{F}} A_i$, if and only if there are sequences $\{x_n\}$ tending to x and $\{i_n\}$ being $\mathcal{F}$-convergent for which $x_n \in A_{i_n}$. In this case, the following statements are equivalent:

(i) $x \in \mathrm{Li}^{\tau}_{\mathcal{F}} A_i$;

(1.4) (ii) for every $\mathcal{F}$-convergent $\{i_n\}_{n=1}^{\infty}$ there is a sequence $\{x_n\}_{n=1}^{\infty}$ convergent to x and such that $x_n \in A_{i_n}$.

(iii) there exists a function $m: I \to X$ such that $m(i) \in A_i$ for $i \in I$ and for each $Q \in \mathcal{N}(x)$ there is $F \in \mathcal{F}$ with $m(F) \subset Q$.

Limits constitute isotone operations: if $A_i \subset B_i$ for each i, then $\mathrm{Li}\, A_i \subset \mathrm{Li}\, B_i$ and $\mathrm{Ls}\, A_i \subset \mathrm{Ls}\, B_i$, as easily follows from the definitions. Consequently if $\{\{A_{ij}\}_{i\in I}\}_{j\in J}$ is a class of families of sets filtered by $\mathcal{F}$, then

$$\mathrm{Li} \bigcap_{j\in J} A_{ji} \subset \bigcap_{j\in J} \mathrm{Li}\, A_{ji}, \quad \mathrm{Ls} \bigcap_{j\in J} A_{ji} \subset \bigcap_{j\in J} \mathrm{Ls}\, A_{ji},$$

$$\bigcup_{j\in J} \mathrm{Li}\, A_{ji} \subset \mathrm{Li} \bigcup_{j\in J} A_{ji}, \quad \bigcup_{j\in J} \mathrm{Ls}\, A_{ji} \subset \mathrm{Ls} \bigcup_{j\in J} A_{ji}. \tag{1.5}$$

We have also

PROPOSITION 1.6. – Let $\{A_i\}_{i\in I}$, $\{B_i\}_{i\in I}$ be families of subsets of X.

(i) $\mathrm{Ls}\,(A_i \cup B_i) = \mathrm{Ls}\, A_i \cup \mathrm{Ls}\, A_i$;

(ii) $\mathrm{Li}\,(A_i \cup B_i) \subset \mathrm{Li}\, A_i \cup \mathrm{Li}\, B_i \cup (\mathrm{Ls}\, A_i \cap \mathrm{Ls}\, B_i)$;

(iii) $\mathrm{Li}\,(A_i \cup B_i) \subset \mathrm{Li}\, A_i \cup \mathrm{Ls}\, B_i$.

PROOF. – (i) Standard transformations lead to $\mathrm{Ls}_{\mathcal{F}}(A_i \cup B_i) = \bigcap_{F\in\mathcal{F}} (\mathrm{Cl}\, A_F \cup \mathrm{Cl}\, B_F)$ and $\mathrm{Ls}_{\mathcal{F}}\, A_i \cup \mathrm{Ls}_{\mathcal{F}}\, B_i = \bigcap_{,FF'\in\mathcal{F}} (\mathrm{Cl}\, A_F \cup \mathrm{Cl}\, B_F)$ and since $F \cap F' \in \mathcal{F}$, the above expressions are equal.

(ii) Suppose that x belongs to $\mathrm{Li}(A_i \cup B_i)$ but not to $\mathrm{Ls}\, A_i \cap \mathrm{Ls}\, B_i$, say is not in $\mathrm{Ls}\, A_i$. Accordingly for every $Q \in \mathcal{N}(x)$ there is $F \in \mathcal{F}$ such that $Q \cap (A_i \cup \cup B_i) \neq \emptyset$ for $i \in F$ and, on the other hand, there is $Q' \in \mathcal{N}(x)$ and $F' \in \mathcal{F}$ for which $Q' \cap A_i = \emptyset$ if $i \in F'$. We conclude that $Q \cap B_i \neq \emptyset$ for $i \in F \cap F'$, that is $x \in \mathrm{Li}\, B_i$. (ii) implies (iii).

KURATOWSKI [20] proved (ii) for sequences in metric spaces.

It follows from (1.5) that the upper (lower) limit of the boundaries $\{\mathrm{Fr}\, A_i\}_{i\in I}$ is included in the intersection of the upper (lower) limits of $\{A_i\}_{i\in I}$ and $\{A_i^c\}_{i\in I}$.

If the space is connected

$$\mathrm{Li}\,\mathrm{Fr}\, A_i = \mathrm{Li}\, A_i \cap \mathrm{Li}\, A_i^c\,. \tag{1.6}$$

The question under what conditions the limits of the (finite) intersection become equal to the intersection of the limits is one of the central problems of convergence theory. We shall provide a sufficient condition for lower limits.

PROPOSITION 1.7. – Let $\{A_i\}_{i\in I}$, $\{B_i\}_{i\in I}$ families filtered by $\mathcal{F}$ be *equi attracted*, that is such that for every x in X every $Q \in \mathcal{N}(x)$ there are $W \in \mathcal{N}(x)$ and $F \in \mathcal{F}$ such that for $i \in F$

$$\text{if } W \cap A_i \neq \emptyset \text{ and } W \cap B_i \neq \emptyset, \text{ then } Q \cap A_i \cap B_i \neq \emptyset. \tag{1.7}$$

Then

$$\mathrm{Li}\,(A_i \cap B_i) \supset \mathrm{Li}\, A_i \cap \mathrm{Li}\, B_i$$

thus the inclusion is actually the equality.

PROOF. – Let x be in $\mathrm{Li}\, A_i \cap \mathrm{Li}\, B_i$ and let $Q \in \mathcal{N}(x)$. Take $W \in \mathcal{N}(x)$ and $F \in \mathcal{F}$ which satisfy (1.7). Since x belongs to $\mathrm{Li}\, A_i \cap \mathrm{Li}\, B_i$, there is $F' \subset F$ such that both $W \cap A_i \neq \emptyset$ and $W \cap B_i \neq \emptyset$. By (1.7), x is in $\mathrm{Li}\,(A_i \cap B_i)$.

The condition of Proposition 1.7 is satisfied, if $\{A_i\}_{i\in I}$ $\{B_i\}_{i\in I}$ are families of subsets of a uniform space $(X, \mathcal{U})$, which *separate equi-decisively*: for every $U \in \mathcal{U}$ there are $V \in \mathcal{U}$ and $F \in \mathcal{F}$ such that $V(A_i) \cap V(B_i) \subset U(A_i \cap B_i)$ as $i \in F$.

We shall give now a special result to be used in Section 5. We shall call the *interior* limit of $\{A_i\}_{i\in I}$ (in (X, τ) filtered by $\mathcal{F}$) the set

$$(\mathrm{Ls}_{\mathcal{F}}^{\tau}\, A_i^c)^c = \bigcup_{F\in\mathcal{F}} \mathrm{Int}_\tau \bigcap_{i\in F} A_i\,. \tag{1.8}$$

We note that the interior limit is included in the interior of the lower limit with respect to the discrete topology.

PROPOSITION 1.8.

$$\mathrm{Li}\,(A_i \cap B_i) \supset (\mathrm{Ls}\, A_i^c)^c \cap \mathrm{Li}\, B_i\,.$$

PROOF. – If h is in $(\mathrm{Ls}\, A_i^c)^c \cap \mathrm{Li}\, B_i$, then on one hand, there is $Q_0 \in \mathcal{N}(h)$ and $F \in \mathcal{F}$ such that for every $i \in F$, $Q_0 \subset A_i$ and on the other for every $Q \in \mathcal{N}(h)$,

$Q_0 \subset Q_0$) there is $F' \in \mathcal{F}$, $(F' \subset F)$, such that $Q \cap B_i \neq \emptyset$ as $i \in F'$. Thus $Q \cap A_i \cap B_i \neq \emptyset$ as $i \in F'$ and therefore h belongs to $\mathrm{Li}\,(A_i \cap B_i)$.

We digress to say that a family $\{A_i\}_{i\in I}$ *uniformly superconverges* to A, if for every $U \in \mathcal{U}$ there exists $F \in \mathcal{F}$ such that $U(A_i) \supset A$ for $i \in F$. This amounts to the condition that for every $U \in \mathcal{U}$ there is $F \in \mathcal{F}$ such that for each $i \in F$ and every x in A, $A_i \cap U(x) \neq \emptyset$.

We conclude by a very special but useful case of families $\{A_t\}_{t>0}$ such that $t' \geqslant t''$ implies $A_{t'} \subset A_{t''}$. Then

$$\mathrm{Li}_{t\to 0} A_t = \mathrm{Ls}_{t\to 0} A_t = \mathrm{Cl} \bigcup_{t>0} A_t \,. \tag{1.9}$$

2. – Families of relations and functions.

Of particular interest are families of subsets of product spaces, say $X \times Y$. Then, of course, the sets constitute relations (or, if one prefers, multifunctions) from X into Y. Special cases are furnished by families of mappings, epigraphs and hypographs.

Let $\{f_i\}_{i\in I}$ be a family of extended-real-valued functions on a topological space (X, τ) and let $\mathcal{F}$ be a filter in I. Consider limits of the epigraphs of this family: $\mathrm{epi}\ f_i = \{(x, r) \colon r \geqslant f_i(x)\}$ (which are epigraphs).

Define the *limit inferior* of $\{f_i\}$ by

$$(\mathrm{li}^\tau_{\mathcal{F}} f_i)(x) = \sup_{Q\in\mathcal{N}_\tau(x)} \ \sup_{F\in\mathcal{F}} \ \inf_{i\in F} \ \inf_{w\in Q} f_i(w) \tag{2.1}$$

and the *limit superior* as

$$(\mathrm{ls}^\tau_{\mathcal{F}} f_i)(x) = \sup_{Q\in\mathcal{N}_\tau(x)} \ \inf_{F\in\mathcal{F}} \ \sup_{i\in F} \ \inf_{w\in Q} f_i(w) \,. \tag{2.2}$$

Then we have that

$$\mathrm{Ls}^{\tau\times\nu}(\mathrm{epi}\, f_i) = \mathrm{epi}\,(\mathrm{li}^\tau f_i) \,, \qquad \mathrm{Li}^{\tau\times\nu}(\mathrm{epi}\, f_i) = \mathrm{epi}\,(\mathrm{ls}^\tau f_i) \tag{2.3}$$

where ν is the natural topology of R (see e.g. [8] [14]).

If $\mathrm{li}\, f_i \geqslant f$, we say that the family $\{f_i\}_{i\in I}$ *subconverges* to f; when $\mathrm{ls}\, f_i \leqslant f$, we say that it *superconverges* to f.

Treated jointly (2.1) and (2.2) form the *infimal limit* of WIJSMAN [37] and (2.3) has been essentially recognized there [37, Thm. 6.1], for $X = R^n$. Separately they appear in [19] by JOLY where one finds (2.3); see also MOSCO [23].

On the other hand the limits (2.1) (2.2) are examples of Γ-limits of DE GIORGI and FRANZONI [9] [7] and (2.3) is reflected in the relationship between Γ-limits and G-limits of DE GIORGI [8].

Analogous facts for hypographs (also special cases of the theory of Γ and G limits) have been established by BUTTAZZO [2].

Let (X, τ), (Y, σ) be topological spaces and $\{A_i\}$ be a family of relations from X into Y we note that

$$(2.4) \qquad \mathfrak{D}(\mathrm{Li} A_i) \subset \mathrm{Li}\, \mathfrak{D}(A_i)\,, \quad \mathfrak{D}(\mathrm{Ls}\, A_i) \subset \mathrm{Ls}\, \mathfrak{D}(A_i)$$

where $\mathfrak{D}(A)$ is the domain of A $(= \{x: Ax \neq \emptyset\})$ or, in another terminology, the projection of A into X. This follows most easily from, e.g., $\mathrm{Li}^{\tau\times o} A_i = \mathrm{Li}^{\tau}\, \mathfrak{D}(A_i) \times Y$ where o is the chaotic topology, because $\tau \times o \subset \tau \times \sigma$.

Observe that (2.4) applied to relations of type $\{(\mathrm{epi}\, f_i)^{-1}\}^{i\in I}$ implies that ([2, Prop. 2.7.])

$$(2.5) \qquad \inf(\mathrm{ls}^{\tau} f_i) \geqslant \overline{\lim}\,(\inf f_i)\,, \quad \inf(\mathrm{li}^{\tau} f_i) \geqslant \underline{\lim}\,(\inf f_i)$$

because $\mathrm{Cl}\, \mathfrak{D}(\mathrm{epi}\, f)^{-1} = [\inf f, \infty)$. These inequalities must not be inverted in general; the equality in the latter requires a very « stable » behavior of functions (Dolecki-Rolewicz [13], Joly [19]).

Suppose that $\{A_i\}_{i\in I}$ subconverges (superconverges) to A. When does the family $\{A_i x\}_{i\in I}$ sub- (super-) converge to Ax for a given $x \in X$? We observe that σ-subconvergence (σ-superconvergence) of $\{A_i x\}$ to Ax for every x, amounts to the corresponding convergence of $\{A_i\}_{i\in I}$ to A in the product topology $\iota\times\sigma$, where ι stands, as usual, for the discrete topology. In other words, we are interested in « pointwise convergence » of relations. Since $\iota\times\sigma$ is finer than $\tau\times\sigma$, we always have

$$\mathrm{Li}^{\sigma} A_i x \subset (\mathrm{Li}^{\tau\times\sigma} A_i)\, x\,, \quad Ls^{\sigma} A_i x \subset (\mathrm{Ls}^{\tau\times\sigma} A_i)\, x\,.$$

Therefore the subconvergence of $\{A_i\}_{i\in I}$ to A implies the subconvergence of $\{A_i x\}_{i\in I}$ to Ax for each x. But this is not true about superconvergence.

We shall give later a general condition for the equality of limits with respect to various topologies and filters. Here we shall consider the special case of relations $\{(\mathrm{epi}\, f_i)^{-1}\}_{i\in I}$ for extended-real-valued function on X. Let $r_0 \in R$. Then the above becomes the problem of convergence of the level sets

$$(2.6) \qquad (\mathrm{epi}\, f_i)^{-1}(r_0) = \{x: f_i(x) \leqslant r_0\}\,.$$

On specializing the definitions we obtain

Proposition 2.1. – The level sets (2.6) superconverge to $(\mathrm{epi}\, f)^{-1} r_0$, if for each x such that $f(x) \leqslant r_0$ and for every $W \in \mathcal{N}(x)$ there is $F \in \mathcal{F}$ such that $\inf_{w\in W} f_i(w) < r_0$ as $i \in F$.

Note that if f is a convex function, $r_0 > \inf f$ and $\mathrm{ls}\, f_i = f$, then the above is:

(2.7) for every x such that $f(x) \leqslant r_0$ for every $W \in \mathcal{N}(x)$ there is $\varepsilon > 0$ and $F \in \mathcal{F}$ such that $\inf_{w\in W} f_i(w) \leqslant r_0 - \varepsilon$ as $i \in F$.

Indeed, by the convexity of f, for every $W \in \mathcal{N}(x)$ there is $\varepsilon > 0$ such that $\inf_{w \in W} f(w) < r_0 - \varepsilon$. Take an x' from W for which $f(x') < r_0 - \varepsilon$ and by superconvergence find $F \in \mathcal{F}$ such that for $i \in F$, $\inf_{w \in W'} f_i(w) \leqslant f(x') + \varepsilon/2$.

COROLLARY (WIJSMAN [37, Thm. 7.1]). – If $\{f_n\}$ (infimally) converges on R^k to a convex function f and $\inf f < r_0$, then the level sets $(\operatorname{epi} f_i)^{-1} r_0$ converge (that is both subconverge and superconverge) to $(\operatorname{epi} f)^{-1} r_0$.

It is interesting that Condition (2.7) is a specialization of a notion of convergence of relations. The ***hyperlimit*** of $\{A_i\}_{i \in I}$ is the set of all these pairs (x, y) which satisfy: for every $V \in \mathcal{N}_\sigma(y)$ there are $Q \in \mathcal{N}_\tau(x)$ and $F \in \mathcal{F}$ such that $V \cap A_i x' \neq \emptyset$ as $x' \in Q$ and $i \in F$. The hyperlimit

$$\operatorname{Lh}_{\mathcal{F}}^{\tau/\sigma} A_i$$

is a subset of $\operatorname{Li}_{\mathcal{F}}^{\iota \times \sigma} A_\iota = \operatorname{Lh}_{\mathcal{F}}^{\iota/\sigma} A_i$, since we may regard hyperlimits as lower limits of $\{A_i x'\}_{i \in I, x' \in X}$.

$$(\operatorname{Lh}_{\mathcal{F}}^{\tau/\sigma} A_i)\, x = \operatorname{Li}_{\mathcal{F} \times \mathcal{N}_\tau(x)}^{\sigma} A_i x' \,. \tag{2.8}$$

Hyperlimits are not closed in general. When the filter is discrete (or in other words, a family of relations reduces to one relation A), $\operatorname{Lh}^{\tau/\sigma} A$ is equal to the set of lower-semicontinuity points of A.

It is straightforward, that (2.7) holds, if and only if (x, r_0) is in the hyperlimit $(\operatorname{epi} f)^{-1}$ of $\{(\operatorname{epi} f_i)^{-1}\}_{i \in I}$.

Hyperconvergence provides a sufficient condition for the superconvergence of $\{A_i x_i\}$ to Ax, where both $\{A_i\}$ and $\{x_i\}$ are filtered by $\mathcal{F}$ and x_i tends to x.

PROPOSITION 2.2. – Let $\{A_i\}_{i \in I}$ hyperconverge to A and let $\{x_i\}_{i \in I}$ converge to x. Then $\{A_i\, x_i\}_{i \in I}$ superconverges to Ax.

It is instructive to think about the special case of the above scheme in which $\{A\}_{i \in I}$ is a constant family of relations. Then the subconvergence of $\{Ax_i\}_{i \in I}$ to Ax for every $\{x_i\}_{i \in I}$ coresponds to the graph-closedness of A at x, while the superconvergence amounts to the lower semicontinuity of A at x. These properties may occur locally (on subsets of Ax).

A classical example is furnished by fibers of a mapping $F: X \to Y$. The assumption of the Lusternik theorem guarantees locally both the superconvergence and subconvergence of the values of the relation $F^{-1}: Y \to X$.

We equip X with a topology τ and Z with a uniformity $\mathcal{U}$. It will be instrumental to observe

PROPOSITION 2.3. – A pair (x, z) is in $\operatorname{Ls}_{\mathcal{F}}^{\tau \times \mathcal{U}} A_i$, if and only of for every U in $\mathcal{U}$ every Q in $\mathcal{N}_\tau(x)$ and each F from $\mathcal{F}$

$$z \in U(A_F Q)\,.$$

A pair (x, z) lies in $\mathrm{Li}_{\mathcal{F}}^{\tau\times\mathcal{U}} A_i$, if and only if for every U in $\mathcal{U}$ and each Q in $\mathcal{N}_\tau(x)$ there is F in $\mathcal{F}$ such that

$$z \in U(A_i Q)\,, \qquad i \in F.$$

3. – Comparison of limits.

In this section we provide conditions for the equality of upper (and lower) limits when considered with respect to various topologies. Our condition of quasi (τ/σ)-equi semicontinuity for families of relations is an extension and refinement of (τ/σ)-equi semicontinuity of DOLECKI, SALINETTI and WETS [14] for families of functions and our results extend (to relations) and slightly refine the analogous ones therein.

Let $\{A_i\}_{i\in I}$ be a family of relations in $X\times Z$ filtered by $\mathcal{F}$ in Y. We consider a uniformity $\mathcal{U}$ in Z and topologies τ, σ in X.

$\{A_i\}$ is said to be *quasi* (τ/σ)*-equi semicontinuous at* x, if for every $U \in \mathcal{U}$ every $W \in \mathcal{N}_\sigma(x)$ there are $V \in \mathcal{U}$, $Q \in \mathcal{N}_\tau(x)$ and $F \in \mathcal{F}$ such that

$$U(A_i W) \supset V(A_i Q)\,, \qquad i \in F. \tag{3.1}$$

This condition is akin to the definition of hyperconvergence, but has the advantage of being expressed in terms of the family $\{A_i\}$ not of the limiting relation.

In the case in which A_i are epigraphs of functions f_i the above condition becomes: for every ε and every $W \in \mathcal{N}_\sigma(x)$ there are δ and $Q \in \mathcal{N}_\tau(x)$ and $F \in \mathcal{F}$ such that for each i in F

$$\inf_{v\in Q} f_i(v) - \delta \geqslant \inf_{w\in W} f_i(w) - \varepsilon\,, \tag{3.2}$$

which is equivalent to (for some other ε)

$$\inf_{v\in Q} f_i(v) \geqslant \inf_{w\in W} f_i(w) - \varepsilon\,. \tag{3.3}$$

When the topology σ is discrete W is substituted by $\{x\}$ and the infimum of the right-hand side by $f_i(x)$ (BUTTAZZO [2, Prop. 2.1]).

In [14] a family of functions $\{f_i\}$ is said to be (τ/σ)*-equi lower semicontinuous*, if there is a subset D of X (reference set) such that $\{\mathrm{epi}\, f_i\}$ is quasi (τ/σ)-equi semicontinuous at every x in $\mathcal{C}$ and for each $x \notin D$, for every M there are $Q \in \mathcal{N}_\tau(x)$ and $F \in \mathcal{F}$ with

$$\inf_{v\in Q} f_i(v) \geqslant M\,, \qquad i \in F.$$

PROPOSITION 3.1. – A filtered family $\{f_i\}$ is (τ/σ)-equi semicontinuous, if and only if $\{\mathrm{epi}\, f_i\}$ is quasi (τ/σ)-equi semicontinuous at each x of

$$\mathcal{D}(\mathrm{li}^\tau f_i) = \mathcal{D}(\mathrm{Ls}^{\tau\times\nu}\, \mathrm{epi}\, f_i)\,.$$

PROOF. – Observe that the latter condition in the definition of (τ/σ)-equi lower semicontinuity amounts to the requirement that if $x \notin D$ then $x \notin \mathfrak{D}(\mathrm{li}^{\tau} f_i) = \{x: (\mathrm{li}^{\tau} f_i)(x) < \infty\}$.

We recall that the coarser the topology the larger are the upper and lower limits.

THEOREM 3.1. – If $\{A_i\}$ is quasi (τ/σ)-semicontinuous at x and $(x, z) \in \mathrm{Li}^{\tau\times\mathcal{U}} A_i$, then $(x, z) \in \mathrm{Li}^{\sigma\times\mathcal{U}} A_i$.

PROOF. – Let (x, z) be in $\mathrm{Li}_{\mathcal{F}}^{\tau\times\mathcal{U}} A_i$: for every $V \in \mathcal{U}$ every $Q \in \mathcal{N}_{\tau}(x)$ there is $F \in \mathcal{F}$ so that for each i in F, $z \in V(A_i Q)$. Take any U from $\mathcal{U}$ and $W \in \mathcal{N}_{\sigma}(x)$. Then, by quasi (τ/σ)-equi semicontinuity, there are $V \in \mathcal{U}$, $Q \in \mathcal{N}_{\tau}(x)$ and $F' \in \mathcal{F}$ such that (3.1) holds. On taking V and Q from the former condition, we have that $z \in U(A_i W)$ as $i \in F \cap F'$, thus (x, z) is in $\mathrm{Li}_{\mathcal{F}}^{\sigma\times\mathcal{U}} A_i$ by virtue of Proposition 2.3.

THEOREM 3.2. – Let $\{A_i\}$ be quasi (τ/σ)-equi semicontinuous at x and $(x, z) \ni \in \mathrm{Ls}^{\tau\times\mathcal{U}} A_i$; then $(x, z) \in \mathrm{Ls}^{\sigma\times\mathcal{U}} A_i$.

PROOF. – An (x, z) is in $Ls_{\mathcal{F}}^{\tau\times\mathcal{U}} A_i$, whenever for every $V \in \mathcal{U}$ every $Q \in \mathcal{N}_{\tau}(x)$ there is $H \in \ddot{\mathcal{F}}$ such that for each $i \in H$, $z \in V(A_i Q)$. By our assumption, for every $U \in \mathcal{U}$ and every $W \in \mathcal{N}_{\sigma}(x)$ there is $F \in \mathcal{F}$ such that $z \in U(A_i W)$ as $i \in H \cap F$. Since $H \cap F$ is in the grill of $\mathcal{F}$, the proof is completed.

THEOREM 3.3. – Suppose that there are $V_0 \in \mathcal{U}$, $Q_0 \in \mathcal{N}_{\tau}(x)$ and $F \in \mathcal{F}$ such that

$$\mathrm{Cl}(V_0(A_F Q_0) \setminus \mathrm{Li}_{\mathcal{F}}^{\sigma\times\mathcal{U}} A_i)$$

is compact. If

$$\mathrm{Ls}_{\mathcal{F}}^{\tau\times\mathcal{U}} A_i \subset \mathrm{Li}_{\mathcal{F}}^{\sigma\times\mathcal{U}} A_i\,,$$

then $\{A_i\}$ is quasi (τ/σ)-equi semicontinuous on $\mathfrak{D}(\mathrm{Ls}_{\mathcal{F}}^{\tau\times\mathcal{U}} A_i)$.

PROOF. – Suppose that the conclusion does not hold: there are $U \in \mathcal{U}$, x in the domain of $\mathrm{Ls}_{\mathcal{F}}^{\tau\times\mathcal{U}} A_i$ and $W \in \mathcal{N}_{\sigma}(x)$ such that for every $V \in \mathcal{U}$ and $Q \in \mathcal{N}_{\tau}(x)$ and each F there is i with

$$Z \setminus U(A_i W) \cap V(A_i Q) \neq \emptyset\,. \tag{3.4}$$

Choose $z(V, Q, F)$ from (3.4). The net $\{z(V, Q, F): (V, Q, F) \in \mathcal{U} \times \mathcal{N}_{\tau}(x) \times \mathcal{F}\}$ is disjoint from $U(\mathrm{Li}_{\mathcal{F}}^{\sigma\times\mathcal{U}} A_i)$ and, by compactness, has a cluster point z. Denoting by $Z(V, Q, F)$ the tail of the discussed net, we have that for every $V \in \mathcal{U}$ every $Q \in \mathcal{N}_{\tau}(x)$ and every $F \in \mathcal{F}$

$$\emptyset \neq V(z) \cap Z(V, Q, F) \cap V(A_F Q)\,.$$

Hence z is in $\mathrm{Ls}_{\mathcal{F}}^{\tau\times\mathcal{U}} A_i$, contrary to the assumptions.

In the case of epigraphs of functions our compactness condition takes form: $\mathrm{ls}^\sigma f_i > -\infty$. Therefore we have

COROLLARY ([14]). – If $-\infty < \mathrm{ls}^\sigma f_i \leqslant \mathrm{li}^\tau f_i$, then $\{f_i\}$ is (τ/σ)-equi lower semiconnuous.

We shall specialize equi semicontinuity for a filtered family of $\{(\mathrm{epi}\, f_i)^{-1}\}_{i\in I}$.

PROPOSITION 3.4. – A family $\{(\mathrm{epi}\, f_i)^{-1}\}_{i\in I}$ is quasi (ν/ι)-equi semicontinuous at $r \in R$ (ν-natural topology of R), if and only if

(3.5) for every $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ and $\varepsilon > 0$, $F \in \mathcal{F}$ such that for each $i \in F$ if $\inf\limits_{V(v)} f_i < r + \varepsilon$, then there is

$$v_i \in U(y) \qquad \text{with } f_i(v_i) \leqslant r.$$

PROOF. – Let y belong to the right-hand side of (3.1): $V(y) \cap (\mathrm{epi}\, f_i)^{-1} B(r, \varepsilon) \neq \emptyset$, then, equivalently, $\mathrm{epi}\, f_i V(y) \cap B(r, \varepsilon) \neq \emptyset$; in other words $\inf f_i < r + \varepsilon$. Similarly, we translate the left hand side.

Observe that *indicator functions* (assuming only the values $0, +\infty$) always satisfy the condition of Proposition 3.4. As well in normed spaces *distance functions* fulfil that condition; for every ball (relation) $U = B_r$ one may take $V = B_{r/2}$ and $\varepsilon = r/2$. The distance functions are also lipschitzian with the constant 1, therefore, by Theorems 3.1 and 3.2, for distance functions metric and pointwise convergences coincide.

We recall that quasi ν/ι-equi semicontinuity is a sufficient condition for the the equivalence of the $\nu \times \mathcal{U}$-sub- (super-) convergence of the epigraphs and the $\mathcal{U}$-sub- (super-) convergence of the corresponding level sets. Thus we deduce

PROPOSITION 3.5. – Let $\{f_i\}_{i\in I}$ be a filtered family of functions that fulfils (3.5). Then $\{f_i\}$ superconverges (subconverges) to f, if and only if the level sets $\{y: f_i(y) \leqslant r\}$ super (sub-) converge to $\{y: f(y) < r\}$ for every $r \in \boldsymbol{R}$.

COROLLARY (compare WIJSMAN [37, Thm 3.1]). – Let (X, ϱ) be a normed space. A family $\{A_i\}_{i\in I}$ superconverges to A, if and only if $\{\mathrm{dist}\,(\cdot, A_i)\}_{i\in I}$ superconverges to $\mathrm{dist}\,(\cdot, A)$ pointwise $\{\mathrm{dist}\,(\cdot, A_i)\}_{i\in I}$ subconverges to $\mathrm{dist}\,(\cdot, A)$, if and only if $\{B_r(A_i)\}_{i\in I}$ subconverges to $B_r(A)$ for every $r > 0$.

4. – Γ and G limits.

Consider n sets $X_1, X_2, \ldots, X_n$ and an extended—real—valued function f on $X_1 \times \ldots \times X_n$. Given filters $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_n$ in $X_1, X_2, \ldots, X_n$, respectively, and a sequence of signs $+$ or $-$: $\alpha_1, \alpha_2, \ldots, \alpha_n$, we define, following DE GIORGI [8],

$$(4.1)\qquad \Gamma(\mathcal{N}_1^{\alpha_1}, \mathcal{N}_2^{\alpha_2}, \ldots, \mathcal{N}_n^{\alpha_n})f = \\ = \operatorname*{ext}^{-\alpha_n}_{N_n\in\mathcal{N}_n} \ldots \operatorname*{ext}^{-\alpha_2}_{N_2\in\mathcal{N}_2} \operatorname*{ext}^{-\alpha_1}_{N_1\in\mathcal{N}_1} \operatorname*{ext}^{\alpha_1}_{\xi_1\in N_1} \operatorname*{ext}^{\alpha_2}_{\xi_2\in N_2} \ldots \operatorname*{ext}^{\alpha_n}_{\xi_n\in N_n} f(\xi_1, \xi_2, \ldots, \xi_n)$$

where $\mathrm{ext}^+ = \sup$ and $\mathrm{ext}^- = \inf$.

We have abbreviated here the original notation on dropping «lim».

The above limit is a (possibly infinite) number. More generally, given topologies $\tau_1, \tau_2, \ldots, \tau_n$ on $X_1, X_2, \ldots, X_n$, we write

$$(4.2)\qquad [\Gamma(\tau_1^{\alpha_1}, \tau_2^{\alpha_2}, \ldots, \tau_n^{\alpha_n})f](x_1, x_2, \ldots, x_n) = \Gamma(\mathcal{N}_{\tau_1}(x_1), \ldots, \mathcal{N}_{\tau_n}(x_n))f,$$

and, of course, the limit $\Gamma(\tau_1^{\alpha_1}, \ldots, \tau_n^{\alpha_n})f$ is a function on $X_1 \times \ldots \times X_n$. It is known [8] that

(i) if ι_k is discrete, then the Γ-limit fixes the k-th coordinate (whatever α_k is):

$$\Gamma(\ldots, \mathcal{N}_{k-1}^{\alpha_{k-1}}, \mathcal{N}_{\iota_k}^{\alpha_k}(x_k), \mathcal{N}_{k+1}^{\alpha_{k+1}}, \ldots)f = \Gamma(\ldots, \mathcal{N}_{k-1}^{\alpha_{k-1}}, \mathcal{N}_{k+1}^{\alpha_{k+1}}, \ldots)f(\ldots, x_k, \ldots)$$

ii) $\Gamma(\ldots, \mathcal{N}_i^-, \ldots)f \leqslant \Gamma(\ldots, \mathcal{N}_i^+, \ldots)f$

(iii) if $\mathcal{N}_i \subset \mathcal{M}_i$, then

$$\Gamma(\ldots, \mathcal{N}_i^-, \ldots)f \leqslant \Gamma(\ldots, \mathcal{M}_i^-, \ldots)f \quad\text{and}\quad \Gamma(\ldots, \mathcal{N}_i^+, \ldots)f \geqslant \Gamma(\ldots, \mathcal{M}_i^+, \ldots)f$$

We observe that

(iv) $\Gamma(\ldots, \mathcal{N}_i^+, \mathcal{N}_{i+1}^-, \ldots)f \leqslant \Gamma(\ldots, \mathcal{N}_{i+1}^-, \mathcal{N}_i^+, \ldots)f$

where we change the order of the i and $i+1$ variable.

PROOF. – Note that if we have «irreducible» sings $\alpha_i \neq \alpha_{i+1}$, the operations

$$\mathrm{ext}^{-\alpha_{i+1}}\,\mathrm{ext}^{-\alpha_i} = \mathrm{ext}^{\alpha_i}\,\mathrm{ext}^{\alpha_{i+1}},$$

that is, are of the same type. Therefore (iv) follows, since always $\sup_{\xi_1}\inf_{\xi_2} g(\xi_1, \xi_2) \leqslant \inf_{\xi_2}\sup_{\xi_1} g(\xi_1, \xi_2)$.

It is a simple observation that «sup» and «inf» operations are examples of Γ-limits:

$$\mathrm{ext}_B^{\alpha} f(\xi) = \Gamma(\mathcal{N}_\iota^{\alpha}(B))$$

where $\mathcal{N}_\iota(B)$ is the filter of all supersets of B; if B is the whole space we may also use the chaotic topology o. Limits of infima (2, 5) are also Γ-limits, $\Gamma(\mathcal{N}^+, o^-)$, $\Gamma(\mathcal{N}^-, o^-)$ respectively.

As we mentioned, it is known [7] [8] that

$$\text{(4.3)} \qquad \mathrm{ls}^{\tau}_{\mathcal{F}} f_i = \Gamma(\mathcal{F}^+, \tau^-) f\,, \quad \mathrm{li}^{\tau}_{\mathcal{F}} f_i = \Gamma(\mathcal{F}^-, \tau^-) f$$

where $f(i, x) = f_i(x)$. For example, we prove the first inequality of (2.5) using Γ-limits. Since $\overline{\lim}\,(\inf f_i) = \inf\limits_{F\in\mathcal{F}} \sup\limits_{i\in F} \inf\limits_{x\in X} f(i, x)$ is equal to $\Gamma(\mathcal{F}^+, o^-)f$ and the latter, by (iii), is less than $\Gamma(\mathcal{F}^+, \tau^-)f = \mathrm{ls}^{\tau}_{\mathcal{F}} f_i$, (2.5) follows.

For completeness's sake, notice that $\Gamma(\tau^-)f$ is the closure (lower semicontinuous hull) of f and $\Gamma(\tau^+)f$ is the least upper semicontinuous function that majorizes f [8].

We present now a criterion for equality of Γ-limits. Let $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_{n-1}$ be filters in $X_1, \ldots, X_{n-1}$ and let $\mathcal{N}$ and $\mathcal{M}$ be filters in X_n.

A function $f\colon X_1\times X_2\times \ldots X_n \to R$ is said to be *equi-$\mathcal{N}/\mathcal{M}$-semicontinuous* (in the last variable), if for every $\varepsilon > 0$ and every $M \in \mathcal{M}$, there are $N\in\mathcal{N}$ and $\tilde{N}_i \in \mathcal{N}_i$, $i = 1, \ldots, n-1$, such that

$$\inf_{\xi_n\in N} f(\xi_1, \ldots, \xi_n) \geqslant \inf_{\xi_n\in M} f(\xi_1, \ldots, \xi_n) - \varepsilon\,, \qquad \text{as } \xi_i \in \tilde{N}_i\,.$$

THEOREM 4.1. – If f is equi-$(\mathcal{N}/\mathcal{M})$-semicontinuous, then

$$\Gamma(\mathcal{N}_1^{\alpha_1}, \mathcal{N}_2^{\alpha_2}, \ldots, \mathcal{M}^-) f \leqslant \Gamma(\mathcal{N}_1^{\alpha_1}, \mathcal{N}_2^{\alpha_2}, \ldots, \mathcal{N}^-) f\,.$$

PROOF. – First we shall show that given $\tilde{N}_k \in \mathcal{N}_k$ we may replace in (4.1) $\operatorname*{ext}\limits_{N_k\in\mathcal{N}_k}^{-\alpha_k}$ by $\operatorname*{ext}\limits_{N_k\in\mathcal{N}_k,\, N_k\subset\tilde{N}_k}^{-\alpha_k}$. To this end consider a function $g\colon X_1\times X_2\times\ldots\times X_n \to \bar{R}$ and for a given $\varepsilon > 0$ and $N_n \in \mathcal{N}_n$ find $\xi_n(\varepsilon, N_n, \xi_1, \ldots, \xi_{n-1})$ such that

$$\operatorname*{ext}_{\xi_n\in N_n}^{\alpha_n} g(\xi_1, \xi_2, \ldots, \xi_n)$$

differs by ε/n from $g(\xi_1, \ldots, \xi_{n-1}, \xi_n(\varepsilon, N_n, \xi_1, \ldots, \xi_{n-1}))$. Continuining this process we shall get an element of $N_1\times N_2\times\ldots\times N_n$ such that the value of g at this element:

$$\text{(4.4)} \qquad g\Big[\xi_1(\varepsilon, N_1), \xi_2(\varepsilon, N_2, \xi_1(\varepsilon, N_1)), \ldots, \xi_n\big(\xi_1(\varepsilon, N_1), \ldots, \xi_{n-1}(\xi_1(\varepsilon, N_1)\ldots)\big)\Big]$$

differs by ε from $\operatorname*{ext}\limits_{\xi_1\in N_1}^{\alpha_1} \ldots \operatorname*{ext}\limits_{\xi_n\in N_n}^{\alpha_n} g(\xi_1, \ldots, \xi_n)$.

Consider (4.4) with N_1 replaced by $N_1 \cap \tilde{N}_1$. If $\alpha_1 = +$, then the value of (4.4) for $N_1 \cap \tilde{N}_1$ will be not greater than its value for N_1 plus ε/n.

Therefore we may pick $N_1(N_2, \ldots, N_n)$ that (ε/n)-attains the infimum of (4.4) over $\mathcal{N}_1\Big(\operatorname*{ext}\limits_{N_1\in\mathcal{N}_1}^{-\alpha_1}\Big)$ from subsets of $\tilde{N}_1$. Similarly we argue if $\alpha_1 = -$. We proceed the same way with $N_2, N_3, \ldots, N_n$. Therefore, the value of (4.1) differs by 2ε from the corresponding values computed with restriction that N_k be a subset of $\tilde{N}_k$. To conclude recall that ε was taken arbitrarily.

Suppose now equi-$(\mathcal{N}/\mathcal{M})$-semicontinuity. Then given $\varepsilon > 0$ and $M \in \mathcal{M}$ there will be $N_1, \dots, N_{n-1}, N$ such that

$$\inf_{\xi_k \in N} f(\cdot, \cdot, \dots, \xi_n) \geqslant \inf_{\xi_k \in M} f(\cdot, \cdot, \dots, \xi_n) - \varepsilon \quad \text{ on } N_1 \times N_2 \times \dots \times N_{n-1} .$$

Therefore $\Gamma(\mathcal{N}_1^{\alpha_1}, \mathcal{N}_2^{\alpha_2}, \dots, \mathcal{N}_{n-1}^{\alpha_{n-1}})$ limits of the above functions: $g(N)$ and $g(M)$ satisfy

$$g(N) \geqslant g(M) - \varepsilon .$$

Consequently $\sup\limits_{M \in \mathcal{M}} g(M) \leqslant \sup\limits_{N \in \mathcal{N}} g(N)$ and the proof is accomplished.

REMARK 4.2. – It follows from the above theorem that if f is $(\mathcal{N}/\mathcal{M})$-equi semicontinuous in the k-th variable, then

$$\Gamma(\mathcal{N}_1^{\alpha_1}, \dots, \mathcal{M}^-, \mathcal{N}_{k+1}^{\alpha_{k+1}}, \dots) f \leqslant \Gamma(\mathcal{N}_1^{\alpha_1}, \dots, \mathcal{N}^-, \mathcal{N}_{k+1}^{\alpha_{k+k}}, \dots) f$$

because we may apply the theorem to the function $\operatorname{ext}^{\alpha_{k+1}}_{\xi_{k+1} \in N_{k+1}} \dots \operatorname{ex}^{\alpha_n}_{\xi_n \in N_n} f(\cdot, \cdot, \dots, \xi_{k-1}, \dots, \xi_n)$ and then proceed with extremization over $\mathcal{N}_{k+1}, \dots, \mathcal{N}_n$.

REMARK 4.3. – The above theorem has been proved in DOLECKI-SALINETTI-WETS [14, Thm 2.3] in two special cases: for $\Gamma(-,-)$ and $\Gamma(+,-)$ limits. See also our Theorems 3.2 and 3.3.

We say that f is *upper equi* $(\mathcal{N}/\mathcal{M})$*-semicontinuous* in the k-th variable if for every $\varepsilon > 0$, $M \in \mathcal{M}$ there are $N_i \in \mathcal{N}_i$ $i \neq k$ and $N \in \mathcal{N}_k$ such that

$$\sup_{\xi_k \in N} f(\xi_1, \dots, \xi_k, \dots, \xi_n) \leqslant \sup_{\xi_k \in M} f(\xi_s, \dots, \xi_k, \dots, \xi_n) + \varepsilon \quad \text{ as } \xi_i \in N_i, \ i \neq k.$$

Analogously to Theorem 4.1, having in mind Remark 4.2, we have

THEOREM 4.4. – If f is upper equi $(\mathcal{N}/\mathcal{M})$-semicontinuous, then

$$\Gamma(\mathcal{N}_1^{\alpha_1}, \dots, \mathcal{N}^+, \dots) f \geqslant \Gamma(\mathcal{N}_1^{\alpha_1}, \dots, \mathcal{M}^+, \dots) f .$$

Let A be a subset of $X_1 \times \dots \times X_n$ (a relation of n-variables). Let $\mathcal{N}_1, \dots, \mathcal{N}_{k-1}$ be filters in $X_1, \dots, X_{k-1}$ and let $\tau_k, \tau_{k+1}, \dots, \tau_n$ be topologies in $X_k, \dots, X_n$, respectively.

The Γ-limit of the indicator function of A

$$\Gamma(\mathcal{N}_1^{\alpha_1}, \dots, \mathcal{N}_{k-1}^{\alpha_{k-1}} \tau_k^{\alpha_k}, \dots, \tau_n^{\alpha_n}) \chi_A \tag{4.5}$$

is the indicator function of a subset of $X_k \times \dots \times X_n$.

Therefore (4.5) is a limiting set of a collection $\{A_{x_1,\dots,x_{k-1}}\}$ of relations (subsets of $X_k\times\dots\times X_n\}$ indexed by $X_1\times, \dots, X_{k-1}$. The collection is filtered (in a nonclassical way) by $\mathcal{N}_1, \dots, \mathcal{N}_{k-1}$.

The level set of (4.5) represents what is called [8] the $G(\mathcal{N}_1^1, \dots, \mathcal{N}_{k-1}^{\alpha_{k-1}}; \tau_k^{\alpha_k}, \dots, \tau_n^{\alpha_n})$ limit (*G-limit*) of the family $\{A_{x_1,\dots,x_{k-1}}\}$.

It is known [8] (and may be easily checked directly) that

$$\mathrm{Li}_{\mathcal{F}}^{\vartheta} A_i = G(\mathcal{F}^+; \vartheta^-) A_i\,, \qquad \mathrm{Ls}_{\mathcal{F}}^{\vartheta} A_i = G(\mathcal{F}^-; \vartheta^-) A_i\,. \tag{4.6}$$

But it is interesting to observe that (in uniform spaces), (4.6) is a consequence of Proposition 3.6.

A seemingly most interesting non-classical G-limit is $G(\mathcal{N}^+, \tau^+, \sigma^-)$ which turns out to be the hyperlimit. Let $\{A_i\}_{i\in I}$ be a family of relations in $X\times Y$ filtered by $\mathcal{N}$ and let τ, σ be topologies in X and Y.

PROPOSITION 4.5.

$$G(\mathcal{N}^+; \tau^+, \sigma^-) A_i = \mathrm{Lh}_{\mathcal{N}}^{\tau/\sigma} A_i\,.$$

PROOF. – Apply the first formula of (4.6) for $\vartheta = \sigma$ and $\mathcal{F} = \mathcal{N}\times\mathcal{N}_\tau(x)$. Thus by (2.8)

$$(\mathrm{Lh}_{\mathcal{N}}^{\tau/\sigma} A_i)(x) = G([\mathcal{N}\times\mathcal{N}_\tau(x)]^+; \delta^-) A_i = [G(\mathcal{N}^+; \tau^+, \sigma^-) A_i](x)\,.$$

Note that if the topology τ is discrete then the lower limit and the hyperlimit coincide. This explains why in Proposition 2.2. in the case of the constant family $\{x\}_{i\in I}$ the hyperconvergence of $\{A_i\}$ may be replaced by its superconvergence (with respect to the discrete topology in X).

Consider now hyperconvergence of families of epigraphs.

PROPOSITION 4.6. – The hyperlimit of the epigraphs of $\{f_i\}_{i\in I}$ is equal to the epigraph of the $\Gamma(+, +)$ limit of f_i:

$$\mathrm{Lh}_{\mathcal{F}}^{\tau/\nu} \operatorname{epi} f_i = \operatorname{epi}\left(\Gamma(\mathcal{F}^+, \tau^+) f\right),$$

where $f(i, x) = f_i(x)$.

PROOF. – Let (x, r) belong to the hyperlimit: for every $s > 0$ there are $W \in \mathcal{N}(x)$ and $F\in\mathcal{F}$ such that for every i in F and w in W, $(\operatorname{epi} f_i)w \cap B_s(r) \neq \emptyset$, that is

$$\sup_{i\in F}\ \sup_{w\in W} f_i(w) \leqslant r + s\,.$$

Equivalently, r is greater than

$$\Gamma(\mathcal{F}^+, \tau^+) f(x) = \inf_{W\in\mathcal{N}_\tau(x)}\ \inf_{F\in\mathcal{F}}\ \sup_{i\in F}\ \sup_{w\in W} f_i(w)\,. \tag{4.6}$$

The above result sheds a new light on the nature of some Γ limits. So far it was known [2] that the hypograph of $\Gamma(+,+)$ limit is equal to the upper limit of the hypographs.

To conclude this section, observe that the interior limit (1.8) of a family of epigraphs $\{\operatorname{epi} f_i\}_{i\in I}$ (filtered by $\mathcal{N}$) may be represented with the aid of

$$\Gamma(\mathcal{N}^-, \tau^+)f(x) = \inf_{W\in\mathcal{N}_\tau(x)} \sup_{F\in\mathcal{N}} \inf_{i\in F} \sup_{w\in W} f_i(w)\,,$$

namely

$$(\operatorname{Ls}(\operatorname{epi} f_i)^c)^c = \{(x, r) : r > \Gamma(\mathcal{N}^-, \tau^+)f(x)\}\,.$$

TANGENCY

5. – Approximating cones.

Several bibliographical accounts of approximating cones have been given ([29] [35] [25] [16] [36]) and we do not intend to compete with them. We shall only say that a great variety of cones that approximate (locally) sets have been studied since the beginning of the century (SEVERI [33] BOULIGAND [1] and others) and that now we witness considerable interest in conical approximations.

We shall discuss principal approximating cones. All of them will be defined as limits of a single relation said homothety. Homothety has been already used (without being named) by HIRIART-URRUTY [16], ROCKAFELLAR [29], PENOT [25]; we are going to deploy it in a more systematic way. The advantage of this approach lies in capitalizing on convergence theory; often, what so far used to be an involved proof becomes an easy consequence of the preceding sections.

The *homothety* in a linear space X is the following relation (multivalued mapping from $2^X\times X\times(0,\infty)$ to X):

$$(C, x, t) \to \frac{1}{t}(C - x)\,. \tag{5.1}$$

If the set C is fixed, (5.1) is called the *homothety of* C and, if needed, will be denoted by $\mathcal{H}_C$. If, moreover, x is fixed, the relation $\mathcal{H}_{C,x}$ is called the *homothety of* C *about* x.

We note that

$$\left\{\begin{array}{ll} \text{(i)} & \text{if } C\subset D, \text{ then } \frac{1}{t}(C-x)\subset\frac{1}{t}(D-x) \\[2mm] \text{(ii)} & \frac{1}{t}(C\cup D - x) = \frac{1}{t}(C-x)\cup\frac{1}{t}(D-x) \\[2mm] \text{(iii)} & \frac{1}{t}(C\cap D - x) = \frac{1}{t}(C-x)\cap\frac{1}{t}(D-x) \\[2mm] \text{(iv)} & \mathfrak{D}\frac{1}{t}\big(C-(x,y)\big) = \frac{1}{t}\big(\mathfrak{D}(C) - x\big) \qquad \text{for every } y\in Cx. \end{array}\right. \tag{5.2}$$

Topologies we are going to consider on X are not necessarily compatible with the linear structure.

The *contingent* of C at x (BOULIGAND [1]) is the upper limit of the homothety of C about x as t tends to 0:

$$K_C^\tau(x) = \mathrm{Ls}^\tau_{t\to 0}\frac{1}{t}(C-x) = \bigcap_{s>0}\mathrm{CL}_\tau\bigcup_{t<s}\frac{1}{t}(C-x)\,. \tag{5.3}$$

The *semitangent* of C at x (of SEVERI [33]) may by introduced by

$$S_C^\tau(x) = \bigcap_{V\in\mathcal{N}_\tau(x)}\mathrm{Cl}_\tau\bigcup_{t>0}\frac{1}{t}(C\cap V-x)\,. \tag{5.4}$$

PROPOSITION 5.1. – If X is a normed spaces, then

$$S_C(x) = K_C(x)\,.$$

PROOF. – An h belongs to the semitangent (of C at x), if and only if for every $V\in\mathcal{U}$ every $Q\in\mathcal{N}_\tau(h)$ there is $t>0$ such that $x+tQ\cap C\cap V(x)\neq\emptyset$. If $\{s_n\}$ tends to zero and we choose $(s_nV)(x)$ for the above formula then the resulting $\{t_n\}$ tends to 0, hence h is in the contingent.

If h belongs to the contingent (for every $Q\in\mathcal{N}_\tau(h)$ and every t_0 there is $t\leqslant t_0$ with $x+tQ\cap C\neq 0$), then as for every $V\in\mathcal{N}_\tau(x)$ we may find t_0 such that $x+tQ\subset V$ for $t\leqslant t_0$, it belongs also to the semitangent.

Obviously $S_C(x)=S_{C\cap V}(x)$ for every $V\in\mathcal{N}(x)$, thus in our case also $K_C(x) = K_{C\cap V}(x)$, that is, the above cones approximate C locally.

The *tangent* of C at x (DUBOVITZKII-MILYUTIN [15]) is the lower limit of the homothety of C about x as t tends to 0:

$$T_C^\tau(x) = \mathrm{Li}^\tau_{t\to 0}\frac{1}{t}(C-x)\,. \tag{5.5}$$

We say that a set E is *directionally open about* x, if for every $h\in X$ there is $V\in\mathcal{N}(h)$ and t_0 such that for each $t\leqslant t_0$, $x+tV\subset E$. Every open set is directionally open provided that neighborhood bases are composed of radial (absorbing) sets (in particular, if X with its topology constitutes a topological vector space); the intersection of two sets directionally open about x is also such. If the topology is discrete, then a set is directionally open at x, if and only if x is its internal point.

Both the tangent and the contingent at x of a set directionally open at x are the whole of X.

PROPOSITION 5.2. – If E is directionally open at x, then

$$T_{C\cap E}(x) = T_C(x)\,,\qquad K_{C\cap E}(x) = K_C(x)\,.$$

PROOF. – Let $h \in T_C(x)$. Take $Q_0 \in \mathcal{N}(h)$ such that for $t \leqslant t_0$ $x + tQ_0 \subset E$. We have that for every $Q \in \mathcal{N}(h)$, $Q \subset Q_0$ there is $t_1 \leqslant t_0$ such that $x + tQ \cap C \neq \emptyset$ for $t \leqslant t_1$; hence $x + tQ \cap C \cap E \neq \emptyset$ and the first formula is demonstrated. The latter one admits a similar proof.

It follows from general facts about limits that the contingent and the tangent are closed; the coarser the topology the larger they are; the contingent includes the tangent.

If C is radial about x (for each w in C the interval $[w, x]$ is a subset of C), then, by (1.9), the contingent is equal to the tangent. This happens, in particular, when C is convex.

It is easy to notice that the following statements are equivalent (e.g. [35, Lemma 7]): (i) $0 \in T_C(x)$; (ii) $0 \in K_C(x)$; (iii) $x \in \operatorname{Cl} C$.

Like limits, tangents and contingents are isotone. In particular

$$T_{C\cap D}(x) \subset T_C(x) \cap T_D(x)\,, \qquad K_{C\cap D}(x) \subset K_C(x) \cap K_D(x)\,. \tag{5.6}$$

We are now concerned with sufficient conditions which imply the equality in (5.6). One example has been already furnished by Proposition 5.2; it is enough that one of the sets C, D be directionally open.

We may use Proposition 1.7 to derive a more general condition for tangents. Specialized for the homotheties of C and D about x it yields this obvious requirement:

(5.7) for every h in x and every $Q \in \mathcal{N}(h)$ there are $W \in \mathcal{N}(h)$, t_0 such that for $t \leqslant t_0$, if $x + tW \cap C \neq \emptyset$ and $x + tW \cap D \neq \emptyset$, then $x + tQ \cap C \cap D \neq \emptyset$.

Note that if one set is directionally open about x then the above condition is satisfied.

A family $\mathcal{G}$ of is called a *directional covering about* x, if for every $h \in X$ that are $Q \in \mathcal{N}(h)$ and $t_h > 0$ such that $x + (0, t_h)\cdot Q$ is a subset of an element of $\mathcal{G}$ and such that every $G \in \mathcal{G}$ is the union of such sets.

PROPOSITION 5.3. – Let C and D be subsets of a normed space X. Let $\mathcal{G}$ be a directional covering about 0 such that for every $G \in \mathcal{G}$ there is $k = k(G)$ such that

$$\operatorname{dist}(g, C \cap D - x) \leqslant k[\operatorname{dist}(g, C - x) + \operatorname{dist}(g, D - x)] \qquad g \in G, \tag{5.8}$$

Then

$$T_{C\cap D}(x) = T_C(x) \cap T_D(x)\,. \tag{5.9}$$

PROOF. – We shall show that (5.7) holds. Let $h \in X$, $r > 0$ and let t_h be such that $x + (0, t_h)\,G$ is in $\mathcal{G}$. We take for $Q = B_r(h)$, $W = B_{r/2k}(h)$. If $x + tW \cap C \neq \emptyset$ and $x + tW \cap D \neq \emptyset$, then, by (5.8), $\operatorname{dist}(th, C \cap D - x)$ is smaller than rt, in other words, $x + tQ \cap C \cap D \neq \emptyset$ as $t \leqslant t_h$.

The condition (5.7) is satisfied, in the special case in which C and D separate decisively at a linear rate (see [11]): there are k and m such that

$$\operatorname{dist}(g, C \cap D) \leqslant k[\operatorname{dist}(g, C) + \operatorname{dist}(g, D)]\,, \quad g \in X\,, \quad \operatorname{dist}(g, C)\,, \quad \operatorname{dist}(g, D) \leqslant m\,.$$

PROPOSITION 5.4. – When C and D are convex (5.7) becomes also necessary for the equality $T_{C\cap D}(x) = T_C(x) \cap T_D(x)$.

PROOF. – Suppose that (5.7) does not hold: there is h and $Q \in \mathcal{N}(h)$ such that for every $W \in \mathcal{N}(h)$ and every t_0 there is $t \leqslant t_0$ for which $x + tW \cap C \neq \emptyset$ and $x + tW \cap \cap D \neq \emptyset$, but $x + tQ$ misses $C \cap D$. Therefore $h \in K_C(x) \cap K_D(x)$, thus, by convexity, is in $T_C(x) \cap T_D(x)$ but not in $T_{C\cap D}(x)$.

When the topology considered in X is discrete, then $K^\iota_C(x)$ is called the *radial contingent* of C at x and $T^\iota_C(x)$ the *radial tangent* (or, simply the *radial cone*) of C at x. Accordingly, h is in the radial contingent (of C at x) if and only of there is a sequence $\{t_n\}_{n=1}^\infty$ convergent to 0 with $x + t_n h \in C$; h belongs to the radial cone of C at x, whenever there is t_0 such that $x + th \in C$ at $t \leqslant t_0$.

Let τ, θ be topologies on X.

The *hypertangent* of C at x is the lower limit of the homethety $\mathcal{H}_C(x', t)$ of C as x' tends to x within C in θ and t tends to 0:

$$\mathcal{T}^{\tau/\theta}_C(x) = \operatorname{Li}^\tau_{\mathcal{N}_\theta(x,C)\times\mathcal{N}(0)} \mathcal{H}_C\,. \tag{5.10}$$

The neighborhood filter $\mathcal{N}_\theta(x, C)$ is for the topology, induced on C by θ.

This cone has been studied (in the case in which θ is the topology induced on C by τ) by ROCKAFELLAR [29] [30] under the name of tangent, by HIRIART-URRUTY [16] [17] and PENOT [25] [26] under the name of peritangent, and by others. The term « hypertangent » has been used in [29] [30] for the *radial hypertangent*, that is the approximating cone (5.10) when τ is discrete. It follows from (1.3) that

$$\mathcal{T}^{\tau/\theta}_C(x) \subset T^\tau_C(x)\,, \qquad \mathcal{T}^{\tau/\iota}_C(x) = T^\tau_C(x)\,.$$

We shall define again the hypertangent starting from considering homothety as the family of relations from C into X indexed by $(0, \infty)$.

Then the hypertangent $\mathcal{T}_C$ (relation from C into X) is the hyperlimit of the homothety (θ is understood to be restricted to C)

$$\mathcal{T}^{\tau/\theta}_C(x) = \left(\operatorname{Lh}^{\theta/\tau}_{t\to 0}\left\{(x', y') \in C\times X : y' \in \frac{1}{t}(C - x')\right\}\right)(x)\,.$$

We conclude that the hypertangent $\mathcal{T}^{\tau/\theta}_C(x)$ is always τ-closed. The hypertangent relation is not in general closed ($\theta\times\tau$).

It is known that if (X, τ) is a topological vector space, then $\mathcal{T}^{\tau/\tau}_C(x)$ is convex for every C and x [28] [34], while available proofs were given in normed spaces.

It is also known that the radial hypertangent $\mathcal{T}^{i/\tau}(x)$ is then convex. We present a generalization of those results. After I had established the result to follow I became acquainted with an essentially equivalent result, although proved differently, by PENOT [38].

We say that (X, τ) is an *almost topological vector space* (almost t.v.s.), whenever the multiplication $(x, t) \mapsto tx$ is continuous in X for every $t \in R$ and the addition $(x, y) \mapsto x + y$ is continuous on $X \times X$.

The discrete topology in a vector space gives rise to an almost topological vector space which is not a topological vector space. More generally, topologies given by invariant uniformities are almost topological vector spaces. Neighborhoods of zero in such spaces need not be radial.

THEOREM 5.5. – Let (X, τ) be almost t.v.s. and (X, θ) be a t.v.s. such that $\theta \subset \tau$. Then for every C and x, the hypertangent $\mathcal{T}_C^{\tau/\theta}(x)$ is convex.

PROOF. – Let h_1, h_2 be in $\mathcal{T}_C^{\tau/\theta}(x)$ and let $0 < \lambda < 1$. For every $W_2 \in \mathcal{N}_\theta(x)$ there are $W_1 \in \mathcal{N}_\theta(x)$, $Q_1 \in \mathcal{N}_\tau(h_1)$ and $t_1 > 0$ such that

$$W_1 + (0, t_1) Q_1 \subset W_2 , \tag{5.11}$$

because θ is compatible with the linear structure of X and τ is finer than θ.

Since h_2 is in the hypertangent, for every $Q_2 \in \mathcal{N}_\tau(h_2)$ there are $W_2 \in \mathcal{N}_\theta(x)$ and $t_2 > 0$ such that for $t < t_2$, $x' \in W_2 \cap C$

$$x' + tQ_2 \cap C \neq \emptyset . \tag{5.12}$$

Let $Q \in \mathcal{N}_\tau(\lambda h_1 + (1 - \lambda) h_2)$. Since (X, τ) is almost t.v.s., there are $Q_1 \in \mathcal{N}_\tau(h_1)$ and $Q_2 \in \mathcal{N}_\tau(h_2)$ such that

$$\lambda Q_1 + (1 - \lambda) Q_2 \subset Q . \tag{5.13}$$

To that Q_2 choose $W_2 \in \mathcal{N}_\theta(x)$ and $t_2 > 0$ such that (5.12) holds. To the above W_2 choose W_1, t_1 and Q_1 such that (5.11) holds and (5.13) continues to hold. For that Q_1, there are $W_1' \in \mathcal{N}_\theta(x)$ and $t_1' > 0$ such that $W_1' \subset W_1$ and $t_1' \leqslant t_1$ for which

$$x'' + tQ_1 \cap C \neq \emptyset \quad \text{as} \quad x'' \in W_1' \cap C \quad \text{and } t < t_1'.$$

Consequently there is $x' \in x'' + \lambda t Q_1 \cap C \subset W_2 \cap C$ and by (5.12)

$$x'' + \lambda t Q_1 + (1 - \lambda) t Q_2 \cap C \neq \emptyset$$

provided $x'' \in W_1' \cap C$ and $t < \min(t_2, t_1')$, which in view of (5.13) yields

$$x'' + tQ \cap C \neq \emptyset .$$

COROLLARY. – The radial hypertangent is convex.

Expanding on the definition of ROCKAFELLAR [30] we say that a set C is *epi-Lipschitzian* at x towards h, if there are $Q \in \mathcal{N}_\tau(h)$, $W \in \mathcal{N}_\theta(x, C)$ and $t_0 > 0$ such that $Q \subset (1/t)(C - x')$ for $x' \in W$ and $t < t_0$. Note that two topologies are involved.

By analogy to our previous definition we say that C is *directionally equi-open about* x, if C is epi Lipschitzian at x towards every h. One realizes that C is epi-Lipschitzian at x towards h, if and only if $h \in \{\mathrm{Ls}^\tau_{\mathcal{N}_\theta(x,C)\times\mathcal{N}(0)}[(1/t)(C - x')]^c\}^c$.

PROPOSITION 5.6. – If C is epi-Lipschitzian at x towards h and $h \in \mathcal{C}_D(x)$ then $h \in \mathcal{C}_{C\cap D}(x)$.

PROOF. – Since C is epi-Lipschitizan at x towards h, h belongs to

$$\left\{\mathrm{Ls}^\tau_{\mathcal{N}_\theta(x,\,C\cap D)\times\mathcal{N}(0)}\left[\frac{1}{t}(C - x')\right]^c\right\}^c,$$

$\mathcal{N}_\vartheta(x, C \cap D)$ being finer than $\mathcal{N}_\vartheta(x, C)$. As well,

$$\mathcal{C}^{\tau/\theta}_D(x) \subset \mathrm{Li}^\tau_{\mathcal{N}_\theta(x,\,C\cap D)\times\mathcal{N}(0)}\frac{1}{t}(D - x)\,.$$

By Proposition 1.8, h belongs to $\mathcal{C}_{C\cap D}(x)$.

An immediate consequence of Proposition 5.6 is the local character of the hypertangent in a topological space (X, θ) when τ is finer than θ.

From the above proposition it follows that if C is epi-Lipschitzian at x towards some h and $\mathrm{Int}\,\mathcal{C}_C(x) \cap \mathcal{C}_D(x) \neq \emptyset$ then $\mathcal{C}_{C\cap D}(x) \supset \mathcal{C}_C(x) \cap \mathcal{C}_D(x)$.

It was proved [29, Thm. 3] that in that case $\mathrm{Int}\,\mathcal{C}_C(x)$ is equal to the interior limit of the homothety (the set of epi-Lipschitizan directions) thus by Prop. 1.8 $\mathcal{C}_{C\cap D}(x) \supset \supset \mathrm{Int}\,\mathcal{C}_C(x) \cap \mathcal{C}_D(x)$, and by convexity one drops «Int» in this formula. (see Appendix).

This result, formulated in the language of functions constitutes the main part of [30, Thm. 2].

We shall pass to a condition which is analoguous to (5.7) and will be derived from Proposition 1.7. Observe that this condition involves only x' from $C \cap D$.

PROPOSITION 5.7. – Suppose that

(5.14) for every $h \in X$ every $Q \in \mathcal{N}_\tau(h)$ there are $V \in \mathcal{N}_\tau(h)$, $W \in \mathcal{N}_\theta(x, C \cap D)$, $t_0 > 0$ such that if each $x' \in W$, $t \leq t_0$, $x' + tV \cap C \neq \emptyset$ and $x' + tV \cap D \neq \emptyset$. then $x' + tQ \cap C \cap D \neq \emptyset$.

Then,

$$\mathcal{C}^{\tau/\theta}_{C\cap D}(x) \supset \mathcal{C}^{\tau/\theta}_C(x) \cap \mathcal{C}^{\tau/\theta}_D(x)\,.$$

PROOF. – We have that

$$\mathrm{Li}_{\mathcal{N}_\theta(x,C)\times\mathcal{N}(0)}\frac{1}{t}(C-x')\subset \mathrm{Li}_{\mathcal{N}_\theta(x,C\cap D)\times\mathcal{N}(0)}\frac{1}{t}(C-x)\,,$$

thus

$$\mathcal{C}_C(x)\cap\mathcal{C}_D(x)\subset \mathrm{Li}_{\mathcal{N}_\theta(x,C\cap D)\times\mathcal{N}(0)}\left[\frac{1}{t}(C-x')\cap\frac{1}{t}(D-x')\right].$$

Now apply Proposition 1.7.

A condition analogous to that of Proposition 5.3 may be inferred from (5.14). There should be a directional covering about 0 on which (5.8) holds uniformly for $x=x'$ from a neighborhood W of x.

The *paratingent* (BOULIGAND [1]) of C at x is the upper limit of the homothety of C as x' tends to x in θ and t tends to 0.

$$\mathcal{K}_C^{\tau/\theta}(x)=\mathrm{Ls}^\tau_{\mathcal{N}_\theta(x,C)\times\mathcal{N}(0)}\mathcal{H}_C \tag{5.15}$$

the *improper chord* of SEVERI [33] is an element of

$$\mathfrak{C}_C^\tau(x)=\bigcap_{V\in\mathcal{N}_\tau(x)}\mathrm{CL}_\tau\bigcup_{t>0,\,x'\in C\cap V}\frac{1}{t}(C\cap V-x')$$

which is a subset of the paratingent $\mathcal{K}_C^{\tau/\theta}(x)$ and if $\tau=\theta$ is metrizable the two coincide.

Let X be a normed space. We observe that

$$\mathrm{dist}\left(h,\frac{1}{t}(C-x)\right)=\frac{1}{t}\,\mathrm{dist}\,(x+th,\,C)\,.$$

Therefore, by Corollary of Proposition 3.5,

$$\begin{aligned}T_C(x)&=\left\{h:\ \mathrm{ls}^\tau_{t\to0}\frac{1}{t}\big(\mathrm{dist}\,(x+th,\,C)\big)=0\right\}\\&=\left\{h:\ \mathrm{ls}^t_{t\to0}\frac{1}{t}\big(\mathrm{dist}\,(x+th,\,C)\big)=0\right\}\end{aligned}$$

and similarly for contingents, hypertangents and paratingents. Such formulae were recognized by PENOT [26] and URSESCU [35] where a reference to a FEDERER's paper may be found.

One may also define approximating cones of C by differentiating its indicator function.

To conclude this section we gather some statements in

$$\mathcal{C}_C^{\tau/\theta}(x)\subset T_C^\tau(x)\subset K_C^\tau(x)\subset\mathcal{K}_C^{\tau/\theta}(x)\,. \tag{5.16}$$

6. – Approximating cones of relations. Generalized derivatives.

Now we consider subsets of $X \times Y$, hence relations from X into Y. Let τ be a topology in X and ϑ in Y.

It follows from (2.4) that

$$\mathfrak{D}(K_C^{\tau\times\vartheta}(x, y)) \subset K_{\mathfrak{D}(C)}^{\tau}(x)\,, \quad \mathfrak{D}(T_C^{\tau\times\vartheta}(x, y)) \subset T_{\mathfrak{D}(C)}^{\tau}(x)\,. \tag{6.1}$$

Analogous statements for hypertangents are not valid in general as in the example of C in R^2

$$C = \{(x, y): x \geqslant 0, y \geqslant 0\} \cup \{(x, y): x \leqslant 0, y \leqslant 0\}$$

and its point $(0, 0)$.

We may rephrase the definitions of approximating cones using the fact that C is a relation, for instance, (h, k) is in $T_C^{\tau\times\vartheta}(x, y)$, if and only if for every $Q \in \mathcal{N}_\vartheta(k)$ and $V \in \mathcal{N}_\tau(h)$ there is t_0 such that

$$y + tQ \cap C(x + tV) \neq \emptyset\,, \qquad t \leqslant t_0\,. \tag{6.2}$$

In this case we shall occasionally use the notation $C^T(x, y)$ rather then $T_C(x, y)$. Of particular interest is the situation where the topology in X is discrete. An (h, k) is in $C^{T, i\times\vartheta}(x, y)$, if and only if for every $Q \in \mathcal{N}_\vartheta(k)$ there is t_0 such that for $t < t_0$

$$y + tQ \cap C(x + th) \neq \emptyset\,, \tag{6.3}$$

that is, when $Q \cap (1/t)[C(x + th) - y] \neq \emptyset$.

Approximating cones of relations are also relations in $X \times Y$; for example, (h, k) is in the contingent of C at (x, y), if and only if $k \in C^K(x, y)h$. The fact that C is now a relation enables us to consider new types of approximations.

Call the *Hadamard cone* of C at (x, y) the hyperlimit of the homothety of C at (x, y) as t converges to 0 (the terminology will be explained later)

$$C^H(x, y) = H_C^{\tau/\vartheta}(x, y) = \mathrm{Lh}_{t\to 0}^{\tau/\vartheta} \frac{1}{t}[C - (x, y)]\,.$$

Consequently $k \in C^H(x, y)\,h$, if and only if for every $Q \in \mathcal{N}_\vartheta(k)$ there is $V \in \mathcal{N}_\tau(h)$ and $t_0 > 0$ such that

$$y + tQ \cap C(x + th') \qquad \text{as } t \leqslant t_0,\ h' \in V.$$

Consider now the important case where relations are single-valued (thus are identified with mappings). The homothety of f at $(x, f(x))$ is for t fixed the mapping $(1/t)\big(f-(x, f(x))\big)$ and we have

$$\mathcal{H}_{x,t}(h) = \frac{1}{t}\,[f-(x, f(x))]\,(h) = \frac{f(x+th)-f(x)}{t} \tag{6.3}$$

which is the difference quotient of f at x.

By specializing general definitions we say that k is in the contingent $f^K(x)\,h$ of f at x towards h if for every $Q \in \mathcal{N}_\vartheta(k)$ and every $V \in \mathcal{N}_\tau(h)$ for each t_0 there is $t < t_0$ such that

$$Q \cap \frac{f(x+tV)-f(x)}{t} \neq \emptyset \tag{6.4}$$

and if the topology in X is discrete, then of course, (6.4) becomes

$$\frac{f(x+th)-f(x)}{t} \in Q\,. \tag{6.5}$$

If (6.4) (respectively (6.5)) holds for all $t < t(Q)$, then we obtain the tangent $f^T(x)$ (*discrete tangent* which is the *Gâteaux* differential if it is linear and continuous (as the function of h)).

A vector k in $f^H(x)\,h$, if and only if for every $Q \in \mathcal{N}_\vartheta(k)$ there are $V \in \mathcal{N}_\vartheta(h)$ and t_0 such that for $t \leqslant t_0$

$$\frac{f(x+tV)-f(x)}{t} \subset Q\,.$$

The classical Hadamard derivative of f at x may be defined as the linear continuous mapping A such that tor every h in X and every function $p\colon (0, \infty) \to X$ such that $\lim_{t\to 0} p(t) = h$,

$$\lim_{t\to 0} \frac{f(x+tp(t))-f(x)}{t} = Ah \qquad \text{(see NASHED [24])}.$$

PROPOSITION 6.1. – If X is a metric space, then the Hadamard approximation of f at $(x, f(x))$ is the (graph of the) Hadamard derivative (provided it is linear and continuous as the function of h).

PROOF. – First note that $f^H(x)\,h$ is at most a singleton as the lower limit of a single-valued family in a Hausdorff space (hence the limit).

Let $k \in f^H(x)\,h$: for every $Q \in \mathcal{N}(k)$ there are $W \in \mathcal{N}(h)$ and t_0 such that for $t < t_0$,

$$\frac{f(x+tW)-f(x)}{t} \subset Q\,.$$

On the other hand, there is $t' \leqslant t_0$ so that $p(t) \in W$ for $t < t'$. Thus $k = Ah$.

Suppose that $k = Ah$. We shall use the dual definition of the lower limit describing $f^{H}(x)h$. G is in the grill of $\mathcal{N}(h)\times\mathcal{N}(0)$ if and only if there exists a sequence (h_n, t_n) in G, $t_n \geqslant t_{n+1}$ converging to $(h, 0)$. Define $p: p(t) = h_n$ if $t_{n+1} < t \leqslant t_n$. Since $k = Ah$, thus, in particular, for every $Q \in \mathcal{N}(k)$ and every G there is $(t, p(t))$ in G so that

$$\frac{f(x + tp(t)) - f(x)}{t} \quad \text{is in } Q.$$

Thus $k \in f^{H}(x)h$.

Consider now topologies τ in X, σ in Y and another topology ϑ on $X\times Y$.

A vector k belongs to the hypertangent $f^{\uparrow}(x)h$ if for every $Q \in \mathcal{N}_\sigma(k)$, $V \in \mathcal{N}_\tau(h)$ there are $t > 0$ and $W \in \mathcal{N}_\vartheta(x, f(x))$ such that

$$\frac{f(x' + tV) - f(x')}{t} \cap Q \neq \emptyset, \qquad (x', f(x')) \in W,\ t \leqslant t_0.$$

We may simplify the above formulation by introducing the «graph topology» in X. If $\vartheta = \vartheta_1 \times \vartheta_2$, then we denote by ϑ_f the supremum of ϑ_1 and of the coarsest topology in X for which f (into (Y, ϑ_2)) is continuous.

In particular, if τ is discrete, then $k \in f^{H}(x)h$, whenever for every $Q \in \mathcal{N}_\sigma(k)$ there are $W \in \mathcal{N}_{\vartheta_f}(x)$ and $t_0 > 0$ such that

$$\frac{f(x' + th) - f(x')}{t} \in Q, \qquad \text{for } t < t_0 \text{ and } x' \in W.$$

This is a property akin to *strict differentiability* which is not uniform in h.

The above concepts concerning functions and the analogous infinitesimal concepts for general relations show strong resemblance. Thus one may call them derivatives: *outer derivative* in the case of contingent, *inner derivative* for tangent, and similarly *hyperderivative*, *paraderivative* and *Hadamard derivative*. Since terminology in the area has not been yet consolidated, I am careful not to introduce the above names formally. I am uncertain whether they fit more for the case of the discrete topology in X or maybe they should have some other requirements like convexity or semicontinuity.

Relations we have been investigating may admit empty values; in particular, the resulting mappings may be implicitly defined on a proper subset $\mathfrak{D}(f)$ of X.

One may also propose definitions that deploy the domain explicitly. For instance given a mapping f defined in a neighborhood of a set D we may define an approximating cone at $(x, f(x))$ as the restriction of $f^{T}(x)$ to $T_D(x)$.

Here we discover a variety of possibilities as the approximating cone of the domain and that of the mapping may be of different type.

This approach is very well adapted to applications in optimality theory.

Already the total differential of SEVERI [33] is defined along these lines. Another example is furnished by the derivative of URSESCU [35] which is the Hadamard derivative (in our sense f^H) of a mapping defined on $D = \mathfrak{D}(f)$ such that $\mathfrak{D}(f^H(x)) = K_{\mathfrak{D}(f)}(x)$.

In general, we have

$$(f|_D)^T(x) \subset f^T(x)|_{T_D(x)}, \qquad (f|_D)^K(x) \subset f^K(x)|_{K_D(x)}, \tag{6.6}$$

This is by virtue of (1.5) (having in mind (5.2)), since the relation $f|_D$ is equal to $f \cap \{(x, y): x \in D\}$.

Surely, hypertangents will not in general enjoy a similar property.

One may use the results of Section 5 to provide sufficient conditions for the opposite inclusion to hold, but we are not going to discuss this here.

Another very important class of relations are epigraphs. We observe that the homothety of the epigraph of f at $(x, f(x))$ is the epigraph of the homothety (6.3). More generally, we shall consider the homothety of the epigraph of at (x, r) in epi f. There will result the epigraph of

$$\ell_{f,(x,r),}(h) = \frac{f(x+th) - r}{t}. \tag{6.7}$$

Before we face the infinitesimal concepts for this case, it is useful to propose a change of notation: ${}^{\geqslant}f = \operatorname{epi} f$.

Apart from its brevity, the new symbol reflects the fact that the epigraph ${}^{\geqslant}f$ is the composition of the relation f and of the order relation $\geqslant$ in R. The inverse of the epigraph is the level relation and it is natural to put $({}^{\geqslant}f)^{-1} = f^{-1\leqslant}$.

The *contingent, Hadamard, tangent, hypertangent epi-derivatives* then result from the general definitions:

$$\begin{aligned}{}^{\geqslant}f^K(x)h &= \left[\Gamma(\mathcal{N}(0)^-, \tau^-)\left(\frac{1}{t}(f(x+th') - f(x))\right)\right](h) = \\ &= \sup_{Q\in\mathcal{N}_\tau(h)} \sup_{t_0} \inf_{t\leqslant t_0} \inf_{h'\in Q} \frac{1}{t}(f(x+th') - f(x))\end{aligned} \tag{6.8}$$

$$\begin{aligned}{}^{\geqslant}f^H(x)h &= \left[\Gamma(\mathcal{N}(0)^+, \tau^+)\left(\frac{1}{t}(f(x+th') - f(x))\right)\right](h) = \\ &= \inf_{Q\in\mathcal{N}_\tau(h)} \inf_{t_0} \sup_{t\leqslant t_0} \sup_{h'\in Q} \frac{1}{t}(f(x+th') - f(x))\end{aligned} \tag{6.9}$$

$$\begin{aligned}{}^{\geqslant}f^T(x)h &= \left[\Gamma(\mathcal{N}(0)^+, \tau^-)\left(\frac{1}{t}(f(x+th') - f(x)\right)\right](h) = \\ &= \sup_{Q\in\mathcal{N}_\tau(h)} \inf_{t_0} \sup_{t\leqslant t_0} \inf_{h'\in Q} \frac{1}{t}(f(x+th') - f(x))\end{aligned} \tag{6.10}$$

$$\begin{aligned}{}^{\geqslant}f^\dagger(x)h &= \left[\left(\Gamma\mathcal{N}(0)^+, \mathcal{N}_\vartheta\big(x, f(x)\big), {}^{\geqslant}f\right)^+, \tau^-\right)\frac{1}{t}(f(x'+th') - r')\right](h) = \\ &= \sup_{Q\in\mathcal{N}_\tau(h)} \inf_{W\in\mathcal{N}_\vartheta((x,f(x)),\geqslant f)} \inf_{t_0} \sup_{t<t_0} \sup_{(x',r')\in W} \inf_{h'\in Q} \frac{1}{t}(f(x'+th') - r').\end{aligned} \tag{6.11}$$

Of course, the epigraph of the contingent (tangent hypertangent, etc.) epi-desivative is the contingent (tangent hypertangent, ...) of the epigraph. In ROCKAFELLAR's notation

$$^{\geqslant}f^{\uparrow}(x) = f^{\uparrow}(x) \tag{6.11'}$$

and we shall call (6.11) the *Rockafellar* (*directional*) derivative When the topology τ is discrete, we have the (*generalized*) *Clarke* (*directional*) derivative; it is customarily denoted by $f^0(x)$.

By introducing the epigraph topology in $\mathfrak{D}(f)$ (graph topology of the epigraph) we may replace $\mathcal{N}_\vartheta\big((x, f(x)), {}^{\geqslant}f\big)$ in (6.11) by more convenient $\mathcal{N}_{(\geqslant f)\vartheta}(x)$ and by using the notation of Section 2 we have the simpler

$$^{\geqslant}f^{\uparrow}(x)\,h = \operatorname{ls}_{\substack{h' \xrightarrow{\tau} h \\ t \to 0,\ x' \xrightarrow[(\geqslant f)\vartheta]{} x}} \frac{1}{t}\big(f(x' + th') - f(x')\big)\,. \tag{6.11''}$$

provided f is ϑ-lower semicontinuous at x. We notice that the epigraph topology determined by f and ϑ_1, ϑ_2 is the coarsest topology finer than ϑ_1 for which f is upper semicontinuous.

The contingent and tangent epi-derivatives are called (*generalized*) *Dini upper* (*lower*) derivatives. For instance, we have that $k \geqslant f^T(x)h$, if and if for each ε, for every $Q \in \mathcal{N}(h)$ there is t_0 such that for $t < t_0$

$$\inf_{h' \in Q} \big(f(x + th') - f(x) - tk\big) \leqslant \varepsilon t$$

and for the discrete topology in X

$$f(x + th) - f(x) - th \leqslant \varepsilon t\,.$$

It follows from convergence theory that

$$^{\geqslant}f^{k}(x) \leqslant {}^{\geqslant}f^{T}(x) \leqslant {}^{\geqslant}f^{\uparrow}(x) \leqslant f^{0}(x)\,. \tag{6.12}$$

After ROCKAFELLAR [28] call f *subdifferentially regular at x towards h* whenever $^{\geqslant}f^{K}(x)\,h = {}^{\geqslant}f^{\uparrow}(x)\,h$. In this case the two are also equal to $^{\geqslant}f^{T}(x)h$.

The *radial subdifferential regulality* (τ is discrete) was studied by CLARKE [5]. Equivalent concept of quasidifferentiability is due to PSHENICHNII [27] (see also MIFFLIN [21]).

In the radial case if $^{\geqslant}f^{K}(x) = {}^{\geqslant}f^{T}(x)$, then both are actually equal to the directional Gâteaux derivative (without continuity and linearity)

$$f'(x)\,h = \lim_{t \to 0} \frac{1}{t}\big(f(x + th) - f(x)\big)\,.$$

Though it is possible to use Theorem 4.4 to furnish conditions for subdifferential regularity we shall give one directly.

PROPOSITION 6.2. – An l. sc. function f is radially subdifferentially regular at x towards h, if and only if $f'(x)h$ exists and for every $\varepsilon > 0$ there are $W \in \mathcal{N}_{(\geqslant f)\vartheta}(x)$ and t_0 such that for $t \leqslant_0$ and $x' \in W$

$$f(x' + th) - f(x') - f'(x)h \leqslant \varepsilon t .$$

The above condition is a unilateral strict differentiability (not uniform in h).

The *Levitin-Milyutin-Osmolovskii approximation* (see IOFFE[18]) amounts to the radial subdifferential regularity of f at x uniformly for x in bounded sets. In fact, the original definition of the L.M.O.-approximation involves a family $f'(x)(\cdot)$ of approximations. However if $\{f'(x')(\cdot)\}$ is equi upper semicontinuous as x' tends to x, then it may be replaced by the single function $f'(x)(\cdot)$.

7. – Approximations which are radial in the domain.

What most recalls classical directional derivatives are approximating cones of relations in $X \times Y$ considered with respect to the discrete topology of X. If $(h, k) \in \in X \times Y$ belongs to such an approximating cone of C at (x, y) then $t \cdot h$ is forced to stay within, or to return frequently into, $\mathfrak{D}(C) - x$ as t tends to 0. For instance, given $C: X \to Y$, $k \in \mathcal{C}_C^{(\iota \times \tau)/\theta}(x, y)h$ whenever for every $Q \in \mathcal{N}_\tau(k)$ there are t_0 and $W \in \mathcal{N}_\vartheta((x, y), C)$ such that for every $t \leqslant t_0$ and $(x', y') \in W$

$$y' + tQ \cap C(x' + th) \neq \emptyset .$$

Due to general properties of limits, an approximating cone radial in the domain is a subset of the corresponding approximating cone with respect to any topology (in the domain). The objective of this section is to establish additional conditions under which the two cones are equal. Such conditions enable one to cope with general concepts with the aid of simplified formulae.

By virtue of Theorems 4.2 and 4.3 the above cones are equal provided that the homothety (of C at x for classical cones and of C for hypercones) in quasi τ-equi semicontinuous. By direct checking we conclude that

LEMMA 7.1. – Let $(Y, \mathcal{U})$ be a uniform space. The homothety of a relation C at (x, y) is quasi τ-equi semicontinuous at h, if and only if for $U \in \mathcal{U}$ there are $V \in \mathcal{U}$, $Q \in \mathcal{N}(h)$ and $s > 0$ such that for every $t < s$

$$(tU)[C(x + th) - y] \supset (tV)[C(x + tQ) - y] . \tag{7.1}$$

In particular the difference quotient at x is τ-equi lower semicontinuous at h if and only if for every $\varepsilon > 0$ there are $Q \in \mathcal{N}_\tau(h)$ and $s > 0$ such that for each $t < s$

$$\sup_{h' \in Q} \big(f(x + th) - f(x + th')\big) \leqslant \varepsilon t \,. \tag{7.2}$$

LEMMA 7.2. – The homothety of C is quasi τ-equi semicontinuous at h (as the family filtered by $\mathcal{N}((x_0, y_0), C) \times \mathcal{N}(0)$ if and only if (7.1) holds for every $t < s$) and $(x, y) \in W$ (a neighborhood in $\mathcal{N}((x_0, y_0), C)$ dependent on U).

In particular, the difference quotient of f is τ-equi lower semicontinuous at h, as a family filtered by $\mathcal{N}_\vartheta((x, f(x)), {\geqslant}f) \times \mathcal{N}(0)$, if and only if for every $\varepsilon > 0$ there, are $Q \in \mathcal{N}_\tau(h)$, $s > 0$ and $W \in \mathcal{N}_\vartheta((x, y), {\geqslant}f)$ such that

$$\sup_{x' \in \mathfrak{D}(W)} \sup_{h' \in Q} \big(f(x' + th) - f(x' + th')\big) \leqslant \varepsilon t \,, \qquad t < s. \tag{7.3}$$

Formalzing what we have already said

THEOREM 7.3. – Suppose that (7.1) holds on the domain of the $\tau \times \mathfrak{U}$-contingent (tangent) of C at (x, y). Then the $\tau \times \mathfrak{U}$-contingent (tangent) and $i \times \mathfrak{U}$-contingent (tangent) are equal.

Of course, the above theorem applied to the epigraph of a function f provides the condition (7.2) under which Dini upper (lower) derivative in the generalizad sense and the strict sense are equal at h.

We say that f is *directionally Lipschitzian at x towards h*, if for every $\varepsilon > 0$ there are neighborhood Q of h and $s > 0$ such that for every $t < s$

$$\sup_{h' \in Q} |f(x + th) - f(x + th')| \leqslant \varepsilon t \,. \tag{7.4}$$

Observe that if f is locally Lipschitzian at x then it is directionally Lipschitzian at x and that the latter entails Property (7.2).

In view of Theorem 4.3 and Lemma 7.2 we have

THEOREM 7.4. – If (7.3) holds, then the Rockafellar derivative and the (generalized) Clarke derivative are equal at h

$$f^{\uparrow}(x)\, h = f^{0}(x)\, h \,. \tag{7.5}$$

Formula (7.5) has been proved by Rockafellar [30, Thm. 3] provided f is *directionally Lipschitzian at x with respect to h.* This condition amounts to the existence of $Q \in \mathcal{N}_\tau(h)$, $W \in \mathcal{N}_\vartheta((x, f(x)), {\geqslant}f)$ $t_0 > 0$ and a constant M such that

$$\sup_{x' \in \mathfrak{D}(W)} \sup_{h' \in Q} \big(f(x' + th') - f(x')\big) \leqslant Mt \,, \qquad t \leqslant t_0 \,. \tag{7.6}$$

The following proposition provides a criterion for (7.3) to hold. Note that the functions considered in this proposition are not, in general, directionally Lipschitzian.

Let f be a real-valued function on X, C a subset of X. We denote $f_C = f + \chi_C$, χ_C the indicator function.

PROPOSITION 7.5. – Let X be a normed space. Let f be locally Lipschitz on a set C. If h belongs to the radial hypertangent of C at x, then f_C satisfies (7.3).

PROOF. – For ε take Q to be a ball of radius δ with center h and choose s and $W \in \mathcal{N}(x)$ so that $x' + th \in C$ for $x' \in W \cap C$ and $t < s$. If needed δ, W and s are reduced so that f is Lipschitzian on $(W \cap C + (0, s)Q)$ with the constant c and so that $c\delta < \varepsilon$. Then if $x' + th'$ is in C (with $x' \in C \cap W$, $t < s$, $h \in Q$), then

$$f_C(x' + th) - f_C(x' + th') \leqslant c\delta t < \varepsilon t\,.$$

and otherwise the difference is $-\infty$.

Appendix.

I shall give here a geometrical and quantitative proof of the fact (which is known) that

THEOREM. – Let A, B be convex sets in a Hausdorff topological vector space X for which $A \cap \operatorname{Int} B \neq \emptyset$. Then

$$\operatorname{Cl}(A \cap \operatorname{Int} B) = \operatorname{Cl}(A \cap B) = \operatorname{Cl} A \cap \operatorname{Cl} B\,.$$

Let x, v be different elements of X and let W be a neighborhood of 0 in X; $\varkappa$, μ, λ, ν positive reals.

By straightforward checking one establishes

LEMMA 1. – If $x' \in x + \mu W$ and $t \geqslant \mu/(\mu + \lambda)$, then

$$(1-t)x + t[v + (1-\lambda)W] \subset (1-t)x' + t(v + W)\,.$$

LEMMA 2. – If $t \geqslant \nu/(\varkappa + \nu)$, then

$$(1-t)(x + \nu W) + tv \subset (1-t)x + t(v + \varkappa W)\,.$$

LEMMA 3 (corollary of Lemmae 1, 2). – If $t \geqslant \max\left(\nu/(1-\lambda+\nu), \mu/(\mu+\lambda)\right)$ and $x' \in x + \nu W$ then

$$(1-t)(x + \nu W) + tv \subset (1-t)x' + t(v + W)\,.$$

In particular by putting $\lambda = \frac{1}{2}$ and $\nu = \mu$ we get the condition $t \geqslant 2\mu/(1 + 2\mu)$.

PROOF OF THEOREM. – We need prove only one inclusion. Let $x \in \mathrm{Cl}\, A \cap \mathrm{Cl}\, B$ and let v and W be such that $v \in A$ and $v + W \subset B$. For every zero neighborhood $V \subset W$ and $\nu > 0$ we have that

$$(x + \nu V) \cap A \neq \emptyset \quad \text{and} \quad x(\nu) \in x + \nu V \cap B .$$

of course,

$$(1-t)x(\nu) + t(v + W) \subset \mathrm{Int}\, B, \text{ as } 0 < t \leqslant 1 \text{ and by Lemma 3 for } t \geqslant \frac{2\nu}{2\nu + 1}$$
$$(1-t)(x + \nu V) + tv \subset \mathrm{Int}\, B .$$

On the other hand $(1-t)(x + \nu V) + tv$ meets A for every $0 < t \leqslant 1$, thus it meets, $A \cap \mathrm{Int}\, B$ if $t \geqslant 2\nu/(2\nu + 1)$.

Let $\beta > 0$ be such that $v - x \in \beta V$. Since

$$(*) \qquad (1-t)(x + \nu V) + tv = x + t(v - x) + (1-t)\nu V ,$$

the set $(*)$ is included in $x + sV$ provided that $s \geqslant t\beta + (1-t)$. By setting $t = 2\nu/(2\nu + 1)$, we get the condition that $s \geqslant ((1 + 2\beta)\lambda\nu)/(1 + 2\nu)$.

Therefore for every V we may find $U \subset V \cap W$ and $s(\nu)$ such that $(1-t)(x + \nu U) + tv \subset sV$, where $t = 2\nu/(2\nu + 1)$.

Therefore sV meets $A \cap \mathrm{Int}\, B$.

Since s tends to 0 with ν, every neighborhood of x meets $A \cap \mathrm{Int}\, B$.

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