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A de Finetti-type Theorem with m-Dependent States

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Summary. In this paper certain translation invariant states on the infinite tensor product C^*-algebra $\mathscr{A} = \bigotimes_{i=1}^{\infty} \mathscr{B}_i$ are considered. For $m \in \mathbb{Z}^+$ a state φ on $\mathscr{A}$ is m-dependent if

$$\varphi(a_1 \otimes a_2 \otimes \ldots \otimes a_l) = \varphi(a_1 \otimes \ldots \otimes a_k)\varphi(a_{k+m+1} \otimes \ldots \otimes a_l)$$

whenever $l > k + m$ and $a_{k+1} = a_{k+2} = \ldots = a_{k+m} = I$. The closed convex hull of the stationary m-dependent states is characterized by a symmetry condition. The case of $m = 0$ corresponds to independence and the result reduces to a C^*-algebraic version, due to E. Størmer, of the classical de Finetti's theorem on exchangeable sequences.

Introduction

De Finetti's celebrated theorem asserts that any exchangeable process is an average of independent identically distributed processes [9]. More precisely, let $\xi_1, \xi_2, \ldots$ be a sequence of $\{0, 1\}$-valued random variables such that

$$P(\xi_1 = e_1, \xi_2 = e_2, \ldots, \xi_n = e_n) = P(\xi_1 = e_{\pi(1)}, \xi_2 = e_{\pi(2)}, \ldots, \xi_n = e_{\pi(n)})$$

holds for all $n \in \mathbb{N}$, for all permutations π of $\{1, 2, \ldots, n\}$ and for every $e_1, e_2, \ldots, e_n \in \{0, 1\}$. Then there exists a unique probability measure μ on $[0, 1]$ such that

$$P(\xi_1 = e_1, \xi_2 = e_2, \ldots, \xi_n = e_n) = \int p^s(1-p)^{n-s}\, d\mu(p)$$

where $s = e_1 + e_2 + \ldots + e_n$ [8].

In the last two decades there has been a new strong interest in exchangeability and several extensions of de Finetti's original theorem have been obtained. Concerning this development we refer to the survey articles [2, 14] and below we describe results directly related to our generalization. Probabilists being averse to the C^*-algebraic language of the present paper may consult [1] to make themselves familiar with the algebraic formulation of de Finetti's theorem.

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Hewitt and Savage extended de Finetti's theorem from $\{0, 1\}$ to any compact Hausdorff space Ω [10]. For our purpose it is more suitable to consider measures on the infinite product space $X = \Omega \times \Omega \times \dots$. Any permutation of the natural numbers gives rise to a transformation of X through the permutation of the coordinates. This transformation is continuous and induces a mapping on the space of continuous functions on X. The functional analytic formulation prefers positive functionals on the function space to measures. Denote by $\mathscr{B}(\mathscr{A})$ the C^*-algebra of all continuous functions on $\Omega(X)$. Then $\mathscr{A} = \mathscr{B} \otimes \mathscr{B} \otimes \dots$ and the result of Hewitt and Sevage tells us in this language that every symmetric (i.e., permutation invariant) positive functional is a unique mixture of product ones. Allowing arbitrary C^*-algebra $\mathscr{B}$ Størmer proved in 1969 that the symmetric states form a simplex with a closed extreme boundary consisting of product states [17]. His result inspired similar theorems and found applications also in quantum statistical mechanics [5, 6, 7, 11, 15].

Our aim in the present article is to carry out a generalization. In order to have the extremal states to be m-dependent we use a weaker notion of symmetry. If $m \in \mathbb{Z}^+$ then we say that a stationary state φ on $\mathscr{A}$ is m-dependent if

$$\varphi(a_1 \otimes a_2 \otimes \dots \otimes a_l) = \varphi(a_1 \otimes \dots \otimes a_k)\varphi(a_{k+m+1} \otimes \dots \otimes a_l)$$

whenever $l > k + m$ and $a_{k+1} = a_{k+2} = \dots = a_{k+m} = I$. This notion comes from probability theory where m-dependent sequences of random variables are frequently studied as slight extension of independence [12].

The method we use is based on results from harmonic analysis on abelian semigroups and originated from the beautiful paper of Ressel [16].

Preliminaries

Let $\mathscr{B}$ be a unital C^*-algebra and for each $n \in \mathbb{N}$ let $\mathscr{B}_n$ a copy of $\mathscr{B}$. Write $\mathscr{A}$ for the infinite projective tensorproduct $\bigotimes_{i=1}^{\infty} \mathscr{B}_i$ [13, 18]. If I is a subset of $\mathbb{N}$ then we denote by $\mathscr{A}_I$ the C^*-subalgebra generated by $\cup\{\mathscr{B}_n : n \in I\}$. If π is a permutation of $\mathbb{N}$ then there is an automorphism α_π of $\mathscr{A}$ such that

$$\alpha_\pi(i_k(b)) = i_{\pi(k)}(b) \qquad (b \in \mathscr{B}, \quad k \in \mathbb{N})$$

where $i_k : \mathscr{B} \to \mathscr{B}_k \subset \mathscr{A}$ is the embedding of $\mathscr{B}$ as a k^{th} factor. After Hewitt, Savage and Størmer we call a state φ of $\mathscr{A}$ symmetric if $\varphi \circ \alpha_\pi = \varphi$ for all finite permutations of $\mathbb{N}$. Evidently, a symmetric state is shift invariant. We write α for the right shift endomorphism of $\mathscr{A}$.

Since the permutation group is generated by transpositions, the permutation invariance may be formulated in a slightly weaker way.

$$\varphi(a_1 a\alpha(b) a_2) = \varphi(a_1 b\alpha(a) a_2) \tag{1}$$

whenever $a_1 \in \mathscr{A}_{[1, l-1]}$, $a, b \in \mathscr{B}_l$ and $a_2 \in \mathscr{A}_{[l+2, \infty]}$. Condition (1) requires the invariance of φ under the permutation exchanging l with $l + 1$ and leaving all other coordinates fixed.

Let $m \in \mathbb{Z}^+$ be fixed. We say that the state φ of $\mathscr{A}$ is m-dependent if

$$\varphi(ab) = \varphi(a)\varphi(b) \tag{2}$$

whenever $a \in \mathscr{A}_{[1,n]}$ and $b \in \mathscr{A}_{[n+m+1,\infty)}$. In probability theory m-dependent sequences of random variables are easily obtained by taking functions of an independent sequence [12]. (For example, if (η_n) is independent then $\xi_n = \eta_n + \eta_{n+1} + \ldots + \eta_{n+m}$ forms an m-dependent sequence.) In the theory of operator algebras it is a bit more complicated to show an m-dependent state. In this paper we shall consider only α-invariant m-dependent states.

We say that φ is m-symmetric if the following two conditions are satisfied.

$$\varphi(a_1 a\alpha^{k+m}(b)a_2) = \varphi(a_1 b\alpha^{k+m}(a)a_2) \tag{3}$$

if $a_1 \in \mathscr{A}_{[1,l-1]}$, $a, b \in \mathscr{A}_{[l+m,l+m+k]}$, $a_2 \in \mathscr{A}_{[l+2m+2k+1,\infty]}$ and

$$\varphi(a_1\alpha^k(a)) \quad \text{is independent of } k \tag{4}$$

if $a \in \mathscr{A}_{[1,l-1]}$, $b \in \mathscr{A}_{[l+m,\infty]}$ and $k, l \in \mathbb{Z}^+$. Below we refer to conditions (3) and (4) simply as symmetry conditions. Note that an m-symmetric state is necessarily α-invariant. Indeed, (4) tells that it is so on $a \in \mathscr{A}_{[m+1,\infty]}$ and combining this with (3) we get the stationarity. We shall see that the appropriate symmetry conditions needed to characterize the closed convex hull of m-dependent states are (3) and (4).

Now we review a few things from harmonic analysis on semigroups. Let S be an abelian semigroup, written additively, with neutral element written 0. A function $f: S \to \mathbb{R}$ is completely positive definite if

$$\sum_{j,k=1}^{n} c_j c_k f(s + s_j + s_k) \geqq 0 \tag{5}$$

for all $n \geqq 1$, $s, s_1, s_2, \ldots, s_n \in S$ and $c_1, c_2, \ldots, c_n \in \mathbb{R}$. A semicharacter $\rho: S \to \mathbb{R}$ means a function such that

$$\rho(s+t) = \rho(s)\rho(t) \quad \text{and} \quad \rho(0) = 1 \,.$$

A nonnegative semicharacter is completely positive definite. An introduction to positive definite functions on abelian semigroups is found in [3]. For our purpose a theorem of Ressel has central importance [16]. Let F be a completely positive definite bounded function on S. Then there exists a unique Radon measure μ over the space of nonnegative bounded semicharacters S_+^* such that

$$F(s) = \int \rho(s)\, d\mu(\rho) \,.$$

Result

Let $\mathscr{A}_{[1,n]}$, $\mathscr{A}$ and α be as above. We denote by $\mathscr{S}_m$ the set of all m-symmetric states. On the state space of $\mathscr{A}$ we consider the weak* topology.

Theorem. *The closed extremal boundary of the compact convex set $\mathcal{S}_m$ consists of the (α-invariant) m-dependent states. Every $\psi \in \mathcal{S}_m$ admits an integral decomposition*

$$\psi(a) = \int \rho(a)\, d\mu(\rho) \qquad (a \in \mathcal{A})$$

with a unique probability Radon measure on the (α-invariant) m-dependent states.

Lemma. *If $\psi \in \mathcal{S}_m$ and $a_1 \in \mathcal{A}^+_{[1,n]}$, $a_2 \in \mathcal{A}_{[1,k]}$ then*

$$\psi(a_1 \alpha^{n+m}(a_2^*)\alpha^{n+k+2m}(a_2)) \geqq 0 .$$

Since

$$a_1 \left(\sum_{l=1}^{t} \alpha^{l(k+m)+n}(a_2^*) \right) \left(\sum_{l=1}^{t} \alpha^{l(k+m)+n}(a_2) \right) \geqq 0 .$$

we have

$$T = \psi\left(a_1 \sum_{l=1}^{t} \alpha^{l(k+m)+n}(a_2^*) \sum_{l=1}^{t} \alpha^{l(k+m)+n}(a_2) \right) \geqq 0 .$$

Due to the symmetry conditions (3) and (4)

$$T = t\psi(a_1 \alpha^{n+m}(a_2^* a_2)) + (t^2 - t)\psi(a_1 \alpha^{n+m}(a_2^*)\alpha^{n+k+2m}(a_2))$$

and division by t^2 and letting $t \to \infty$ gives the Lemma.

To prove the Theorem we set S for the free abelian semigroup generated by the positive contractions in $\mathcal{A}_\infty = \cup\{\mathcal{A}_{[1,n]} : n \in \mathbb{N}\}$. A typical element of S is a formal sum

$$s = a_1 \dotplus a_2 \dotplus \ldots \dotplus a_k \qquad (a_i \in \mathcal{A}_\infty^{+,1}\,, \quad k \in \mathbb{N})$$

where the sequence $(a_1, a_2, \ldots, a_k)$ is determined by s up to a permutation. For a finite sequence $(a_1, a_2, \ldots, a_k)$ in $\mathcal{A}_\infty^{+,1}$ we set

$$G_0(a_1, a_2, \ldots, a_k) = \alpha^{m+l}(a_1) \ldots \alpha^{k(m+l)}(a_k)$$

if l is the smallest integer such that $a_1, a_2, \ldots, a_k \in \mathcal{A}_{[1,l]}$. If $s = a_1 \dotplus a_2 \dotplus \ldots \dotplus a_k$ and $\psi \in \mathcal{S}_m$ then we may define

$$F(s) = \psi(G_0(a_1, a_2, \ldots, a_k))$$

because the symmetry conditions on ψ provide that the right hand side is independent of the ordering of $(a_1, a_2, \ldots, a_k)$. For the sake of a simpler notation we choose and fix for every $s \in S$ an ordered decomposition $s = a_1 \dotplus a_2 \dotplus \ldots \dotplus a_k$ and we set

$$G(s) = G_0(a_1, a_2, \ldots, a_k) .$$

We have $G(s) \in \mathcal{A}_{[1,k(m+l)+l]}$ with l described above and we call the number $k(m+l) + l = L(s)$ the rank of s. The symmetry conditions yield that

$$F(s_1 \dotplus s_2 \dotplus s_3) = \psi(G(s_1)\alpha^t(G(s_2))\alpha^u(G(s_3)))$$

whenever $t \geqq L(s_1)$, $u \geqq t + L(s_2)$ and $s_1, s_2, s_3 \in S$.

Now we are going to check that F is completely positive definite. We have to show that

$$\sum_{j,k=1}^{n} c_j c_k F(s \dotplus s_j \dotplus s_k) \geqq 0$$

for all $n \geqq 1$, $s, s_1, s_2, \ldots, s_n \in S$ and $c_1, c_2, \ldots, c_n \in \mathbb{R}$. We choose $l \in \mathbb{N}$ such that

$$l \geqq L(s), L(s_1), \ldots, L(s_n)$$

and have

$$F(s \dotplus s_i \dotplus s_j) = \psi(G(s)\alpha^{il}(G(s_i))\alpha^{(j+n)l}(G(s_j)))\,.$$

Taking $b_0 = G(s)$ and $b_i = \alpha^{il}(g(s_i))$ we obtain

$$\sum_{j,k=1}^{n} c_j c_k F(s \dotplus s_j \dotplus s_k) = \psi\left(b_0\left(\sum_{i=1}^{n} c_i b_i\right)\alpha^{nl}\left(\sum_{j=1}^{n} c_j b_j\right)\right)$$

which is nonnegative thanks to the Lemma. Indeed, $b_0, b_1, \ldots, b_n$ are positive operators and $b_1, \ldots, b_n$ are contained in $\mathscr{A}_{[m+l, nl]}$.

The function F: $S \to [0, 1]$ is proven to be completely positive definite. According to Proposition 1 of [16] there is a unique Radon probability measure μ on the compact subset $K = \{\rho \in S^*: 0 \leqq \rho \leqq 1\}$ of the semicharacters such that

$$F(s) = \int \rho(s)\, d\mu(\rho) \qquad (s \in S)\,.$$

To each m-dependent state φ there corresponds a semicharacter $j(\varphi)$ in an obvious way:

$$j(\varphi)(a_1 \dotplus a_2 \dotplus \ldots \dotplus a_k) = \prod_{i=1}^{k} \varphi(a_i)$$

So j is an embedding of the m-dependent states into K. We have to prove that the measure μ is actually concentrated on the range of j.

Following Ressel we fix $a_1, a_2 \in \mathscr{A}_\infty^{+,1}$ such that $a_1 + a_2 = I$. Then

$$\begin{aligned}
\int (\rho(a_1) + \rho(a_2))^n d\mu(\rho) &= \sum_{k=0}^{n} \binom{n}{k} \int \rho(a_1)^k \rho(a_2)^{n-k} d\mu(\rho) \\
&= \sum_{k=0}^{n} \binom{n}{k} \int \rho(ka_1 \dotplus (n-k)a_2)\, d\mu(\rho) \\
&= \sum_{k=0}^{n} \binom{n}{k} F(ka_1 \dotplus (n-k)a_2) \\
&= \psi((a_1 + a_2)\alpha^{l+m}(a_1 + a_2) \ldots \alpha^{(n-1)(l+m)}(a_1 + a_2)) = 1
\end{aligned}$$

as a consequence of symmetry. So we find that $\int (\rho(a_1) + \rho(a_2))^n d\mu(\rho) = 1$ for every $n \in \mathbb{N}$, which implies easily that

$$H^2(a_1, a_2) = \{\rho \in K: \rho(a_1) + \rho(a_2) = 1\}$$

if of measure 1. Similarly, for $b_1, b_2, b_3 \in \mathscr{A}_\infty^{+,1}$ with $b_1 + b_2 + b_3 = I$ we have $\rho(b_1) + \rho(b_2) + \rho(b_3) = 1$ for almost all $\rho \in K$. Since

$$H^3(b_1, b_2, b_3) = \{\rho \in K: \rho(b_1) + \rho(b_2) + \rho(b_3) = 1\}$$

is closed we obtain that

$$K_1 = \cap\{H^2(a_1, a_2): a_1 + a_2 = I\} \cap \{H^3(b_1, b_2, b_3): b_1 + b_2 + b_3 = I\}$$

is a closed set of full measure.

Now let $\rho \in K_1$. If $a+b+c=I$ then $\rho(a)+\rho(b)+\rho(c)=1=\rho(a+b)+\rho(c)$. This gives $\rho(a+b)=\rho(a)+\rho(b)$ and, in particular, ρ is monotone, $\rho(0)=0$ and $\rho(I)=1$. By induction $\rho(\lambda a)=\lambda\rho(a)$ for all rational $\lambda \in [0,1]$. Due to the monotonicity this must hold also for irrational λ. It turns out that ρ comes from the restriction of a state on $\mathscr{A}$ to the positive contractions $\mathscr{A}_\infty^{+,1}$ of $\mathscr{A}_\infty$.

Let us fix an $m' \geqq m$ and $a \in \mathscr{A}_\infty^{+,1}$. We define three endomorphisms of S. Being S free any mapping given on the generators extends to a homomorphism in a unique way. Set

$$h(b)=\alpha(b) \qquad (b \in \mathscr{A}_\infty^{+,1})$$

$$h_a(b)=b \dot{+} a \qquad (b \in \mathscr{A}_\infty^{+,1})$$

$$k_a^{m'}(b)(a)=b\alpha^{l+m'}(a) \qquad (b \in \mathscr{A}_{[l,\infty)}^{+,1} \text{ but } b \notin \mathscr{A}_{[l-1,\infty)}^{+,1}) .$$

It is straightforward to see that

$$F \circ h = F \quad \text{and} \quad F \circ h_a = F \circ k_a^{m'} .$$

From the uniqueness of the decomposing measure we obtain that μ must be concentrated on the closed set

$$K_2=\{\rho \in K_1 \colon \rho \circ h_a = \rho \circ k_a^{m'} \quad \text{for every} \quad m' \geqq m,\, a \in \mathscr{A}_\infty^{+,1} \text{ and } \rho \circ h = \rho\}$$

For $\rho \in K_2$ the α-invariance is evident and

$$\rho(a_1\alpha^{l+m'}(a)) = \rho \circ k_a^{m'}(a_1) = \rho \circ h_a(a_1) = \rho(a \dot{+} a_1) = \rho(a)\rho(a_1)$$

provides the m-dependency (2). So we proved that the range of the embedding j is contained in K_2 and the proof of the theorem is complete.

Discussion

First we formulate our theorem in terms of random variables. Let $\xi_1, \xi_2, \ldots$ be a stationary sequence of random variables. In order to make clear the m-symmetry we introduce a convenient notation. G with some subscript will always be an open set. If $I=\{n, n+1, \ldots, n+t\}$ and $J=\{k, k+1, \ldots, k+t\}$ are intervals in $\mathbb{N}$ of the same length then we write $\xi(I) \in G(J)$ for the set

$$\bigcap_{i=0}^{t} \{\xi_{n+i} \in G_{k+i}\} .$$

The notion of m-symmetry includes two kinds of invariance. On the one hand,

$$P(\xi(I_1) \in G(I_1),\, \xi(I_2) \in G(J_2),\, \xi(I_3) \in G(J_3),\, \xi(I_4) \in G(I_4)) =$$

$$= P(\xi(I_1) \in G(I_1),\, \xi(I_2) \in G(J_3),\, \xi(I_3) \in G(J_2),\, \xi(I_4) \in G(I_4))$$

whenever $I_1 < I_2 < I_3 < I_4$ are intervals in $\mathbb{N}$ such that there is a gap of length not smaller than m between the consecutive ones and J_2, J_3 are also intervals with $|I_i|=|J_i|$. On the other hand, it is required that

$$P(\xi(I_1) \in G(J_1),\, \xi(I_2+t) \in G(J_2))$$

does not depend on $t \in \mathbb{Z}^+$ whenever $I_1 < I_2, |I_i| = |J_i|$ and there is a gap of at least m numbers between I_1 and I_2. The Theorem asserts that a stationary m-symmetric sequence is a mixture of stationary m-dependent sequences. We have not seen this result in the literature of probability theory.

If we compare symmetry with m-symmetry then the essential point is the following. While symmetry is characterized by transposition of single coordinates in the definition of m-symmetry transposition of blocks of coordinates occurs (under the condition that there is an interval of length m between the two blocks).

Now we show an example of an m-dependent stationary state on an infinite tensorproduct of C^*-algebras. Let $l^2(\mathbb{Z})$ be the complex Hilbert space of the double infinite sequences. Then there is a unique C^*-algebra $\tilde{\mathscr{A}}$ determined by the following conditions

(i) for every $f \in l^2(\mathbb{Z})$ a unitary $W(f)$ in $\tilde{\mathscr{A}}$ and the linear span of $\{W(f): f \in l^2(\mathbb{Z})\}$ is dense in $\tilde{\mathscr{A}}$.
(ii) $W(-f) = W(f)^* \qquad (f \in l^2(\mathbb{Z}))$
(iii) $W(f)W(g) = W(f+g)\exp(i\mathrm{Im}\langle f, g\rangle) \qquad (f, g \in l^2(\mathbb{Z}))$.

We fix a function $h: \mathbb{Z} \to \mathbb{C}$ such that $\mathrm{supp}\, h \subset [0, m]$ and the Fourier transform of h is nonnegative. Define the convolution operator A on $l^2(\mathbb{Z})$ by

$$(Af)(n) = \sum_{i=-\infty}^{\infty} h(n-i)g(i)\,.$$

Then

$$\alpha(f, g) = \mathrm{Re}(\langle f, g\rangle + \langle Af, g\rangle) \qquad (f, g \in l^2(\mathbb{Z}))$$

gives a positive semidefinite real bilinear form on $l^2(\mathbb{Z})$ and there is a state φ on $\tilde{\mathscr{A}}$ such that

$$\varphi(W(f)) = \exp(-\tfrac{1}{2}\alpha(f, f)) \qquad (f \in l^2(\mathbb{Z}))$$

(Concerning the details of the construction of $\tilde{\mathscr{A}}$ and φ we refer to 5.2 of [4].) For $n \in \mathbb{Z}^+$ let $\mathscr{B}_n$ be the subalgebra generated by $\{W(f): \mathrm{supp} f \subset \{n\}\}$. Then

$$\bigotimes_{n=1}^{\infty} \mathscr{B}_n = \mathscr{A} = C^*\{W\{f\}: \mathrm{supp} f \subset \mathbb{N}\}$$

and φ is m-dependent on $\mathscr{A}$. Indeed, by straightforward checking

$$\varphi(W(f)W(g)) = \varphi(W(f))\varphi(W(g))$$

if $\mathrm{supp} f \subset [1, l]$ and $\mathrm{supp}\, g \subset [l+m', \infty)$ with $m' > m$.

Using the notation of the preliminaries, we assume now that $\mathscr{B} = B(\mathscr{H})$ for a complex Hilbert space $\mathscr{H}$. Hudson and Moody showed that if the symmetric state φ of $\mathscr{A}$ restricted to every $\mathscr{A}_{[1,n]}$ is normal (that is, φ is locally normal), then the decomposing measure is concentrated on the normal states [10]. Their proof goes through without change in the m-symmetric generalization.

In [7] Fannes, Lewis and Verbeure consider states on infinite tensorproduct $\mathscr{C} \otimes \mathscr{B} \otimes \mathscr{B} \otimes \ldots$ which are symmetric in the $\mathscr{B}$ factors. From the Størmer theorem they deduced a characterization of these states of $\mathscr{C} \otimes \mathscr{B} \otimes \ldots$. The m-symmetric version of their result may be obtained similarly.

In our proof the stationarity of all states was exploited very much. Fannes treated nonstationary states in a de Finetti-type theorem [5, 6]. We can imagine that by a stronger control of the permutations his theorem may be recaptured by this analytical approach and also an m-symmetric version might be obtained.

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