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Probability Theory and Related Fields

L_2-lower bounds for a special class of random walks

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Summary. We investigate the L_2-speed of convergence to stationarity for a certain class of random walks on a compact connected Lie group. We give a lower bound on the number of steps k necessary such that the k-fold convolution power of the original step distribution has an L_2-density. Our method uses work by Heckman on the asymptotics of multiplicities along a ray of representations. Several examples are presented.

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1 Introduction

Given a random walk on a compact group G with step distribution ν one is frequently interested in the speed of convergence of this random walk to its stationary distribution (under mild conditions, the normalized Haar measure on G). In this paper we investigate the distance $\|\nu_k - \lambda_G\|_2$ in $L_2(G)$ between the k-fold convolution power ν_k of a certain natural probability measure ν on a compact Lie group G and normalized Haar measure λ_G. Besides being of interest in its own right, such an L_2-estimate gives an upper bound on total variation distance and can frequently be derived with more precision than for other types of estimates. This, of course, presupposes that from some k_0 onward, ν_k does have an L_2-density with respect to Haar measure for $k > k_0$, even though the original measure ν, in many cases, is singular.
Given a measure ν on G, we are therefore faced with the following problems:

- How many steps k does it take for ν_k to be in $L_2(G)$?
- If ν_k is in $L_2(G)$, find an estimate for $\|\nu_k - \lambda_G\|_2$. In particular, how many steps k does it take for $\|\nu_k - \lambda_G\|_2$ to become small?

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We cannot answer these questions in this generality. There are of course examples for which ν_k never is in $L_2(G)$ (trivially, take ν to be unit mass at the identity element e). For many random walk problems we can give a precise answer to the second question (see [3] and references therein for examples on finite groups, and [7, 8] for examples on continuous groups, among many other references). Here we will narrow our focus to the first question and a special class of measures on a compact connected Lie group.

We will always assume G to be a compact connected Lie group. We then consider the following class of probability measures on G: For a given closed connected subgroup K of G, we take $\nu^{G,K}$ to be the probability measure concentrated on the set

$$S^{G,K} := \{gkg^{-1}: g \in G, k \in K\}$$

and induced from λ_G, Haar measure on G, and λ_K, Haar measure on K, via the map

$$f^{G,K}: G \times K \to G, \qquad f^{G,K}(g,k) = gkg^{-1}$$

for all $g \in G$ and $k \in K$.

The goal of this paper is to give a general lower bound for the number of steps k necessary such that the k-fold convolution power $\nu_k^{G,K}$ of $\nu^{G,K}$ has an L_2-density with respect to λ_G. To determine whether $\nu_k^{G,K}$ *is* in $L_2(G)$ for k larger than some k_0 and, if so, to estimate k_0 remain open problems.

A precise statement (Theorem 3.1) and proof of our result can be found in Sect. 3.

In proving Theorem 3.1, we will use a special feature of the L_2-norm, namely, the Plancherel theorem. For a given irreducible representation ρ of G, we denote the restriction of ρ from G to K by $\mathrm{res}_K^G(\rho)$. As will be shown later, the Fourier coefficient $\hat{\nu}^{G,K}(\chi)$ is equal to the multiplicity of the trivial representation of K occurring in $\mathrm{res}_K^G(\rho)$. There is a general formula for the multiplicity of an irreducible representation of K in $\mathrm{res}_K^G(\rho)$ (see Lemma 3.1 in [5]), but it is too complicated to use for our purposes. We therefore resort to the so-called *asymptotic multiplicity function* as defined by Heckman (1982). This will be introduced in Sect. 2, where we also present its relevant properties. Finally, in Sect. 4 we work out several examples.

2 Heckman's asymptotic multiplicity function $M_\mu^{G,K}$

Our main tool for the proof of Theorem 3.1 is Heckman's asymptotic multiplicity function $M_\mu^{G,K}$ for the restriction of an irreducible representation of a compact connected Lie group G to a closed Lie subgroup K. The main reference throughout this section is [5]. Here we introduce $M_\mu^{G,K}$ and cite several results from [5]. For background on the representation theory of compact, connected Lie groups see, for example, [2] or [6].

From now on let G be a compact connected Lie group and K be a closed connected subgroup of G. We fix maximal tori T_K in K and T_G in G with $T_K \subset T_G$. We will use the following notation:

Let $\mathbf{g}, \mathbf{k}, \mathbf{t}_G, \mathbf{t}_K$ denote the various corresponding Lie algebras and let $\mathbf{g}^*, \mathbf{k}^*, \mathbf{t}_G^*, \mathbf{t}_K^*$ denote their duals. There exists an inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{g}$ which is invariant under the adjoint representation. This inner product induces an isomorphism between $\mathbf{g}$ and $\mathbf{g}^*$, $\mathbf{k}$ and $\mathbf{k}^*$, $\mathbf{t}_G$ and $\mathbf{t}_G^*$, and $\mathbf{t}_K$ and $\mathbf{t}_K^*$. From now on we will identify $\mathbf{g}$ with $\mathbf{g}^*$, etc., via $\langle \cdot, \cdot \rangle$. We denote the orthogonal projection from $\mathbf{g}$ to $\mathbf{t}_K$ by q.

Let R_G be the root system of G (corresponding to $\mathbf{t}_G$) and R_K the root system of K (corresponding to $\mathbf{t}_K$). Furthermore, let Q_G be the sublattice of $\mathbf{t}_G$ generated by R_G and W_G the Weyl group of G. We define Q_K and W_K analogously. We choose sets of positive roots R_G^+ and R_K^+ in the following way. Consider the so-called parabolic root system $R_H = \{\alpha \in R_G\colon q(\alpha) = 0\}$; here our notation follows Heckman [5]. Choose a set of positive roots $R_H^+ \subset R_H$. Choose $H_0 \in \mathbf{t}_K$ such that $\alpha(H_0) \neq 0$ for all $\alpha \in R_G \backslash R_H$. It can then be shown that $R_K^+ := \{\alpha \in R_K\colon \alpha(H_0) > 0\}$ is a set of positive roots for R_K, and $R_G^+ := R_H^+ \cup \{\alpha \in R_G\colon \alpha(H_0) > 0\}$ is a set of positive roots for R_G.

Corresponding to R_G^+ and R_K^+ we have C_G^+ and C_K^+, the fundamental Weyl chambers. We denote the set of integer lattice points in C_G^+ by P_G^+ and the set of integer lattice points in C_K^+ by P_K^+. Recall that P_G^+ is in one-to-one correspondence with the irreducibles of G and P_K^+ is in one-to-one correspondence with the irreducibles of K. Take $\mu \in P_G^+$ and consider the corresponding irreducible representation ρ_μ of G. The restriction of ρ_μ from G to K decomposes into

$$\operatorname{res}_K^G(\rho_\mu) = \bigoplus_{\sigma \in P_K^+} m_\mu^{G,K}(\sigma) \pi_\sigma \tag{1}$$

where $m_\mu^{G,K}\colon P_K^+ \to \mathbf{N}_0$ is the multiplicity function and π_σ denotes the irreducible representation of K corresponding to σ. As pointed out in the introductory Sect. 1, there exists a formula for $m_\mu^{G,K}(\sigma)$, but it is too complicated to use for our purposes. We will use the so-called asymptotic multiplicity function $M_\mu^{G,K}$, as defined by Heckman [5], instead. Let $A = q(R_G^+ \backslash R_H^+) \backslash R_K^+$. We treat A as a multiset; that is, we allow each element $\alpha \in A$ to occur with a multiplicity m_α in A. (The multiplicities arise from different elements in $R_G^+ \backslash R_H^+$ having the same orthogonal projection onto $\mathbf{t}_k$.) Assume that $\mathbf{g}$ and $\mathbf{k}$ have no simple ideals in common and that $\operatorname{rank}(A \backslash \{\alpha\}) = \operatorname{rank}(A)$ for all $\alpha \in A$. (A consists of lattice points; by the rank of a lattice we mean the dimension of the subspace spanned by the lattice points.)
Assuming these conditions, Heckman defines the function

$$M_\mu^{G,K}\colon \mathbf{t}_K \to \mathbf{R}$$

(see [5, (3.16)]), which is called the asymptotic multiplicity function in light of the following theorem. Recall that by the support of a function we mean the closure of the set of points on which the function is unequal to zero.

Theorem 2.1 [5] *There exists a constant $C_{G,K} > 0$ such that for $\mu \in P_G^+$ and $\sigma \in q(\mu + Q_G) \cap P_K^+$*

$$|m_\mu^{G,K}(\sigma) - M_\mu^{G,K}(\sigma)| \leqq C_{G,K}(1 + |\mu|)^{s-1}, \tag{2}$$

where $s = \text{card}(R_G^+) - \text{card}(R_K^+) - \text{rank}(A)$ *and* $|\mu|$ *denotes the Euclidean norm of* μ.

The function $M_\mu^{G,K}$ *has compact support and is piecewise polynomial on* $\mathbf{t}_K$ *and satisfies the homogeneity relation*

$$M_{r\mu}^{G,K}(r\sigma) = r^s M_\mu^{G,K}(\sigma) \tag{3}$$

for all $r > 0$, $\mu \in \mathbf{t}_G$ *and* $\sigma \in \mathbf{t}_K$.

The homogeneity property (3) of $M_\mu^{G,K}$ will be crucial here. We also need to understand the support of $M_\mu^{G,K}$, denoted by $\text{supp}(M_\mu^{G,K})$. It can be seen from the exact definition of $M_\mu^{G,K}$ (see [5, (3.16)]) that $M_\mu^{G,K}$ is always zero in case μ lies on a wall of the fundamental Weyl chamber C_G^+. We will need the following results from [5] (see Lemmas 7.2 and 7.3 and Corollary 7.4 therein).

Lemma 2.2 [5] *For* $\mu \in C_G^+$ *and* $\sigma_1, \sigma_2 \in C_K^+$ *we have*

$$M_\mu^{G,K}\left(\frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_2\right) \geqslant 2^{-s}\min\{M_\mu^{G,K}(\sigma_1), M_\mu^{G,K}(\sigma_2)\}, \tag{4}$$

where s is defined as in Theorem 2.1.

Lemma 2.3 [5] *For* μ *in the intersection of* P_G^+ *and the interior of* C_G^+ *we have:*
(a)

$$\{\text{supp}(m_\mu^{G,K}) \cap P_K^+\} \subset \{\text{supp}(M_\mu^{G,K}) \cap P_K^+\}. \tag{5}$$

(b) *The set* $\text{supp}(M_\mu^{G,K}) \cap C_K^+$ *is a convex polytope.*

3 Statement and Proof of Results

Let G be a compact connected Lie group, K a closed connected subgroup and $\nu^{G,K}$ the measure defined in Sect. 1. We will use the notation introduced in Sect. 2 for the remainder of this chapter. Also, recall that $A = q(R_G^+ \setminus R_H^+) \setminus R_K^+$ is a multiset.

Theorem 3.1 *Suppose that* $\mathbf{g}$ *and* $\mathbf{k}$ *have no simple ideals in common and that* $\text{rank}(A \setminus \{\alpha\}) = \text{rank}(A)$ *for all* $\alpha \in A$. *If there is at least one* $\mu \in P_G^+ \cap$(*interior of* C_G^+) *for which* $M_\mu^{G,K}(0) \neq 0$ *and* $0 \in q(\mu + Q_G)$, *then it takes at least*

$$k = \frac{\text{card}(R_G^+) + 1}{\text{card}(R_K^+) + \text{rank}(A)}$$

steps for the k-fold convolution power $\nu_k^{G,K}$ *of* $\nu^{G,K}$ *to have an* L_2*-density with respect to Haar measure* λ_G.

Proof. The measure $\nu^{G,K}$ is conjugacy invariant, i.e., $\nu^{G,K}(U) = \nu^{G,K}(gUg^{-1})$ for all $g \in G$ and all measurable subsets $U \subseteq G$. Therefore, by Schur's lemma, the Fourier transform

$$\hat{\nu}^{G,K}(\rho_i) := \int_G \rho_i(g)\, d\nu^{G,K}(g)$$

is a scalar for each irreducible representation ρ_i of G and equal to $(1/d_i)\hat{\nu}^{G,K}(\chi_i)I_{d_i}$ (where d_i denotes the dimension of ρ_i, and χ_i denotes the corresponding character). Thus, by the Plancherel theorem, the measure $\nu_k^{G,K}$ has an L_2-density with respect to Haar measure λ_G if and only if

$$\sum_{i=0}^{\infty} |\hat{\nu}_k^{G,K}(\chi_i)|^2 = \sum_{i=0}^{\infty} \left|\frac{\hat{\nu}^{G,K}(\chi_i)}{d_i}\right|^{2k} d_i^2 < \infty . \tag{6}$$

We first compute $\hat{\nu}^{G,K}(\chi)$ for a fixed irreducible character χ (corresponding to ρ) of G. From the definition of $\nu^{G,K}$ we see that

$$\hat{\nu}^{G,K}(\chi) = \int_K \chi(k)\,d\lambda_K(k).$$

But $\mathrm{res}_K^G(\rho) = \oplus_{\sigma\in P_K^+} m^{G,K}(\sigma)\pi_\sigma$ (recall (1)), so that

$$\hat{\nu}^{G,K}(\chi) = \sum_{\sigma\in P_K^+} m^{G,K}(\sigma) \int_K \chi_\sigma\,d\lambda_K,$$

where the χ_σ on the right-hand side are irreducible characters of K. By the orthonormality property of the irreducible characters of a compact Lie group, we clearly get

$$\hat{\nu}^{G,K}(\chi) = m^{G,K}(0). \tag{7}$$

From now on we use the points in P_G^+ (the highest weights) to index the irreducibles of G. We will analyze

$$\sum_{\mu\in P_G^+} \left(\frac{m_\mu^{G,K}(0)}{d_\mu}\right)^{2k} d_\mu^2$$

by considering a particular *ray of representations*, i.e., we will pick an appropriate $\mu \in P_G^+$ and show that

$$\sum_{r=1}^{\infty} \left(\frac{m_{r\mu}^{G,K}(0)}{d_{r\mu}}\right)^{2k} d_{r\mu}^2 = \infty \quad \text{for } k < \frac{\mathrm{card}(R_G^+) + 1}{\mathrm{card}(R_K^+) + \mathrm{rank}(A)} \quad \text{with } k \in \mathbf{N}_0. \tag{8}$$

We now make use of the asymptotic multiplicity function $M^{G,K}(\cdot)$, whose existence is guaranteed by the conditions we impose in Theorem 3.1 (see [5]). For a fixed $\mu \in P_G^+ \cap$(interior of C_G^+) for which $M_\mu^{G,K}(0) \neq 0$ and $0 \in q(\mu + Q_G)$, it follows from Theorem 2.1 that

$$m_{r\mu}^{G,K}(0) = r^s M_\mu^{G,K}(0) + O(r^{s-1})$$

for all $r > 0$; here $s = \mathrm{card}(R_G^+) - \mathrm{card}(R_K^+) - \mathrm{rank}(A)$. On the other hand, the dimension d_μ for $\mu \in P_G^+ \cap$(interior of C_G^+) is a polynomial of degree exactly $\mathrm{card}(R_G^+)$ in the components of μ. This can easily be seen with the use of *Weyl's dimension polynomial*:

$$d_\mu = \prod_{\alpha\in R_G^+} \frac{\langle\alpha, \mu + \psi\rangle}{\langle\alpha, \psi\rangle},$$

where $\psi = \frac{1}{2}\sum_{\alpha \in R_G^+} \alpha$. We thus see that

$$\left(\frac{m_{r\mu}^{G,K}(0)}{d_{r\mu}}\right)^{2k} d_{r\mu}^2 \asymp r^{2(\mathrm{card}(R_G^+) - k(\mathrm{card}(R_K^+) + \mathrm{rank}(A)))}$$

for large r and fixed $\mu \in P_G^+ \cap$(interior of C_G^+). From this (8), and hence also the statement of Theorem 3.1, follow directly. □

4 Examples

Here we present several examples illustrating the use of Theorem 3.1. In each case, G is a subgroup of the unitary group $U(N)$ and the corresponding Lie algebra $\mathbf{g}$ is therefore a subalgebra of $\mathbf{u}(N)$, the Lie algebra of skew-Hermitian $N \times N$ matrices. An inner product on $\mathbf{g}$ which is invariant under the adjoint representation can be taken to be $\langle A, B \rangle := -(2\pi)^{-2}\mathrm{tr}(AB)$ for $A, B \in \mathbf{g}$. In order to show that there exists at least one $\mu \in P_G^+ \cap$(interior of C_G^+) for which $M_\mu^{G,K}(0) \neq 0$, we will make use of several so-called *branching theorems*. They describe simply, for specific G and K and for a given irreducible ρ of G, how $\mathrm{res}_K^G \rho$ decomposes into irreducibles of K. References are, for example, [1, 4, 9]. Inductive use of these branching laws is justified by the transitivity of restriction:

$$\mathrm{res}_H^G(\rho) = \mathrm{res}_H^K(\mathrm{res}_K^G(\rho)) \quad \text{for subgroups } H \subset K \subset G.$$

We can easily check in all examples discussed here that $\mathbf{g}$ and $\mathbf{k}$ have no ideals other than $\{0\}$ in common, i.e., that for each nonzero $B \in \mathbf{k}$ there is always $A \in \mathbf{g}$ such that $[A, B] = AB - BA \notin \mathbf{k}$. From now on we will use the following notation: $P_G^{+,0} := P_G^+ \cap$(interior of C_G^+).

Example 1 (Random rotations) $G = SO(N)$ for $N \geqq 4$, $K = SO(2) \cong S^1$. K is embedded in G via

$$\begin{pmatrix} \cos 2\pi t & -\sin 2\pi t & 0 & \cdots & 0 \\ \sin 2\pi t & \cos 2\pi t & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{with } t \in \mathbf{R}/\mathbf{Z}.$$

Consider

$$SO(2) \cong S^1 \cong \left\{ \begin{pmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{pmatrix} : t \in \mathbf{R}/\mathbf{Z} \right\}.$$

We take the standard choice of maximal torus T_G, namely the subgroup $SO(2) \times \cdots \times SO(2) \times 1$ (with n factors of $SO(2)$) for $N = 2n + 1$ and the subgroup $SO(2)\times \cdots \times SO(2)$ (with n factors of $SO(2)$) for $N=2n$. The set of positive roots of $SO(N)$ (for $N \geqq 4$) is

$$R^+_{SO(N)} = \begin{cases} \{e_j \pm e_i : 1 \leqq i < j \leqq n\} \cup \{e_i : 1 \leqq i \leqq n\} & \text{for } N = 2n+1, \\ \{e_j \pm e_i : 1 \leqq i < j \leqq n\} & \text{for } N = 2n. \end{cases}$$

Therefore,

$$\operatorname{card} R^+_{SO(N)} = \begin{cases} n^2 & \text{for } N = 2n+1, \\ n^2 - n & \text{for } N = 2n. \end{cases}$$

Also, $SO(2)$ has no roots, so that $\operatorname{card} R^+_{SO(2)} = 0$.

We now describe the Lie algebras involved in this example. The Lie algebra $\mathbf{g}$ of $SO(N)$ consists of the skew-symmetric real $N \times N$ matrices. The Lie algebra $\mathbf{t}_G$ of the maximal torus T_G can be seen to be the block diagonal matrices with ith block equal to $2\pi \begin{pmatrix} 0 & -t_i \\ t_i & 0 \end{pmatrix}$:

$$\mathbf{t}_G = \left\{ 2\pi \begin{pmatrix} 0 & -t_1 & 0 & \cdots & 0 \\ t_1 & 0 & & & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & & & 0 & -t_n \\ 0 & \cdots & 0 & t_n & 0 \end{pmatrix} : t_i \in \mathbf{R} \text{ for } 1 \leqq i \leqq n \right\} \cong \mathbf{R}^n$$

for $N = 2n$ and similarly (with an additional bottom row and last column both consisting entirely of zeroes) for $N = 2n+1$. We also have

$$\mathbf{k} = \mathbf{t}_K = \left\{ 2\pi \begin{pmatrix} 0 & -t & 0 & \cdots & 0 \\ t & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} : t \in \mathbf{R} \right\} \cong \mathbf{R}.$$

We can easily verify that $\langle A, B \rangle = -(2\pi)^{-2}\operatorname{tr}(AB) = 2\sum_{i=1}^n t_i s_i$ for all $A, B \in \mathbf{t}_G \cong \mathbf{R}^n$, where $A = (t_1, \ldots, t_n)$ and $B = (s_1, \ldots, s_n)$. Therefore the orthogonal projection $q : \mathbf{t}_G \to \mathbf{t}_K$ is given by projection onto the first component: $q(\vec{x}) = (x_1, 0, \ldots, 0)$ for all $\vec{x} \in \mathbf{R}^n \cong \mathbf{t}_G$. From this it follows that

$$R^+_{SO(N)} \backslash R^+_H = \begin{cases} \{e_i \pm e_1 :\ 1 < i \leqq n\} \cup \{e_1\} & \text{for } N = 2n+1, \\ \{e_i \pm e_1 :\ 1 < i \leqq n\} & \text{for } N = 2n, \end{cases}$$

and that $A = q(R^+_G \backslash R^+_H) \backslash R^+_K$ consists of the element $(1, 0, \ldots, 0)$, with multiplicity n in case $N = 2n+1$ and with multiplicity $n-1$ in case $N = 2n$, and the element $(-1, 0, \ldots, 0)$ with multiplicity $n-1$ in either case. Thus $\operatorname{rank}(A \backslash \{\alpha\}) = \operatorname{rank}(A) = 1$ for all $\alpha \in A$ (since we assume $n \geqq 2$ in either case). Furthermore, it is easy to see that $(0, \ldots, 0) \in q(\mu + Q_G)$ for all $\mu \in P^+_G$ (recall that Q_G is the sublattice of $\mathbf{t}_G$ generated by the roots R_G).

We next prove that $\mathbf{g}$ and $\mathbf{k}$ have no ideals in common. Indeed, for any

$$B = 2\pi \begin{pmatrix} 0 & -b & 0 & \cdots & 0 \\ b & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \in \mathbf{k}$$

with $b \neq 0$ we can take

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -a & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \in \mathbf{g}$$

for any $a \neq 0$ and verify that $[A, B] \notin \mathbf{k}$.

Now choose μ in $P_G^{+,0}$ with $\mu_1 = 1$. The following branching theorem makes it clear that $\mathrm{res}_{SO(2)}^{SO(N)}(\rho_\mu)$ always contains the following irreducible representations of $SO(2)$: the representation of index 0 (i.e., the trivial representation), the representation of index 1, and the representation of index -1.

Theorem 4.1 *Let $N \geqq 3$. Identify $SO(N-1)$ with the subgroup of $SO(N)$ of all elements of the form $\left(\begin{smallmatrix} B & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We then have*

(*a*) $\mathrm{res}_{SO(2n)}^{SO(2n+1)}(\rho_\mu) = \bigoplus \pi_{\bar\mu}$ *where the sum is over all $\bar\mu$ with $|\bar\mu_1| \leqq \mu_1 \leqq \bar\mu_2 \leqq \cdots \leqq \bar\mu_n \leqq \mu_n$, and*

(*b*) $\mathrm{res}_{SO(2n-1)}^{SO(2n)}(\rho_\mu) = \bigoplus \pi_{\bar\mu}$ *where the sum is over all $\bar\mu$ with $|\mu_1| \leqq \bar\mu_1 \leqq \mu_2 \leqq \cdots \leqq \bar\mu_{n-1} \leqq \mu_n$.*

Indeed, let us assume that $\mu = (1, \mu_2, \ldots, \mu_n)$. In case $N = 2n+1$, we see from Theorem 4.1(a) that $\mathrm{res}_{SO(N-1)}^{SO(N)}(\rho_\mu)$ contains the representation of highest weight $(1, \mu_2, \ldots, \mu_n)$ (among others). In case $N = 2n$, we see from Theorem 4.1(b) that $\mathrm{res}_{SO(N-1)}^{SO(N)}(\rho_\mu)$ contains the representation of highest weight $(1, \mu_2, \ldots, \mu_{n-1})$ (among others). From this we see that $\mathrm{res}_{SO(3)}^{SO(N)}(\rho_\mu)$ always contains the irreducible representation of index 1. This representation, further restricted to $SO(2)$, yields the irreducible representations of $SO(2)$ of indices $-1, 0$, and 1 (again by Theorem 4.1(a)).

We now show that $M_\mu^{SO(N),SO(2)}(0) \neq 0$. By Lemma 2.3(a), $\{-1, 0, 1\} \subset \mathrm{supp}(M_\mu^{G,K})$. Since, by Lemma 2.3(b), $\mathrm{supp}(M_\mu^{G,K}) \cap C_K^+$ is a convex polytope, i.e., for this example, a closed interval, and since $M_\mu^{SO(N),SO(2)}$ is a piecewise polynomial nonnegative function (recall Theorem 2.1; the nonnegativity of $M_\mu^{G,K}$ on C_K^+ is a direct consequence of Theorem 6.4

in [5]), we can conclude with the use of Lemma 2.2 that $M_\mu^{SO(N),SO(2)}(0) \neq 0$. It now follows from Theorem 3.1 that it takes at least

$$k = \begin{cases} n^2 + 1 & \text{for } N = 2n + 1, \\ n^2 - n + 1 & \text{for } N = 2n \end{cases}$$

steps for $\nu_k^{SO(N),SO(2)}$ to have an L_2-density with respect to Haar measure $\lambda_{SO(N)}$.

Remark 4.2 If we slightly change the definition of the measure $\nu^{SO(N),SO(2)}$ by changing Haar measure on $SO(2)$ to the probability measure on $SO(2)$ with density proportional to $(\sin \pi t)^{N-2}$, the result is quite different from that in Example 1: The L_2-norm of the kth convolution power of the new measure is close to 1 after $k = \frac{1}{2} n \log n + cn$ steps for $c \geqq c_0$, where c_0 is some universal positive constant, and for both $N = 2n$ and $N = 2n + 1$. See [7] for more details.

Example 2(a) (Random complex reflections) $G = U(N)$ with $N \geqq 3$, $K = U(1) \cong S^1$. K is embedded in G via

$$\begin{pmatrix} e^{i2\pi t} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \quad \text{with } t \in \mathbf{R}/\mathbf{Z}.$$

We take the standard choice of maximal torus T_G, namely the subgroup of diagonal matrices of $U(N)$. The set of positive roots is

$$R_{U(N)}^+ = \{e_j - e_i \colon 1 \leqq i < j \leqq N\}.$$

Therefore, card $R_{U(N)}^+ = \frac{1}{2}(N^2 - N)$. Furthermore,

$$\mathbf{t}_G = \{\mathrm{diag}(i2\pi t_1, \dots, i2\pi t_N) \colon t_k \in \mathbf{R} \text{ for } 1 \leqq k \leqq N\} \cong \mathbf{R}^N$$

and

$$\mathbf{k} = \mathbf{t}_K = \{\mathrm{diag}(i2\pi t, 0, \dots, 0) \colon t \in \mathbf{R}\} \cong \mathbf{R}.$$

We can easily verify that our choice of inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{g}$, restricted to $\mathbf{t}_G \cong \mathbf{R}^N$, is the standard Euclidian inner product. As in Example 1, the orthogonal projection $q \colon \mathbf{t}_G \to \mathbf{t}_K$ is projection onto the first component. From this it follows that $R_{U(N)}^+ \backslash R_H^+ = \{e_k - e_1 \colon 1 < k \leqq N\}$ and that A consists of the element $(-1, 0, \dots, 0)$ with multiplicity $N - 1$.

Now choose $\mu \in P_G^{+,0}$ with $\mu_1 \leqq -1$ and $\mu_N \geqq 1$. The following branching theorem makes it clear that $\mathrm{res}_{U(1)}^{U(N)}(\rho_\mu)$ always contains the following irreducible representations of $U(1)$: the representation of index 0 (i.e., the trivial representation), the representation of index 1, and the representation of index -1.

Theorem 4.3 *Let $N \geqq 2$. Identify $U(N-1)$ with the subgroup of $U(N)$ of all elements of the form $\left(\begin{smallmatrix} B & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We then have*

$$\operatorname{res}^{U(N)}_{U(N-1)}(\rho_\mu) = \oplus \pi_{\bar{\mu}},$$

where the sum is over all $\bar{\mu}$ with $\mu_1 \leqq \bar{\mu}_1 \leqq \mu_2 \leqq \cdots \leqq \bar{\mu}_{N-1} \leqq \mu_N$.

For example, let us choose $\mu = (-1, 1, \mu_3, \ldots, \mu_N)$. From Theorem 4.3 we see that $\operatorname{res}^{U(N)}_{U(N-1)}(\rho_\mu)$ contains the representation of index $(-1, 1, \mu_3, \ldots, \mu_{N-1})$ (among others). Therefore, $\operatorname{res}^{U(N)}_{U(2)}(\rho_\mu)$ contains the representation of index $(-1, 1)$, which, further restricted to $U(1)$, yields exactly the representations of indices $0, 1$, and -1.

We can apply the same reasoning as used in Example 1 to show that $M^{U(N),U(1)}_\mu(0) \neq 0$. The rest of the conditions of Theorem 3.1 are easily checked. It follows that it takes at least $k = \frac{1}{2}(N^2 - N) + 1$ steps for $v^{U(N),U(1)}_k$ to have an L_2-density with respect to Haar measure $\lambda_{U(N)}$.

Example 2(b) $G = U(N)$ with $N \geqq 3$, $K = U(1) \cong S^1$. K is embedded in G via

$$\begin{pmatrix} e^{i2\pi t} & & & & 0 \\ & e^{i2\pi t} & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

with $t \subset \mathbf{R}/\mathbf{Z}$. The projection $q: \mathbf{t}_G \to \mathbf{t}_K$ is the orthogonal projection from $\mathbf{R}^N$ onto the one dimensional subspace of $\mathbf{R}^N$ spanned by $(1, 1, 0, \ldots, 0)$. From this it follows that

$$R^+_{U(N)} \backslash R^+_H = \{e_i - e_1 \colon 3 \leqq i \leqq N\} \cup \{e_i - e_2 \colon 3 \leqq i \leqq N\}$$

and that A consists of the element $(-\frac{1}{2}, -\frac{1}{2}, 0, \ldots, 0)$ with multiplicity $2N - 4$.

Now choose $\mu \in P^{+,0}_{U(N)}$ for which $\mu_1 = -1$, $\mu_2 = 0$, and $\mu_3 = 1$. We claim that for such a μ we have: $m^{U(N),U(1)}_\mu(x) \neq 0$ for $x = -1, 0, 1$. Indeed, μ and any vector resulting from permuting the elements in μ are weights of ρ_μ. Thus we see that $\chi_{\rho_\mu}(\operatorname{diag}(e^{i2\pi t}, e^{i2\pi t}, 1, \ldots, 1))$ (which is a sum of powers of $e^{i2\pi t}$) must contain the terms $e^{i2\pi t}, e^{-i2\pi t}$, and 1 with positive coefficients. Now the same arguments as before apply to prove that $M^{U(N),U(1)}_\mu(0) \neq 0$. The rest of the conditions of Theorem 3.1 are easily checked. Therefore, as for Example 2(a), it takes at least $k = \frac{1}{2}(N^2 - N) + 1$ steps for $v^{U(N),U(1)}_k$ to have an L_2-density with respect to Haar measure $\lambda_{U(N)}$.

Example 3 $G = U(N)$ with $N \geqq 5$, $K = T^2 \cong S^1 \times S^1$. K is embedded in G via

$$\begin{pmatrix} e^{i2\pi t_1} & & & & 0 \\ & e^{i2\pi t_2} & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

with $t_1, t_2 \in \mathbf{R}/\mathbf{Z}$. The projection $q: \mathbf{t}_G \to \mathbf{t}_K$ is the orthogonal projection from $\mathbf{R}^N$ onto the two dimensional subspace of $\mathbf{R}^N$ spanned by $(1, 0, \ldots, 0)$ and $(0, 1, 0, \ldots, 0)$. Here we have

$$R^+_{U(N)} \backslash R^+_H = \{e_i - e_1 \colon 2 \leqq i \leqq N\} \cup \{e_i - e_2 \colon 3 \leqq i \leqq N\}$$

and A consists of the element $(-1, 1, 0, \ldots, 0)$ appearing with multiplicity 1 and the elements $(-1, 0, \ldots, 0)$ and $(0, -1, 0, \ldots, 0)$, each appearing with multiplicity $N - 2$. Note that rank $A = 2$ in this case.

Now choose $\mu \in P^{+,0}_{U(N)}$ for which $\mu_1 < -1$, $\mu_2 = -1$, $\mu_3 = 0$, $\mu_4 = 1$, and $\mu_5 > 1$. From the branching theorem for $U(N)$ (Theorem 4.3) we can see that $\mathrm{res}^{U(N)}_{U(2)}(\rho_\mu)$ contains the following irreducibles of $U(2)$ (among others): the irreducible of highest weight $(0, 0)$ (i.e., the trivial representation), the irreducible of highest weight $(-1, -1)$, and the irreducible of highest weight $(1, 1)$. Since T^2 is a maximal torus of $U(2)$, it follows that $m^{U(N), T^2}_\mu(x) \neq 0$ for $x = (0, 0), (-1, -1), (1, 1)$. We can thus prove in the same manner as before that $M^{U(N), T^2}_\mu(0) \neq 0$.

The rest of the conditions of Theorem 3.1 are easily checked. We can now conclude that it takes at least $k = \frac{1}{4}(N^2 - N) + \frac{1}{2}$ steps for $\nu^{U(N), T^2}_k$ to have an L_2-density with respect to Haar measure $\lambda_{U(N)}$.

Example 4 $G = Sp(n)$ with $n \geqq 2$, $K = U(1) \cong S^1$. K is embedded in G via

$$\begin{pmatrix} z & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \in \mathrm{Gl}(n, \mathbf{H}), \quad z \in U(1).$$

We now view $Sp(n)$ as a group of $2n$-dimensional complex matrices via its natural representation. The standard choice of maximal torus T_G is the subgroup of diagonal matrices. The set of positive roots is

$$R^+_{Sp(n)} = \{e_j \pm e_i \colon 1 \leqq i < j \leqq n\} \cup \{2e_i \colon 1 \leqq i \leqq n\}.$$

Therefore, $\mathrm{card}(R^+_{Sp(n)}) = n^2$. Furthermore,

$$\mathbf{t}_G = \{\mathrm{diag}(i2\pi t_1, \ldots, i2\pi t_n, -i2\pi t_1, \ldots, -i2\pi t_n) \colon t_k \in \mathbf{R} \text{ for } 1 \leqq k \leqq n\} \cong \mathbf{R}^n$$

and

$$\mathbf{k} = \mathbf{t}_K = \{\mathrm{diag}(i2\pi t, 0, \ldots, 0, -i2\pi t, 0, \ldots, 0) \colon t \in \mathbf{R}\} \cong \mathbf{R}.$$

Thus the projection $q:\mathbf{t}_G \to \mathbf{t}_K$ is given by projection onto the first quaternionic component. It follows that

$$R^+_{Sp(n)} \backslash R^+_H = \{e_k \pm e_1 : 2 \leqq k \leqq n\} \cup \{2e_1\}$$

and that A consists of the elements $(\pm 1, 0, \ldots, 0)$, each with multiplicity $n-1$, and the element $(2, 0, \ldots, 0)$ with multiplicity 1. Therefore $\mathrm{rank}(A) = 1$.

We now show that there exists $\mu \in P_G^{+,0}$ for which $\mathrm{res}^{Sp(n)}_{U(1)}(\rho_\mu)$ contains the following irreducible representations of $U(1)$: the representation of index 0, the representation of index 1, and the representation of index -1.

Choose $\mu \in P_G^{+,0}$ for which $\mu_1 = 1$. The following branching theorem, used inductively, makes it clear that $\mathrm{res}^{Sp(n)}_{Sp(1)}(\rho_\mu)$ always contains the following representations of $Sp(1)$: the representation of index 0 and the representation of index 1 (i.e., the natural representation).

Theorem 4.4 *Identify $Sp(n-1)$ with the subgroup of $Sp(n)$ of all elements of the form $\left(\begin{smallmatrix} B & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We then have*

$$\mathrm{res}^{Sp(n)}_{Sp(n-1)}(\rho_\mu) = \bigoplus N_{\mu\bar{\mu}} \pi_{\bar{\mu}}$$

where the sum is over all $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_{n-1})$ with $0 \leqq \bar{\mu}_1 \leqq \cdots \leqq \bar{\mu}_{n-1}$, and the multiplicity $N_{\mu\bar{\mu}}$ is the number of sequences $p_1, \ldots, p_n$ of integers satisfying

$$0 \leqq p_1 \leqq \mu_1 \leqq p_2 \leqq \cdots \leqq p_n \leqq \mu_n$$

and

$$p_1 \leqq \bar{\mu}_1 \leqq p_2 \leqq \cdots \leqq \bar{\mu}_{n-1} \leqq p_n.$$

Indeed, let us assume that $\mu = (1, \mu_2, \ldots, \mu_n)$. From Theorem 4.4 we see that $\mathrm{res}^{Sp(n)}_{Sp(n-1)}(\rho_\mu)$ contains the representations of indices $(1, \mu_2, \ldots, \mu_{n-1})$ and $(0, \mu_2, \ldots, \mu_{n-1})$ (among others). It follows that $\mathrm{res}^{Sp(n)}_{Sp(1)}(\rho_\mu)$ contains the representations of indices 0 and 1 (among others).

We now restrict the representation of index 0 and the representation of index 1 of $Sp(1)$ further to the maximal torus of $Sp(1)$ (i.e., to the subgroup of diagonal matrices of the form $\mathrm{diag}(e^{i2\pi t}, e^{-i2\pi t})$ with $t \in \mathbf{R}/\mathbf{Z}$). This maximal torus is our subgroup $K = U(1)$ and we now can see that the representations of index $-1, 0$, and 1 occur in $\mathrm{res}^{Sp(n)}_{U(1)}(\rho_\mu)$. As before, we conclude that $M_\mu^{Sp(n), U(1)}(0) \neq 0$. The rest of the conditions of Theorem 3.1 are easily checked. It thus follows that it takes at least $k = n^2 + 1$ steps for $\nu_k^{Sp(n), U(1)}$ to have an L_2-density with respect to Haar measure $\lambda_{Sp(n)}$.

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