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Percolation in half-spaces: equality of critical densities and continuity of the percolation probability*

David J. Barsky[1], Geoffrey R. Grimmett[2], and Charles M. Newman[3]

[1] Department of Mathematics, University of California, Davis, CA 95616, USA
[2] School of Mathematics, University of Bristol, Bristol BS8 1TW, England, UK
[3] Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

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Summary. Renormalization arguments are developed and applied to independent nearest-neighbor percolation on various subsets $\mathbb{L}$ of $\mathbb{Z}^d$, $d \geqq 2$, yielding:

- Equality of the critical densities, $p_c(\mathbb{L})$, for $\mathbb{L}$ a half-space, quarter-space, etc., and (for $d > 2$) equality with the limit of slab critical densities.
- Continuity of the phase transition for the half-space, quarter-space, etc.; i.e., vanishing of the percolation probability, $\theta_{\mathbb{L}}(p)$, at $p = p_c(\mathbb{L})$.

Corollaries of these results include uniqueness of the infinite cluster for such $\mathbb{L}$'s and sufficiency of the following for proving continuity of the full-space phase transition: showing that percolation in the full-space at density p implies percolation in the half-space at the *same* density.

0 Introduction

In recent years, renormalization techniques have begun to provide a microscope capable of focusing on the critical region of the percolation phase transition. Such techniques have been applied to percolation in two dimensions [e.g., Russo (1978); Seymour and Welsh (1978); Durrett (1984); Kesten (1987)], three and more dimensions [e.g., Aizenman et al. (1983)], and in long-range models [e.g., Newman and Schulman (1986); Aizenman and Newman (1986)]. This paper is concerned with short-range percolation in more than two dimensions, and has two main purposes: first, to develop new methods in renormalization technology, and secondly, to apply these new methods to the question of the continuity of the phase transition – i.e., to show that there is no percolation at the critical point.

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We are partly successful in our last aim. In particular, we prove the continuity of the percolation probability for independent nearest-neighbor (bond or site) percolation on various subsets $\mathbb{L}$ of $\mathbb{Z}^d$, $d \geqq 2$, such as the half-space ($\mathbb{L} = \mathbb{Z}^{d-1} \times \mathbb{Z}_+$ where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$), quarter-space, etc. This implies that to show continuity for the full-space, it would suffice to prove that if there is percolation in the full-space at density p, then there is percolation in the half-space at the same p. A weaker result, in which "the same p" is replaced by "any larger p," has been obtained by Grimmett and Marstrand (1990). Their result shows that the critical densities coincide for full-spaces and half-spaces, thus extending our result here that critical densities coincide for half-spaces, quarter-spaces, etc. The Grimmett-Marstrand (1990) result and another result of Bezuidenhout and Grimmett (1990), who resolve the continuity issue for the contact process, are based in part on extensions of the methods presented here.

We remark that it is already known that the percolation probability for the full-space $\mathbb{Z}^d$ vanishes at the critical density if either $d=2$ [Harris (1960); Kesten (1980); Russo (1981)] or d is sufficiently large [Hara and Slade (1989, 1990)]; these results have been reached by arguments quite different from those presented here. See Grimmett (1989) for a general account of percolation theory and for other material in the background of this paper.

In the next section of the paper, we state precisely our main results and discuss the related literature. Included there is an introduction to the renormalization techniques used later. We end this section with a brief discussion of the general approach underlying such "block" arguments.

Imagine an algorithmic process for growing the cluster of the origin for a (possibly dependent) nearest-neighbor site percolation model on the quadrant $\mathbb{Z}_+^2$ of the plane, such as one might perform on a personal computer. First assign a deterministic order to all the sites in $\mathbb{Z}_+^2$. At time 0, check the occupation status of the origin. At time n, choose the site x_n with the lowest order from the set of unchecked neighbors of sites already found to be occupied, and check the occupation status of x_n. If the set of unchecked neighbors is empty, terminate the process. Note that both x_n and λ_n, the conditional probability that x_n is occupied given the previous history of the process, are random since they depend on this history. If $\lambda_n \geqq \lambda_c + \varepsilon$ (a.s.) for some $\varepsilon > 0$ and all $n \geqq 1$, where λ_c is the critical density for the independent site model on $\mathbb{Z}_+^2$, then the original model percolates. To see why this is so, extend the algorithmic process by declaring a random subset of the sites found to be occupied to be "red," as follows. Let $W_1, W_2, \dots$ be i.i.d. random variables uniformly distributed on $[0, 1]$ which are independent of the original site percolation model. Declare x_n to be red if it is occupied *and* $W_n < (\lambda_c + \varepsilon)/\lambda_n$. It is evident, after a moment's reflection, that the connected component of the set of all red sites which contains the origin is distributed exactly as the set of occupied sites connected to the origin from an independent site percolation model at density $\lambda_c + \varepsilon$. When the red sites percolate (from the origin), then so do the occupied ones.

Let us now begin with, say, independent bond percolation in three dimensions, for which we wish to show that (at a certain bond density p') percolation occurs in the quarter-slice $\mathbb{L}' = \{-L', \dots, L'\} \times \mathbb{Z}_+^2$. We do this by partitioning $\mathbb{L}'$ into rectangular regions which are translates of $\{-L', \dots, L'\} \times \{-K, \dots, K\}^2$ (i.e., cubes if $L' = K$); each such region may be identified with a site in $\mathbb{Z}_+^2$ in a natural way. The "cube" containing the origin is said to be occupied if the set of sites in its east and north faces – i.e., those faces having normal vectors (0, 1, 0) and (0, 0, 1), respectively – which are connected to the origin by open bonds in the cube are sufficiently "good." We then run an algorithmic process paralleling the one described above except that here the elementary step is to choose a cube neighboring one already found to be occupied, and then check whether the new cube is occupied – in the sense that the good set of sites in the neighboring face of the old cube is connected to good sets of sites in the other faces of the new cube by open bonds of the new cube. (When the new cube has more than one occupied neighbor, the one with the lowest deterministic order is chosen to be the "old" cube.) If the conditional probability λ_n of this event exceeds $\lambda_c+\varepsilon$, uniformly over all past histories, then bond percolation occurs in $\mathbb{L}'$.

This type of renormalization argument may be applied to the question of continuity as follows. One begins by assuming that, for a given p, there is percolation in say the orthant $\mathbb{L}=\mathbb{Z}^d_+$. One then shows that L' and K can be chosen in such a way that the above conditional probabilities λ_n exceed $\lambda_c+\varepsilon$ for some $\varepsilon>0$. It follows that for the same L' and K and some *smaller* density p', these conditional probabilities still exceed $\lambda_c+\varepsilon$. This shows that percolation in the orthant $\mathbb{L}$ at density p implies percolation in the quarter-slice $\mathbb{L}'$ at a smaller density p'. Consequently – since there are translates of $\mathbb{L}'$ which are proper subsets of $\mathbb{L}$ – one obtains (i) $\theta_{\mathbb{L}}(p_c(\mathbb{L}))=0$, and (ii) $p_c(\mathbb{L})=\lim_{L'\to\infty} p_c(\mathbb{L}')$, where $\theta_{\mathbb{L}}$ and $p_c(\mathbb{L})$ are the percolation probability and critical density, respectively, for the subset $\mathbb{L}$. This argument is made rigorous by Barsky et al. (1991), with limited geometrical complications. These complications are somewhat greater when the assumption of percolation in the *orthant* is replaced by a similar assumption for the *half-space*.

1 Statement of results

Throughout this paper we will be concerned with nearest-neighbor percolation on certain subsets of $\mathbb{Z}^d$. For concreteness, we restrict our attention to bond percolation, although the arguments and results can clearly be modified for site percolation. We thus consider the subgraph of $\mathbb{Z}^d$ with vertex set $\mathbb{L}$ and edge set consisting of all nearest-neighbor edges in $\mathbb{L}$ (i.e., pairs of elements $\{x, y\}$ from $\mathbb{L}$ separated by Euclidean distance one). Edges can be either open or closed. Each edge is open with probability $p\in[0,1]$ independently of the states of all other edges; p is called the bond density. We write P_p (or often just P) to denote the ensuing probability measure.

In percolation theory, one considers the random subgraph obtained by deleting all closed edges; its connected components are called (open) clusters. For any $\mathbb{L}$, the event that there exists an infinite cluster is clearly a tail event, and thus has probability zero or one – percolation is said to occur when this probability is one. We will only consider $\mathbb{L}$'s which are connected (before deletion of closed edges) and which contain the origin, denoted by 0. The percolation probability, $\theta_{\mathbb{L}}(p)$, is then defined to be the probability that the cluster of the origin is infinite, or equivalently,

$$\theta_{\mathbb{L}}(p)=P_p\ \text{(there is a nearest-neighbor path of open bonds starting from the origin and passing through infinitely many distinct sites).}$$

It is easily seen that percolation in $\mathbb{L}$ at bond density p occurs if and only if $\theta_{\mathbb{L}}(p)>0$.

In each of the subsets $\mathbb{L}$ which we shall consider, there is a critical density $p_c(\mathbb{L})$ strictly between 0 and 1 such that $\theta_{\mathbb{L}}(p)=0$ for $p<p_c(\mathbb{L})$ and $\theta_{\mathbb{L}}(p)>0$ for $p>p_c(\mathbb{L})$. The theorem following guarantees for certain $\mathbb{L}$'s that $\theta_{\mathbb{L}}$ vanishes exactly at the appropriate critical density; as discussed following the theorem, this leads to the continuity of $\theta_{\mathbb{L}}$ (at all densities p). The $\mathbb{L}$'s for which this continuity is proved include the half-space

$$\mathbb{H}=\mathbb{Z}^{d-1}\times\mathbb{Z}_+,$$

the quarter-space, etc. We also consider slabs

$$\mathbb{S}_L=\mathbb{Z}^{d-1}\times\{0,\ldots,L\}$$

and quarter-slices

$$\mathbb{Q}_L=\{-L,\ldots,L\}^{d-2}\times\mathbb{Z}_+^2,$$

as well as the limits of their critical densities:

$$p_c(\mathbb{S})=\lim_{L\to\infty}p_c(\mathbb{S}_L)\quad\text{and}\quad p_c(\mathbb{Q})=\lim_{L\to\infty}p_c(\mathbb{Q}_L).$$

The first part of the theorem is already known in two dimensions following the work of Harris (1960); Russo (1978); Seymour and Welsh (1978); Kesten (1980). However, our arguments provide a new proof.

Theorem 1.1 i) *Suppose $d\geqq 2$. The percolation probability of the half-space vanishes at the half-space critical density:*

$$\theta_{\mathbb{H}}(p_c(\mathbb{H}))=0. \tag{1.1}$$

ii) *Suppose $d\geqq 3$. There is equality between the critical density of the half-space and the limits of the critical densities of quarter-slices:*

$$p_c(\mathbb{H})=p_c(\mathbb{Q}). \tag{1.2}$$

Comments. 1) It has not yet been proved that the full-space percolation probability vanishes at the appropriate critical density in all dimensions (as is commonly believed), although there has been some progress. If $d=2$, this conclusion follows from the combination of the Harris (1960) result that $\theta_{\mathbb{Z}^2}(\frac{1}{2})=0$ and the Kesten (1980) proof that the critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. For site percolation in two dimensions, where the value of the critical density is not known exactly, the corresponding result was proved in Russo (1981). For the case of large dimensions, it was shown in Barsky and Aizenman (1991) [see also Barsky (1987)] that $\theta_{\mathbb{Z}^d}(p_c(\mathbb{Z}^d))=0$ if a certain "triangle condition" – believed to be valid for $d>6$ – is satisfied [(this result was later strengthened in Aizenman et al. (1987)]. Hara and Slade (1989,

1990) have recently shown that the triangle condition is indeed satisfied in sufficiently high dimensions – $d \geqq 7$ for models with "spread-out" finite-range bonds, and much larger d for the nearest-neighbor model. We further mention that there exist some special percolation models for which it has been shown that θ is strictly positive at the appropriate critical density. Aizenman and Newman (1986) demonstrated this property for a particular one-dimensional model with long-range bonds and Chayes and Chayes (1986) showed that nearest-neighbor models on logarithmic wedges of $\mathbb{Z}^2$ can also exhibit this behavior.

2) It is immediate from (1.2) and monotonicity that the critical densities of quadrants, octants, ..., and orthants must coincide with both $p_c(\mathbb{H})$ and $p_c(\mathbb{Q})$. Additionally, these critical densities must also coincide with $p_c(\mathbb{S})$. We note that the equality of $p_c(\mathbb{S})$ and $p_c(\mathbb{Q})$ has already been proved in Kesten (1989). It was conjectured in Aizenman et al. (1983) that $p_c(\mathbb{Z}^d) = p_c(\mathbb{S})$ for $d \geqq 3$ dimensions. This equality is of considerable interest as there are a number of results [see the discussion in Grimmett and Marstrand (1990)] which show that behavior which is expected for all $p > p_c(\mathbb{Z}^d)$ does in fact occur for $p > p_c(\mathbb{S})$. We note that this problem has recently been solved by Grimmett and Marstrand (1990).

3) Combining the first sentence of Comment 2 with equality (1.1), it is readily observed that there cannot be percolation in the quadrant, octant, ..., or orthant at their common critical density.

4) In the full-space $\mathbb{L} = \mathbb{Z}^d$, the question of whether the percolation probability, $\theta_{\mathbb{Z}^d}$, is a continuous function on $(0, 1)$ reduces to deciding if it is continuous at the critical density, $p_c(\mathbb{Z}^d)$. First, $\theta_{\mathbb{Z}^d} \equiv 0$ for $p < p_c(\mathbb{Z}^d)$ [by definition]. Secondly, it is easy to see that $\theta_{\mathbb{Z}^d}$ is a nondecreasing upper semicontinuous function of p, and hence it must be right-continuous. [Note for future reference that these first two observations are geometry-independent, and thus they both hold for $\theta_{\mathbb{L}}$ [with $p_c(\mathbb{Z}^d)$ replaced by $p_c(\mathbb{L})$, of course] for *any* subset $\mathbb{L}$ of $\mathbb{Z}^d$.] Thirdly, it was shown by van den Berg and Keane (1984) that $\theta_{\mathbb{Z}^d}$ is left-continuous strictly above $p_c(\mathbb{Z}^d)$ provided that the infinite cluster is a.s. unique, i.e., provided that w.p.1 there can be no more than one infinite cluster. [Their result extends immediately to a large class of subsets of $\mathbb{Z}^d$ including half-spaces, quadrants, ..., orthants, slabs and quarter-slices.] Finally, it is known (see Comment 5 below) that there is (a.s.) uniqueness of the infinite cluster in $\mathbb{Z}^d$. The combination of these four facts yields the conclusion that $\theta_{\mathbb{Z}^d}$ is continuous at $p_c(\mathbb{Z}^d)$, and indeed on all of $(0, 1)$, if and only if $\theta_{\mathbb{Z}^d}(p_c(\mathbb{Z}^d)) = 0$. Although we are unable to resolve the continuity issue for the full-space, the argument outlined in this comment can be used to settle the question for various partial-spaces – once we have proved that there is uniqueness of the infinite cluster in those spaces.

5) For the full-space (and also the half-space and the quadrant) in two dimensions, Harris (1960) [see also Fisher (1961)] showed that there could be at most one infinite cluster on the square lattice. More generally, Aizenman et al. (1987) showed that the infinite cluster is unique in the full-space for all dimensions; a simplified version of their argument was produced by Gandolfi et al. (1988); Gandolfi (1989) has extended the argument to some dependent percolation models. The recent uniqueness result of Burton and Keane (1989) subsumes all of the above-mentioned (full-space) results for finite-range models. Various extensions of Burton and Keane (1989) can be found in Gandolfi et al. (1991).

6) It follows from Comments 2 and 3 that percolation in a half-space, quadrant, ..., or orthant implies percolation in some quarter-slice. It was proven

in Aizenman et al. (1983) [see Theorem 4.4 and the discussion on page 60 there] that, in $d=3$ dimensions, percolation in a quarter-slice implies that the corresponding full-slice (and all of $\mathbb{Z}^3$) contains a unique infinite cluster. Essentially the same argument shows that under the same hypothesis, there is uniqueness of the infinite cluster in full-spaces, half-spaces, quadrants, ..., and orthants (as well as half-slabs, half-slices, ...) for any $d \geqq 3$. We note that Kesten (1989) has already shown that the infinite cluster in the half-space is unique.

We summarize the observations made in Comments 2, 3, 4 and 6 in the following result.

Corollary to Theorem 1.1 *Suppose* $d \geqq 3$. i) *There is equality between the critical densities of the subspaces* $\mathbb{Z}^{d-e} \times \mathbb{Z}^e_+$ $(1 \leqq e \leqq d)$ *and the limits of the critical densities of slabs and quarter-slices:*

$$p_c(\mathbb{H}) = p_c(\mathbb{Z}^{d-2} \times \mathbb{Z}^2_+) = \ldots = p_c(\mathbb{Z}^d_+) = p_c(\mathbb{S}) = p_c(\mathbb{Q}). \tag{1.3}$$

ii) *The percolation probabilities of the subspaces* $\mathbb{Z}^{d-e} \times \mathbb{Z}^e_+$ $(1 \leqq e \leqq d)$ *all vanish at their common critical density:*

$$\theta_{\mathbb{H}}(p_c(\mathbb{H})) = \ldots = \theta_{\mathbb{Z}^d_+}(p_c(\mathbb{Z}^d_+)) = 0. \tag{1.4}$$

iii) *When they exist, the infinite clusters in* $\mathbb{Z}^{d-e} \times \mathbb{Z}^e_+$ $(1 \leqq e \leqq d)$ *are a.s. unique.*

iv) *The percolation probabilities of the subspaces* $\mathbb{Z}^{d-e} \times \mathbb{Z}^e_+$ $(1 \leqq e \leqq d)$ *are continuous functions at all densities p.*

Comment. 7) The results listed in the corollary are already known to be true if $d=2$, as explained previously, provided that (1.3) is modified by removing $p_c(\mathbb{S})$ and $p_c(\mathbb{Q})$. Our methods may be used to provide new derivations in two dimensions of (1.3) [again with $p_c(\mathbb{S})$ and $p_c(\mathbb{Q})$ removed] and (1.4).

The next theorem is the heart of the paper. In addition to providing an immediate proof of Theorem 1.1, it also makes some progress towards proving that percolation in the full-space implies percolation in some sufficiently "thick" two-dimensional slice of the space. Further progress is made in Grimmett and Marstrand (1990).

Theorem 1.2 i) *If* $\theta_{\mathbb{H}}(p) > 0$, *then there exist* $\delta > 0$ *and* $L' \in \mathbb{Z}_+$ *such that* $\theta_{\mathbb{Q}_{L'}}(p - \delta) > 0$. *In words, if there is percolation in the half-space at a particular density of open bonds, then a quarter-slice can be found for which there is percolation at some lower density of open bonds.*

ii) *More generally, if* $\theta_{\mathbb{L}}(p) > 0$ *for some* $\mathbb{L} = \left(\bigtimes_{i=1}^{f} \{0, \ldots, h_i\} \right) \times \mathbb{Z}^{d-e-f} \times \mathbb{Z}^e_+$ *with* $e \geqq 1$ *and* $d - f \geqq 2$, *then there exist* $\delta > 0$ *and* $L' \in \mathbb{Z}_+$ *such that* $\theta_{\mathbb{L}'}(p - \delta) > 0$ *for* $\mathbb{L}' = \left(\bigtimes_{i=1}^{f} \{0, \ldots, h_i\} \right) \times \{-L', \ldots, L'\}^{d-f-2} \times \mathbb{Z}^2_+$.

We remark that part ii) of Theorem 1.2 can be used to prove slab-like versions of Theorem 1.1. For example, it can be shown that i) the percolation probability of the half-"slab" $\{0, \ldots, h\} \times \mathbb{Z}^{d-2} \times \mathbb{Z}_+$ is zero at its critical density, and ii) the limit as $L \to \infty$ of the critical density of the quarter-"slice" $\{0, \ldots, h\} \times \{-L, \ldots, L\}^{d-3} \times \mathbb{Z}_+^2$ equals the critical density of this half-slab.

We shall prove Theorem 1.2 i), using renormalization methods, which can be extended with slight modifications (not presented here) to also prove part ii).

Here is a sketch of our main argument. Impose a grid of $2K \times 2K \times \ldots \times 2K$ cubes on the subset $\mathbb{Q}_K$ of $\mathbb{H}$. (In this sketch $K = L'$; in the actual argument of Sect. 4, we impose on $\mathbb{Q}_{L'}$ a grid of $(2L')^{d-2} \times (2K)^2$ "cubes" with $K > L'$.) There is a natural identification of these $2K$-cubes in $\mathbb{Q}_K$ with sites in $\mathbb{Z}_+^2$. Based on the percolation model on $\mathbb{H}$, we shall construct a type of dependent percolation process on $\mathbb{Z}_+^2$ in such a way that the clusters of the latter process correspond in a certain way to clusters of the former. Thus, if the latter process is supercritical, then so is the former. We shall then show that, subject to a suitable initial hypothesis on the original model [such as that $\theta_{\mathbb{H}}(p) > 0$], the integer K may be chosen large enough that the corresponding (dependent) site percolation process is supercritical. It will follow that there is percolation in thick quarter-slices (even after slightly reducing the bond density) whenever there is percolation in half-spaces. Our main goal is then to describe a suitable way of constructing such a renormalized site percolation process.

The idea is as follows. First we examine the cube C_0 containing the origin. If the origin is joined within C_0 by open paths to "many" sites on (or at least "near" to) the north and east faces of C_0, we declare C_0 to be "good" and the site in $\mathbb{Z}_+^2$ corresponding to C_0 (say 0) to be occupied. (Throughout this sketch we shall use such terms as "many" and "near" without formal explanations.) If C_0 is found to be "bad" (i.e., not good), then the process is stopped. If C_0 is good, then we use the algorithm for growing clusters outlined in the preceding section to choose the first neighbor, say x, of 0. The cube C_x corresponding to the site x is now examined, and declared to be good (and x is declared to be occupied) if one of the sites reached previously on (or near) the face common with C_0 is joined (mostly) within C_x to many sites on (or near) the other three of the north, south, east and west faces of C_x. (As we shall see later in Sect. 3, we will actually not declare C_x to be good unless these many sites are reached by a specific geometric construction.) This procedure is now iterated in the natural way – at each stage one chooses a new cube neighboring some good cube (according to our cluster-growing algorithm) and declares this new cube to be either "good" or "bad." We shall make this procedure rigorous, and show that, under the hypothesis that $\theta_{\mathbb{H}}(p) > 0$, the associated (dependent) percolation process is supercritical. It will follow that there is a positive probability of being able to find an infinite network of good cubes; the existence of such a network implies that the original percolation model *in the quarter-slice* $\mathbb{Q}_K$ is supercritical.

Most of the work lies in showing that the hypothesis $\theta_{\mathbb{H}}(p) > 0$ is sufficient to build (with large probability) appropriate open paths across a large cube. In doing this, we shall first work on a smaller scale, showing the existence of "occupied bricks" which may be used to construct such paths (see Figs. 7 and 10 for an indication of what this means). More particularly, we use the hypothesis in question to understand the geometry of open paths within a parallelopiped of dimensions $2L \times 2L \times \ldots \times 2L \times H$ for some L and H; we shall have no control over the ratio L/H for an occupied brick, a fact which will later turn out to be significant. We shall then build paths by carefully positioning a sequence of bricks within a larger cube.

Two difficulties arise in building such paths. First, for geometrical reasons, we require different strategies depending on whether L/H is small or large; we shall end up using different arguments depending on whether $L \leq H$ or not. Secondly, in putting the bricks together there is a degree of uncertainty about the exact placement of particular bricks. As a consequence of this we cannot normally expect to reach sites *exactly on* a specified face of a specified cube. Instead we make do with reaching near the face by defining a "target region" near the center of the face and stopping the brick construction when we arrive in this target area.

We have now reached the end of the outline of the argument. In Sect. 2 it is established that occupied bricks exist with high probability. Sect. 3 contains the geometrical constructions for connecting the bricks to build the cube crossing events. In Sect. 4 we combine these results to give the proof of part i) of Theorem 1.2.

2 Existence of occupied bricks

By the term "brick" we mean a d-dimensional box $B(L, H)$ in $\mathbb{Z}^d$ with side-lengths $2L, 2L, \ldots, 2L$ and H, for some L and H. Our aim in this section is to describe an event which is measurable on the interior of the brick $B(L, H)$ and useful in constructing long open paths, and to show the existence of (deterministic) L and H for which this event has large probability whenever $\theta_{\mathbb{H}}(p) > 0$. We begin with some notation. Bricks come in four varieties: north, south, east and west, which are translates of the boxes

$$\begin{aligned} B_{\text{North}}(L, H) &= \{-L, \ldots, L\}^{d-1} \times \{0, \ldots, H\}, \\ B_{\text{South}}(L, H) &= \{-L, \ldots, L\}^{d-1} \times \{-H, \ldots, 0\}, \\ B_{\text{East}}(L, H) &= \{-L, \ldots, L\}^{d-2} \times \{0, \ldots, H\} \times \{-L, \ldots, L\} \quad \text{and} \\ B_{\text{West}}(L, H) &= \{-L, \ldots, L\}^{d-2} \times \{-H, \ldots, 0\} \times \{-L, \ldots, L\}, \end{aligned}$$

respectively. (The terminology arises from ignoring the first $d-2$ coordinates. The distinction between north and south bricks, and between east and west bricks, will become clearer after the notion of "occupied brick" is introduced.) For simplicity we restrict our attention (for the time being) to north bricks, and while operating under this restriction we will generally drop the 'North' subscript.

The top of the brick $B(L, H)$ can be divided (see Fig. 1) into 2^{d-1} congruent regions:

$$\begin{aligned} T_1(L, H) &= \{0, \ldots, L\}^{d-1} \times \{H\}, \\ T_2(L, H) &= \{0, \ldots, L\}^{d-2} \times \{-L, \ldots, 0\} \times \{H\}, \ldots, \\ T_{2^{d-1}}(L, H) &= \{-L, \ldots, 0\}^{d-1} \times \{H\}. \end{aligned}$$

Similarly, the sides can be divided into $2(d-1)\, 2^{d-2}$ subregions:

$$\begin{aligned} S_1(L, H) &= \{L\} \times \{0, \ldots, L\}^{d-2} \times \{0, \ldots, H\}, \\ S_2(L, H) &= \{L\} \times \{0, \ldots, L\}^{d-3} \times \{-L, \ldots, 0\} \times \{0, \ldots, H\}, \ldots, \\ S_{(d-1)2^{d-1}}(L, H) &= \{-L, \ldots, 0\}^{d-2} \times \{-L\} \times \{0, \ldots, H\}. \end{aligned}$$

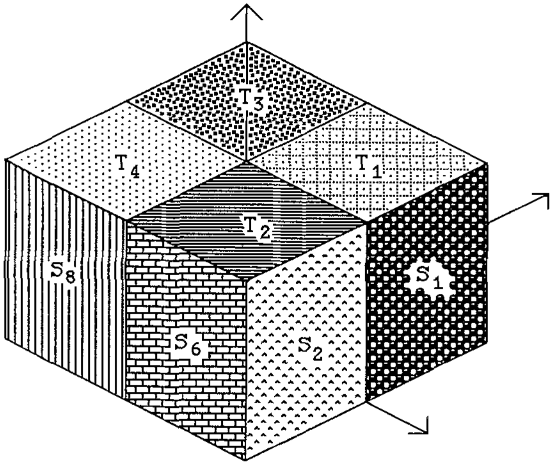

Fig. 1. The brick $B_{\text{North}}(L, H)$ in $d=3$ dimensions

(We will simply refer to the regions as S_i and T_j when the dimensions L and H may be taken from the context.) Letting

$$D=\{-L, \ldots, L\}^{d-1} \times \{0\},$$
$$S=S(L,H)=\cup_i S_i, \quad \text{and} \quad T=T(L,H)=\cup_j T_j$$

denote the bottom, sides and top of the brick, we can write the boundary of the brick as $\partial B(L,H)=D \cup S \cup T$. By a *connection in* $B^*(L,H)$ we will mean an open path between sites in $B(L,H)$ which is accomplished without making any use of "boundary bonds." A *boundary bond* in $B(L,H)$ is a bond between two sites in $\partial B(L,H)$. A *connection in* $B'(L,H)$ will mean a connection which does not use any bond between pairs of sites in the bottom D of the brick.

Let k be a positive integer. A *hyperblock* is a translate of

$$b_m(k)=\{-k, \ldots, k\}^{m-1} \times \{0\} \times \{-k, \ldots, k\}^{d-m} \quad \text{where } m \in \{1, \ldots, d\}.$$

The hyperblock $x+b_m(k)$ is said to be *open* if all of the sites in $x+b_m(k)$ are connected to one another by open bonds lying entirely in the hyperblock. If $x \in S_i$ for a unique value of i, we denote the hyperblock centered at x and parallel to S_i by $x+b_{m(x)}(k)$. If $x \in S_i$ for more than one such i, we arbitrarily choose one such value of i and define $x+b_{m(x)}(k)$ accordingly. Eventually we will remove this arbitrariness by only considering sites in S which are not on any hyperedge. For $y \in T$, we consider the hyperblock $y+b_d(k)$.

Suppose $L \geqq k$ and $H \geqq 2k$. We define the brick $B(L,H)$ to be *occupied* (see Fig. 2) if there exists at least one site $x_i \in S_i$ (and $y_j \in T_j$) for each $i=1, \ldots, (d-1)\,2^{d-1}$ (resp. $j=1, \ldots, 2^{d-1}$) which is the center of an open hyperblock $x_i+b_{m(x_i)}(k)$ contained in S [resp. $y_j+b_d(k)$ contained in T] and which is connected in $B'(L,H)$ to $b_d(k)$. Note that this definition does not require the initial

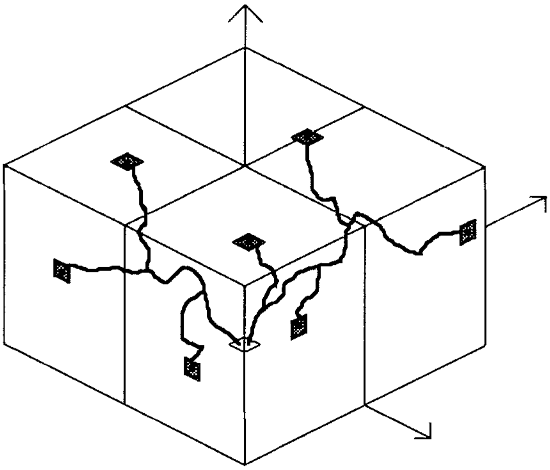

Fig. 2. An occupied north brick in $d=3$ dimensions and some of its attachment sites. Each small shaded region represents an open hyperblock. The bold lines represent paths of open bonds

hyperblock $b_d(k)$ to be open. We refer to the centers of the open hyperblocks on S and T which are reached from $b_d(k)$ as the *attachment sites* of the brick.

The requirement that occupied bricks have many attachment sites will be used in Sect. 3. There we construct long paths of open bonds having certain properties by sequentially placing occupied bricks at appropriately chosen attachment sites of other occupied bricks. The ability to *choose* an attachment site from any S_i or T_j will be essential, even though we will never *use* more than two attachment sites of any brick.

We can now state the existence theorem for occupied (north) bricks.

Proposition 2.1 *Suppose that there is percolation in the half-space (i.e.,* $\theta_{\mathbb{H}}(p)>0$*). Then for each* $\varepsilon>0$ *there exists a hyperblock length* $k(>0)$*, a brick length* $L(\geqq k)$*, and a brick height* $H(\geqq 2k)$ *such that*

$$\text{(2.1a)} \qquad P_p(B_{\text{North}}(L,H) \text{ is occupied}) > 1-\varepsilon.$$

Furthermore, there also exists $\delta>0$ *such that*

$$\text{(2.1b)} \qquad P_{p-\delta}(B_{\text{North}}(L,H) \text{ is occupied}) > 1-\varepsilon.$$

We first note that (2.1b) is an immediate consequence of (2.1a) since the event $\{B_{\text{North}}(L,H)$ is occupied$\}$ depends on the states of only finitely many bonds, and hence its probability is a continuous function of p.

Before plunging into the details we present an overview of the proof. The basic idea is quite simple. By making k very large we can guarantee that there is a large probability that $b_d(k)$ contains a site z which belongs to an infinite cluster in $\mathbb{H}^*$ (i.e., $\mathbb{H}$ with the bonds in $\mathbb{Z}^{d-1}\times\{0\}$ removed). If this event occurs for some z, then (in the limit as $L,H\to\infty$) the number of sites on the union of the sides S and the top T of $B(L,H)$ which are connected to z in $B^*(L,H)$ is a.s. unbounded. An appropriate choice of a sequence of (L,H)'s will then ensure that there are as many of these sites as may be required on both the sides and the top. If there are enough sites $x \in S$ (and $y \in T$) which are connected to z in $B^*(L, H)$, then one can sample enough disjoint sets of bonds in S (resp. T) to ensure that there is high probability of finding open hyperblocks containing some of these sites. Finally, if $b_d(k)$ is connected with sufficiently large probability to an open hyperblock in both S and T, then it will be connected with large probability to at least one open hyperblock in each subregion S_i and T_j. (The last step uses a standard argument based on the Harris-FKG inequality; it is presented in the proof of Proposition 2.1 following Corollary 2.7 below.) Therefore, the basic strategy to prove (2.1a) is first to choose a large k and then to show that $b_d(k)$ is connected in $B^*(L, H)$ to at least M sites in both S and T, where M is large enough that in any collection of M sites in S (resp. T), there will be a huge number of disjoint hyperblocks in S (resp. T) with each hyperblock containing at least one of the M sites.

We next explain exactly how k and M are chosen. There are two separate cases to consider in choosing k (the first of which is, a posteriori, eliminated). Let the $\varepsilon > 0$ of Proposition 2.1 be given and take $\eta \in (0, \frac{1}{2})$; later we shall choose η in terms of ε.

1. Suppose $\theta_{\mathbb{S}_h^*}(p) = 0$ for every $h > 0$. (The asterisk in $\mathbb{S}_h^*$ indicates that "boundary" bonds are not considered in determining the cluster of the origin.) Then choose $k = k(\eta)$ so that

$$P(b_d(k) \leftrightarrow \infty \text{ in } \mathbb{H}^*) > 1 - \eta^2. \tag{2.2a}$$

2. If $\theta_{\mathbb{S}_h^*}(p) > 0$ for some values of h, then arbitrarily pick such a value h_0 and take k so that

$$P(b_d(k) \leftrightarrow \infty \text{ in } \mathbb{S}_{h_0}^*) > 1 - \eta^2. \tag{2.2b}$$

In either case we fix k for the remainder of the proof of Proposition 2.1 and abbreviate $b_m(k)$ to b_m.

We need to find a number $M = M(k, \eta)$ so that given M or more sites $\{x_r\}$ in S, we can choose a set of sites $\{z_r\}$ in S so that $x_r \in z_r + b_{m(z_r)} \subset S$, and such that there are enough disjoint hyperblocks in the collection $\{z_r + b_{m(z_r)}\}$ so that, with large probability, at least one of them is open.

Thus M must satisfy a requirement of the form

$$[1 - P(b_d \text{ is open})]^{N_1} < \eta, \tag{2.3a}$$

where $N_1 = \lceil M/(6k+1)^d \rceil$ with $\lceil x \rceil$ being the smallest integer greater than x. The term $(6k+1)^d$ is a geometric "packing factor" which will be used (see the discussion preceding Lemma 2.6) to guarantee that there are enough sites $\{x_r\}$ so that many *disjoint* hyperblocks $\{z_r + b_{m(z_r)}\}$ are available.

Similar considerations for a collection of sites $\{y_r\}$ in T show that if M satisfies (2.3a), then there will be enough disjoint hyperblocks in T if there are at least M sites in the collection. However, for technical reasons (see the proof of Lemma 2.5), we require also that

$$P\ (\text{fewer than } M \text{ successes in } N_2 \text{ Bernoulli trials with sucess probability } p) < \eta, \tag{2.3b}$$

where $N_2 = \lceil 2M/p \rceil$ and p is the density of open bonds.

The proof of Proposition 2.1 is broken up into several lemmas of which the first follows. Let Y_h be the number of sites on the "top" hypersurface ($\mathbb{Z}^{d-1} \times \{h\}$) of the slab $\mathbb{S}_h$ which are connected strictly within $\mathbb{S}_h$ to sites in b_d by open bonds:

$$(2.4) \qquad Y_h = |\{y \in \mathbb{Z}^{d-1} \times \{h\} : b_d \leftrightarrow y \text{ in } \mathbb{S}_h^*\}|.$$

The first lemma is a sort of zero-one law for the random variables Y_h which is used directly in the proof of Lemma 2.3 below, and whose method of proof will be used in proving Lemma 2.5.

Lemma 2.2 *With probability one, either there exists a (random) height H so that $Y_h = 0$ for all $h \geqq H$, or else $\lim_{h\to\infty} Y_h = \infty$.*

Proof. Define a new random variable

$$Z_h(n) = |\{j \leqq h : 0 < Y_j < n\}|.$$

It suffices to show that $Z_\infty(n) \equiv \lim_{h\to\infty} Z_h(n) < \infty$ a.s. for all $n \geqq 1$.

The idea is as follows: if $Z_\infty(n)$ is infinite then there are infinitely many hypersurfaces $\mathbb{Z}^{d-1} \times \{h\}$ which are connected to b_d in $\mathbb{H}^*$ but which contain fewer than n sites connected to b_d in $\mathbb{S}_h^*$. Each time that such a hypersurface $\mathbb{Z}^{d-1} \times \{h\}$ is reached, there is probability at least $(1-p)^{(2d-1)n}$ that none of these sites are connected to the next hypersurface $\mathbb{Z}^{d-1} \times \{h+1\}$. Therefore, the latter event occurs for some h a.s., so that $P_p(Z_\infty(n) = \infty) = 0$, as required.

More rigorously, we define for each $n \geqq 1$ a sequence of random variables $\{W_r\}_{r\geqq 1}$, where W_r is the smallest height h for which $Z_h(n) = r$ (and $W_r = \infty$ if $Z_\infty(n) < r$). Then for each r

$$\begin{aligned} P(Z_\infty(n) < \infty | Z_\infty(n) \geqq r) &= P(Z_\infty(n) < \infty | W_r < \infty) \\ &\geqq P(Y_{W_r+1} = 0 | W_r < \infty) \geqq (1-p)^{(2d-1)n}. \end{aligned}$$

Letting $r \to \infty$, we see that $P(Z_\infty(n) < \infty) = 1$, since for any random variable Z,

$$\lim_{r\to\infty} P(Z < \infty | Z \geqq r) = \begin{cases} 1, & \text{if } P(Z < \infty) = 1 \\ 0, & \text{if } P(Z < \infty) < 1. \end{cases} \quad \square$$

Lemma 2.3 *If $\theta_{\mathbb{H}}(p) > 0$, and k and N_2 are chosen as above, then there exists an $H_0 = H_0(\eta)$ such that*

$$(2.5) \qquad P(Y_h \geqq 2N_2) > 1 - \eta$$

for all $h \geqq H_0$.

Proof. There are two cases to consider:

a) $\theta_{\mathbb{S}_h^*}(p) = 0$ for all h,

and

b) $\theta_{\mathbb{S}_h^*}(p) > 0$ for some h.

(Again, a posteriori, the first possibility can eventually be disregarded.)

a) From Lemma 2.2 (w.p.1) either $Y_h \to \infty$ as $h \to \infty$ or else $Y_h = 0$ eventually. Because $\theta_{\mathbb{S}_h^*}(p) = 0$ for all h, the events "$Y_h \to \infty$ as $h \to \infty$" and "$b_d \leftrightarrow \infty$ in $\mathbb{H}^*$" can only differ by a set of bond configurations of measure zero. Thus

$$P(Y_h \geqq 2N_2) \geqq P(Y_h \geqq 2N_2 \mid Y_j \to \infty \text{ as } j \to \infty)\, P(b_d \leftrightarrow \infty \text{ in } \mathbb{H}^*).$$

The conditional probability on the right-hand side (RHS) of this inequality can be made larger than $1-\eta^2$ by taking h sufficiently large, and the other probability on the RHS is larger than $1-\eta^2$ by (2.2a). For such values of h, (2.5) holds since $\eta < \frac{1}{2}$.

b) Take $H_0 = h_0$, where h_0 is as in (2.2b). For each $h \geqq h_0$, consider the nested sequence of boxes $\{B_n = B(k+n, h)\}_{n \geqq 1}$. Now algorithmically grow the "cluster of b_d" in $\mathbb{S}_h^*$ using a minor variant of the procedure described in Sect. 0: deterministically order all of the bonds of $\mathbb{S}_h^*$, at time 0 take b_d to be the initial collection of sites, and at later times check the bond with lowest order (not already checked) which could connect another site to the current "cluster" – adding the site to the cluster if the bond is open. Upon reaching the first site in $\partial B_n \backslash \partial B_{n-1}$, call it x_n, observe that none of the bonds along the line

$$\{y = (y(1), \ldots, y(d)) : y(1) = x_n(1), \ldots, y(d-1) = x_n(d-1)\}$$

have yet been examined. With probability p^h *all* of the bonds along this line which also lie in $\mathbb{S}_h^*$ are open, in which case b_d is connected to the site on the top hypersurface $\mathbb{Z}^{d-1} \times \{h\}$ directly "overhead" of x_n. It follows that every time that the "cluster of b_d" reaches the boundary of a new box B_n, it has at least probability p^h of being connected (in $\partial B_n \backslash \partial B_{n-1}$) to the overhead site on $\mathbb{Z}^{d-1} \times \{h\}$. Thus, conditional on the event that $b_d \leftrightarrow \infty$ in $\mathbb{S}_h^*$, $b_d(k)$ must also be connected in $\mathbb{S}_h^*$ w.p.1 to infinitely many sites on $\mathbb{Z}^{d-1} \times \{h\}$. Since k was chosen so that $b_d(k) \leftrightarrow \infty$ in $\mathbb{S}_{h_0}^*$ (and hence also in $\mathbb{S}_h^*$) with at least probability $1-\eta^2$, we have just shown that

$$P(Y_h = \infty) > 1 - \eta^2$$

[and hence (2.5) is valid] for all $h \geqq h_0$. □

Now define $Y_{l,h}$ to be the number of sites on the top of the brick $B(l,h)$ which are connected to (sites in) b_d by open bonds – not using the boundary bonds of $B(l,h)$:

$$Y_{l,h} = |\{y \in T(l,h) : b_d \leftrightarrow y \text{ in } B^*(l,h)\}|.$$

Clearly, $Y_{l,h} \to Y_h$ as $l \to \infty$, and combining this fact with Lemma 2.3 shows that for every $h \geqq H_0$, there exists a length $L_0 = L_0(h)$ for which

$$P(Y_{l,h} \geqq N_2) > 1 - 2\eta \tag{2.6}$$

whenever $l \geqq L_0$. We now "fine-tune" the height h so that there are just barely N_2 sites on the top of $B(l,h)$ connected to b_d in $B^*(l,h)$. This fine-tuning will be used shortly in order to prove that there must be many sites on both the top and the sides of the brick which are connected to b_d in $B^*(l,h)$.

Lemma 2.4 *For every $l \geqq L_0(H_0)$ there exists a height $H_1 = H_1(l) > H_0$ so that*

(2.7a) $$P(Y_{l,H_1-1} \geqq N_2) > 1 - 2\eta$$

and

(2.7b) $$P(Y_{l,H_1} \geqq N_2) \leqq 1 - 2\eta.$$

Furthermore, $H_1(l) \to \infty$ as $l \to \infty$.

Proof. To demonstrate the existence of such an H_1 it suffices to show that $Y_{l,h} \to 0$ as $h \to \infty$, because we already know from (2.6) that (2.7a) is satisfied with $H_1 = H_0 + 1$. If a subsequence of the random variables $\{Y_{l,h}\}_{h > H_0}$ were not to tend to zero for some value of $l \geqq L_0(H_0)$, then there would necessarily be percolation in $\{-l, \ldots, l\}^{d-1} \times \mathbb{Z}_+$, an event with zero probability. Therefore Eqs. (2.7) are satisfied for some $H_1 > H_0$; henceforth let $H_1(l)$ denote the minimal value of H_1 (above H_0) satisfying those equations.

To see that $H_1(l)$ diverges as $l \to \infty$, first observe that H_1 is nondecreasing in l. If H_1 does not diverge then it has a finite limit $H_1(\infty)$. Since $Y_{l,H_1(l)} \to Y_{H_1(\infty)}$ as $l \to \infty$, it follows from (2.7b) that

$$P(Y_{H_1(\infty)} \geqq N_2) \leqq 1 - 2\eta$$

which clearly contradicts (2.5). Therefore $H_1(l)$ must diverge as $l \to \infty$. □

We now introduce the random variable $X_{l,h}$ which counts the number of sites on the sides of $B(l, h)$ which are connected to b_d by open bonds in $B^*(l, h)$:

$$X_{l,h} = |\{x \in S(l, h):\ b_d \leftrightarrow x \text{ in } B^*(l, h)\}|.$$

We shall next show that the fine-tuning of Lemma 2.4 guarantees the existence of values of l and h so that, with high probability, both $X_{l,h}$ and $Y_{l,h}$ are large.

Lemma 2.5 *There exists a length $L_1 > L_0(H_0)$ so that for all $l \geqq L_1$,*

(2.8a) $$P(X_{l,H_1(l)} \geqq M) > 1 - \eta$$

and

(2.8b) $$P(Y_{l,H_1(l)} \geqq M) > 1 - 3\eta.$$

Proof. Since it is simpler, we prove (2.8b) first. From inequality (2.7a) of Lemma 2.4, it follows that for every $l \geqq L_0(H_0)$

$$\begin{aligned} P(Y_{l,H_1(l)} \geqq M) &\geqq P(Y_{l,H_1(l)-1} \geqq N_2)\, P(Y_{l,H_1(l)} \geqq M \mid Y_{l,H_1(l)-1} \geqq N_2) \\ &> (1 - 2\eta)\, P(Y_{l,H_1(l)} \geqq M \mid Y_{l,H_1(l)-1} \geqq N_2). \end{aligned}$$

One way in which the event $\{Y_{l,H_1(l_1)} \geqq M\}$ can occur is for at least M of the sites that are counted in $Y_{l,H_1(l)-1}$ to be directly connected (i.e., by single open bonds) to their nearest-neighbors on the upper hypersurface $\{-l, \dots, l\}^{d-1} \times \{H_1(l)\}$. Thus

$$P(Y_{l,H_1(l)} \geqq M) > (1-2\eta)\, P \text{ (at least } M \text{ successes in } N_2 \text{ Bernoulli trials)}$$
$$\geqq 1-3\eta,$$

where the last inequality was obtained from (2.3b).

For the proof of (2.8a), assume for the moment that there exists an $L_1 > L_0(H_0)$ so that

$$P(X_{l,H_1(l)} + Y_{l,H_1(l)} \geqq M + N_2) > 1 - 2\eta^2, \tag{2.9}$$

for all $l \geqq L_1$. Then taking $l \geqq L_1$ and using the FKG inequality we have that

$$2\eta^2 > P(X_{l,H_1(l)} < M,\ Y_{l,H_1(l)} < N_2) \geqq P(X_{l,H_1(l)} < M)\, P(Y_{l,H_1(l)} < N_2).$$

Combining this inequality with the bound (2.7b) of Lemma 2.4 shows that

$$P(X_{l,H_1(l)} < M) \leqq \eta$$

which is what we wanted to show. It remains only to verify (2.9) for some sufficiently large L_1. For this, note that

$$\hat{Z}(m) = |\{j: 0 < X_{j,H_1(j)} + Y_{j,H_1(j)} < m\}|$$

is a.s. finite (see the related proof of Lemma 2.2) so that the events

$$\{X_{l,H_1(l)} + Y_{l,H_1(l)} \to \infty \text{ as } l \to \infty\} \quad \text{and} \quad \{b_d \leftrightarrow \infty \text{ in } \mathbb{H}^*\}$$

agree up to an event of zero probability. Therefore,

$$\begin{aligned} &P(X_{l,H_1(l)} + Y_{l,H_1(l)} \geqq M + N_2) \\ &\quad \geqq P(X_{l,H_1(l)} + Y_{l,H_1(l)} \geqq M + N_2 \mid X_{j,H_1(j)} + Y_{j,H_1(j)} \to \infty \text{ as } j \to \infty) \\ &\quad \cdot P(b_d \leftrightarrow \infty \text{ in } \mathbb{H}^*). \end{aligned}$$

Letting $l \to \infty$ and using (2.2), we obtain (2.9). □

Now, given a set $\mathfrak{X}$ containing M or more sites in S, it is easy to see that one may find a set $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{X})$ containing $N_1 (= \lceil M/(6k+1)^d \rceil)$ sites in S with the properties (i) for all $z \in \mathfrak{Z}$, there exists $x \in \mathfrak{X}$ such that $\|x - z\|_\infty \stackrel{\text{def}}{=} \max\{|x(i) - z(i)| : i = 1, \dots, d\} \leqq k$, and (ii) the hyperblocks $\{z + b_{m(z)} : z \in \mathfrak{Z}\}$ are disjoint and lie entirely in S.

By property (ii) above, the events $\{\{z + b_{m(z)} \text{ is open}\} : z \in \mathfrak{Z}\}$ are independent. Since there are more than N_1 sites in $\mathfrak{Z}$, (2.3a) implies that

$$\text{P}(\exists \text{ a site } z \in \mathfrak{Z}(\mathfrak{X}) \text{ for which } z + b_{m(z)} \text{ is open in } S) > 1 - \eta, \tag{2.10a}$$

for every collection $\mathfrak{X}$ of at least M sites in S.

The definitions and arguments of the preceding two paragraphs may be repeated for the top of the brick with $\mathfrak{X}$, S, $\mathfrak{Z}$, z, x and $m(z)$ replaced by $\mathfrak{Y}$, T, $\mathfrak{W}$, w, y and d, respectively. Thus we also have that

$$\text{(2.10b)} \qquad P(\exists \text{ a site } w \in \mathfrak{W}(\mathfrak{Y}) \text{ for which } w + b_d \text{ is open in } T) > 1 - \eta,$$

for every collection $\mathfrak{Y}$ of at least M sites in T.

We are nearly ready to prove Proposition 2.1. We introduce a pair of random variables which are analogous to $X_{l,h}$ and $Y_{l,h}$, with the difference being that these new variables count the attachment sites of $B(l, h)$ instead of the sites connected to b_d in $B^*(l, h)$:

$$U_{l,h} = |\{x \in S(l, h): x + b_{m(x)} \text{ is an open hyperblock contained in } S \text{ which is connected to } b_d \text{ in } B'(l, h)\}|$$

and

$$V_{l,h} = |\{y \in T((l, h): y + b_d \text{ is an open hyperblock contained in } T \text{ which is connected to } b_d \text{ in } B'(l, h)\}|.$$

Lemma 2.6 *For all $l \geqq L_1$ (with L_1 as in Lemma 2.5)*

$$\text{(2.11a)} \qquad P(U_{l,H_1(l)} \geqq 1) > 1 - 2\eta$$

and

$$\text{(2.11b)} \qquad P(V_{l,H_1(l)} \geqq 1) > 1 - 4\eta.$$

Proof. The probability on the LHS of (2.11a) can be no smaller than the probability that $U_{l,H_1(l)} \geqq 1$ in the following specific manner: b_d is connected in $B^*(L, H)$ to a collection $\mathfrak{X}$ of at least M sites in S, and there is at least one site $z \in \mathfrak{Z}(\mathfrak{X})$ for which $z + b_{m(z)}$ is open. Conditioning on the set of sites to which b_d is connected in $B^*(L, H)$, we have

$$P(U_{l,H_1(l)} \geqq 1) \geqq \sum_{\mathfrak{X} \subset S: |\mathfrak{X}| \geqq M} P(\exists \text{ a site } z \in \mathfrak{Z}(\mathfrak{X}) \text{ for which } z + b_{m(z)} \text{ is open in } S \mid \mathfrak{X} = \{x \in S: x \leftrightarrow b_d \text{ in } B^*(L, H)\}) \cdot P(\mathfrak{X} = \{x \in S: x \leftrightarrow b_d \text{ in } B^* L, H\}).$$

The conditional probabilities above are all of the form $P(G|F)$ where F depends only on the nonboundary bonds of $B(L, H)$ and G depends only on the boundary bonds; thus F and G are independent and $P(G|F) = P(G)$. Using (2.10a) and (2.8a), we obtain

$$P((U_{l,H_1(l)} \geqq 1) \geqq (1 - \eta) P(X_{l,H_1(l)} \geqq M) \geqq (1 - \eta)^2.$$

This proves (2.11a), and a similar argument using (2.10b) and (2.8b) proves (2.11b). □

As a consequence of Lemma 2.6, we have a result whose proof serves as a model for the proof of Proposition 2.1.

Corollary 2.7 *Suppose that $\theta_{\mathbb{H}}(p)>0$. Then for each $\varepsilon>0$ there exist positive integers k, L and H so that*

(2.12) $P(B_{\text{North}}(L, H)$ *has at least one attachment site in S and one in T*$)>1-\varepsilon$.

Proof. Take $\eta=\varepsilon/6$ and let k, $L=L_1$ and $H=H_1(L_1)$ be as given above in (2.2), Lemma 2.5 and Lemma 2.4, respectively. Then

$$\begin{aligned} &P(B_{\text{North}}(L, H) \text{ has at least one attachment site in } S \text{ and one in } T)\\ &\quad=P(U_{L,H}\geqq 1, V_{L,H}\geqq 1)\geqq 1-P(U_{L,H}=0)-P(V_{L,H}=0)\\ &\quad>1-2\eta-4\eta=1-\varepsilon. \quad \square \end{aligned}$$

Corollary 2.7 is a little weaker than Proposition 2.1. In the next section we shall require some control over the positions of the attachment sites. To this end we define a final set of random variables:

$$\begin{aligned} U_{L,H}^{(i)}=|\{x\in S_i(l,h)\colon x+b_{m(x)} &\text{ is an open hyperblock contained in } S\\ &\text{which is connected to } b_d \text{ in } B'(l,h)\}| \quad \text{for } i=1,\dots,(d-1)\,2^{d-1} \end{aligned}$$

and

$$\begin{aligned} V_{L,H}^{(j)}=|\{y\in T_j(l,h)\colon y+b_d &\text{ is an open hyperblock contained in } T\\ &\text{which is connected to } b_d \text{ in } B'(l,h)\}| \quad \text{for } j=1,\dots,2^{d-1}. \end{aligned}$$

These random variables now enable us to complete the proof of Proposition 2.1 by utilizing a type of argument first introduced by Russo (1978).

Proof of Proposition 2.1 Choose η so that $f(\eta)=\varepsilon$ where

$$f(\eta)=(d-1)\,2^{d-1}(2\eta)^{2^{1-d}/(d-1)}+2^{d-1}(4\eta)^{2^{1-d}}.$$

Then take k, $L=L_1$, and $H=H_1(L_1)$ as in (2.2), Lemma 2.5 and Lemma 2.4, respectively. By symmetry it follows that

$$P(U_{L,H}^{(i)}=n)=P(U_{L,H}^{(1)}=n)$$

for every $n\geqq 0$ and every $i\in\{1,\dots,(d-1)\,2^{d-1}\}$. The key observation is that

$$\begin{aligned} [P(U_{L,H}^{(1)}=0)]^{(d-1)2^{d-1}} &= \prod_{i=1}^{(d-1)2^{d-1}} P(U_{L,H}^{(i)}=0)\\ &\leqq P(U_{L,H}^{(i)}=0 \text{ for all } i\in\{1,\dots,(d-1)\,2^{d-1}\})\\ &=P(U_{L,H}=0)<2\eta, \end{aligned}$$

by the Harris-FKG inequality and Lemma 2.6. Hence

$$P(U_{L,H}^{(i)}=0)<(2\eta)^{2^{1-d}/(d-1)} \tag{2.13a}$$

for each i; a similar argument shows that

$$P(V_{L,H}^{(j)}=0)<(4\eta)^{2^{1-d}} \tag{2.13b}$$

for each $j \in \{1, \ldots, 2^{d-1}\}$. The bounds (2.13) imply that

$$\begin{aligned} &P(B_{\text{North}}(L, H) \text{ is occupied for the hyperblock length } k) \\ &\quad = 1 - P(U_{L,H}^{(i)} = 0 \text{ for some } i \text{ or } V_{L,H}^{(j)} = 0 \text{ for some } j) \\ &\quad > 1 - f(\eta) = 1 - \varepsilon, \end{aligned}$$

which completes the proof. □

We conclude this section by briefly returning to discuss south, east and west bricks. Basically, the south brick $B_{\text{South}}(L, H)$ is a north brick $B_{\text{North}}(L, H)$ which has just been reflected across the hypersurface $x_d = 0$ in $\mathbb{Z}^d$, and an east (or west) brick is a north brick which has just been rotated 90° clockwise (resp. counterclockwise) in the $x_{d-1} - x_d$ plane of $\mathbb{Z}^d$. The "tops" and "sides" of these bricks are the images of the "top" and "sides" of $B_{\text{North}}(L, H)$ under these mappings. We use the natural notation T_{North}, S_{North}, T_{South}, S_{South}, T_{East}, etc. The top (and the sides) of any of these bricks can be divided into 2^{d-1} [resp. $(d-1)2^{d-1}$] congruent subregions $T_{\oplus, j}$ (resp. $S_{\oplus, i}$), where $\oplus$ can be either North, South, East or West. The brick $x + B_{\text{South}}(L, H)$ is said to be occupied (for a given value of k) if the hyperblock $x + b_d$ is connected in $x + B'_{\text{South}}(L, H)$ to attachment sites in every subregion $x + S_{\text{South}, i}$ and $x + T_{\text{South}, j}$, and the brick $x + B_{\text{East}}(L, H)$ [resp. $x + B_{\text{West}}(L, H)$] is said to be occupied (for the hyperblock length k) if the hyperblock $x + b_{d-1}$ is connected in $x + B'_{\text{East}}(L, H)$ [resp. $x + B'_{\text{West}}(L, H)$] to attachment sites in every subregion $x + S_{\text{East}, i}$ and $x + T_{\text{East}, j}$ [resp. $x + S_{\text{West}, i}$ and $x + T_{\text{West}, j}$]. We conclude this section with an immediate consequence of Proposition 2.1.

Proposition 2.8 *If $\theta_{\mathbb{H}}(p) > 0$ and $\varepsilon > 0$, then there exist $\delta > 0$ and positive integers k, L and H so that*

$$P_{p-\delta}(B_{\oplus}(L, H) \text{ is occupied}) > 1 - \varepsilon, \tag{2.14}$$

where $\oplus \in \{$North, South, East, West$\}$.

3 The crossing constructions

3.1 Preliminary remarks and the reduction to two dimensions

In the preceding section we showed that if $\theta_{\mathbb{H}}(p) > 0$, then there is a good probability of being able to produce occupied bricks $B_{\oplus}(L, H)$ [where $\oplus$ can be any of the subscripts: North, South, East or West]. In the first part of this section we will show that these bricks possess convenient 'stacking' properties. The following series of examples illustrates how the bricks may be stacked, introduces some necessary notation, and also explains various conventions which will be used below. In attaching one brick to another to form stacks, we will not allow overlaps or more than one attachment per face. We will also need to control the placement of the new bricks so that the stacks do not stray outside of certain prearranged regions.

i) Suppose that the brick $B_{\oplus}(L, H)$ is occupied. Then the hyperblock $b_{m(\oplus)}$ at the center of the bottom of $B_{\oplus}(L, H)$ [$m(\oplus) = d$ if $\oplus \in \{$North, South$\}$ and $m(\oplus) = d - 1$ if $\oplus \in \{$East, West$\}$] is connected in $B'_{\oplus}(L, H)$ to attachment sites $y_j \in T_{\oplus, j}$

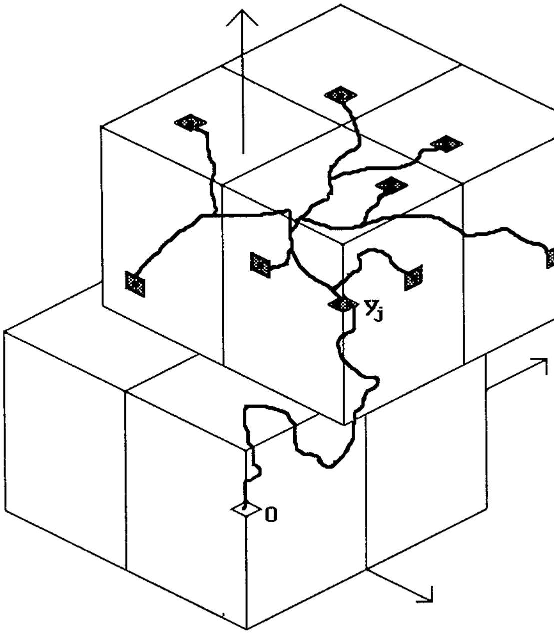

Fig. 3. The brick $y_j+B_{\text{North}}(L,H)$ attached to the brick $B_{\text{North}}(L,H)$

– if there is more than one attachment site in the region $T_{\oplus,j}$, then pick the earliest in some fixed ordering of such sites and call it y_j. For any fixed $j\in\{1,\ldots,2^{d-1}\}$, we can now examine the brick $y_j+B_{\oplus}(L,H)$. If this second brick is also occupied, then the hyperblock $b_{m(\oplus)}$ is connected [through $y_j+b_{m(\oplus)}$] to attachment sites on the top and all of the sides of $y_j+B_{\oplus}(L,H)$. (See Fig. 3 for the case where $\oplus$ is North.)

Now, if y is any fixed site in $T_{\oplus,j}$, then the event E_1, that $B_{\oplus}(L,H)$ is an occupied brick with attachment site $y_j=y$, is independent of the event E_2, that $y+B_{\oplus}(L,H)$ is an occupied brick. Thus if $\theta_{\mathbb{H}}(p)>0$ and k, L and H are as in Proposition 2.8, then by a simple conditioning argument, we see that there is a probability exceeding $(1-\varepsilon)^2$ that $b_{m(\oplus)}$ is connected to the attachment sites of some second $\oplus$-brick thus placed. However if for some $z\in T_{\oplus}(z\neq y)$ we define E_3 to be the event that $z+B_{\oplus}(L,H)$ is an occupied brick, then it is readily seen that E_2 and E_3 are generally dependent with the implication that we shall not be permitted to attempt the placement of two bricks atop the same occupied brick.

ii) So far we have only discussed attaching two bricks by placing a brick on top of an occupied brick of the same type. To simplify later discussions, we introduce the concept of codirection: the two *codirections* of a given direction $\oplus$ are its two orthogonal directions – for example, the codirections of north are east and west. Although we will never try to attach a south brick to a north brick (or vice versa) or the corresponding operation with east and west bricks, we will sometimes find it necessary to attach $\otimes$-bricks to the sides of $\oplus$-bricks (where $\oplus$ and $\otimes$ are codirections). For the sake of definiteness, we shall consider the operation of attaching an east brick to the side of an occupied

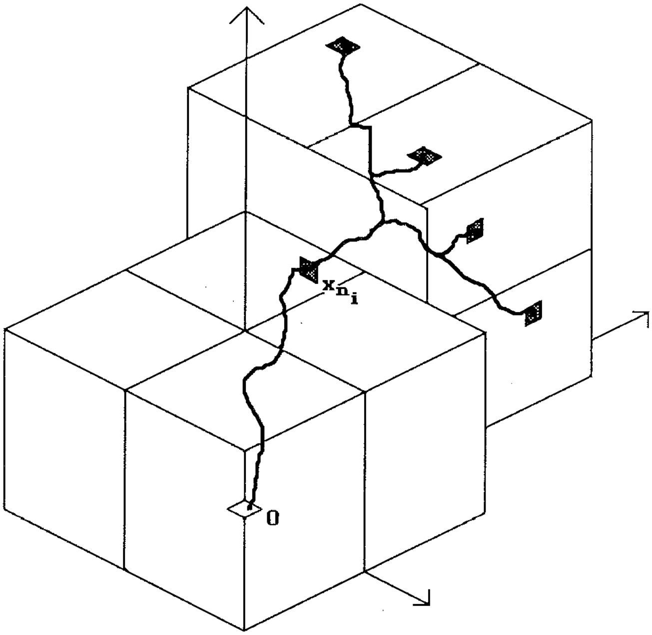

Fig. 4. The brick $x_{n_i}+B_{\text{East}}(L, H)$ attached to the brick $B_{\text{North}}(L, H)$

north brick (see Fig. 4), since each of the other seven ways of attaching two unlike bricks can be obtained by reflecting and/or rotating this procedure.

If the brick $B_{\text{North}}(L, H)$ is occupied, then there are 2^{d-2} subregions S_{North,n_i} of S_{North} lying in the brick's "eastern" face $F_{\text{North,East}}=\{-L, \ldots, L\}^{d-2}\times\{L\}\times\{0, \ldots, H\}$, and each S_{n_i} contains an attachment site x_{n_i}. Again, in the event that there is more than one candidate for x_{n_i}, we will use some given fixed rule to pick the attachment site which will be so identified, and we will only consider brick attachments which are made at these selected sites. If $x_{n_i}+B_{\text{East}}(L, H)$ is also occupied, then the original hyperblock b_d is conneted through the open hyperblock $x_{n_i}+b_{d-1}$ to all of the attachment sites of $x_{n_i}+B_{\text{East}}(L, H)$. As in the case of attaching two bricks of the same type, for any $x\in S_{n_i}$, the events $\{B_{\text{North}}(L, H)$ is occupied with attachment site $x\in S_{n_i}\}$ and $\{x+B_{\text{East}}(L, H)$ is occupied$\}$ are independent, while the events $\{x+B_{\text{East}}(L, H)$ is occupied$\}$ and $\{x'+B_{\text{East}}(L, H)$ is occupied$\}$ are generally dependent for $x\neq x'\in F_{\text{North,East}}$. In general, the same procedure can be used for placing a $\oplus$-brick beside an occupied $\otimes$-brick (where $\oplus$ and $\otimes$ are codirections) with the natural definition of $F_{\otimes,\oplus}$, the $\oplus$ side of the brick $B_{\otimes}(L, H)$.

iii) In order to avoid problems of dependence, we will not attach more than one brick to any given face of a first occupied brick. However, attaching one brick to each of two different faces of a first brick is permissible – provided that the two bricks being attached do not overlap each other. Sometimes it will be necessary to attach to a brick of type $\oplus$ both a brick of the same direction and a brick of one of the two codirections. The pair of attachments can be made without encountering dependency difficulties, if sufficient care is exercised in the placement of the second $\oplus$-brick. For example, suppose that

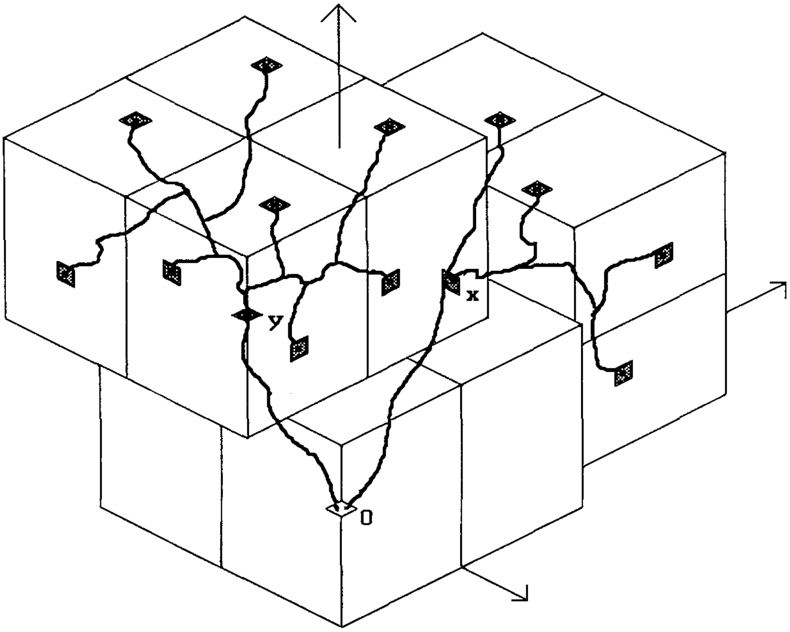

Fig. 5. The bricks $y+B_{\mathrm{North}}(L,H)$ and $x+B_{\mathrm{East}}(L,H)$ attached to the brick $B_{\mathrm{North}}(L,H)$

we wish to attach both a north brick and an east brick to the north brick $B_{\mathrm{North}}(L,H)$ – see Fig. 5. We can make the occupation events for the upper north brick and the east brick independent by requiring that the attachment site in T_{North} for the upper north brick belongs to the western half of the top: $T_{\mathrm{North,West}}=\{-L,\ldots,L\}^{d-2}\times\{-L,\ldots,0\}\times\{H\}$. Simple geometric considerations show that if $y+B_{\mathrm{North}}(L,H)$ is a brick with $y\in T_{\mathrm{North,West}}$ and $x+B_{\mathrm{East}}(L,H)$ is a brick with $x\in F_{\mathrm{North,East}}$, then the intersection of these two bricks (if nonempty) must lie in the hypersurface $\mathbb{Z}^{d-2}\times\{L\}\times\mathbb{Z}$. Hence the events $\{y+B_{\mathrm{North}}(L,H)$ is occupied$\}$ and $\{x+B_{\mathrm{East}}(L,H)$ is occupied$\}$ are independent, since they depend only on the states of bonds which are in the interiors of the bricks concerned. A similar discussion is valid for the more general case of attaching both a $\oplus$- and a $\otimes$-brick to an occupied $\otimes$-brick, where $\oplus$ and $\otimes$ are codirections and $T_{\otimes,\oplus}$ is interpreted as the $\oplus$-half of the top of the brick $B_{\otimes}(L,H)$.

This example (and the "reflected" example of attaching east and north bricks to a north brick) shows the necessity of being able to find attachment sites in either of the two halves $T_{\mathrm{North,West}}$ and $T_{\mathrm{North,East}}$ of the top. In the next example we will exert more control over the positions of the attachment sites by utilizing the subsections T_j of the top and S_i of the sides.

iv) Suppose that starting from the initial brick $B_0=B_{\mathrm{North}}(L,H)$, we were to pile up a sequence of north bricks B_n – one on top of another. For each $n\geq 0$, $B_{n+1}=y_n+B_{\mathrm{North}}(L,H)$ where y_n is an attachment site in the top $T_n=y_{n-1}+T_{\mathrm{North}}$ of the brick B_n. If the y_n's are chosen capriciously, then we have little control over the first $d-2$ coordinates of y_n (for large n) – in the sense that these coordinates are only guaranteed to satisfy the trivial (and not very useful) bounds $|y_n(r)|\leq nL$ for $r=1,\ldots,d-2$. However, there is a simple rule for choosing the attachment sites with the result that $|y_n(r)| \leqq 2L$ for $r=1, \ldots, d-2$ and for every $n \geqq 0$: choose y_n from one of the two subregions in $y_{n-1} + \bigtimes_{r=1}^{d-2} I_r(y_{n-1}) \times A \times \{H\}$ where

$$I_r(y_{n-1}) = \begin{cases} \{-L, \ldots, 0\} & \text{if } y_{n-1}(r) \geqq 0 \\ \{0, \ldots, L\} & \text{if } y_{n-1}(r) < 0, \end{cases}$$

and $A = \{-L, \ldots, L\}$. If it is necessary that the attachment site y_n be chosen from $y_{n-1} + T_{\text{North,East}}$ (resp. $y_{n-1} + T_{\text{North,West}}$), then take A to be $\{0, \ldots, L\}$ (resp. $\{-L, \ldots, 0\}$), in which case the subregion of T_n from which y_n comes is uniquely determined. Of course there are analogous rules for attaching south (or east or west) bricks together so as to never stray outside of the slice $\{-2L, \ldots, 2L\}^{d-2} \times \mathbb{Z}^2$. It is easily seen that these rules can also be extended to ensure that if $\{B_n\}$ is any sequence of occupied bricks with initial brick $B_\oplus(L, H)$, where the bricks are attached as in the above examples, and if $\{z_n\}$ is the sequence of attachment sites at which the attachments are made, then there is no loss of generality in assuming that $|z_n(r)| \leqq 2L$ for $r=1, \ldots, d-2$ and for every $n \geqq 0$.

Henceforth, we will call a sequence of occupied bricks which are attached to one another in accordance with all of the rules and examples found above an *allowed* "sequence" of (occupied) bricks.

We note that if $B_0 = u + B_\oplus(L, H)$ is an occupied brick in $\{-2L, \ldots, 2L\}^{d-2} \times \mathbb{Z}^2$, then there are four particular ways in which another brick B_1 can be attached to B_0 so that $\{B_0, B_1\}$ is an allowed sequence of bricks. For the sake of definiteness, we shall assume that $B_0 = u + B_{\text{North}}(L, H)$, in which case we can

1) attach another north brick $B_1 = y + B_{\text{North}}(L, H)$ at some attachment site y in the eastern half of the top of B_0 (i.e., $y \in u + T_{\text{North,East}}$),
2) attach $B_1 = y + B_{\text{North}}(L, H)$ at some attachment site $y \in u + T_{\text{North,West}}$,
3) attach $B_1 = x + B_{\text{East}}(L, H)$ at some attachment site $x \in u + F_{\text{North,East}}$, or
4) attach $B_1 = x + B_{\text{West}}(L, H)$ at some attachment site $x \in u + F_{\text{North,West}}$.

In each of the four cases, the subregion of S_{North} or T_{North} from which the attachment site is to be chosen is uniquely determined (by the choice of the case, and by the rules for keeping the bricks inside the slice $\{-2L, \ldots, 2L\}^{d-2} \times \mathbb{Z}^2$). Now each of these subregions has only a single attachment site at which an attachment may be made. Observe that if the brick B_0 is projected onto some plane parallel to the $x(d-1)-x(d)$ plane, say $\{0\}^{d-2} \times \mathbb{Z}^2$, then one of these four attachment sites projects onto each of the four line segments $\{\tilde{u} + \tilde{T}_{\text{North},\oplus}, \tilde{u} + \tilde{F}_{\text{North},\oplus} : \oplus \in \{\text{East, West}\}\}$ where the tilde signifies projection onto $\{0\}^{d-2} \times \mathbb{Z}^2$, i.e., $\tilde{u} = (0, \ldots, 0, u(d-1), u(d))$, $\tilde{T}_{\text{North,East}} = \{0\}^{d-2} \times \{0, \ldots, L\} \times \{H\}$, etc. Similar considerations apply when B_0 has a different orientation.

We shall contain our construction process in a two-dimensional slice by projecting bricks onto the $x(d-1)-x(d)$ plane. For this purpose we require more notation.

The *briquette* $\tilde{u} + \tilde{B}_\oplus(L, H)$ is defined to be the projection onto $\{0\}^{d-2} \times \mathbb{Z}^2$ of the brick $u + B_\oplus(L, H)$. There are four important portions of the boundary of $\tilde{u} + \tilde{B}_\oplus(L, H)$: these are the line segments $\tilde{u} + \tilde{T}_{\oplus,\otimes}$ and $\tilde{u} + \tilde{F}_{\oplus,\otimes}$ where $\otimes$ ranges over both codirections of $\oplus$. If $u + B_\oplus(L, H)$ is occupied, then we say that $\tilde{u}+\tilde{B}_{\oplus}(L,H)$ is a *successful* briquette and it has four *connection sites* – one in each of the subregions of the boundary listed above – which are the projections of the four attachment sites previously discussed. We shall say that $\tilde{u}$ is *connected* to each of the connection sites of $\tilde{u}+\tilde{B}_{\oplus}(L,H)$ with the understanding that this is a statement about the brick $u+B_{\oplus}(L,H)$, and not a statement about the actual configuration of bonds in the briquette. Suppose now that $\{B_n\}_{n=0}^{N}$ is an allowed sequence of occupied bricks in $\{-2L,\ldots,2L\}^{d-2}\times\mathbb{Z}^2$ with $B_n=u_n+B_{\oplus_n}(L,H)$. If the brick B_i is attached to B_j, we say that the corresponding briquette $\tilde{B}_i$ is *connected* to $\tilde{B}_j$. A connected sequence of successful briquettes which corresponds to an allowed sequence of occupied bricks is called a *successful briquette sequence.* So if $\tilde{v}$ is a connection site of any briquette in the successful sequence $\{\tilde{B}_n\}_{n=0}^{N}$, then $\tilde{u}_0$ is connected to $\tilde{v}$ – in the sense of briquette connections.

For any fixed configuration of bonds in the slice $\{-2L,\ldots,2L\}^{d-2}\times\mathbb{Z}^2$, the rules for constructing allowed sequences of occupied bricks are such that there is a one-to-one correspondence between the sequences of bricks and their briquettes (given the initial site u_0 of the first brick). It is clear that the bricks uniquely determine the briquettes. To reconstruct the brick sequence from the briquettes, first note that the briquettes and the last two coordinates of u_0 determine the bricks up to their first $d-2$ coordinates. These coordinates are then determined by the remaining coordinates of u_0 and the rules given in i)–iv) above. Thus it is possible to reduce the problem of building connections between the

$$\begin{aligned}
&\text{“south”}\ (\{-2L,\ldots,2L\}^{d-2}\times\{-K,\ldots,K\}\times\{-K\}),\\
&\text{“north”}\ (\{-2L,\ldots,2L\}^{d-2}\times\{-K,\ldots,K\}\times\{K\}),\\
&\text{“east”}\ (\{-2L,\ldots,2L\}^{d-2}\times\{K\}\times\{-K,\ldots,K\})\\
&\text{and “west”}\ (\{-2L,\ldots,2L\}^{d-2}\times\{-K\}\times\{-K,\ldots,K\})\\
&\text{faces of the “cube”}\ \{-2L,\ldots,2L\}^{d-2}\times\{-K,\ldots,K\}^2
\end{aligned}$$

out of allowed sequences of occupied bricks to the problem of building connections between the corresponding faces of the "square" $\{0\}^{d-2}\times\{-K,\ldots,K\}^2$ out of sequences of successful briquettes.

In the light of this reduction to two dimensions, we restate Proposition 2.8 using the terminology of briquettes.

Proposition 3.1 *If there is percolation in the half-space (i.e., $\theta_{\mathbb{H}}(p)>0$), then for every $\varepsilon>0$ there exist $\delta>0$ and integers k, L and H such that*

$$P_{p-\delta}(\tilde{B}_{\oplus}(L,H)\ \textit{is successful})>1-\varepsilon, \tag{3.1}$$

where $\oplus\in\{$North, South, East, West$\}$.

It should be noted that the hyperblock length appears implicitly in (3.1) – in the definition of successful briquette.

Henceforth we assume that $\varepsilon>0$, and that k, L and H are as in Proposition 3.1. We shall ignore the first $d-2$ coordinates, with the consequence that the projection of the "cube" is now the square $\{-K,\ldots,K\}^2$. The first coordinate [formerly the $(d-1)^{\text{th}}$ coordinate] will be refered to as the "x-coordinate," and the second (formerly the d^{th}) will be called the "y-coordinate." In demonstrating percolation in a quarter-slice of the original d-dimensional space, we can restrict out attention to nonnegative x and y coordinates. To abbreviate the notation further, we write z for the site (x, y) in $\mathbb{Z}_+^2$, and (when the dimensions will be clear from the context) $\tilde{B}_\oplus$ for the briquette $\tilde{B}_\oplus(L, H)$.

In the remainder of this section we shall suppose that $\mathbb{Z}_+^2$ has been divided into squares which are translates of $S=\{-K, \ldots, K\}^2$ by vectors $((2n_1+1)K, (2n_2+1)K$ where $n_1, n_2 \in \mathbb{Z}_+$, and K is a function of L and H which will be specified later. Each such square, which can be identified with the site $(n_1, n_2) \in \mathbb{Z}_+^2$, is also the projection of a "cube" in the quarter-slice $\{-2L, \ldots, 2L\}^{d-2} \times \mathbb{Z}_+^2$ onto the $x(d-1)-x(d)$ plane. Every square has at most four neighboring squares and shares one face with each neighbor. We are interested in examining the squares in the context of the cluster-growth algorithm outlined informally in Sect. 1.

Specifically, if a square has a neighbor which has previously been declared to be "good," then we wish to show that the square in question has a high probability of being found to be "good" should the cluster-growth algorithm ever call for it to be checked. In Sect. 4, we shall define what it means for the square with $n_1=n_2=0$ to be good. For the case of a square with $(n_1, n_2) \neq (0, 0)$, we say that the square is *good* if (i) it has a good neighbor [when a square has more than one neighbor already declared to be good, a single one of them (the one having the lowest deterministic order) is treated as *the* good neighbor], (ii) it is called upon to be checked by the cluster-growth algorithm [for the dependent percolation process on $\mathbb{Z}_+^2$], and (iii) a particular type of briquette "network" is successful. A briquette network is a union of briquette sequences connected together in such a way that every connected subsequence would be successful if all of its briquettes were successful; if every briquette in the network is in fact successful, we say that the network is successful. The special network referred to above is called the *crossing network* (from the good neighbor square) and it is contained entirely in the square and a small portion of its good neighbor.

For ease of notation, we will explain the rules for constructing the crossing network for the square S from its southern neighbor

$$S'=\{-K, \ldots, K\} \times \{-2K, \ldots, -K\}.$$

By translation invariance, and invariance under reflections and 90° rotations, it follows that the probability of S being good given that S' is its good neighbor is the same as the probability of any square S'' being good given that S''' is its good neighbor. In this particular example we would like to have a procedure for building a successful network of briquettes that connects a given connection site in the common face of S and S' to connection sites in the north, east and west faces of S. However, in building the briquette sequences which make up the network, although we have some potential for "steering", there will always be a degree of uncertainty about the exact position of the final briquette. We cannot be certain therefore, in building a network to traverse a large square, of being able to hit the target faces exactly; we may undershoot by a small amount. To deal with this, we introduce "target regions," (see Fig. 6) which are parts of the square bordering the faces which the briquette network is trying to reach. We shall be content with proving that there is a high probability

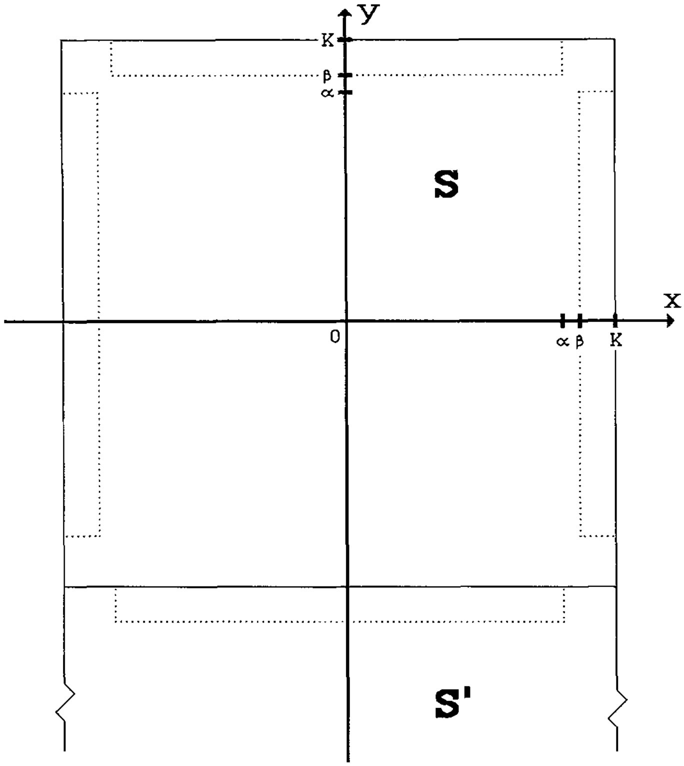

Fig. 6. The square S, its southern neighbor S', and the relevant target regions for a briquette crossing of S from S'. |— Boundary of a square, ⋮··· Boundary of a target region.
$K=5H+7L$, $\alpha=4H+5L$ and $\beta=K-H+1$ when $H \geqq L$.
$K=11L+11H$, $\alpha=9L+9H$ and $\beta=K-L-2H+1$ when $L>H$.
The target regions are of size $(2\alpha)\times(K-\beta)$ or $(K-\beta)\times(2\alpha)$

of being able to construct briquette crossings from a given starting point in the target region of a square S' to all the target regions of one of its neighbors S.

Our principal ingredient for building paths across a large box is the briquette $\tilde{B}_{\oplus}(L, H)$. Unfortunately, we have no control over the ratio H/L, and this lack of knowledge leads to some geometrical complications. It turns out that there are two cases, depending on whether H/L is small or large. For each of these cases we shall describe a construction for building successful crossing networks. An important feature of these geometric constructions is that there exists a finite upper bound R, independent of H and L, on the number of briquettes required for a square-crossing. Since Proposition 3.1 says that a briquette size can be found so that each briquette in the crossing network (conditionally) has probability at least $1-\varepsilon$ of being successful, it follows that the entire crossing network is then successful with probability at least $(1-\varepsilon)^R$ – which of course may be made as close to 1 as desired by an appropriate choice of ε.

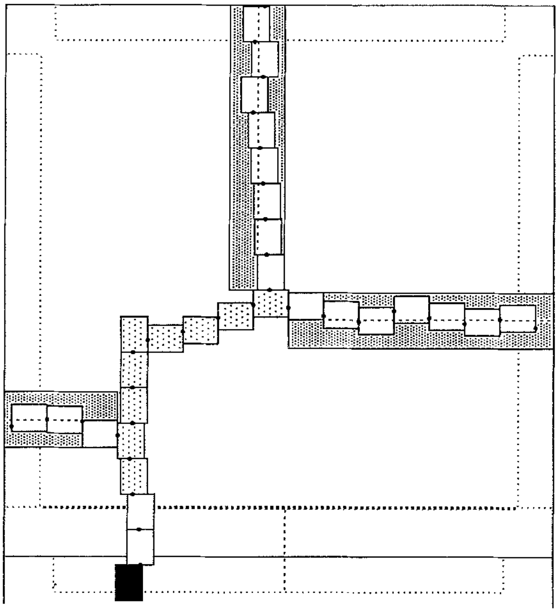

Fig. 7. A sample crossing sequence for the case $H \geqq L$ with $x_n < 0$. ■ The final briquette from the previous crossing sequence; ▯ Briquettes from either of the two north centering sequences; ▭ ▭ Briquettes from the east and west centering sequences; ▯▭ Briquettes from the bifurcation sequence; --¦ Centering lines; ······ Triggering line; Portions of the alleys not filled by briquettes

3.2 *The crossing construction for* $H \geqq L$

In this section we suppose that $H \geqq L$. The reader will find it helpful to occasionally refer to Fig. 7. We take $K = 5H + 7L$ and use target regions which have dimensions $(8H + 10L) \times (H - 1)$. In the particular example of constructing the crossing network from S' to S, we must describe a procedure for building a successful network of briquettes that connects a given connection site in the north target region of S'

$$(\{-4H-5L, \ldots, 4H+5L\} \times \{-K-H+1, \ldots, -K\})$$

to connection sites in the north, east and west target regions of S:

$$\{-4H-5L, \ldots, 4H+5L\} \times \{K-H+1, \ldots, K\},$$
$$\{K-H+1, \ldots, K\} \times \{-4H-5L, \ldots, 4H+5L\}$$

and

$$\{-K, \ldots, -K+H-1\} \times \{-4H-5L, \ldots, 4H+5L\},$$

respectively.

The rules for constructing crossing networks are divided into two sets which we call "centering" rules and "bifurcation" rules. All rules are of the form: connect a specific type of briquette to an earlier briquette at a specified connection site. It is always implicitly assumed that the new briquette is successful;

if at any step an unsuccessful briquette is obtained, the crossing procedure is terminated and the square S is declared to be *bad*.

The construction procedure is as follows (see Fig. 7). First, the centering rules are used to build a briquette sequence which proceeds "northward" from the north target region of S' into the square S, while not wandering too close to the east and west sides of S. After the briquette sequence has progressed sufficiently far into the square S, the centering rules are (temporarily) abandoned and the bifurcation rules are applied to connect a particular sequence of nine briquettes (the bifurcation sequence) to this centering sequence. The bifurcation sequence is such that it will be possible to find three connection sites corresponding to the north, east and west target regions of S, and the centering rules can be used to connect each site with its target region. Finally, the rules guarantee that if the initial centering sequence, the bifurcation sequence and the three final centering sequences are each successful, then their union is a successful briquette network.

We now present the centering rules in the context of their initial usage. Let $z_1=(x_1, y_1)$ denote the initial connection site (in the north target region of S') for the crossing of S, and attach the north briquette $\tilde{B}_1=z_1+\tilde{B}_{\text{North}}$ to the site z_1. (As will be explained below, the site z_1 is uniquely determined. In general, it is the connection site of a successful briquette $\tilde{B}_{S'}$ in S', and the sequence $\{\tilde{B}_{S'}, \tilde{B}_1\}$ is successful if $\tilde{B}_1$ is successful; when S' is the square corresponding to the origin in $\mathbb{Z}^2_+$, z_1 is just the midpoint of the north face of S'.) Now $\tilde{B}_1$ must have (under the assumption that it is successful) a connection site in each of the two halves of its top. If $x_1 \geqq 0$, choose the next connection site $z_2=(x_2, y_2)$ to be the connection site in $z_1+\tilde{T}_{\text{North, West}}$; if $x_1<0$, take z_2 to be the connection site in $z_1+\tilde{T}_{\text{North, East}}$. In either case, the north briquette $\tilde{B}_2=z_2+\tilde{B}_{\text{North}}$ is attached to $\tilde{B}_1$ at z_2. This method for choosing the connection site for connecting the next briquette is referred to as "centering along the line $x=0$." Now continue piling north briquettes on top of one another using the centering criterion ($x_i \geqq 0$ implies $z_{i+1} \in z_i+\tilde{T}_{\text{North, West}}$ and $x_i<0$ implies $z_{i+1} \in z_i+\tilde{T}_{\text{North, East}}$) to decide which connection sites are to be used – until some briquette reaches the "triggering line" $y=-4H-5L$.

The triggering line marks the southern boundaries of the east and west target regions of S. Once the initial centering sequence reaches this line, begin the bifurcation procedure to "split off" sequences which will eventually reach those target regions. Suppose that the briquette crossing the triggering line is $\tilde{B}_n=z_n+\tilde{B}_{\text{North}}$, with $-5H-5L \leqq y_n < -4H-5L$. For the rest of this subsection we will operate under the assumption (for the sake of definiteness) that $x_n<0$; in the complementary case (i.e., $x_n \geqq 0$) one only has to reverse all future references to east and west.

A pictorial version of the bifurcation sequence can be found in Fig. 8. A more formal description follows below. Begin by taking z_{b1} to be the connection site of $\tilde{B}_n$ on the western half of its top (i.e., $z_{b1} \in z_n+\tilde{T}_{\text{North, West}}$), and attaching the first bifurcation briquette $\tilde{B}_{b1}=z_{b1}+\tilde{B}_{\text{North}}$ at z_{b1}. Then connect the second briquette $\tilde{B}_{b2}=z_{b2}+\tilde{B}_{\text{North}}$ to $\tilde{B}_{b1}$ at z_{b2}, where z_{b2} is the connection site of $\tilde{B}_{b1}$ on the western half of its top. For $i \in \{3, 4, 5\}$, take z_{bi} to be the connection site of $\tilde{B}_{b(i-1)}$ in $z_{b(i-1)}+\tilde{T}_{\text{North, East}}$, and attach the next bifurcation briquette $\tilde{B}_{bi}=z_{bi}+\tilde{B}_{\text{North}}$ there. Then choose z_{b6} to be the connection site of $\tilde{B}_{b5}$ on its eastern side, and connect the sixth briquette $\tilde{B}_{b6}=z_{b6}+\tilde{B}_{\text{East}}$ to $\tilde{B}_{b5}$ at z_{b6}.

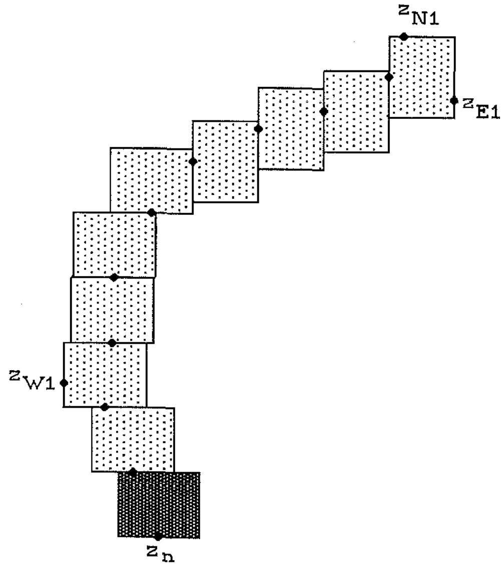

Fig. 8. The bifurcation sequence for the $H \geqq L$ algorithm in the case $x_n < 0$.
Portion of a bifurcation briquette; Portion of the triggering briquette; ● Connection site of a successful briquette

Finally, take $z_{bi}(i \in \{7, 8, 9\})$ to be the connection site in the northern half of the top of $\tilde{B}_{b(i-1)}$ and attach the next bifurcation briquette $\tilde{B}_{bi} = z_{bi} + \tilde{B}_{\text{East}}$ there.

It is easily seen (from Fig. 8), that if each of its briquettes is individually successful, the bifurcation sequence can fail to be successful only if there is a prohibited intersection between one of the five north briquettes and one of the four east briquettes. Now note that if $z = (x, y)$ is any site in any of the last four briquettes, then

$$(3.2) \qquad y \geqq y_{b6} - L \geqq y_{b5} + k - L \geqq y_{b5} + k - H = y_{b4} + k > y_{b4}.$$

(The second inequality uses the fact that if z_{b6} is a connection site of the briquette $\tilde{B}_{b5}$, then it corresponds to a an attachment site of a brick $\tilde{B}_{b5}$ which cannot be any closer to the bottom of $\tilde{B}_{b5}$ than the hyperblock length k.) Thus there cannot possibly be an intersection between the first three north briquettes and the east briquettes. Also observe that – as in example iii) above – no intersection between either of the bricks corresponding to $\tilde{B}_{b4}$ and $\tilde{B}_{b5}$ and the bricks corresponding to the east briquettes is prohibited. Thus, if each of the nine briquettes $\tilde{B}_{b1}, \ldots, \tilde{B}_{b9}$ is successful, then the bifurcation sequence is successful. It is also easily seen that the union of the initial centering sequence and the bifurcation sequence is successful if each of these briquettes is successful, since $\{\tilde{B}_1, \ldots, \tilde{B}_n, \tilde{B}_{b1}, \ldots, \tilde{B}_{b5}\}$ is just a sequence of north briquettes with each briquette attached atop its predecessor, and (3.2) implies that there is no intersection between any of the briquettes in the centering sequence and the east briquettes in the bifurcation sequence.

The last part of the bifurcation process is the identification of the three connection sites from which the final centering briquette sequences will be constructed. The connection sites corresponding to the western, eastern and northern target regions are labeled $z_{W1}=(x_{W1}, y_{W2})$, $z_{E1}=(x_{E1}, y_{E2})$ and $z_{N1}=(x_{N1}, y_{N2})$, respectively. We take z_{W1} to be the connection site on the western face of $\tilde{B}_{b2}$, z_{E1} to be the connection site on the southern half of the top of $\tilde{B}_{b9}$, and z_{N1} to be the connection site on the northern face of $\tilde{B}_{b9}$.

We shall give a careful treatment of the westward centering sequence, followed by sketches of the northward and eastward sequences. (See Fig. 7.) Let $\{\tilde{B}_{W1}, \tilde{B}_{W2}, \ldots, \tilde{B}_{Wn_W}\}$ be a sequence of west briquettes with initial site z_{W1} which is centered along the line $y=y_{W2}+L$, and which stops as soon as some briquette reaches beyond the line $x=-K+H$; the centering procedure is as described above for the initial sequence except that west briquettes are used in place of north briquettes. We note the following.

a) Once the construction has been fully described it will be apparent that the x-coordinate x_1 of the initial site satisfies the bound $x_1 \geqq -4H-4L$. The centering rules (as applied to the initial briquette sequence) then imply that $x_n \geqq -4H-4L$ also, which in turn implies that $x_{W1} \geqq -K+H$. Since the westward sequence does indeed begin to the east of its "finish line" $x=-K+H-1$, it is not vacuous.

b) The centering rules and the choice of the centering line for the westward sequence tell us that every briquette in that sequence lies in the "alley" $A_W=\{(x,y)\in S: x \leqq x_{W1};\, y_{W1}-L \leqq y \leqq y_{W2}+3L\}$ – see Fig. 7. This fact has several implications. First, the bounds just given on the y-coordinates of sites belonging to briquettes in the westward sequence can be combined with the (easily derived) relations

$$y_{b2}<y_{W1}<y_{b2}+H, \qquad y_{b2}=y_{b1}+H \quad \text{and} \quad -4H-5L \leqq y_{b1} < -3H-5L$$

to show that the "top" of the last west briquette is actually contained entirely within the west target region of S.

c) Another consequence of the westward sequence being confined to the alley A_W is that it cannot overlap the initial sequence – since $y>y_{b1}$ for any site (x,y) in the alley, while $y \leqq y_{b1}$ for every site (x,y) belonging to a briquette in the initial sequence. Also, the intersection of the westward sequence with the bifurcation sequence is a subset of the "bottom" of $\tilde{B}_{W1}$; thus as before the brick occupation events are conditionally independent.

Similarly, the northward (resp. eastward) sequence $\{\tilde{B}_{Ni}\}$ (resp. $\{\tilde{B}_{Ei}\}$) is defined by starting at the connection site z_{N1} (resp. z_{E1}), centering north (resp. east) briquettes along the line $x=x_{N1}-L$ (resp. $y=y_{E1}-L$), and stopping when some briquette reaches beyond the "finish-line" $y=K-H$ (resp. $x=K-H$). It is easily verified that the northward (resp. eastward) sequence is not vacuous since $y_{N1}<2H-L$ (resp. $x_{E1}<4H+4L$). The centering rules tell us that the briquettes of the northward (resp. eastward) sequence all lie in the alley

$$A_N=\{(x,y)\in S: x_{N1}-3L \leqq x \leqq x_{N1}+L,\, y \geqq y_{N1}\}$$
$$[\text{resp. } A_E=\{(x,y)\in S: x \geqq x_{E1},\, y_{E1}-3L \leqq y \leqq y_{E1}+L\}],$$

and it is easily checked that neither A_N nor A_E overlaps either the initial briquette sequence or A_W (and hence the westward sequence). Additionally, the pairwise intersections between the bifurcation sequence, the northward sequence and the eastward sequence are subsets of the bottoms of $\tilde{B}_{N1}$ and $\tilde{B}_{E1}$, which is allowed. Thus the union of all five sequences – initial, bifurcation, westward, northward and eastward – is a successful network (call it the crossing network) if all of its briquettes are successful. Furthermore, the alley bounds may be combined with the estimates $-H-5L<x_{N1}<4H+4L$ and $-6L<y_{E1}<2H-2L$ to show that the tops of the final briquettes $\tilde{B}_{Nn_N}$ and $\tilde{B}_{En_E}$ lie inside their corresponding target regions.

The last step in the crossing construction is the determination of the connection sites z_{Wn_W+1}, z_{En_N+1} and z_{En_E+1} which may serve as the initial connection sites for the crossings of the three neighboring squares of S besides S'. The prescription is to choose final connection sites on the tops of the briquettes $\tilde{B}_{Wn_W}$, $\tilde{B}_{Nn_N}$ and $\tilde{B}_{En_E}$ according to the centering rules which would be used to construct the initial briquette sequence in the neighboring square. For example, if $y_{Wn_W}<0$, then take $z_{Wn_W+1}\in z_{Wn_W}+\tilde{T}_{\text{West,North}}$; otherwise take $z_{Wn_W+1}\in z_{Wn_W}+\tilde{T}_{\text{West,South}}$.

Here is a final note about the construction. We claimed at one point that if $x_1<0$, then $x_1\geqq -4H-4L$ – this inequality was used directly to prove that z_{W1} was not already in the west target region, and later used implicitly to show that $x_{N1}>-H-5L$ (which in turn implied that $z_{Nn_N}+\tilde{T}_{\text{North}}$ was contained in the north target region). Now the top of the final briquette in the northward sequence is contained in the north target region, so if $(x, y)\in z_{Nn_N}+\tilde{T}_{\text{North}}$, then $-4H-5L\leqq x\leqq 4H+5L$. Combining this inequality with the centering rules for the choice of z_{Nn_N+1} shows that $-4H-4L\leqq x_{Nn_N+1}\leqq 4H+4L$, which is what we assumed above. For the other target regions, a similar argument shows that $y_{En_E+1}, y_{Wn_W+1}\in\{-4H-4L, \ldots, 4H+4L\}$.

Having reached the end of the description of the crossing construction, we make some final remarks before moving on. First, as we have seen in our examination of the overlaps between the various subsequences, the crossing network is successful if each of its five constituent subsequences is successful. Second, as claimed at the beginning of the description of the construction, it is clear that no briquette used in crossing S intersects any briquette used in crossing S' (except, of course, in the bottom of the first briquette in S). Thus, (i) the union of the crossing network in S with the crossing network in S' is successful if each crossing network is individually successful, and (ii) conditional on ever examining S, the probability that S is declared to be "good" is at least $(1-\varepsilon)^R$ where R is a uniform upper bound on the number of briquettes necessary for a crossing of S. We may take $R=47$ since 24 briquettes suffice to cross from the north target region of S' to the north target region of S and 23 briquettes suffice for crossing from the west target region of S to the east target region. Note that the number of successful briquettes required for a square crossing does not depend on L and H (as long as $H\geqq L$).

We conclude by remarking that there exist simpler algorithms for crossing squares with $2L\times H$ briquettes in the case $H\geqq L$. Our choice of construction was motivated in part by a desire to give a construction which could be modified in a natural way to handle the case $L>H$. For any $c>0$, one can design a variant of the above construction which works when $H\geqq cL$. Unfortunately, under this hypothesis, the number R_c of briquettes required for a briquette crossing diverges as c tends to 0. Since we have no non-trivial bound on the ratio of H to L, this naive approach does not work.

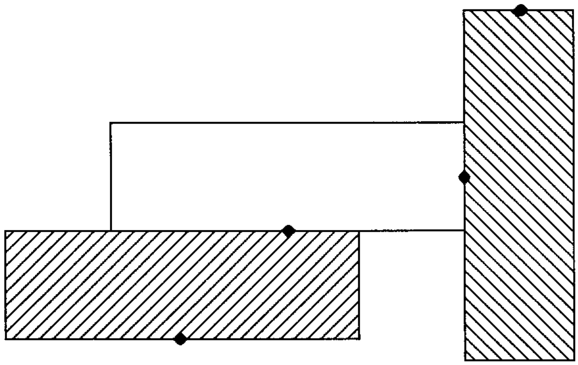

Fig. 9. The crab $C_{\text{North,East}}$. ▨ Portion of the body of the crab; ▧ Portion of the claw of the crab

3.3 The crossing construction for $L>H$

In this section we suppose $L>H$. Since the crossing construction of this subsection is based on the construction of the preceding subsection, we shall be somewhat briefer here in our explanations. The reader is encouraged to refer to Fig. 10. The north, east and west target regions of S are

$$\{-9L-9H, \ldots, 9L+9H\} \times \{K-L-2H+1, \ldots, K\},$$
$$\{K-L-2H+1, \ldots, K\} \times \{-9L-9H, \ldots, 9L+9H\}$$

and

$$\{-K, \ldots, -K+L+2H-1\} \times \{-9L-9H, \ldots, 9L+9H\},$$

respectively, where now $K=11(H+L)$.

To guarantee that we can cross squares with a bounded (independently of the ratio H/L) number of briquettes, we must devise a way of using the *length* of the briquette, and not (as we did in subsection 3.2) its *height* to accomplish the crossing. The method which we use is to combine the bricks into triplets called "crabs" which play the role which was held by the individual briquettes when $H \geqq L$.

Crabs come in eight varieties: labelled $C_{\oplus\otimes}$ for $\oplus \in \{\text{North, East, West, South}\}$ and $\otimes$ a codirection of $\oplus$. To construct the crab of type $C_{\oplus\otimes}$ at the site z_1 (i.e., $z_1+C_{\oplus\otimes}$), begin with the briquette $\tilde{B}_1 = z_1 + \tilde{B}_\oplus$. Then attach another briquette of the same type $\tilde{B}_2 = z_2 + \tilde{B}_\oplus$ at the connection site $z_2 \in z_1 + \tilde{T}_{\oplus,\otimes}$. Finally connect the briquette $\tilde{B}_3 = z_3 + \tilde{B}_\otimes$ to $\tilde{B}_2$ at the connection site $z_3 \in z_2 + \tilde{F}_{\oplus,\otimes}$. The first briquette in the crab is called its "body," and the third briquette its "claw" (see Fig. 9). We say that a crab is successful if each of its three constituent briquettes is successful.

Crabs have stacking properties somewhat like those of briquettes. For example, if $z_1 + C_{\oplus\otimes}$ is any successful crab and $*$ is a codirection of $\oplus$, then taking z_2 to be the connection site on the $\oplus$ face of the claw, one can attach a second crab $z_2 + C_{\oplus *}$ to the first crab so that their intersection is confined to the bottom of the body of the second crab. Thus the union of the two crabs is a successful briquette sequence if each crab is successful.

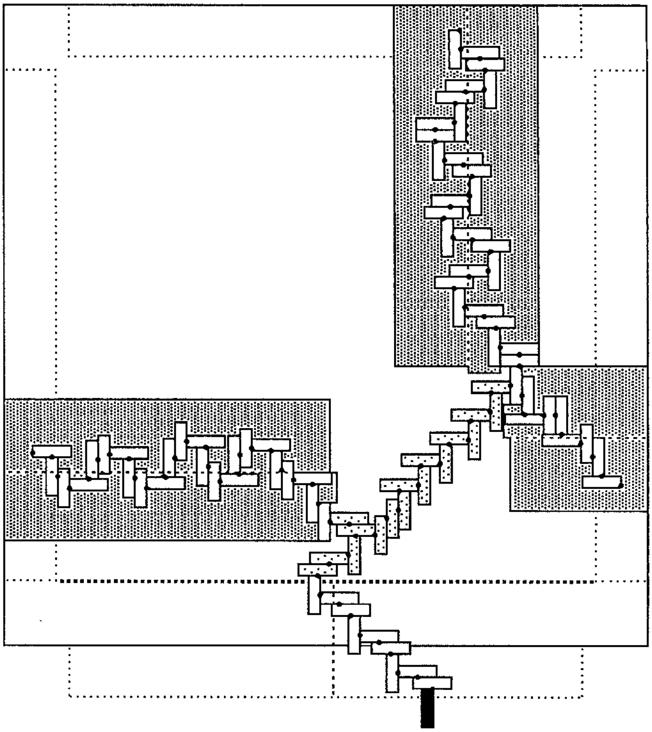

Fig. 10. A sample crossing sequence for the case $L > H$ with $x_{b1} < 0$

Most of the square crossing in the case $H \geq L$ was accomplished by the four centering sequences. A key property of these sequences was that the connection of each additional briquette advanced the sequence a distance H – which was the principal length scale – in the direction along the centering line. Crabs have a similar property in the situation when $L > H$: the attaching of each additional crab in a sequence of $C_{\oplus\otimes}$ and $C_{\oplus *}$ crabs (where $\otimes$ and $*$ are the two codirections of $\oplus$) advances the sequence at least a distance $H+L$ (actually, at least $H+L+k$ – where k is the hyperblock "length") in the $\oplus$ direction.

We define a centering sequence of crabs by example. An eastward sequence of crabs centered along the line $y = \hat{y}$ is a sequence of crabs $\{C_i = z_i + C_{\mathrm{East}\otimes_i}\}_{i=1}^{n}$ with $\otimes_i \in \{\text{North, South}\}$ for $i \geq 1$ and $z_i = (x_i, y_i)$ the connection site in the eastern face of the claw of C_{i-1} for $i > 1$. One centers the sequence through the choice of the $\otimes_i$'s. We take $\otimes_i = \text{North}$ if $y_i < \hat{y}$, and we choose $\otimes_i = \text{South}$ if $y_i \geq \hat{y}$. It is clear how one defines centering sequences of crabs in other directions. A difference between a centering sequence of briquettes when $H \geq L$ and the corresponding sequence of crabs when $L > H$ is that the crabs will generally wander further away from the centering line. More precisely, every site in a centering sequence of briquettes (when $H \geq L$) is within distance $2L$ of the centering line, whereas the sites in a centering sequence of crabs (when $L > H$) could be as far as a distance $3L+H$ (actually, as far as $3L+H-2k$) away from the centering line. (Indeed, the very name "crab" was suggested by the manner in which these centering sequences shuffle from side to side as they slowly advance.) It is because of these larger deviations from the centering lines that we must use a larger square and larger target regions than in Sect. 3.2.

The "overhangs" of the claws of crabs lead to a more fundamental difference between sequences of briquettes and crabs. By way of example, observe that in a centering sequence of north briquettes, every site (except those in the bottom) of the $(n+1)^{\text{th}}$ briquette has a y-coordinate strictly greater than that of any site in the n^{th} briquette. However, the analogous statement for crabs may be false. It is precisely because of this overhang effect that we define a square to be good only if the *particular* crossing network of briquettes defined below is successful. If we only required the existence of *some* network of crabs, then the overhangs could cause dependence difficulties at the point where we would try to link up the crossings in two adjacent squares. We note that this presents no difficulty in the case $H \geqq L$, and so we could relax the requirement that the briquette crossing be accomplished by the particular network described in subsection 3.2, replacing it (for a crossing originating in the south) by the simpler condition that the initial connection site z_1 be connected to some final connection sites $z_{E,\text{final}}$, $z_{W,\text{final}}$ and $z_{N,\text{final}}$ in the three target regions of the square by a successful briquette network contained in $\{(x, y): x_{E,\text{final}} \leqq x \leqq x_{W,\text{final}},\ y_1 \leqq y \ll_{N,\text{final}}\}$. We chose not to use this definition when $H \geqq L$, so that we could give both cases a similar treatment.

We are now ready to present the crossing construction. As in the previous subsection, the crossing network is divided into five subsequences – initial, bifurcation and three final sequences. However, because the centering sequences of crabs display greater lateral motion than the briquettes of the previous subsection, we must use a larger bifurcation sequence to separate the (wider) alleys of the final sequences from each other, and from the initial sequence.

Starting with a connection site z_1 in the north target region of S', an initial sequence of crabs is centered along the line $x=0$. When (the north face of a claw in) the sequence reaches (or passes) the "triggering line," $y=-9L-9H$, the centering sequence is stopped and the bifurcation sequence is begun. Without loss of generality, we may assume that the connection site z_{b1} on the northern face of the claw of the triggering crab has x-coordinate $x_{b1}<0$. (The case $x_{b1} \geqq 0$ will be handled by reversing all references to east and west in the treatment of the case $x_{b1}<0$ below.)

The bifurcation sequence is a sequence of seventeen briquettes, $\{\tilde{B}_{bi}\}_{i=1}^{17}$ (see Fig. 11). The first three briquettes are the briquettes of the crab $z_{b1}+C_{\text{North,East}}$. The fourth briquette is a north briquette connected to that crab at the connection site z_{b4} in the northern face of its claw (i.e., $\tilde{B}_{b4}=z_{b4}+\tilde{B}_{\text{North}}$ with $z_{b4} \in z_{b3} + \tilde{F}_{\text{East,North}}$). The fifth briquette is another north briquette connected to $\tilde{B}_{b4}$ at the connection site z_{b5} in the western half of the top of $\tilde{B}_{b4}$. The sixth briquette is also connected to $\tilde{B}_{b4}$. We take $\tilde{B}_{b6}=z_{b6}+\tilde{B}_{\text{East}}$ where z_{b6} is the connection site on the eastern face of $\tilde{B}_{b4}$. The next two briquettes are also east briquettes; attach $\tilde{B}_{b7}=z_{b7}+\tilde{B}_{\text{East}}$ to $\tilde{B}_{b6}$ at the connection site $z_{b7} \in z_{b6}+\tilde{T}_{\text{East,North}}$, and $\tilde{B}_{b8}=z_{b8}+\tilde{B}_{\text{East}}$ to $\tilde{B}_{b7}$ at the connection site $z_{b8} \in z_{b7}+\tilde{T}_{\text{East,North}}$. The eight briquette is the first of a sequence of ten briquettes which form a "staircase." For $i \in \{1, 2, 3, 4, 5\}$, the briquette $\tilde{B}_{b(2i+7)}$ is a north briquette connected to the northern face of the east briquette $\tilde{B}_{b(2i+6)}$ at the connection site $z_{b(2i+7)}$, and, for $i \in \{2, 3, 4, 5\}$, the briquette $\tilde{B}_{b(2i+6)}$ is an east briquette connected to the eastern face of the north briquette $\tilde{B}_{b(2i+5)}$ at the connection site $z_{b(2i+6)}$.

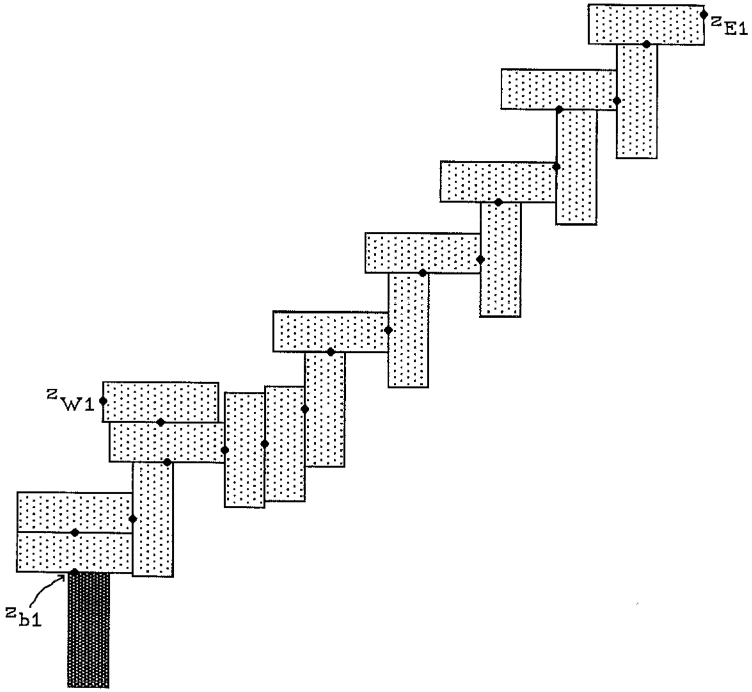

Fig. 11. The bifurcation sequence for the $L>H$ algorithm in the case $x_{b1}<0$. Portion of a bifurcation briquette; Portion of the claw of the triggering crab

We claim that there is no prohibited overlapping of briquettes in the bifurcation sequence. The only part of this last statement which might not be immediately clear is that there is no intersection between the briquettes $\tilde{B}_{b5}$ and $\tilde{B}_{b9}$. To see that such an intersection cannot occur, we argue as follows. Either $L>2H$, or else $L\leqq 2H$. If $L>2H$, there cannot be an intersection because simple geometry shows that for a site (x, y) to be in $\tilde{B}_{b5}$, we must have $y\leqq y_{b4}+2H$; but $(x, y)\in\tilde{B}_{b9}$ implies that $y\geqq y_{b4}+L+k$. On the other hand, in the case $L\leqq 2H$ we see that for (x, y) to be in $\tilde{B}_{b5}$, we must have $x\leqq x_{b4}+L$; but $(x, y)\in\tilde{B}_{b9}$ implies that $x\geqq x_{b4}+2H+k$.

The connection site for the westward centering sequence is $z_{W1}\in z_{b5}+\tilde{F}_{\text{North,West}}$. A West-North crab is attached to the bifurcation sequence at z_{W1}, and thereafter the sequence of crabs is centered along the line $y=y_{W1}+2L+H$ (see Fig. 10). The sequence stops when the (western face of the) claw of some crab reaches the region $\{x<-10L-9H\}$. It is easily verified that every crab in the sequence is contained in the alley

$$A_W=\{(x, y)\in S:\ x\leqq x_{W1}, y_{W1}-L\leqq y\leqq y_{W1}+5L+2H\}$$
$$\cup\{(x, y)\in S: x_{W1}\leqq x\leqq x_{W1}+L-H-k, y_{W1}+L\leqq y\leqq y_{W1}+2L+H\}$$

– where the second part of A_W is an appendage which at most contains the eastern portion (overhang) of the claw of the first crab. The crude bound $x_{b1}\geqq -9L-9H$ implies that $x_{W1}\geqq -10L-9H$ (and hence the sequence begins to the east of the finish line $x=-10L-9H-1$), and the inequality $-9L-9H \leqq y_{b1} \leqq -8L-7H$ implies that $-8L-7H \leqq y_{W1} \leqq -7L-3H$ (and thus the sequence will be in the west target region when it reaches the finish line).

The connection site for the eastward centering sequence is $z_{E1} \in z_{b17} + \tilde{F}_{\text{North, East}}$. We connect an East-South crab to $\tilde{B}_{b17}$ there, and continue the crab sequence by centering along the line $y=y_{E1}-2L-H$ until some crab reaches the finish line $x=10L+9H+1$. This sequence (which begins to the west of the finish line and which ends with a crab that has the eastern face of its claw in the east target region) is contained in the alley

$$A_E=\{(x,y)\in S: x \geqq x_{E1}, y_{E1}-5L-2H \leqq y \leqq y_{E1}+L\}$$
$$\cup \{(x,y)\in S: x_{E1}-L+H+k \leqq x \leqq x_{E1}, y_{E1}-2L-H \leqq y \leqq y_{E1}-L\}.$$

The last centering sequence is different from the other two final centering sequences in that it does not begin with a connection site of some briquette in the bifurcation sequence. Take z_{N1} to be the connection site on the northern face of the east briquette which is the body of the first crab in the eastward sequence. We begin centering a northward sequence of crabs from z_{N1} along the line $x=x_{N1}-2L-H$ – the first crab must be of the North-West type – and we stop when some crab reaches the finish line $y=10L+9H+1$. The sequence of crabs begins south of the finish line, is contained in the alley

$$A_N=\{(x,y)\in S: x_{N1}-5L-2H \leqq x \leqq x_{N1}+L, y_{N1} \leqq y\}$$
$$\cup \{(x,y)\in S: x_{N1}-2L-H \leqq x \leqq x_{N1}-L, y_{N1}-L+H+k \leqq y \leqq y_{N1}\},$$

and ends with the northern face of the claw of the last crab in the north target region.

It is rather obvious how one should choose the final connection sites from which the crossings of the three neighbors of S other than S' may originate. We take the final connection site in, say, the eastward centering sequence to be the connection site on the eastern face of the claw of the last crab in the eastward centering sequence. Since both the final eastward centering sequence in S and the initial centering sequence in $S''=\{K, \ldots, 2K\} \times \{-K, \ldots, K\}$ (the eastern neighbor of S) are sequences of East-North and East-South crabs, it is easily seen that the union of the crossing networks in S and S'' is successful if each of the crossing networks is successful. Similar considerations apply for the final connection sites in the westward and northward sequences.

Finally, one may easily verify that there is no prohibited overlapping of briquettes in the five subsequences, with the consequence that their union (i.e., the crossing network) is successful if each of its briquettes is successful. It can also be shown that the number of briquettes required to cross the square is bounded by some number which is independent of the length scales L and H – provided that $L>H$. One calculation yields $R=125$ as an upper bound for this quantity.

4 Robustness of half-space percolation

We finally turn to the proof of Theorem 1.2i). It has already been explained in Sect. 1 how the main result, Theorem 1.1, follows from this theorem. As the proof has already been outlined in considerable detail, we shall be somewhat brief here. Afterwards, we shall comment on some rather minor extensions of Theorem 1.2. For a pair of major extensions, see Grimmett and Marstrand (1990); Bezuidenhout and Grimmett (1990).

Proof of Theorem 1.2 i) Suppose that there is percolation in the half-space at bond density p, i.e., $\theta_{\mathbb{H}}(p)>0$. Let $\varepsilon>0$ and choose δ, k, L and H (by Proposition 2.8) so that

$$P_{p-\delta}(B_{\oplus}(L,H) \text{ is occupied}) > 1-\frac{\varepsilon}{125}$$

for $\oplus \in \{\text{North, South, East, West}\}$. "Partition" the quarter-slice

$$\mathbb{Q}_{2L}=\{-2L,\ldots,2L\}^{d-2}\times\mathbb{Z}_+^2$$

into the "cubes"

$$C(n_1,n_2)=\{-2L,\ldots,2L\}^{d-2}\times\{2n_1K,\ldots,2(n_1+1)K\}\\ \times\{2n_2K,\ldots,2(n_2+1)K\}$$

where $n_1, n_2 \in \mathbb{Z}_+^2$ and

$$K=\begin{cases}5H+7L, & \text{if } H\geqq L\\ 11H+11L, & \text{if } H<L.\end{cases}$$

Observe that the origin can be connected to the open hyperblocks

$$\{-k,\ldots,k\}^{d-2}\times\{K-k,\ldots,K+k\}\times\{2K\}$$

and

$$\{-k,\ldots,k\}^{d-2}\times\{2K\}\times\{K-k,\ldots,K+k\},$$

using only the open bonds in the region

$$A=\{-2L,\ldots,2L\}^{d-2}\times(\{0,\ldots,K+k\}^2\cup\{K-k,\ldots,K+k\}\\ \times\{K+k,\ldots,2K\}\cup\{K+k,\ldots,2K\}\times\{K-k,\ldots,K+k\}),$$

with some positive probability $\pi_0=\pi_0(p,K,k)$. If such a connection from the origin to the two hyperblocks exists, we say that $C(0,0)$ is good.

Using the crossing algorithms of subsection 3.2 or 3.3 [and assuming, for simplicity, that $C(1,0)$ and $C(0,1)$ are the first two cubes to be examined], we obtain

$$P_{p-\delta}(C(1,0) \text{ is good} \mid C(0,0) \text{ is good}) > \left(1-\frac{\varepsilon}{125}\right)^{125} > 1-\varepsilon$$

and

$$P_{p-\delta}(C(0,1) \text{ is good} \mid C(0,0) \text{ is good}) > 1-\varepsilon$$

since each crossing requires no more than 125 bricks. (Note that the definition of the region A is such that if $L>H$, any overhangs of the claws of the first crabs in the crossing networks for $C(1,0)$ and $C(0,1)$ will not use any bonds that might have been examined in order to determine that $C(0,0)$ was good.)

Now algorithmically grow the cluster (of good cubes) of $C(0,0)$ in $\mathbb{Q}_{2L}$ as described in Sect. 1. From Sects. 2 and 3 we know that if $C(x,y)$ is the neighboring cube of some good cube $C(x', y')$ [i.e., $|x-x'|+|y-y'|=1$], and $C(x, y)$ has not yet been examined, then

$$P_{p-\delta}(C(x, y) \text{ is good} \,|\, \text{the algorithm calls on } C(x, y) \text{ to be checked with } C(x', y') \text{ as its specified good neighbor}) > 1-\varepsilon.$$

Choosing ε so that $1-\varepsilon > \lambda_c(\mathbb{Z}_+^2)$ [where $\lambda_c(\mathbb{Z}_+^2)$ is the critical density for independent site percolation in $\mathbb{Z}_+^2$ with percolation probability $\theta^s_{\mathbb{Z}_+^2}(\lambda)$], and recalling the coloring argument of Sect. 0, we see that there is at least probability $\pi_0(p, K, k)\theta^s_{\mathbb{Z}_+^2}(1-\varepsilon)$ of the cluster of good cubes being infinite. Since our renormalization is such that an infinite connected path of good cubes on the macroscopic (i.e., renormalized) scale implies the existence of a corresponding percolating network of open bonds on the microscopic (i.e., original) scale, we have succeeded in showing that there is bond percolation in the quarter-slice $\mathbb{Q}_{2L}$ at bond density $p-\delta$ for some L and δ if there is bond percolation in the half-space $\mathbb{H}$ at bond density p. □

The proof of Theorem 1.2 ii) is similar to the proof of Theorem 1.2 i). A major change needs to be made back in Sect. 2: we need slightly different notions of open hyperblock and attachment site. If

$$\mathbb{L} = \left(\bigtimes_{i=1}^{f} \{0, \ldots, h_i\} \right) \times \mathbb{Z}^{d-e-f} \times \mathbb{Z}_+^e \quad \text{with } e \geqq 1 \quad \text{and} \quad d-f \geqq 2,$$

take the hyperblocks "of length k" to be translates of

$$\left(\bigtimes_{i=1}^{f} \{0, \ldots, h_i\} \right) \times \{-k, \ldots, k\}^{m-1} \times \{0\} \times \{-k, \ldots, k\}^{d-f-m}$$
$$\text{for } m \in \{1, \ldots, d-f\}.$$

For definiteness, the attachment site corresponding to an open hyperblock may be taken to have all of its first f coordinates equal to zero. These different definitions of course lead to minor adjustments in the proof of the analogue of Proposition 2.1, e.g., the packing factor in the constant N_1 of (2.3 a) will change, but the general structure of the argument remains the same.

Simple extensions of Theorem 1.2 can be obtained by replacing the quarter-slice with a smaller subset (a "sector-slice" or "wedge"). Defining $\Lambda(a)$ – for $a>0$ – to be the sector $\{(x, y): 0 \leqq x, 0 \leqq y \leqq ax\}$ in $\mathbb{Z}_+^2$, a variant of the argument given in the proof above shows that percolation in the half-space at bond density p implies percolation in the wedge $\{-2L, \ldots, 2L\}^{d-2} \times \Lambda(a)$ at bond density $p-\delta$ for some $\delta = \delta(a) > 0$ (and a similar statement holds for half-slabs). The only major modification is that one needs to redefine "good" for cubes on the boundary of the wedge.

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