

# Multiple Wiener-Itô integral expansions for level-crossing-count functionals

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Received June 30, 1989, in revised form May 18, 1990

**Summary.** This paper applies the stochastic calculus of multiple Wiener-Itô integral expansions to express the number of crossings of the mean level by a stationary (discrete- or continuous-time) Gaussian process within a fixed time interval  $[0, T]$ . The resulting expansions involve a class of hypergeometric functions, for which recursion and differential relations and some asymptotic properties are derived. The representation obtained for level-crossing counts is applied to prove a central limit theorem of Cuzick (1976) for level crossings in continuous time, using a general central limit theorem of Chambers and Slud (1989a) for processes expressed via multiple Wiener-Itô integral expansions in terms of a stationary Gaussian process. Analogous results are given also for discrete-time processes. This approach proves that the limiting variance is strictly positive, without additional assumptions needed by Cuzick.

## 1. Introduction

There is now a very well-developed stochastic calculus for smooth functionals of stochastic integrals with respect to Wiener process (Kallianpur 1980), which has been applied extensively to problems on diffusions and counting processes. This calculus could also be applied to the study of nonlinear functionals of stationary Gaussian processes, as has been remarked by Kallianpur (1980, Chap. 6), by using the representation of such functionals as multiple Wiener-Itô integral expansions. The paper of Chambers and Slud (1989b) is one effort in this direction. Indeed, since the celebrated Diagram Theorem of Dobrushin and Major (1979) can be viewed as a representation theorem for polynomials of multiple Wiener-Itô integrals, there is every hope that some *nonsmooth* nonlinear functionals of stationary Gaussian processes could be represented explicitly as multiple Wiener-Itô integral expansions. The nonsmooth functionals studied in this paper are the counts of level-crossings.

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\* Research supported by Office of Naval Research contracts N00014-86-K-0007 and N00014-89-J-1051

There are two reasons for interest in explicitly defined multiple Wiener-Itô integral expansions. First, it has been known at least since the work of Versik (1962) (see also Chap. 13 of Sinai 1977, for clarification) that spectral ergodic-theoretic properties such as weak-mixing and mixing for a functional ("factor" in the language of ergodic theory) of a stationary Gaussian process, can be expressed in terms of the absolute-continuity equivalence class of the underlying spectral measure together with the multiple Wiener-Itô integrands. A second way to exploit Wiener-Itô integral expansions has been developed by Taqqu (1975), Dobrushin and Major (1979), Maruyama (1976) and Chambers and Slud (1989a, b) among others. These authors prove general (functional) central and noncentral limit theorems for such expansions. In the present paper, central limit theorems of Chambers and Slud (1989a) will be applied to the level-crossings counts.

Our general references for multiple Wiener-Itô integrals are the monograph of Major (1981) and Chapter 6 of Kallianpur (1980). The relevant results from the general theory are summarized also in each of the papers of Chambers and Slud (1989a, b).

In this paper,  $X_t$  is a mean-0 and variance-1 stationary Gaussian process, either in discrete or continuous time as specified. Its correlation function will be denoted by  $r(t)$ , and its spectral measure (assumed nonatomic) by  $\sigma$  (either on  $[-\pi, \pi]$  or on  $\mathbb{R}$ ). On the spaces  $L^2(\mathbb{R}^k, \sigma^k, \text{sym})$  of complex square-integrable functions  $f$  which are symmetric in their  $k$  real arguments  $(x_1, \dots, x_k)$  and which satisfy  $f(-x) = \overline{f(x)}$ , the multiple Wiener-Itô integral operators are denoted  $I_k: L^2(\mathbb{R}^k, \sigma^k, \text{sym}) \rightarrow \mathbb{R}$ , and the constant function with value 1 in the domain of  $I_k$  is denoted  $1_k$ .

## 2. Representation results and limit theorems

The starting point is to recognize that the indicator  $I_{[X_t, X_{t+1} < 0]}$  that  $X_t$  and  $X_{t+1}$  are of different signs, is the sum of products  $I_{[X_t < 0]} \cdot I_{[X_{t+1} > 0]}$  and  $I_{[X_t > 0]} \cdot I_{[X_{t+1} < 0]}$  of functionals which depend only on single coordinates of the underlying process  $X_t$ . After expressing the functional  $I_{[X_0 > 0]}$  as a multiple Wiener-Itô integral expansion and expressing products of expansions through the Diagram Theorem, we obtain the first Proposition and Corollary.

**Proposition 1.** Define  $\rho = r(1)$ ,  $S_L^0 \equiv S_L^0 \equiv 1$  for  $L \geq 1$ , and for  $j \geq 1$

$$S_L^j \equiv S_L^j(\lambda_1, \lambda_2, \dots, \lambda_L) \equiv \sum_{1 \leq n_1 < \dots < n_j \leq L} e^{i(\lambda_{n_1} + \dots + \lambda_{n_j})}$$

Then

$$I_{[X_0 X_1 < 0]} = \frac{1}{2} - \frac{1}{\pi} \sum_{m=0}^{\infty} (-1)^m I_{2m} \left( \sum_{j=0}^{2m} S_{2m}^j C_{m,j}(\rho) \right),$$

where

$$C_{m,j}(\rho) \equiv \sum_{\alpha \geq 0: \alpha+j \text{ odd}} (-1)^{\alpha-1} \frac{\rho^\alpha}{\alpha!} \frac{(j+\alpha-1)!(2m-j+\alpha-1)!}{\left(\frac{j+\alpha-1}{2}\right)! \left(\frac{2m-j+\alpha-1}{2}\right)!} 2^{-\alpha-m+1}$$

**Corollary 1.** For positive integers  $T$ ,

$$\sum_{j=0}^{T-1} \left( I_{[X_j X_{j+1} < 0]} - \frac{\arccos(\rho)}{\pi} \right) = \frac{1}{\pi} \sum_{m=1}^{\infty} (-1)^{m-1} \cdot I_{2m} \left( \frac{e^{iT(\lambda_1 + \dots + \lambda_{2m})} - 1}{e^{i(\lambda_1 + \dots + \lambda_{2m})} - 1} \sum_{j=0}^{2m} S_{2m}^j C_{m,j}(\rho) \right).$$

Since the main point of this paper is to give explicit multiple Wiener-Itô integral representations for functionals related to level-crossings counts  $\sum_k I_{[X_k X_{k+1} < 0]}$ , it is important to catalogue some basic properties of the family  $\{C_{m,j}(\cdot)\}$  of special functions. It will be proved in Sect. 3 below that the functions  $C_{m,j}$  are hypergeometric functions, and each of the following properties will be seen to derive from known relationships among such functions.

**Proposition 2.** The functions  $C_{m,j}(\rho)$  arising in Proposition 1 satisfy the following identities and recursion relationships:

(0)  $C_{0,0}(\rho) = \arcsin(\rho)$ .

(1) For all  $0 \leq j \leq 2m$  and  $m \geq 1$ ,  $C_{m,j}(\rho) \equiv C_{m,2m-j}(\rho)$ , and  $(1 - \rho^2)^{m-1/2} C_{m,j}(\rho)$  is a polynomial in  $\rho$  which for  $1 \leq j \leq 2m - 1$  has degree at most  $m - 1$ .

(2) For  $m \geq 1$ ,  $C_{m,2}(\rho) = \frac{(2m-2)!}{(m-1)!2^{m-1}} \frac{\rho}{(1 - \rho^2)^{m-1/2}}$  and

$$C_{m,1}(\rho) = -\frac{(2m-2)!}{(m-1)!2^{m-1}} \frac{1}{(1 - \rho^2)^{m-1/2}}.$$

(3) For all  $0 \leq j \leq 2m - 2$ , with  $m \geq 1$ ,  $\frac{d}{d\rho} C_{m-1,j}(\rho) = -C_{m,j+1}(\rho)$ .

(4) For  $0 \leq j \leq 2m - 1$ ,  $C_{m+1,j}(\rho) = 2(m-j) C_{m,j}(\rho) + C_{m+1,j+2}(\rho)$ .

(5) For  $j \leq m - 1$ ,  $C_{m+1,j+2}(\rho) = (-1)^j \frac{(2(m-j))!}{(m-j)!2^{m-j}} \frac{d^j}{d\rho^j} \frac{\rho}{(1 - \rho^2)^{m-j+1/2}}.$

In the following theorem, asymptotic normality has been proved by Ho and Sun (1987) and also follows easily from Theorems 1 and 2 of Chambers and Slud (1989a) using the representation of Proposition 1. The positive lower bound for the asymptotic variance is apparently new.

**Theorem 1.** Suppose that the covariances  $r(n) = E\{X(0)X(n)\}$  of the discrete-time Gaussian process  $X(t)$  are such that  $\sum_{n=-\infty}^{\infty} r^2(n) < \infty$ . Then

$$\frac{1}{\sqrt{T}} \sum_{j=0}^{T-1} \left( I_{[X_j X_{j+1} < 0]} - \frac{\arccos(\rho)}{\pi} \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \alpha^2) \quad \text{as } T \rightarrow \infty$$

where the asymptotic variance  $\alpha^2$  satisfies

$$\alpha^2 \geq \frac{4}{\pi(1 - \rho^2)} \int |\rho - \cos(\lambda)|^2 f^2(\lambda) d\lambda > 0$$

for  $\rho = r(1)$ , and where the spectral density  $f \in L^2$  is defined a.e. by the mean-square convergent series  $f(x) = (2\pi)^{-1} \sum_{n=-\infty}^{\infty} e^{-inx} r(n)$ .

For the remaining results, take  $t$  to be a continuous time parameter. We assume from now on the condition which Cramer and Leadbetter (1967) and Geman (1972) respectively showed to be sufficient and necessary for the number of zeroes of  $X_t$  in  $[0, 1]$  to have finite variance, namely that  $r''(0)$  is finite and that

$$\int_0^\varepsilon \frac{r''(t) - r''(0)}{t} dt < \infty \quad \text{for some } \varepsilon > 0 \quad (2.1)$$

**Theorem 2.** *For a mean-0, variance-1, continuous-time stationary Gaussian process  $X_t$  satisfying (2.1), the number of axis crossings within the unit interval  $[0, 1]$  has the representation*

$$\frac{\sqrt{-r''(0)}}{\pi} \left\{ 1 + \sum_{m=1}^{\infty} I_{2m} \left( \frac{e^{i(\lambda_1 + \dots + \lambda_{2m})} - 1}{i(\lambda_1 + \dots + \lambda_{2m})} \cdot \sum_{l=0}^m \left( \frac{-2}{r''(0)} \right)^l \frac{\binom{1/2}{l} l! (2m-2l)!}{(m-l)! 2^{m-l}} \sum_{1 \leq n_1 < \dots < n_{2l} \leq 2m} \lambda_{n_1} \dots \lambda_{n_{2l}} \right) \right\}$$

An immediate but very interesting consequence of this representation is that, under the hypotheses of the last theorem, the number  $N_X(T)$  of axis-crossings in  $[0, T]$  by the process  $X_t$  is an integral functional:

**Corollary 2.** *Under the hypotheses of Theorem 2, for each  $T \in (0, \infty)$*

$$N_X(T) = \int_0^T \theta_s \circ Z_0 ds,$$

where  $\theta_s$  is the time-shift map on  $(\Omega, \mathcal{F}, P)$  and  $Z_0$  is the square-integrable functional of  $\{X_t\}$  with the explicit representation

$$Z_0 = \frac{\sqrt{-r''(0)}}{\pi} \left\{ 1 + \sum_{m=1}^{\infty} I_{2m} \left( \sum_{l=0}^m \left( \frac{-2}{r''(0)} \right)^l \frac{\binom{1/2}{l} l! (2m-2l)!}{(m-l)! 2^{m-l}} \sum_{1 \leq n_1 < \dots < n_{2l} \leq 2m} \lambda_{n_1} \dots \lambda_{n_{2l}} \right) \right\} \quad (2.2)$$

**Corollary 3.** *The random variable  $Z_0$  of (2.2) is equal to*

$$\frac{\sqrt{-r''(0)}}{\pi} \left\{ 1 - \sum_{m=1}^{\infty} \sum_{l=0}^m \frac{2^{-m+1}}{2l-1} \frac{1}{l!(m-l)!} H_{2l} \left( \frac{X'_0}{\sqrt{-r''(0)}} \right) H_{2m-2l}(X_0) \right\} \quad (2.3)$$

where  $H_k(x)$  denotes the  $k$ 'th Hermite polynomial normalized to have leading coefficient 1, and where  $X'_0$  denotes the mean-square derivative of the process  $X_t$  at  $t = 0$ :

$$X'_0 \equiv \left. \frac{d}{dt} X_t \right|_{t=0} \equiv I_1(i\lambda).$$

Just as we obtained Theorem 1 from Proposition 1 by means of the general central limit theorem of Chambers and Slud (1989a), so from Theorem 2 follows the central limit theorem of Cuzick (1976) for numbers of axis-crossings in large intervals by continuous-time Gaussian processes.

**Theorem 3.** Suppose that  $X_t$  is a mean-0, variance-1, continuous-time stationary Gaussian process satisfying (2.1), and assume in addition that both  $r(t)$  and  $r''(t)$  are Lebesgue square-integrable functions on  $\mathbb{R}$ . If  $N_x(T)$  denotes the number of axis-crossings by  $X_t$  on the interval  $[0, T]$ , then

$$\frac{1}{\sqrt{T}} \left( N_x(T) - T \frac{\sqrt{-r''(0)}}{\pi} \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2) \quad \text{as } T \rightarrow \infty,$$

where the asymptotic variance  $\sigma^2$  is finite and satisfies

$$\sigma^2 \geq \frac{(-r''(0))}{\pi} \int \left( 1 + \frac{\lambda^2}{r''(0)} \right)^2 f^2(\lambda) d\lambda > 0$$

with the  $L^2$  function  $f$  defined by the mean-square convergent integral  $(2\pi)^{-1} \int e^{-i\lambda t} r(t) dt$ .

Again the conclusion of positive asymptotic variance is new. Cuzick (1976, condition (A3) of Theorem 1) needed additional assumptions (such as his Lemma 5) to obtain it. In this continuous-time case, an explicit integral formula for the asymptotic variance  $\sigma^2$  in terms of the covariance function  $r(t)$  is known, due to Schultheiss et al. (1955) and Cramer and Leadbetter (1967). Indeed, these authors provide a corresponding formula for the exact variance of  $N_x(T)/\sqrt{T}$ , and the assumption of square-integrability for  $r''(t)$  then ensures that the asymptotic variance remains bounded as  $T \rightarrow \infty$ .

### 3. Proofs

*Proof of Proposition 1.* Observe first that if

$$H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} (e^{-x^2/2})$$

denotes the  $k$ 'th Hermite polynomial (normalized to have leading coefficient 1), then  $EH_k^2(X_0) = k!$  and  $I_k(1_k) = H_k(X_0)/k!$ . Since the polynomials  $H_k(X_0)/\sqrt{k!}$  form an orthonormal basis of  $L^2(X_0)$ ,

$$I_{[X_0 \geq 0]} = \sum_{k=0}^{\infty} H_k(X_0) \frac{1}{\sqrt{k!}} \int_0^{\infty} \frac{1}{\sqrt{k!}} H_k(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Now for  $k \geq 1$ ,

$$\begin{aligned} \int_0^{\infty} H_k(x) e^{-x^2/2} dx &= \int_0^{\infty} (-1)^k \frac{d^k}{dx^k} (e^{-x^2/2}) dx \\ &= (-1)^{k-1} \frac{d^{k-1}}{dx^{k-1}} (e^{-x^2/2})|_{x=0} = H_{k-1}(0) \end{aligned}$$

which is 0 unless  $k$  is odd. Moreover, since

$$e^{xt - t^2/2} = \sum_{m=0}^{\infty} \frac{H_m(x)}{m!} t^m \quad \text{implies} \quad H_{2n}(0) = (-1)^n \frac{(2n)!}{2^n n!}$$

we conclude that

$$\begin{aligned} I_{[X_0 \geq 0]} &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} H_{2n+1}(X_0) \frac{(-1)^n (2n)!}{(2n+1)! 2^n n!} \\ &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^n n!} I_{2n+1}(1_{2n+1}) \end{aligned}$$

Now

$$\begin{aligned} I_{[X_0 X_1 < 0]} &= \frac{1}{2} - 2 \left( I_{[X_0 \geq 0]} - \frac{1}{2} \right) \left( I_{[X_1 \geq 0]} - \frac{1}{2} \right) = \frac{1}{2} - \frac{1}{\pi} \sum_k \sum_n (-1)^{k+n} \frac{(2k)!(2n)!}{2^{k+n} k! n!} \\ &\quad \cdot I_{2k+1}(1_{2k+1}) I_{2n+1}(1_{2n+1}) e^{i(\lambda_1 + \dots + \lambda_{2n+1})} \end{aligned}$$

which we re-express by the Diagram Formula (Major 1981, p. 42). The double-sum must have finite expected square because  $I_{[X_0 X_1 < 0]}$  does. In what follows, the free interchange of orders of summation can be justified rigorously by first replacing  $\sum_k$  by  $\sum_{k=1}^K$  and  $\sum_n$  by  $\sum_{n=1}^N$ , next projecting  $\sum_{k=1}^K \sum_{n=1}^N$  onto the range-space of the  $2m$ th order multiple Wiener-Itô integral operator  $I_{2m}$ , and finally taking limits as  $K$  and  $N$  go to  $\infty$ . With  $S_L^I$  as defined in Proposition (2.1),

$$\begin{aligned} \pi \left( \frac{1}{2} - I_{[X_0 X_1 < 0]} \right) &= \sum_k \sum_n (-1)^{k+n} \frac{(2k)!(2n)!}{2^{k+n} k! n!} \sum_{\alpha=0}^{2(k \wedge n)+1} \binom{2k+1}{\alpha} \\ &\quad \cdot \binom{2n+1}{\alpha} \alpha! \left( \int e^{i(x_1 + \dots + x_n)} \sigma(dx_1) \cdots \sigma(dx_n) \right) \\ &\quad \cdot \frac{(2(k+n+1-\alpha))!}{(2k+1)!(2n+1)!} I_{2(k+n+1-\alpha)}(e^{i(\lambda_1 + \dots + \lambda_{2n+1-\alpha})}) \\ &= \sum_k \sum_n \sum_{\alpha=0}^{2(k \wedge n)+1} \frac{\rho^\alpha}{\alpha!} \binom{2(k+n+1-\alpha)}{2n+1-\alpha} (-1)^{k+n} \\ &\quad \cdot \frac{(2k)!(2n)!}{2^{k+n} k! n!} I_{2(k+n+1-\alpha)} \left( S_{2k+2n+2-2\alpha}^{2n+1-\alpha} \left/ \binom{2(k+n+1-\alpha)}{2n+1-\alpha} \right. \right) \end{aligned}$$

which, after the change of summation-indices  $p = 2n + 1 - \alpha$ ,  $q = 2k + 1 - \alpha$ , becomes

$$\sum_{\alpha=0}^{\infty} \left( -\frac{1}{2} \right)^{\alpha-1} \frac{\rho^\alpha}{\alpha!} \sum_p' \sum_q' \left( -\frac{1}{2} \right)^{(p+q)/2} \frac{(p+\alpha-1)!(q+\alpha-1)!}{\left( \frac{p+\alpha-1}{2} \right)! \left( \frac{q+\alpha-1}{2} \right)!} I_{p+q}(S_{p+q}^p)$$

where the summations  $\sum'$  run over all nonnegative integers  $p, q$  for which  $p + \alpha$  and  $q + \alpha$  are odd. Now with a new summation index defined as  $m = (p + q)/2$ , the assertion of Proposition 1 follows.  $\square$

*Proof of Proposition 2.* We first give explicit formulas showing that the functions  $C_{m,j}(\rho)$  are of hypergeometric type. Indeed, substitute into the series defining

$C_{m,j}(\rho)$  the “duplication formula”

$$\frac{z!}{(z/2)!} = \frac{\Gamma(z+1)}{\Gamma\left(\frac{z}{2}+1\right)} = \frac{2^z}{\sqrt{\pi}} \Gamma\left(\frac{z+1}{2}\right)$$

respectively for  $z = j + \alpha - 1$ ,  $2m - j + \alpha - 1$ , and  $\alpha$ , to find

$$C_{m,j}(\rho) = \frac{1}{\sqrt{\pi}} \sum_{\alpha \geq 0: \alpha+j \text{ odd}} (-1)^{\alpha-1} 2^{m-1} \frac{\Gamma((j+\alpha)/2) \Gamma(m-(j-\alpha)/2)}{\Gamma((\alpha+2)/2) \Gamma((\alpha+1)/2)} \rho^\alpha \quad (3.1)$$

When  $j$  is odd, let the running index  $\alpha$  in (3.1) be  $2n$  for  $n = 0, 1, 2, \dots$ ; when  $j$  is even, we take  $\alpha = 2n + 1$ . In both cases, the infinite sum has a simple expression in terms of the classical hypergeometric series

$$F(a, b, c, z) \equiv \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(b+n)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+n)} \frac{z^n}{n!}$$

for parameter  $c$  not equal to a negative integer, namely

$$C_{m,j}(\rho) = \begin{cases} -\frac{2^{m-1}}{\pi} \Gamma\left(\frac{j}{2}\right) \Gamma\left(m-\frac{j}{2}\right) F\left(\frac{j}{2}, m-\frac{j}{2}, \frac{1}{2}, \rho^2\right) & j \text{ odd} \\ \frac{2^m}{\pi} \Gamma\left(\frac{j+1}{2}\right) \Gamma\left(m-\frac{j-1}{2}\right) \rho F\left(\frac{j+1}{2}, m-\frac{j-1}{2}, \frac{3}{2}, \rho^2\right) & j \text{ even} \end{cases} \quad (3.2)$$

Here we adopt the convention that  $\Gamma(\alpha+n)/\Gamma(\alpha) \equiv 1$  for any  $\alpha$  when  $n = 0$ , and  $\Gamma(\alpha+n)/\Gamma(\alpha) = 0$  if  $\alpha$  is a negative integer with  $\alpha+n \geq 1$ .

Our general reference for hypergeometric functions is the Bateman Manuscript Project (1953), vol. 1, chap. 2, which we cite as BMP. We prove properties (0) to (5) of the Proposition by reference to known relationships among hypergeometric functions, given by formula numbers from Sects. 2.8 and 2.9 of BMP.

(0) By (3.2) and formula (2.8.13) of BMP,

$$C_{0,0}(\rho) = \rho F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \rho^2\right) = \arcsin(\rho).$$

(1) Formula (3.2), like the formula defining  $C_{m,j}(\rho)$  in Proposition 1, is evidently unchanged if  $j$  is replaced by  $2m-j$ . Moreover, formulas (2.9.1–2) of BMP, with  $z = \rho^2$  and  $a = j/2$ ,  $b = m-j/2$ ,  $c = 1/2$  for  $j$  odd and  $a = (j+1)/2$ ,  $b = m-(j-1)/2$ ,  $c = 3/2$  for  $j$  even, show that

$$(1 - \rho^2)^{m-1/2} C_{m,j}(\rho) =$$

$$\begin{cases} -\frac{2^{m-1}}{\pi} \Gamma\left(\frac{j}{2}\right) \Gamma\left(m-\frac{j}{2}\right) F\left(\frac{1-j}{2}, \frac{j+1}{2} - m, \frac{1}{2}, \rho^2\right) & j \text{ odd} \\ \frac{2^m}{\pi} \Gamma\left(\frac{j+1}{2}\right) \Gamma\left(m-\frac{j-1}{2}\right) \rho F\left(1-\frac{j}{2}, 1+\frac{j}{2} - m, \frac{3}{2}, \rho^2\right) & j \text{ even} \end{cases} \quad (3.3)$$

Since  $F(a, b, c, z)$  is a polynomial of degree (at most)  $k$  in  $z$  whenever  $a$  or  $b$  is a negative integer  $-k$ , the right-hand sides of (3.3) for  $j = 0, 1, 2, \dots, 2m$  are

polynomials in  $\rho$ , and for all values of  $j$  other than 0 and  $2m$ , the degree in  $\rho$  is at most  $m - 1$ .

(2) follows immediately from (3.3) upon substituting  $j = 1$  or  $2$ , since  $F(a, b, c, z) \equiv 1$  when  $a$  or  $b = 0$ .

(3) For  $j$  odd, we have

$$\frac{d}{d\rho} C_{m-1,j}(\rho) = 2 \sum_{n=0}^{\infty} \frac{\rho^{2n}}{(2n)!} \frac{(2n+j)!}{\left(n+\frac{j}{2}\right)!} \frac{(2n+2m-2-j)!}{\left(n+m-1-\frac{j}{2}\right)!} 2^{-2n-m} = -C_{m,j+1}(\rho)$$

which can also be seen via (3.2) and formula (2.8.20) of BMP. For  $j$  even, put  $z = \rho^2$  and use the chain rule  $\frac{d}{d\rho} = 2\rho \frac{d}{dz}$  in formula (2.8.22) of BMP, to obtain

$$\begin{aligned} \frac{d}{d\rho} C_{m,j}(\rho) &= \frac{2^m}{\pi} \Gamma\left(\frac{j+1}{2}\right) \Gamma\left(m - \frac{j-1}{2}\right) 2z^{1/2} \\ &\quad \cdot \frac{d}{dz} \left( z^{1/2} F\left(\frac{j+1}{2}, m - \frac{j-1}{2}, \frac{3}{2}, z\right) \right) \\ &= \frac{2^m}{\pi} \Gamma\left(\frac{j+1}{2}\right) \Gamma\left(m - \frac{j-1}{2}\right) F\left(\frac{j+1}{2}, m - \frac{j-1}{2}, \frac{1}{2}, z\right) \\ &= -C_{m+1,j+1}(\rho) \end{aligned}$$

(4) When  $j$  is odd and less than  $2m$ , it is easy to check via (3.2) that Property (4) is equivalent to:

$$\begin{aligned} \left(m - \frac{j}{2}\right) F\left(\frac{j}{2}, m + 1 - \frac{j}{2}, \frac{1}{2}, \rho^2\right) &= (m-j) F\left(\frac{j}{2}, m - \frac{j}{2}, \frac{1}{2}, \rho^2\right) \\ &\quad + \frac{j}{2} F\left(\frac{j}{2} + 1, m - \frac{j}{2}, \frac{1}{2}, \rho^2\right) \end{aligned}$$

and this relation holds by formula (2.8.32) of BMP with  $a = j/2$ ,  $b = m - j/2$ ,  $c = 1/2$ . Similarly, for even  $j < 2m$ , Property (4) is equivalent to formula (2.8.32) of BMP with  $a = (j+1)/2$ ,  $b = m - (j-1)/2$ ,  $c = 3/2$ .

(5) For  $0 \leq j \leq m-1$ ,  $C_{m+1,j+2}(\rho) = (-1)^j \frac{d^j}{d\rho^j} C_{m+1-j,2}(\rho)$  by (3), and the result follows immediately by formula (2) for  $C_{m+1-j,2}(\rho)$ .  $\square$

*Proof of Theorem 1.* For the Central Limit assertion, we apply Theorem 2 of Chambers and Slud (1989a) to the representation given in Proposition 1, with

$$Y \equiv I_{[X_0, X_1 < 0]} - \frac{\arccos(\rho)}{\pi} \equiv \sum_{k \geq 2} I_k(f_k), \text{ and with Hermite rank } m \text{ of } 2. \text{ It is}$$

important to remark here that, although the assumption (A.1) of Chambers and Slud (1989a, p. 326) apparently requires that  $\sigma$  be absolutely continuous with respect to Lebesgue measure on  $[-\pi, \pi]$ , in fact the proof of Theorem 2 of Chambers and Slud makes use (via their Lemma 4) only of the existence of *some* bounded positive functions  $g_M(\cdot)$  on  $[-\pi, \pi]$  whose  $m$ 'th convolutions converge uniformly as  $M \rightarrow \infty$  to the density of  $\sigma^{*m}$ . But under the assumption

$\sum_{n=-\infty}^{\infty} |r(n)|^m < \infty$ , which is here in force, the density of  $\sigma^{*m}$  is bounded and



continuous and is given by  $(2\pi)^{-1} \sum_{-\infty}^{\infty} r^m(n) e^{-inx}$ . Then the functions

$$g_M(x) \equiv (2\pi)^{-1} \sum_{-\infty}^{\infty} r(n) e^{-inx - \frac{1}{2M} n^2}$$

are bounded and continuous, and by the Dominated Convergence Theorem  $g_M^{*m}$  converges uniformly as  $M \rightarrow \infty$  to the density of  $\sigma^{*m}$ .

The functions  $f_k$  in the multiple Wiener-Itô integral expansion for  $Y \equiv I_{[X_0 X_1 < 0]}$  —  $\pi^{-1} \arccos(\rho)$  are  $f_k(\underline{\lambda}) = 0$  for odd  $k$ , and

$$f_k(\underline{\lambda}) = \pi^{-1} (-1)^{n-1} \sum_{j=0}^{2n} S_{2n}^j C_{n,j}(\rho) \quad \text{for } k = 2n, n \geq 1$$

Thus  $f_k(\underline{\lambda})$  is bounded and continuous. The hypothesis (A.3) of Theorems 1 and 2 of Chambers and Slud (1989a) follows easily, and by Lemma 2.3 of Chambers and Slud (1989a), for  $k = 2n$

$$\begin{aligned} 2\pi a_k &\equiv \lim_{T \rightarrow \infty} \text{Var} \left( T^{-1/2} I_k \left( \frac{e^{iT(\lambda_1 + \dots + \lambda_k)} - 1}{e^{i(\lambda_1 + \dots + \lambda_k)} - 1} f_k(\underline{\lambda}) \right) \right) \\ &= \frac{2}{k! \pi} \int \left( \sum_{j=0}^{2n} S_{2n}^j C_{n,j}(\rho) \right)^2 f(\lambda_1) \dots f(\lambda_k) d\lambda_1 \dots d\lambda_{k-1} \Big|_{\lambda_k = -\lambda_1 - \dots - \lambda_{k-1}} \end{aligned}$$

with the last integral taken over  $[-\pi, \pi]^{k-1}$ . Moreover, a general argument of Ho and Sun (1987, (2.6)–(2.9)) using only the property that  $I_{[X_0 X_1 < 0]}$  depends on finitely many coordinates of  $X_t$ , implies that for all large  $T$ , all  $k \geq 2$ , and some constant  $M_1$  not depending on  $k$  or on  $T$ ,

$$\text{Var} \left( T^{-1/2} I_k \left( \frac{e^{iT(\lambda_1 + \dots + \lambda_k)} - 1}{e^{i(\lambda_1 + \dots + \lambda_k)} - 1} f_k(\underline{\lambda}) \right) \right) \leq M_1 \text{Var}(I_k(f_k))$$

An immediate consequence of the last two displayed formulas is the assumption (A.2') of Theorem 2 of Chambers and Slud (1989a). That theorem now implies the asserted central limit convergence of the present Theorem.

To prove the lower bound on the limiting variance  $\alpha^2$ , observe first that the pair of random variables

$$\frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} I_2(e^{it(\lambda_1 + \lambda_2)} f_2(\underline{\lambda})), \quad \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} \sum_{n=2}^{\infty} I_{2n}(e^{it(\lambda_1 + \dots + \lambda_{2n})} f_{2n}(\underline{\lambda}))$$

are jointly asymptotically normal and independent as  $T \rightarrow \infty$  by the central limit theorems of Chambers and Slud, with asymptotic variances adding to  $\alpha^2$ . By Lemma 2.3 of Chambers and Slud (1989a), the asymptotic variance of the first of the displayed variables is

$$\begin{aligned} &2\pi \lim_{h \rightarrow 0+} \frac{1}{2h} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{1}{2} |f_2(\underline{\lambda})|^2 I_{[|\lambda_1 + \lambda_2| \leq h]} \sigma(d\lambda_2) \sigma(d\lambda_1) \\ &= \pi \int_{-\pi}^{\pi} |f_2(\lambda_1, -\lambda_1)|^2 f(-\lambda_1) f(\lambda_1) d\lambda_1 \end{aligned}$$

Now we make use of the special form for  $f_2(\underline{\lambda})$  at  $(\lambda_1, -\lambda_1)$ , namely

$$\begin{aligned} f_2(\lambda_1, -\lambda_1) &= \frac{1}{\pi} \{C_{1,0}(\rho) + (e^{i\lambda_1} + e^{-i\lambda_1})C_{1,1}(\rho) + C_{1,2}(\rho)\} \\ &= \frac{1}{\pi\sqrt{1-\rho^2}} \{2\rho - 2\cos(\lambda_1)\} \end{aligned}$$

where we apply Proposition 2.(1)–(2) to find that  $C_{1,0}(\rho) = C_{1,2}(\rho) = \rho(1-\rho^2)^{-1/2}$  and  $C_{1,1}(\rho) = -(1-\rho^2)^{-1/2}$ . Substituting for  $f_2(\lambda_1, -\lambda_1)$ , we have proved that

$$\alpha^2 \geq \frac{1}{\pi(1-\rho^2)} \int_{-\pi}^{\pi} 4\{\rho - \cos(\lambda)\}^2 f^2(\lambda) d\lambda$$

which is strictly positive because  $f$  is a well-defined and nontrivial Lebesgue square-integrable function on  $[-\pi, \pi]$ .  $\square$

*Proof of Theorem 2.* Letting

$$S_L^j(\varepsilon) \equiv \sum_{1 \leq n_1 < \dots < n_j \leq L} e^{i\varepsilon(\lambda_{n_1} + \dots + \lambda_{n_j})}$$

we find

$$\begin{aligned} 0 &= L^2\text{-}\lim_{\varepsilon \rightarrow 0+} I_{[X_0 X_\varepsilon < 0]} \\ &= L^2\text{-}\lim_{\varepsilon \rightarrow 0+} \left( \frac{1}{2} - \frac{1}{\pi} \sum_{m=0}^{\infty} (-1)^m I_{2m} \left( \sum_{j=0}^{2m} S_{2m}^j(\varepsilon) C_{m,j}(r(\varepsilon)) \right) \right) \end{aligned}$$

which implies that for all  $m > 0$ ,

$$\begin{aligned} 0 &= L^2\text{-}\lim_{\varepsilon \rightarrow 0+} I_{2m} \left( \sum_{j=0}^{2m} S_{2m}^j(\varepsilon) C_{m,j}(r(\varepsilon)) \right) \\ &= I_{2m} \left( \lim_{\varepsilon \downarrow 0} \sum_{j=0}^{2m} S_{2m}^j(\varepsilon) C_{m,j}(r(\varepsilon)) \right) \end{aligned} \quad (3.4)$$

where the limit inside  $I_{2m}(\cdot)$  is taken in the sense of  $L^2(\sigma^{2m}, \text{sym}) = \{f \in L^2(\mathbb{R}^{2m}, \sigma^{2m}) : f \text{ symmetric in } \lambda_1, \dots, \lambda_{2m} \text{ and } f(-\underline{\lambda}) = \overline{f(\underline{\lambda})}\}$ . Define now for  $\rho \in (0, 1)$  and  $\underline{\lambda} \equiv (\lambda_1, \dots, \lambda_{2m}) \in \mathbb{R}^{2m}$

$$G_m(\rho; \underline{\lambda}) \equiv \sum_{j=0}^{2m} S_{2m}^j C_{m,j}(\rho)$$

Then our results of Proposition 2 on the form of the functions  $C_{m,j}(\rho)$  imply that  $G_m(\rho; t\underline{\lambda})$  is analytic and bounded in the real argument  $t$  and is equal, for each fixed  $t$  and  $\underline{\lambda}$ , to  $(1-\rho^2)^{-m+1/2}$  multiplied by a polynomial in  $\rho$ . The main part of our proof consists of two Propositions describing the top-order (i.e., order  $\varepsilon$ ) behavior of  $G_m(r(\varepsilon); \varepsilon\underline{\lambda})$  for each  $\underline{\lambda}$  as  $\varepsilon \rightarrow 0$ .

Since  $C_{m,j}(\rho)$  is a meromorphic function of  $\rho$ , we can define

$$h_m \equiv \lim_{\rho \rightarrow 1-} (1-\rho^2)^{1/2} \sum_{j=0}^{2m} \binom{2m}{j} C'_{m,j}(\rho) \quad \text{for } m \geq 0$$

It follows immediately from Proposition 2.(0) that  $h_0 = 1$ , and direct calculation using the recursion relations in Proposition 2.(4) shows that  $h_1 = 1$ ,  $h_2 = 3$ , and  $h_3 = 15$ .

**Proposition 3.** For  $m \geq 1$ ,  $h_m = (2m-1)(2m-3) \dots 1 = \frac{2^m}{\sqrt{\pi}} \Gamma(m + \frac{1}{2})$ .

**Lemma 1.**  $\lim_{\rho \rightarrow 1^-} (1 - \rho^2)^{(k-1)/2} \sum_{j=0}^{2m} \binom{j}{k} \binom{2m}{j} C_{m,j}(\rho)$  exists for each  $k = 0, \dots, 2m$  and has the value 0 for odd  $k$ ,  $-h_m$  for  $k = 0$ , and  $-\binom{2m}{k} (-2)^l h_{m-l} \left(\frac{1}{2}\right)_l l!$  for  $k = 2l$ ,  $l \geq 1$ .

**Proposition 4.**

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sum_{j=0}^{2m} S_{2m}^j(\varepsilon) C_{m,j}(r(\varepsilon)) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} G_m(r(\varepsilon); \varepsilon \underline{\lambda}) \\ &= - \sum_{l=0}^m (-r''(0))^{1/2-l} 2^l h_{m-l} \left(\frac{1}{2}\right)_l l! \\ &\quad \times \sum_{1 \leq p_1 < \dots < p_{2l} \leq 2m} \lambda_{p_1} \dots \lambda_{p_{2l}}. \end{aligned}$$

We proceed to prove Proposition 3, Lemma 1, Proposition 4, and then return to Theorem 2.

*Proof of Proposition 3.* When  $\underline{\lambda} = \underline{0}$ ,  $S_{2m}^j = \binom{2m}{j}$ , so that

$$G_m(\rho; \underline{0}) = \sum_{j=0}^{2m} \binom{2m}{j} C_{m,j}(\rho)$$

By Proposition 2.(1),  $(1 - \rho^2)^{m-1/2} G_m(\rho; \underline{0})$  is a polynomial in  $\rho$ , but by (3.4),  $G_m(\rho; \underline{0})$  has the value 0 at  $\rho = 1$ . Therefore the function

$$R_m(\rho) \equiv (1 - \rho^2)^{m-1/2} (1 - \rho)^{-m} G_m(\rho; \underline{0}) = (1 + \rho)^m (1 - \rho^2)^{-1/2} G_m(\rho; \underline{0})$$

is a polynomial in  $\rho$ . By l'Hôpital's rule applied to  $G_m(\rho; \underline{0})/\sqrt{1 - \rho^2}$ ,

$$\begin{aligned} -2^{-m} R_m(1-) &= \lim_{\rho \rightarrow 1^-} (1 - \rho^2)^{1/2} \frac{d}{d\rho} G_m(\rho; \underline{0}) \\ &= \lim_{\rho \rightarrow 1^-} (1 - \rho^2)^{1/2} \sum_{j=0}^{2m} \binom{2m}{j} C'_{m,j}(\rho) = h_m. \end{aligned}$$

But since  $(1 - \rho)^m R_m(\rho) = (1 - \rho^2)^{m-1/2} G_m(\rho; \underline{0})$ , we have

$$m! (-1)^m R_m(1-) = \frac{d^m}{d\rho^m} \{(1 - \rho^2)^{m-1/2} G_m(\rho; \underline{0})\}_{\rho=1-}$$

and

$$h_m = -2^{-m} (-1)^m \frac{1}{m!} \sum_{j=0}^{2m} \binom{2m}{j} \frac{d^m}{d\rho^m} \{(1 - \rho^2)^{m-1/2} C_{m,j}(\rho)\}_{\rho=1-}$$

By Proposition 2.(1) and (2), the only terms which contribute to the last sum are the two (equal) terms with  $j = 0$  and  $j = 2m$ . Thus

$$h_m = 2^{-m+1}(-1)^{m+1} \frac{1}{m!} \frac{d^m}{d\rho^m} \{(1 - \rho^2)^{m-1/2} C_{m,0}(\rho)\}_{\rho=1} \quad (3.5)$$

In the rest of this proof, we refer again to formula numbers in the Bateman Manuscript Project (BMP) chapter on hypergeometric functions. Combining (3.3) and (3.5), we find that

$$\frac{h_m}{2^m \Gamma(m + \frac{1}{2})/\sqrt{\pi}} = 2^{-m+1} \frac{(-1)^{m+1}}{m!} \frac{d^m}{d\rho^m} \left\{ \rho F\left(1, 1 - m, \frac{3}{2}, \rho^2\right) \right\}_{\rho=1}$$

which by BMP formula (2.11.9) is

$$\begin{aligned} &= \frac{2^{-m}(-1)^{m+1}}{m!(2m-1)} \frac{d^m}{d\rho^m} \left\{ F\left(1, 1 - 2m, \frac{3}{2} - m, \frac{1-\rho}{2}\right) \right. \\ &\quad \left. - F\left(1, 1 - 2m, \frac{3}{2} - m, \frac{1+\rho}{2}\right) \right\}_{\rho=1} \end{aligned}$$

Using BMP formula (2.8.20) to re-express the derivatives yields

$$\begin{aligned} &\frac{1}{2}(-2)^m \{(-2)^{-m} F(m+1, 1 - m, \frac{3}{2}, 0) - 2^{-m} F(m+1, 1 - m, \frac{3}{2}, 1)\} \\ \text{(by BMP (2.8.33))} &= \frac{1}{2} \left\{ 1 - (-1)^m \frac{m + \frac{1}{2}}{-1/2} F\left(m+1, -m, \frac{3}{2}, 1\right) \right\} \\ \text{(by BMP (2.8.46))} &= \frac{1}{2} \left\{ 1 + (-1)^m (2m+1) \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - m)\Gamma(\frac{3}{2} + m)} \right\} \end{aligned}$$

and this final expression is equal to 1, completing the proof.  $\square$

*Proof of Lemma 1.* The Lemma is true by definition of  $h_m$  for  $k = 0$ . We proceed by induction on  $k$ . Assume that we have already proved the Lemma for all values of  $k \leq 2l, l \geq 1$ . Then by the identity  $C_{m,j} \equiv C_{m,2m-j}$ , for  $k = 2l + 1$

$$\sum_{j=0}^{2m} \binom{j}{k} \binom{2m}{j} C_{m,j}(\rho) = \frac{1}{2} \sum_{j=0}^{2m} \binom{2m}{j} C_{m,j}(\rho) \left\{ \binom{j}{k} + \binom{2m-j}{k} \right\}$$

is the sum of polynomials in  $j$  of even degree  $2l$  multiplied by  $\binom{2m}{j} C_{m,j}(\rho)$ , and is therefore of order  $(1 - \rho^2)^{-l+1/2}$  as  $\rho$  increases to 1. It follows that as  $\rho \rightarrow 1 -$ ,

$$(1 - \rho^2)^{(k-1)/2} \sum_{j=0}^{2m} \binom{j}{k} \binom{2m}{j} C_{m,j}(\rho) \rightarrow 0 \quad \text{for } k = 2l + 1.$$

Next, for  $k = 2l + 2$ ,  $(1 - \rho^2)^{l+1/2} \sum_{j=0}^{2m} \binom{j}{k} \binom{2m}{j} C_{m,j}(\rho)$  differs from

$$\begin{aligned} &(-1)^{l+1} (1 - \rho^2)^{l+1/2} \sum_{j=0}^{2m} \frac{j(j-1) \dots (j-l)(2m-j)(2m-j-1) \dots (2m-j-l)}{k!} \\ &\quad \times \binom{2m}{j} C_{m,j}(\rho) \end{aligned} \quad (3.6)$$

by a term of order  $(1 - \rho^2)$ ; thus our Lemma will be proved if we show that as  $\rho$  increases to 1, (3.6) converges to

$$= - \binom{2m}{k} (-2)^{l+1} h_{m-l-1} \left( \frac{1/2}{l+1} \right) (l+1)!$$

Still maintaining  $k = 2l + 2$ , we find that (3.6) is equal to

$$(-1)^{l+1} (1 - \rho^2)^{l+1/2} \sum_{j=l+1}^{2m-l-1} \frac{(2m)(2m-1) \dots (2m-k+1)}{k!} \binom{2m-k}{j-l-1} C_{m,j}(\rho)$$

which by Proposition 2.(3) applied  $l+1$  times is equal to

$$(1 - \rho^2)^{l+1/2} \binom{2m}{k} \frac{d^{l+1}}{d\rho^{l+1}} \sum_{j=l+1}^{2m-l-1} \binom{2m-k}{j-l-1} C_{m-l-1,j-l-1}(\rho) \quad (3.7)$$

With the index of summation changed to  $r = j - l - 1$ , (3.7) becomes

$$(1 - \rho^2)^{l+1/2} \binom{2m}{k} \frac{d^{l+1}}{d\rho^{l+1}} \sum_{r=0}^{2m-k} \binom{2m-k}{r} C_{m-l-1,r}(\rho) \quad (3.8)$$

Finally, we know for all  $n$  that for  $\rho$  in the neighborhood of 1,

$$\sum_{r=0}^{2n} \binom{2n}{r} C_{n,r}(\rho) + h_n (1 - \rho^2)^{1/2}$$

is equal to  $(1 - \rho)^{3/2}$  multiplied by some smooth function of  $\rho$ . It follows that for  $j \leq n$ , as  $\rho$  increases to 1

$$(1 - \rho^2)^{j-1/2} \frac{d^j}{d\rho^j} \sum_{r=0}^{2n} \binom{2n}{r} C_{n,r}(\rho) \rightarrow -2^{j-1/2} 2^{1/2} h_n (-1)^j \left( \frac{1/2}{j} \right) j!$$

Substituting  $m - l - 1$  for  $n$  and  $l + 1$  for  $j$ , we find that the limit of (3.8) as  $\rho$  increases to 1 is

$$- \binom{2m}{k} (-2)^{l+1} h_{m-l-1} \left( \frac{1/2}{l+1} \right) (l+1)!$$

and our induction is completed, proving the Lemma.  $\square$

*Proof of Proposition 4.* Recall that the function  $(1 - \rho^2)^{m-1/2} G_m(\rho; t \underline{\lambda})$  is a polynomial in  $\rho \in (0, 1)$  and analytic in  $t$  for fixed  $\underline{\lambda}$ . Now for small  $\varepsilon$ ,  $1 - r^2(\varepsilon)$  is asymptotic to  $-r''(0) \varepsilon^2$ , so that  $\varepsilon^{2m} G_m(r(\varepsilon); t \underline{\lambda})$  tends to 0 as  $\varepsilon$  does. Since we have proved in Lemma 1 that  $(1 - \rho^2)^{-1/2} \cdot G_m(\rho; \underline{0})$  converges to  $-h_m$  as  $\rho$  increases to 1, we expand the terms  $S_{2m}^j(\varepsilon)$  in powers of  $\varepsilon$  to find

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} G_m(r(\varepsilon); \varepsilon \underline{\lambda}) = -(-r''(0))^{1/2} h_m + \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \sum_{j=0}^{2m} \sum_{k=1}^{2m} i^k (\lambda_{n_1} + \dots + \lambda_{n_j})^k \frac{\varepsilon^k}{k!} C_{m,j}(r(\varepsilon))$$

$$1 \leq n_1 < \dots < n_j \leq 2m$$

Each term  $(\lambda_{n_1} + \dots + \lambda_{n_j})^k$  can be expanded as a sum of monomials in  $\lambda_i$  of the form  $K(d, k; \underline{a}) \lambda_{p_1}^{a_1} \dots \lambda_{p_d}^{a_d}$  where  $0 \leq p_1 < \dots < p_d \leq 2m$ , where the positive integer powers  $a_1, \dots, a_d$  sum to  $k$ , and where the integers  $K(d, k; \underline{a})$  do not

depend on  $j$ . Moreover, within the sum  $\sum_{1 \leq n_1 < \dots < n_j \leq 2m} (\lambda_{n_1} + \dots + \lambda_{n_j})^k$ , each

term  $K(d, k; \underline{a}) \lambda_{p_1}^{a_1} \dots \lambda_{p_d}^{a_d}$  which appears must appear precisely  $\binom{2m-d}{j-d} = \binom{2m}{j} \binom{j}{d} / \binom{2m}{d}$  times. Thus we know, for each fixed  $j, k$ , that

$$\begin{aligned} & \sum_{1 \leq n_1 < \dots < n_j \leq 2m} (\lambda_{n_1} + \dots + \lambda_{n_j})^k \\ &= \sum_{\underline{d}; \underline{a}} \binom{2m}{j} \binom{j}{d} K(d, k; \underline{a}) \sum_{1 \leq p_1 < \dots < p_d \leq 2m} \lambda_{p_1}^{a_1} \dots \lambda_{p_d}^{a_d} / \binom{2m}{d} \end{aligned}$$

where the outer summation ranges over all  $d = 1, \dots, k$  and all positive integers  $a_1, \dots, a_d$  which sum to  $k$ . Finally, by Lemma 1 we learn that for fixed  $d \leq k$ , if either  $d < k$  or  $k$  is odd then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{k-1} \sum_{j=0}^{2m} \binom{2m}{j} \binom{j}{d} C_{m,j}(r(\varepsilon)) = 0.$$

Now change the summation-index for even  $k$  to  $l$ , where  $k = 2l$ ; observe that the constant  $K(d, k; \underline{a})$  for  $\underline{a} = (1, \dots, 1)$  and  $d = k$  reduces to  $k!$ ; and apply Lemma 1 once more to find

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} G_m(r(\varepsilon); \varepsilon \underline{\lambda}) &= -(-r''(0))^{1/2} h_m + \sum_{l=1}^m (-1)^l \sum_{1 \leq p_1 < \dots < p_{2l} \leq 2m} \lambda_{p_1} \dots \lambda_{p_{2l}} \\ &\cdot \lim_{\varepsilon \rightarrow 0} \varepsilon^{2l-1} \sum_{j=0}^{2m} \binom{2m}{j} \frac{\binom{j}{2l}}{\binom{2m}{2l}} C_{m,j}(r(\varepsilon)) \\ &= - \sum_{l=0}^m (-1)^l \sum_{1 \leq p_1 < \dots < p_{2l} \leq 2m} \lambda_{p_1} \dots \lambda_{p_{2l}} \\ &\cdot (-r''(0))^{1/2-l} (-2)^l \binom{1/2}{l} h_{m-l} l! \end{aligned}$$

Our Proposition is proved.  $\square$

*Conclusion of proof of Theorem 2.* Under the condition (2.1), the sequence of random variables

$$\begin{aligned} \sum_{j=0}^{\varepsilon^{-1}-1} I_{[X_{j\varepsilon} X_{(j+1)\varepsilon} < 0]} &= \varepsilon^{-1} \left( \frac{1}{2} - \frac{1}{\pi} \arcsin(r(\varepsilon)) \right) + \frac{1}{\pi} \sum_{m=1}^{\infty} (-1)^m I_{2m} \\ &\cdot \left( \frac{e^{i(\lambda_1 + \dots + \lambda_{2m})} - 1}{e^{i\varepsilon(\lambda_1 + \dots + \lambda_{2m})} - 1} \sum_{j=0}^{2m} S_{2m}^j(\varepsilon) C_{m,j}(\rho) \right) \end{aligned}$$

increases monotonically for  $\varepsilon$  along the sequence  $\varepsilon = 2^{-n}$  to the square-integrable random variable equal to the number  $N_t(0, 1)$  of axis-crossings by the process  $X_t$  for  $t \in [0, 1]$ . It follows that convergence takes place also in mean square, and that the same sequence of variables orthogonally projected onto the subspace range  $(I_{2m}) \equiv H_{2m}$  must correspondingly be uniformly square-integrable and converge in mean square. This reasoning shows that for each  $m$ , and for  $\varepsilon = 2^{-n}$ , the functions

$$\frac{e^{i(\lambda_1 + \dots + \lambda_{2m})} - 1}{e^{i\varepsilon(\lambda_1 + \dots + \lambda_{2m})} - 1} \sum_{j=0}^{2m} S_{2m}^j(\varepsilon) C_{m,j}(\rho)$$

converge as  $\varepsilon \rightarrow 0$  ( $n \rightarrow \infty$ ) in  $L^2(\sigma^{2m}, \text{sym})$ . To identify the limit, it suffices to find a  $\sigma^{2m}$  a.e. limit (for fixed  $\lambda_1, \dots, \lambda_{2m}$ ). First observe that the function  $\varepsilon(e^{i(\lambda_1 + \dots + \lambda_{2m})} - 1)/(e^{i\varepsilon(\lambda_1 + \dots + \lambda_{2m})} - 1)$  is bounded, uniformly for  $\varepsilon$  in a small neighborhood of 0, and converges pointwise as  $\varepsilon \rightarrow 0$  to  $-i(e^{i(\lambda_1 + \dots + \lambda_{2m})} - 1)/(\lambda_1 + \dots + \lambda_{2m})$ . Thus the Theorem follows from the pointwise convergence of

$$\varepsilon^{-1} G_m(r(\varepsilon); \underline{\varepsilon} \underline{\lambda}) \equiv \varepsilon^{-1} \sum_{j=0}^{2m} S_{2m}^j(\varepsilon) C_{m,j}(r(\varepsilon))$$

proved in Proposition 3.  $\square$

*Proof of Corollaries 2 and 3.* By inspection, if  $Z_0$  is given by (2.2), then  $\int_0^T \theta_s \circ Z_0 ds$  agrees for all  $T$  with  $N_X(T)$  as expressed in Theorem 2. Next observe that  $X_0$  and  $X'_0/\sqrt{-r''(0)}$  are jointly Gaussian and independent with unit variances. All Hermite polynomials  $H_k(X_0)$  and  $H_j(X'_0/\sqrt{-r''(0)})$  are therefore uncorrelated, and are given respectively by  $k! I_k(1_k)$  and  $j! I_j(\lambda_1 \dots \lambda_j i^j (-r''(0))^{-j/2})$ . By a simple case of the Diagram formula,

$$\begin{aligned} I_{2m} \left( \sum_{1 \leq n_1 < \dots < n_{2l} \leq 2m} \lambda_{n_1} \dots \lambda_{n_{2l}} / (-r''(0))^l \right) \\ = \frac{1}{(2m)!} \binom{2m}{2l} (2l)! I_{2l}(\lambda_1 \dots \lambda_{2l} (r''(0))^{-l}) (2m-2l)! I_{2m-2l}(1_{2m-2l}) \\ = \frac{1}{(2l)!} H_{2l}(X'_0/\sqrt{-r''(0)}) \frac{1}{(2m-2l)!} H_{2m-2l}(X_0). \end{aligned}$$

To complete the proof of (2.3), substitute into (2.2) using the identity

$$\binom{1/2}{l} = \frac{1}{l!} (1/2)(-1/2) \dots (- (2l-3)/2) = (-1)^{-l+1} \frac{(2l-2)!}{2^{2l-2}(l-1)!l!}. \quad \square$$

*Proof of Theorem 3.* The asserted asymptotic normality follows from Theorem 2 of Chambers and Slud (1989a), now in a continuous-time setting and applied to the functional  $Z_0 \equiv \sum_{m=0}^{\infty} I_{2m}(\gamma_{2m})$  of (2.2). Assumption (A.1) of Chambers and Slud holds in modified form exactly as in the first paragraph of the proof of Theorem 1 above, and (A.3) again follows without difficulty from continuity of the integrands  $\gamma_{2m}$ . Finally, (A.2') follows as in the proof of Theorem 1 from Lemma 2.3 of Chambers and Slud together with the key observation that  $T^{-1} \text{Var}(\int_0^T I_{2m}(\gamma_{2m}(\underline{\lambda})) \exp(it(\lambda_1 + \dots + \lambda_{2m})) dt)$  is bounded by a constant  $M_1$  times  $\text{Var} I_{2m}(\gamma_{2m}(\underline{\lambda}))$ , uniformly in  $m$  and  $T$ . This last fact follows from the previously-cited argument of Ho and Sun (1987, (2.3), (2.6)–(2.9)): in their argument, replace  $j$  by  $2m$ ,  $dG$  by  $d\sigma$ ,  $\alpha_j(x_1, \dots, x_j)$  by  $\gamma_{2m}(x_1, \dots, x_{2m})$ ,  $d$  and  $\delta$  by 2, and put  $h_1(x) = 1$ ,  $h_2(x) = ix/\sqrt{-r''(0)}$ .

Just as in the proof of Theorem 1, we find

$$\lim_{T \rightarrow \infty} \text{Var}(N_X(T))/T \geq \lim_{T \rightarrow \infty} \text{Var} \left( I_2 \left( \frac{e^{iT(\lambda_1 + \lambda_2)} - 1}{i(\lambda_1 + \lambda_2)\sqrt{T}} \gamma_2(\lambda_1, \lambda_2) \right) \right)$$

which by Lemma 2.3 of Chambers and Slud is equal to

$$2\pi \cdot \frac{1}{2} \cdot \int |\gamma_2(\lambda_1, -\lambda_1)|^2 f(\lambda_1) f(-\lambda_1) d\lambda_1$$

and is precisely the lower bound given in Theorem 3.  $\square$

*Acknowledgments.* I am grateful to the referee for the suggestion to relate the functions  $C_{m,j}$  to known special functions, and to Professor Frank Olver for the observation that they are of hypergeometric type.

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