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Probability Theory and Related Fields

Strong approximations of semimartingales by processes with independent increments

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Summary. Strong approximation theorems for continuous time semimartingales are obtained by combining some techniques of the general theory of stochastic processes with some of the direct approximation of dependent random variables by independent ones. Continuous processes with independent increments whose variance functions increase polynomially or exponentially are considered as approximating processes. The basic assumptions of the main results only contain rates of convergence for certain probabilities. In particular, moment assumptions are not required. Some almost sure invariance principles for partial sum processes with nonlinear growth of variance and for functionals of Markov processes are derived by applying the main results.

Introduction

Almost sure invariance principles for stochastic processes have been derived under various dependence assumptions. Most of the results concentrate on discrete time processes, in particular on the approximation of partial sum processes by a Brownian motion with variance parameter σ^2. With a sufficiently small error term, such as a term of the form $O(t^{\frac{1}{2}-\lambda})$, such an almost sure invariance principle essentially implies all classic fluctuation results (cf. [27], Chap. 1).

In this paper, strong approximation theorems for continuous time semimartingales are derived. This class contains most of the stochastic processes relevant for applications such as partial sum processes, diffusions, point processes and local martingales (cf. [18, 23, 30]). As reference processes, i.e. as approximating processes, we consider continuous processes with independent increments, whose variance functions are polynomial or exponential. The error term $O(t^{\frac{1}{2}-\lambda})$ is replaced by $O(f(t)^{\frac{1}{2}} t^{-\lambda})$. In this term, the natural coupling of the "inner time scale", i.e. of the variance function f of the reference process, with the approximation rate is expressed.

The basic assumptions which allow the approximation of the semimartingales are given in convergence rates for certain probabilities only. Having the canonical decomposition of semimartingales in mind we actually consider three terms: First, the probability for the big jumps, second, the probability for the deviation of the two time scales, and third, the probability for the deviation of the truncated trend parts. In particular, no moment assumptions are required, which is an essential feature distinctive from many strong approximation theorems known so far (cf. [1, 5, 25, 27]).

Our main result concerning polynomial time scales (Theorem 1) is stated in the next chapter. The fundamental approach in deriving this theorem is based upon approximating dependent random variables by independent ones according to a result of Berkes and Philipp [1] on the one hand, and upon some ideas taken from the proof of the weak invariance principle for semimartingales by Liptser and Shiryaev [21] on the other hand. From this result we deduce some conclusions.

Chapter 2 is devoted to the discussion of results for exponential time scales (Theorem 3). In Chaps. 3 and 4, we prove Theorem 1 and Theorem 3.

In Chap. 5, a strong invariance principle for discrete time partial sum processes with nonlinear growth of variance is derived by applying Theorem 1. This result is a substantial improvement of the almost sure invariance principle for martingales by Morrow and Philipp [24] and thus emphasizes the right choice of the assumptions in our main results. In verifying the conditions of Theorem 1, it is essential to consider probabilities only and not more global conditions such as statements on the order of moments.

Using Theorem 2 an almost sure invariance principle for functionals of Markov processes is deduced in Chap. 6.

The theorems are phrased by means of the general theory of stochastic processes. Therefore, the required terms and definitions are shortly explained in the following. For this, the notations from [18] and [23] are mainly used. For further studies see [6, 7, 13, 15].

Let $(\Omega, \mathfrak{A}, P)$ be a complete probability space and $\mathscr{F}=(\mathscr{F}_t)_{t\geqq 0}$ a nondecreasing right-continuous filtration. We use the notation $X=(X, \mathscr{F})$ for an $\mathscr{F}$-adapted stochastic process $X=(X_t)_{t\geqq 0}$ with right-continuous paths having limits from the left. As usual, $\mathscr{M}^1_{\text{loc}}(\mathscr{F})(\mathscr{M}^2_{\text{loc}}(\mathscr{F}))$ denotes the class of all local (locally square-integrable) martingales $M=(M, \mathscr{F})$ with respect to $\mathscr{F}$. We also write $\mathscr{V}(\mathscr{F})$, respectively $\mathscr{A}_{\text{loc}}(\mathscr{F})$ for the set of processes $V=(V, \mathscr{F})$ with paths of finite, respectively locally integrable variation over every compact interval of $\mathbb{R}^+ := [0, \infty)$.

A stochastic process $X=(X, \mathscr{F})$ is called a *semimartingale with respect to* $\mathscr{F}$ if it admits a representation $X=M+V$ where $M\in\mathscr{M}^1_{\text{loc}}(\mathscr{F})$ and $V\in\mathscr{V}(\mathscr{F})$. If a semimartingale has a decomposition $X=N+V$ where $N\in\mathscr{M}^2_{\text{loc}}(\mathscr{F})$ and $V\in\mathscr{A}_{\text{loc}}(\mathscr{F})$ it is called a *special semimartingale* (with respect to $\mathscr{F}$).

We denote the *predictable σ-field* of subsets of $\mathbb{R}^+\times\Omega$ by $\mathscr{P}=\mathscr{P}(\mathscr{F})$ which is generated by the continuous $\mathscr{F}$-adapted processes. Moreover, a process X is called *predictable* if the mapping $(t, \omega)\to X_t(\omega)$ is $\mathscr{P}$-measurable.

For every special semimartingale $X=(X, \mathscr{F})$ a uniquely determined predictable process $A\in\mathscr{A}_{\text{loc}}(\mathscr{F})$ with $A_0=0$ exists such that $X-A\in\mathscr{M}^1_{\text{loc}}(\mathscr{F})$. This process A is called the *compensator of* X. Here and in all the sequel we identify processes which are P-indistinguishable.

In particular, if $M\in\mathscr{M}^2_{\text{loc}}(\mathscr{F})$ we denote the compensator of M^2 by $\langle M\rangle$ and call it *quadratic characteristic of* M.

Now let $X=(X, \mathscr{F})$ be a real semimartingale with respect to $\mathscr{F}$. Write $\mathscr{B}^+$ $(\mathscr{B})$ for the Borel-σ-field of $\mathbb{R}^+(\mathbb{R})$ and define

$$\mu(\cdot, A) := \sum_{s>0} 1_{\{\Delta X_s \neq 0\}}\, \varepsilon_{(s, \Delta X_s)}(A)$$

for $A \in \mathscr{B}^+ \otimes \mathscr{B}$. Here ε_a denotes the Dirac measure at a and $\Delta X_s := X_s - \lim_{u \uparrow s} X_u$ the *jump of* X *at time* $s \in \mathbb{R}^+$. The random measure μ is called the *random measure associated to the jumps of* X. For this random measure μ a unique, positive, predictable random measure v called the *compensator of* μ exists such that for every $\mathscr{P} \otimes \mathscr{B}$-measurable mapping $W\colon \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$

$$E\left[\int_0^\infty \int W(s,x)\,\mu(ds,dx)\right] = E\left[\int_0^\infty \int W(s,x)\,v(ds,dx)\right]$$

and for all $\omega \in \Omega$, $t > 0$ $v(\omega, \mathbb{R}^+ \times \{0\}) = 0$ and $0 \leqq v(\omega, \{t\} \times \mathbb{R}) \leqq 1$ (see [18], Theorem II.1.8 and Proposition II.1.17(b)).

We consider the following process associated to the semimartingale X:

$$X_t^{(1)} := \sum_{s \leqq t} (\Delta X_s)\, 1_{\{|\Delta X_s| > 1\}} = \int_0^1 \int_{\{|x|>1\}} x\,\mu(ds,dx), \qquad t \geqq 0.$$

The process $X - X^{(1)}$ is also a semimartingale, has jumps bounded by 1 and is therefore a special semimartingale. Thus the compensator $B = (B, \mathscr{F}) \in \mathscr{A}_{\mathrm{loc}}(\mathscr{F})$ of $X - X^{(1)}$ exists.

Further, we denote the continuous local martingale part of X by $X^c = (X^c, \mathscr{F})$ (see [18], Proposition I.4.27) and its quadratic characteristic by $\langle X^c \rangle$.

This triple $(B, \langle X^c \rangle, v)$ uniquely determined by the semimartingale X is called the *local characteristics of* X (associated to the truncation function $h(x) := x\,1_{\{|x| \leqq 1\}}$, $x \in \mathbb{R}$).

In terms of this triple and the random measure μ, the semimartingale X has the following decomposition (cf. [15], Corollary III.3.78):

$$X = X_0 + X^c + \int \int_{\{|x| \leqq 1\}} x(\mu - v)(ds,dx) + \int \int_{\{|x|>1\}} x\,\mu(ds,dx) + B.$$

This decomposition is called the *canonical representation of the semimartingale* X.

If a semimartingale X is a process with independent increments there is a version of the triple $(B, \langle X^c \rangle, v)$ which is not random and, conversely, if a semimartingale has deterministic local characteristics this process has independent increments.

Especially, if $Z = (Z, \mathscr{F})$ denotes a continuous process with independent increments, two deterministic functions W and f on $\mathbb{R}^+$ exist such that the local characteristics are given by $(W, f, 0)$, the process $M = Z - W$ is a martingale, and its quadratic characteristic $\langle M \rangle$ is f. The function W is called the *drift of* Z and f is called the *variance function of* Z.

Finally, we introduce two processes which we use in a suitable way in order to control the inner time scale of the semimartingale considered in the following chapters. Therefore, let a constant $\rho > 0$ and a positive increasing function f

on $\mathbb{R}^+$ be given. We define the processes $V(X,f,\rho)$ and $V(X)$ (with values in $[0,\infty]$) for $t\in\mathbb{R}^+$ by

$$V(X,f,\rho)_t := \langle X^c\rangle_t + \int_0^t \int_{\{x^2\leq f(s)s^{-\rho}\}} x^2\,\nu(ds,dx) - \sum_{s\leq t}\Big[\int_{\{x^2\leq f(s)s^{-\rho}\}} x\,\nu(\{s\},dx)\Big]^2$$

and

$$V(X)_t := \langle X^c\rangle_t + \int_0^t \int x^2\,\nu(ds,dx) - \sum_{s\leq t}\Big[\int x\,\nu(\{s\},dx)\Big]^2.$$

These processes are of course quadratic characteristics of suitably truncated local martingale parts of X.

1. Results for polynomial time scales

Let $W=(W_t)_{t\geq 0}$ be a real-valued function and f a nondecreasing one on $\mathbb{R}^+$ with $f_0=0$. In this section we assume that f grows polynomially, i.e. there is some $\lambda_0\in(0,1]$ and some sufficiently large $t^*\in\mathbb{R}^+$ such that for $t\geq t^*$ the mappings $t\to f(t)t^{-\lambda_0}$ and for all $\kappa>0$ $t\to t^\kappa-\log f(t)$ are increasing. In order to avoid trivial problems, let us assume for the rest of this paper that $f(1)>0$, $X_0=0$ and $\Delta X_s=0$ for all $s\in[0,1]$.

Using " $\ll$ " instead of the Landau symbol "$O(\dots)$" we have the following strong approximation theorem for semimartingales:

Theorem 1. *Let $X=(X,\mathscr{F})$ be a real semimartingale with respect to $\mathscr{F}$, $(B,\langle X^c\rangle,\nu)$ its local characteristics and μ the random measure associated to its jumps. Suppose constants $\delta,\psi,\vartheta,\gamma>0$ and $\rho\in(0,\lambda_0)$ exist such that*

$$(1.1)\qquad P\left(\int_1^t \int_{\{x^2>f(s)s^{-\rho}\}} |x|\,\mu(ds,dx)\geq f(t)^{\frac{1}{2}}t^{-\delta}\right)\ll t^{-\gamma},$$

$$(1.2)\qquad P(|V(X,f,\rho)_t-f(t)|\geq f(t)t^{-\psi})\ll t^{-\gamma}$$

and

$$(1.3)\qquad P\left(\sup_{s\leq t}\left|B_s+\int_0^s \int_{\{1<x^2\leq f(u)u^{-\rho}\}} x\,\nu(du,dx)-W_s\right|\geq f(t)^{\frac{1}{2}}t^{-\vartheta}\right)\ll t^{-\gamma}.$$

Then the semimartingale X can be redefined on a richer probability space together with a continuous process $Z=(Z_t)_{t\geq 0}$ with independent increments, drift W and variance function f such that with probability 1

$$X_t-Z_t\ll f(t)^{\frac{1}{2}}t^{-\lambda}$$

for all $\lambda\in(0,\min\{\delta,\vartheta,\gamma,117\rho/278,13\psi/34,13\lambda_0/34\}/26)$.

Often one wishes to formulate all assumptions in terms of the local characteristics. For this reason, condition (1.1) is checked under this aspect in the following proposition. For the sake of brevity, we use notations of the form "$(n.k)$ is valid" for n, $k=1,2,3,\dots$ instead of "$(n.k)$ is valid for appropriate constants $\psi,\delta,\rho,\vartheta,\gamma>0$".

Proposition 1.1. (i) *If there are constants* $\rho, \delta, \gamma > 0$ *such that*

$$(1.4)\qquad P\left(\int_1^t \int_{\{x^2 > f(s)s^{-\rho}\}} |x|\, \nu(ds, dx) \geqq f(t)^{\frac{1}{2}} t^{-\delta}\right) \ll t^{-\gamma},$$

then (1.1) *is valid.*
(ii) *Assumption* (1.1) *implies* (1.4), *if*

$$(1.5)\qquad E[\sup_{s \leqq t} |\Delta X_s|] \ll f(t)^{\frac{1}{2}} t^{-\varepsilon}$$

holds for some $\varepsilon > 0$.

Proof of Proposition 1.1. From [18], Theorem II.1.8(i), for all finite $\mathscr{F}$-stopping times τ follows

$$(1.6)\qquad E\left[\int_1^\tau \int_{\{x^2 > f(s)s^{-\rho}\}} |x|\, \mu(ds, dx)\right] = E\left[\int_1^\tau \int_{\{x^2 > f(s)s^{-\rho}\}} |x|\, \nu(ds, dx)\right].$$

Thus the stochastic integral $\iint |x|\, 1_{\{x^2 > f(s)s^{-\rho}\}}\, \mu(ds, dx)$ is dominated in the sense of Lenglart by the corresponding integral with respect to ν and vice versa.
(i) The Lenglart inequality (cf. [18], Lemma I.3.30(a)) and (1.4) yield

$$P\left(\int_1^t \int_{\{x^2 > f(s)s^{-\rho}\}} |x|\, \mu(ds, dx) \geqq f(t)^{\frac{1}{2}} t^{-\delta'}\right)$$
$$\ll t^{-(\delta-\delta')} + P\left(\int_1^t \int_{\{x^2 > f(s)s^{-\rho}\}} |x|\, \nu(ds, dx) \geqq f(t)^{\frac{1}{2}} t^{-\delta}\right) \ll t^{-\gamma'}$$

for $0 < \delta' < \delta$, $0 < \gamma' \leqq \min(\delta - \delta', \gamma)$ and large t, i.e. (1.1) is valid.
(ii) Assume (1.1) and (1.5). By the definition of the random measure μ and (1.5) we get

$$E[\sup_{s \leqq t} \int_{\{x^2 > f(s)s^{-\rho}\}} |x|\, \mu(\{s\}, dx)] \leqq E[\sup_{s \leqq t} |\Delta X_s|] \ll f(t)^{\frac{1}{2}} t^{-\varepsilon}.$$

Because of (1.6) we can use the Lenglart inequality again, but now in the form of [18], Lemma I.3.30(b). Thus we obtain from (1.1)

$$P\left(\int_1^t \int_{\{x^2 > f(s)s^{-\rho}\}} |x|\, \nu(ds, dx) \geqq f(t)^{\frac{1}{2}} t^{-\delta'}\right)$$
$$\leqq (f(t)^{\frac{1}{2}} t^{-\delta'})^{-1} \{f(t)^{\frac{1}{2}} t^{-\delta} + E[\sup_{s \leqq t} \int_{\{x^2 > f(s)s^{-\rho}\}} |x|\, \mu(\{s\}, dx)]\}$$
$$+ P\left(\int_1^t \int_{\{x^2 > f(s)s^{-\rho}\}} |x|\, \mu(ds, dx) \geqq f(t)^{\frac{1}{2}} t^{-\delta}\right)$$
$$\ll t^{-\gamma'}$$

for $0 < \delta' < \delta \wedge \varepsilon$, $0 < \gamma' \leqq \min(\delta - \delta', \varepsilon - \delta', \gamma)$ and large t, i.e. (1.4) holds.

After condition (1.4) has in the following proposition been replaced by a more global assumption which is a little stronger but often satisfied in applications the two other conditions of Theorem 1 can equivalently be rephrased.

Proposition 1.2. *Let the jumps of the semimartingale X satisfy the regularity assumption*

$$E\left[\int_1^\infty \int_{\{x^2>f(s)s^{-\rho}\}} x^2 f(s)^{-1} s^\rho \mu(ds, dx)\right] < \infty \tag{1.7}$$

for some $\rho > 0$, *then:*

(i) *Condition* (1.4) *is fulfilled.*

(ii) *Assumption* (1.2) *is equivalent to*

$$P(|V(X)_t - f(t)| \geqq f(t)\, t^{-\psi}) \ll t^{-\gamma}. \tag{1.8}$$

(iii) *Assumption* (1.3) *is equivalent to*

$$P\left(\sup_{s\leqq t}\left|B_s + \int_0^s \int_{\{x^2>1\}} x\,\nu(du, dx) - W_s\right| \geqq f(t)^{\frac{1}{2}}\, t^{-\vartheta}\right) \ll t^{-\gamma}. \tag{1.9}$$

Remark 1.3. Assumption (1.7) is equivalent to

$$E\left[\int_1^\infty \int_{\{y^2>f(s)s^{-\rho}\}} y^2 f(s)^{-1} s^\rho \nu(ds, dy)\right] < \infty. \tag{1.10}$$

This remark is an immediate consequence of [18], Theorem II.1.8 (i).

Proof of Proposition 1.2. Let t be sufficiently large and, without loss of generality, $\rho \leqq \lambda_0$.

(i) By the Markov inequality and (1.10) we obtain

$$\begin{aligned}
&P\left(\int_1^t \int_{\{x^2>f(s)s^{-\rho}\}} |x|\,\nu(ds, dx) \geqq f(t)^{\frac{1}{2}}\, t^{-\delta}\right)\\
&\quad\leqq P\left(\int_1^t \int_{\{x^2>f(s)s^{-\rho}\}} |x|\,[f(s)^{\frac{1}{2}} s^{-\rho/2}]^{-1}\,\nu(ds, dx) \geqq t^{-\delta+\rho/2}\right)\\
&\quad\leqq t^{\delta-\rho/2}\, E\left[\int_1^\infty \int_{\{x^2>f(s)s^{-\rho}\}} x^2 f(s)^{-1} s^\rho\,\nu(ds, dx)\right]\\
&\quad\ll t^{-\gamma} \quad \text{for } \delta\in(0, \rho/2) \text{ and } \gamma\in(0, \rho/2-\delta].
\end{aligned}$$

(ii) In an analogous way we get

$$P\left(\int_0^t \int_{\{x^2>f(s)s^{-\rho}\}} x^2\,\nu(ds, dx) \geqq f(t)\, t^{-\psi}\right) \ll t^{-\gamma} \tag{1.11}$$

from (1.10) for $\psi\in(0,\rho)$ and $\gamma\in(0,\rho-\psi]$. Since

$$
\begin{aligned}
&\Big|\sum_{s\leq t}\Big[\int x\,\nu(\{s\},dx)\Big]^2-\sum_{s\leq t}\Big[\int_{\{x^2\leq f(s)s^{-\rho}\}} x\,\nu(\{s\},dx)\Big]^2\Big|\\
&\quad\leq\sum_{s\leq t}\Big\{\Big[\int_{\{x^2>f(s)s^{-\rho}\}} x\,\nu(\{s\},dx)\Big]^2\\
&\qquad+2\Big|\int_{\{x^2>f(s)s^{-\rho}\}} x\,\nu(\{s\},dx)\Big|\cdot\Big|\int_{\{x^2\leq f(s)s^{-\rho}\}} x\,\nu(\{s\},dx)\Big|\Big\}\\
&\quad\leq\sum_{s\leq t}\Big\{\int_{\{x^2>f(s)s^{-\rho}\}} x^2\,\nu(\{s\},dx)+2\int_{\{x^2>f(s)s^{-\rho}\}}|x|\,f(s)^{\frac12}s^{-\rho/2}\,\nu(\{s\},dx)\Big\}\\
&\quad\leq 3\int_0^t\int_{\{x^2>f(s)s^{-\rho}\}} x^2\,\nu(ds,dx),
\end{aligned}
$$

we conclude the equivalence claimed in (ii) from the triangle inequality and (1.11).

(iii) For the equivalence of (1.3) and (1.9) it is sufficient to show that for appropriate $\vartheta,\gamma>0$

$$P\Big(\sup_{s\leq t}\Big|\int_0^s\int_{\{x^2>f(u)u^{-\rho}\}} x\,\nu(du,dx)\Big|\geq f(t)^{\frac12}t^{-\vartheta}\Big)\ll t^{-\gamma}.$$

Since

$$\sup_{s\leq t}\Big|\int_0^s\int_{\{x^2>f(u)u^{-\rho}\}} x\,\nu(du,dx)\Big|\leq\int_0^t\int_{\{x^2>f(s)s^{-\rho}\}}|x|\,\nu(ds,dx),$$

the inequality above immediately follows from (1.4) for $\vartheta=\delta\in(0,\rho/2)$ and $\gamma\in(0,\rho/2-\vartheta]$ (see also Proof of (i)).

As we are going to see in the next theorem, the regularity assumption (1.7) does not only guarantee the convergence of the big jumps in the sense of (1.4) but it is also the condition that yields the local square-integrability of semimartingale X (cf. [18], Definition II.2.27). The reason for this is the fact that a semimartingale is a locally square-integrable semimartingale if and only if the process $\sup_{s\leq\cdot}X_s^2$ is located in $\mathscr{A}_{\text{loc}}(\mathscr{F})$ (see [18], Lemma II.2.28), which is implied by (1.7) in addition to condition (1.4).

Using the special semimartingale property, the assumptions (1.8) and (1.9) are rewritten in a more compact manner which helps phrasing the following strong approximation theorem.

Theorem 2. *Let $X=(X,\mathscr{F})$ be a real semimartingale with respect to $\mathscr{F}$. Suppose that the random measure μ associated to its jumps satisfies* (1.7) *for some $\rho>0$.*

Then X is a locally square-integrable semimartingale, i.e. a uniquely determined predictable process $A=(A_t)_{t\geq 0}\in\mathscr{A}_{\text{loc}}(\mathscr{F})$ with $A_0=0$ and a square-integrable local martingale $N=(N,\mathscr{F})$ exist such that $X=N+A$.

Furthermore, assume there are constants ψ, ϑ, $\gamma>0$ such that

(1.12) $$P(|\langle N\rangle_t-f(t)|\geq f(t)\,t^{-\psi})\ll t^{-\gamma}$$

and

(1.13) $$P(\sup_{s\leq t}|A_s-W_s|\geq f(t)^{\frac12}t^{-\vartheta})\ll t^{-\gamma}.$$

Then the semimartingale X can be redefined on a richer probability space together with a continuous process $Z=(Z_t)_{t\geqq 0}$ with independent increments, drift W and variance function f such that with probability 1

$$X_t - Z_t \ll f(t)^{\frac{1}{2}} t^{-\lambda}$$

for some $\lambda > 0$.

Proof of Theorem 2. Without loss of generality we can assume $\rho \leqq \lambda_0$.

(a) Defining for each $n \in \mathbb{N}$ the $\mathscr{F}$-stopping time R_n by

$$R_n := \inf\{t \geqq 0 \colon \sup_{s \leqq t} X_s^2 + \sum_{s \leqq t} (\Delta X_s)^2 \geqq n\} \wedge n,$$

we get from (1.7) and the definition of μ

$$\begin{aligned} E[\sup_{s \leqq R_n} X_s^2] &\leqq E[\sup_{s \leqq R_n} 2\{X_{s-}^2 + (\Delta X_s)^2\}] \\ &\leqq 2E[\sup_{s < R_n} X_s^2 + \sup_{s \leqq R_n} \{(\Delta X_s)^2 1_{\{(\Delta X_s)^2 \leqq f(s) s^{-\rho}\}} + (\Delta X_s)^2 1_{\{(\Delta X_s)^2 > f(s) s^{-\rho}\}}\}] \\ &\leqq 2n + 2f(n) n^{-\rho} + 2f(n) n^{-\rho} E\left[\int_1^\infty \int_{\{x^2 > f(s) s^{-\rho}\}} x^2 f(s)^{-1} s^\rho \mu(ds, dx)\right] < \infty. \end{aligned}$$

Consequently, the process $\sup_{s \leqq \cdot} X_s^2$ is locally integrable. Hence, we see from [18], Lemma II.2.28, that X is a locally square-integrable semimartingale, i.e. it admits a unique decomposition $X = N + A$ with $N \in \mathscr{M}_{\mathrm{loc}}^2(\mathscr{F})$ and predictable $A \in \mathscr{A}_{\mathrm{loc}}(\mathscr{F})$ with $A_0 = 0$.

(b) Denote the local characteristics of X by $(B, \langle X^c \rangle, \nu)$. From [18], Proposition II.2.29(a), Eqs. 2.30 and 2.31, we know that the process A is given by

$$A_t = B_t + \int_0^t \int_{\{x^2 > 1\}} x \, \nu(ds, dx) \tag{1.14}$$

and that the quadratic characteristic of N can be calculated thus

$$\langle N \rangle_t = \langle X^c \rangle_t + \int_0^t \int x^2 \nu(ds, dx) - \sum_{s \leqq t} [\int x \, \nu(\{s\}, dx)]^2 = V(X)_t. \tag{1.15}$$

Introducing (1.14), respectively (1.15) in (1.13), respectively (1.12), we obtain the validity of (1.8) and (1.9), respectively. From Theorem 1 and the equivalences of Proposition 1.2 we conclude the assertion of Theorem 2.

Remark 1.4. For local martingales, Theorem 2 is especially useful. Because of its uniqueness the predictable process A of Theorem 2 equates the compensator of process X and therefore it vanishes identically in the case of local martingales.

So, the validity of (1.7) and (1.12) suffices for a strong invariance principle for local martingales. If in addition X is continuous the jump condition (1.7) is trivially satisfied.

Remark 1.5. The relevance of the assumptions (1.7), respectively (1.10) and therefore of condition (1.1) in particular becomes clear if a partial sum process of the form $X_t = \sum_{n \leqq t} \zeta_n$, $t \geqq 0$, is considered for a sequence $(\zeta_n)_{n \geqq 1}$ of real-valued random variables.

In the case of almost-sure invariance principles for such time-discrete processes with Brownian motion as reference process and error term $O(t^{\frac{1}{2}-\lambda})$, a $(2+\delta)$-moment $(\delta > 0)$ of these random variables being uniformly bounded is a standard condition (cf. [5, 10, 19, 25, 27]). This boundedness implies assumption (1.7) (with $f(t) = t$).

If $(\zeta_n)_{n \geqq 1}$ even is a sequence of identically distributed random variables this moment condition on the other hand ensues from (1.7), i.e. in this case the regularity assumption (1.7) is equivalent to the existence of a $(2+\delta)$-moment.

In this context one should not leave unmentioned that this moment condition is necessary for such a strong invariance principle (see [4], Theorem 2.6.4).

Proof of Remark 1.5. (a) Let $(\zeta_n)_{n \geqq 1}$ be an arbitrary sequence of random variables with

$$\sup_{n \geqq 1} E[|\zeta_n|^{2+\delta}] \leqq C < \infty$$

for some $\delta > 0$. Then we have for $\rho := \delta(4 + 2\delta)^{-1}$ and all $n \in \mathbb{N}$

$$E[\zeta_n^2 1_{\{\zeta_n^2 > n^{1-\rho}\}} n^{-(1-\rho)}] \leqq E[|\zeta_n|^{2+\delta}] n^{-(1-\rho)(1+\delta/2)} \leqq C n^{-(1+\delta/4)}.$$

Summing over n yields (1.7).

(b) Let $(\zeta_n)_{n \geqq 1}$ be a sequence of identically distributed random variables satisfying (1.7) for some $\rho \in (0, 1)$. Then for $\delta := 2\rho/(1-\rho)$

$$\begin{aligned} E[|\zeta_1|^{2+\delta}] &\leqq \sum_{n \geqq 0} P(\zeta_1^2 > n^{2/(2+\delta)}) \\ &\leqq 2 + \sum_{n \geqq 2} E[\zeta_1^2 1_{\{\zeta_1^2 > n^{2/(2+\delta)}\}} n^{-2/(2+\delta)}] \\ &\leqq 2 + E\left[\int_1^\infty \int_{\{x^2 > s^{1-\rho}\}} x^2 s^{-(1-\rho)} \mu(ds, dx)\right] < \infty. \end{aligned}$$

2. Results for exponential time scales

In this chapter, we denote a real-valued function on $\mathbb{R}^+$ by W and an increasing one by f with $f_0 = 0$. Further, we assume that f grows exponentially, i.e. there are constants $\beta \geqq \alpha > 0$, $C > 0$ and $t^* \geqq 0$ such that for all $t \geqq t^*$ the function $t \to \exp\{t^\alpha\} f(t)^{-1}$ is decreasing and the inequality $\exp\{t^\beta\} f(t)^{-1} \geqq C$ is satisfied.

Theorem 3. *Let $X=(X,\mathscr{F})$ be a real semimartingale with respect to $\mathscr{F}$, $(B, \langle X^c\rangle, \nu)$ its local characteristics and μ its random measure associated to the jumps. Suppose that constants $\delta, \vartheta \geqq 6\beta$, $\gamma > 6\beta$ and $\psi, \rho \geqq 16\beta$ exist such that*

$$(2.1) \qquad P\left(\int_1^t \int_{\{x^2 > f(s)s^{-\rho}\}} |x|\,\mu(ds, dx) \geqq f(t)^{\frac{1}{2}} t^{-\delta}\right) \ll t^{-\gamma},$$

$$(2.2) \qquad P(|V(X,f,\rho)_t - f(t)| \geqq f(t)\, t^{-\psi}) \ll t^{-\gamma}$$

and

$$(2.3) \qquad P\left(\sup_{s\leqq t}\left|B_s + \int_0^s \int_{\{1<x^2\leqq f(u)u^{-\rho}\}} x\,\nu(du, dx) - W_s\right| \geqq f(t)^{\frac{1}{2}} t^{-\vartheta}\right) \ll t^{-\gamma}.$$

Then the semimartingale X can be redefined on a richer probability space together with a continuous process $Z=(Z_t)_{t\geqq 0}$ with independent increments, drift W and variance function f such that with probability 1

$$X_t - Z_t \ll f(t)^{\frac{1}{2}} t^{-\lambda}$$

for some $\lambda > 0$.

In analogy to Chap. 1, corresponding propositions and remarks on Theorem 3 can be phrased. Furthermore, strong invariance principles for semimartingales with Brownian motion as reference process and logarithmic approximation rate can be derived from Theorem 3 by means of time changes.

3. Proof of Theorem 1

The proof follows a method developed by Berkes and Philipp [1] based on the approximation of dependent random variables by independent ones with prescribed distributions. We are going to approximate the semimartingale X at suitable chosen time points by a continuous process with independent increments and after that we include the intermediate points with appropriate maximal inequalities.

Let the process $X=(X_t)_{t\geqq 0}$ satisfy the assumptions made in Theorem 1. Without loss of generality we can assume that $\delta, \psi \in (0, \lambda_0)$, $\gamma, \vartheta \in (0, \frac{1}{2}\lambda_0)$ and $t^* = 1$.

We put $\alpha := 6{,}5(\min\{\delta, \vartheta, \gamma, 117\rho/278, 13\psi/34\})^{-1}$ and define for $k \in \mathbb{N}$

$$t_k := k^\alpha \quad (t_0 := 0), \qquad l_k := t_k - t_{k-1} \quad \text{and} \quad h_k := f(t_k) - f(t_{k-1}).$$

Lemma 3.1. *As $k \to \infty$ we have*

$$k^{-1} \ll h_k f(t_k)^{-1} \ll k^{-\frac{1}{2}}.$$

Proof of Lemma 3.1. Since the function $t \to f(t)t^{-\lambda}$ is increasing for all $\lambda \in (0, \lambda_0)$, the definition of t_k yields

$$f(t_{k-1}) f(t_k)^{-1} \leqq (t_k/t_{k-1})^{-\lambda} = [k/(k-1)]^{-\lambda\alpha} = (1-k^{-1})^{\lambda\alpha}$$

for all $k \geqq 2$. Therefore, we have

$$h_k f(t_k)^{-1} = 1 - f(t_{k-1}) f(t_k)^{-1} \geqq 1 - (1-k^{-1})^{\lambda\alpha} \gg k^{-1}.$$

Furthermore, if we set $\kappa := (2\alpha)^{-1}$ the function $t \to t^\kappa - \log f(t)$ is nondecreasing and we get $f(t_{k-1}) f(t_k)^{-1} \geqq \exp[t_{k-1}^\kappa - t_k^\kappa]$. Together with the definitions of t_k and κ we conclude

$$h_k f(t_k)^{-1} = 1 - f(t_{k-1}) f(t_k)^{-1} \leqq 1 - \exp[t_{k-1}^\kappa - t_k^\kappa] \leqq t_k^\kappa - t_{k-1}^\kappa \ll k^{-\frac{1}{2}}.$$

Denoting the continuous local martingale part of X by $X^c = (X^c, \mathscr{F})$, we get the following decomposition of X:

Lemma 3.2. *For all $t \in \mathbb{R}^+$, X_t admits the representation*

$$X_t = X_t^c + \int_0^t \int_{\{x^2 \leqq f(s) s^{-\rho}\}} x(\mu - \nu)(ds, dx)$$

$$+ B_t + \int_0^t \int_{\{1 < x^2 \leqq f(s) s^{-\rho}\}} x\, \nu(ds, dx)$$

$$+ \int_0^t \int_{\{x^2 > f(s) s^{-\rho}\}} x\, \mu(ds, dx) \quad a.s.$$

This decomposition is a slightly more general form of the well known canonical representation of the semimartingale X (see [18], Theorem II.2.34). Since the proof follows analogous lines, it is left to the reader.

In the following let $k \in \mathbb{N}$ and $t \in \mathbb{R}^+$ be arbitrary. We define the filtration $\mathscr{F}^k = (\mathscr{F}_t^k)_{t \geqq 0}$ by

(3.1) $$\mathscr{F}_t^k := \mathscr{F}_{(t_{k-1}+t) \wedge t_k}$$

and we set

(3.2) $$X_t^k := h_k^{-\frac{1}{2}} \{X_{(t_{k-1}+t) \wedge t_k} - X_{t_{k-1}}\}$$

and

(3.3) $$X_t^{k,c} := h_k^{-\frac{1}{2}} \{X^c_{(t_{k-1}+t) \wedge t_k} - X^c_{t_{k-1}}\}.$$

Therefore, the process $X^k = (X_t^k)_{t \geqq 0}$ is a semimartingale and $X^{k,c} = (X_t^{k,c})_{t \geqq 0}$ is a continuous local martingale with respect to $\mathscr{F}^k$. Further, by μ^k, respectively ν^k denote the random measure associated to the jumps of X^k, respectively the compensator of μ^k and define the function $g(k, \cdot)$ by

(3.4) $$g(k, s) := h_k^{-1} f((t_{k-1}+s) \wedge t_k) [(t_{k-1}+s) \wedge t_k]^{-\rho}$$

for $s \geqq 0$. Thus, because of Lemma 3.2, (3.2) and (3.3) we obtain the decomposition

(3.5) $$X_t^k = X_t^{k,c} + \int_0^t \int_{\{x^2 \leqq g(k,s)\}} x(\mu^k - \nu^k)(ds, dx)$$

$$+ h_k^{-\frac{1}{2}} \{B_{(t_{k-1}+t) \wedge t_k} - B_{t_{k-1}}\} + \int_0^t \int_{\{h_k^{-1} < x^2 \leqq g(k,s)\}} x\, \nu^k(ds, dx)$$

$$+ \int_0^t \int_{\{x^2 > g(k,s)\}} x\, \mu^k(ds, dx) \quad \text{a.s.}$$

If we write

$$B_t^k := h_k^{-\frac{1}{2}} \{B_{(t_{k-1}+t)\wedge t_k} - B_{t_{k-1}}\}, \tag{3.6}$$

$$V_t^k := B_t^k + \int_0^t \int_{\{h_k^{-1} < x^2 \leqq g(k,s)\}} x\, \nu^k(ds, dx) \tag{3.7}$$

and

$$M_t^k := X_t^{k,c} + \int_0^t \int_{\{x^2 \leqq g(k,s)\}} x(\mu^k - \nu^k)(ds, dx), \tag{3.8}$$

(3.5) yields the representation

$$X_t^k = M_t^k + V_t^k + \int_0^t \int_{\{x^2 > g(k,s)\}} x\, \mu^k(ds, dx) \quad \text{a.s.} \tag{3.9}$$

Lemma 3.3. *The process $M^k = (M_t^k)_{t \geqq 0}$ is a square-integrable local martingale with respect to $\mathscr{F}^k$ and its quadratic characteristic at time t is given by*

$$\begin{aligned} \langle M^k \rangle_t = \langle X^{k,c} \rangle_t &+ \int_0^t \int_{\{x^2 \leqq g(k,s)\}} x^2\, \nu^k(ds, dx) \\ &- \sum_{s \leqq t} \Big[\int_{\{x^2 \leqq g(k,s)\}} x\, \nu^k(\{s\}, dx)\Big]^2. \end{aligned} \tag{3.10}$$

Having in mind the strong orthogonality of the continuous and purely discontinuous local martingale part of M^k, the assertion of this lemma can easily be deduced by applying [18], Theorem II.1.33(a), to the function $U^k(t, x) := x\, 1_{\{x^2 \leqq g(k,t)\}}$ $(t \geqq 0,\ x \in \mathbb{R})$. Therefore, we omit the proof.

Let us define the $\mathscr{F}^k$-stopping time τ_k by

$$\tau_k := \inf\{t \geqq 0 \colon \langle M^k \rangle_t \geqq 1\} \wedge l_k.$$

The quadratic characteristic of the stopped local martingale $M^{k,\tau_k} := (M_t^{k,\tau_k})_{t \geqq 0} := (M_{t\wedge\tau_k}^k)_{t \geqq 0} \in \mathscr{M}_{\mathrm{loc}}^2(\mathscr{F}^k)$ at time $t \in \mathbb{R}^+$ is in view of (3.10) given by

$$\begin{aligned} \langle M^{k,\tau_k} \rangle_t = \langle X^{k,c} \rangle_{t\wedge\tau_k} &+ \int_0^{t\wedge\tau_k} \int_{\{x^2 \leqq g(k,s)\}} x^2\, \nu^k(ds, dx) \\ &- \sum_{s \leqq t\wedge\tau_k} \Big[\int_{\{x^2 \leqq g(k,s)\}} x\, \nu^k(\{s\}, dx)\Big]^2. \end{aligned} \tag{3.11}$$

We eventually apply [1], Theorem 1, to the sequence $(X_{l_k}^k - W_{l_k}^k)_{k \geqq 1}$ where the deterministic process $W^k = (W_t^k)_{t \geqq 0}$ is given by $W_t^k := h_k^{-\frac{1}{2}} \{W_{(t_{k-1}+t)\wedge t_k} - W_{t_{k-1}}\}$ $(t \geqq 0)$. If we put $T_k := k^{49/48}$, we need a suitable estimate for

$$\lambda_k := \sup_{|u| \leqq T_k} E[|E[\exp\{iu(X_{l_k}^k - W_{l_k}^k)\} - \exp\{-\tfrac{1}{2}u^2\} \,|\, \mathscr{F}_0^k]|]. \tag{3.12}$$

For this purpose let k be a fixed and sufficiently large integer.

First we consider the term

$$\lambda_k^{(1)} := \sup_{|u| \leq T_k} E[|E[\exp\{iu(X_{l_k}^k - W_{l_k}^k)\} - \exp\{iuM_{\tau_k}^k\} | \mathscr{F}_0^k]|]. \tag{3.13}$$

Defining $\varepsilon_k := k^{-5,25}$ and $F_k := \{|X_{l_k}^k - W_{l_k}^k - M_{\tau_k}^k| \leq 3\varepsilon_k\} \in \mathscr{F}_{l_k}^k$, we get

$$\begin{aligned} \lambda_k^{(1)} &\leq \sup_{|u| \leq T_k} E[|\exp\{iu(X_{l_k}^k - W_{l_k}^k)\} - \exp\{iuM_{\tau_k}^k\}|] \\ &= \sup_{|u| \leq T_k} [\int 1_{F_k^c} |\exp\{iu(X_{l_k}^k - W_{l_k}^k - M_{\tau_k}^k)\} - 1| \, dP \\ &\quad + \int 1_{F_k} |\exp\{iu(X_{l_k}^k - W_{l_k}^k - M_{\tau_k}^k)\} - 1| \, dP] \\ &\leq 2P(F_k^c) + T_k \int 1_{F_k} |X_{l_k}^k - W_{l_k}^k - M_{\tau_k}^k| \, dP \\ &\leq 2P(F_k^c) + 3\varepsilon_k T_k. \end{aligned} \tag{3.14}$$

To evaluate $P(F_k^c)$ we split $X_{l_k}^k$ according to (3.9). Hence follows

$$\begin{aligned} P(F_k^c) &\leq P(|V_{l_k}^k - W_{l_k}^k| > \varepsilon_k) + P(|M_{l_k}^k - M_{\tau_k}^k| > \varepsilon_k) \\ &\quad + P\left(\left|\int_0^{l_k} \int_{\{x^2 > g(k,s)\}} x\,\mu^k(ds, dx)\right| > \varepsilon_k\right). \end{aligned} \tag{3.15}$$

Now we estimate the summands of the right hand side separately.

From (3.4), (3.6) and (3.7) we obtain

$$\begin{aligned} &P(|V_{l_k}^k - W_{l_k}^k| > \varepsilon_k) \\ &= P\left(\left|B_{t_k} - B_{t_{k-1}} + \int_{t_{k-1}}^{t_k} \int_{\{1 < x^2 \leq f(s)s^{-\rho}\}} x\,\nu(ds, dx) - W_{t_k} + W_{t_{k-1}}\right| > h_k^{\frac{1}{2}} \varepsilon_k\right) \\ &\leq P\left(\left|B_{t_k} + \int_0^{t_k} \int_{\{1 < x^2 \leq f(s)s^{-\rho}\}} x\,\nu(ds, dx) - W_{t_k}\right| > \tfrac{1}{2} h_k^{\frac{1}{2}} \varepsilon_k\right) \\ &\quad + P\left(\left|B_{t_{k-1}} + \int_0^{t_{k-1}} \int_{\{1 < x^2 \leq f(s)s^{-\rho}\}} x\,\nu(ds, dx) - W_{t_{k-1}}\right| > \tfrac{1}{2} h_k^{\frac{1}{2}} \varepsilon_k\right). \end{aligned}$$

Since the definition of t_k and Lemma 3.1 together with $\alpha\vartheta \geq 6,5$ imply

$$\varepsilon_k h_k^{\frac{1}{2}} f(t_k)^{-\frac{1}{2}} t_k^{\vartheta} \gg k^{-5,25} k^{-\frac{1}{2}} k^{\alpha\vartheta} = k^{\alpha\vartheta - 5,75} \gg k^{3/4},$$

we have $\frac{1}{2}\varepsilon_k h_k^{\frac{1}{2}} \geq f(t_k)^{\frac{1}{2}} t_k^{-\vartheta} \geq f(t_{k-1})^{\frac{1}{2}} t_{k-1}^{-\vartheta}$. Consequently, from assumption (1.3) we get

$$P(|V_{l_k}^k - W_{l_k}^k| > \varepsilon_k) \ll t_{k-1}^{-\gamma}. \tag{3.16}$$

To estimate the second term we apply [16], Corollary I.3.21 to $S=\tau_k$, $T=l_k$, $\varepsilon=\varepsilon_k$ and $\eta=2\varepsilon_k^3$. We have

$$(3.17)\qquad \begin{aligned} &P(\sup_{\tau_k\leqq t\leqq l_k}|M_t^k-M_{\tau_k}^k|>\varepsilon_k)\\ &\leqq 2\varepsilon_k+P(\langle M^k\rangle_{l_k}-\langle M^k\rangle_{\tau_k}\geqq 2\varepsilon_k^3)\\ &\leqq 2\varepsilon_k+P(|\langle M^k\rangle_{l_k}-1|\geqq\varepsilon_k^3)+P(|\langle M^k\rangle_{\tau_k}-1|\geqq\varepsilon_k^3).\end{aligned}$$

Because of Lemma 3.3, Eq. (3.10), follows

$$\begin{aligned} &P(|\langle M^k\rangle_{l_k}-1|\geqq\varepsilon_k^3)\\ &=P\left(\left|\langle X^{k,c}\rangle_{l_k}+\int_0^{l_k}\int_{\{x^2\leqq g(k,s)\}} x^2\,\nu^k(ds,dx)\right.\right.\\ &\qquad\left.\left.-\sum_{s\leqq l_k}\Big[\int_{\{x^2\leqq g(k,s)\}} x\,\nu^k(\{s\},dx)\Big]^2-1\right|\geqq\varepsilon_k^3\right)\\ &=P(|V(X,f,\rho)_{t_k}-V(X,f,\rho)_{t_{k-1}}-h_k|\geqq h_k\,\varepsilon_k^3)\\ &\leqq P(|V(X,f,\rho)_{t_k}-f(t_k)|\geqq\tfrac12 h_k\,\varepsilon_k^3)\\ &\quad+P(|V(X,f,\rho)_{t_{k-1}}-f(t_{k-1})|\geqq\tfrac12 h_k\,\varepsilon_k^3).\end{aligned}$$

Since Lemma 3.1, $\alpha\psi\geqq 17$ and the definitions of t_k and ε_k imply

$$h_k\,\varepsilon_k^3\,f(t_k)^{-1}\,t_k^{\psi}\gg k^{-16,75}\,k^{\psi\alpha}\gg k^{\frac14}$$

and thus $\frac12 h_k\,\varepsilon_k^3\geqq f(t_k)\,t_k^{-\psi}\geqq f(t_{k-1})\,t_{k-1}^{-\psi}$, from the inequality above together with (1.2) we get

$$(3.18)\qquad \begin{aligned} &P(|\langle M^k\rangle_{l_k}-1|\geqq\varepsilon_k^3)\\ &\leqq P(|V(X,f,\rho)_{t_k}-f(t_k)|\geqq f(t_k)\,t_k^{-\psi})\\ &\quad+P(|V(X,f,\rho)_{t_{k-1}}-f(t_{k-1})|\geqq f(t_{k-1})\,t_{k-1}^{-\psi})\\ &\ll t_k^{-\gamma}+t_{k-1}^{-\gamma}\ll t_{k-1}^{-\gamma}.\end{aligned}$$

Furthermore, taking into account the definition of $\tau_k\leqq l_k$ we have

$$\begin{aligned} &P(|\langle M^k\rangle_{\tau_k}-1|\geqq\varepsilon_k^3)\\ &=P(\langle M^k\rangle_{\tau_k}-1\geqq\varepsilon_k^3,\ \tau_k<l_k)+P(|\langle M^k\rangle_{l_k}-1|\geqq\varepsilon_k^3,\ \tau_k=l_k)\\ &\leqq P(\langle M^k\rangle_{l_k}-1\geqq\varepsilon_k^3,\ \tau_k<l_k)+P(|\langle M^k\rangle_{l_k}-1|\geqq\varepsilon_k^3,\ \tau_k=l_k)\\ &\leqq P(|\langle M^k\rangle_{l_k}-1|\geqq\varepsilon_k^3).\end{aligned}$$

Because of (3.18) this means

$$(3.19)\qquad P(|\langle M^k\rangle_{\tau_k}-1|\geqq\varepsilon_k^3)\ll t_{k-1}^{-\gamma}.$$

Introducing (3.18) and (3.19) in (3.17) we conclude

$$(3.20)\qquad P(\sup_{\tau_k\leqq t\leqq l_k}|M_t^k-M_{\tau_k}^k|>\varepsilon_k)\ll\varepsilon_k+t_{k-1}^{-\gamma}.$$

Now we consider the third summand of (3.15). We have

$$P\left(\left|\int_0^{l_k}\int_{\{x^2>g(k,s)\}} x\,\mu^k(ds,dx)\right|>\varepsilon_k\right)\leqq P\left(\int_1^{t_k}\int_{\{x^2>f(s)s^{-\rho}\}} |x|\,\mu(ds,dx)>h_k^{\frac{1}{2}}\varepsilon_k\right).$$

Lemma 3.1 and the relation $\alpha\delta\geqq 6{,}5$ imply $h_k^{\frac{1}{2}}\varepsilon_k f(t_k)^{-\frac{1}{2}}t_k^{\delta}\gg k^{3/4}$, and, consequently, $f(t_k)^{\frac{1}{2}}t_k^{-\delta}\leqq h_k^{\frac{1}{2}}\varepsilon_k$. Together with assumption (1.1) we obtain

$$P\left(\left|\int_0^{l_k}\int_{\{x^2>g(k,s)\}} x\,\mu^k(ds,dx)\right|>\varepsilon_k\right)\ll t_k^{-\gamma}. \tag{3.21}$$

Finally, inserting (3.16), (3.20) and (3.21) into (3.15) for the probability of F_k^c follows

$$P(F_k^c)\ll t_{k-1}^{-\gamma}+\varepsilon_k. \tag{3.22}$$

Recalling (3.14), this means for $\lambda_k^{(1)}$

$$\lambda_k^{(1)}\ll t_{k-1}^{-\gamma}+\varepsilon_k T_k. \tag{3.23}$$

Next, we consider the term

$$\lambda_k^{(2)}(u):=E[|E[\exp\{iuM_{\tau_k}^k\}\,|\mathscr{F}_0^k]-\exp\{-\tfrac{1}{2}u^2\}|]$$

for $|u|\leqq T_k$, $u\in\mathbb{R}$. For this purpose, let $u\in\mathbb{R}$ with $|u|\leqq T_k$ be arbitrary, but fixed.

A main tool in our proof is:

Lemma 3.4. (a) *With probability* 1 *we have*

$$u^2\langle M^k\rangle_{\tau_k}\leqq[1+g(k,l_k)]\,T_k^2\leqq 2\,T_k^2<\infty \tag{3.24}$$

and

$$\sup_{t\geqq 0}|u|\,|\Delta M_t^{k,\tau_k}|\leqq 2\,T_k\,g(k,l_k)^{\frac{1}{2}}\leqq\tfrac{1}{4}. \tag{3.25}$$

(b) *Let η^k be the compensator of the random measure associated to the jumps of M^{k,τ_k}, then the process $A^k=(A_t^k)_{t\geqq 0}$, defined by*

$$A_t^k:=\tfrac{1}{2}u^2\langle X^{k,c}\rangle_{t\wedge\tau_k}+\int_0^t\int(e^{iux}-1-iux)\,\eta^k(ds,dx), \tag{3.26}$$

is predictable with respect to $\mathscr{F}^k$ and has paths of finite variation over each compact interval of $\mathbb{R}^+$.

Furthermore, the process $Z^k=(Z_t^k)_{t\geqq 0}$, given by $Z_0^k:=1$ and

$$Z_t^k:=\exp\{iuM_t^{k,\tau_k}\}\,\mathscr{E}_t^{-1}(A^k)\qquad(t>0), \tag{3.27}$$

represents a uniformly integrable $\mathscr{F}^k$-martingale with

$$E[Z_{l_k}^k\,|\mathscr{F}_0^k]=1\quad a.s. \tag{3.28}$$

Here the process $\mathscr{E}(A^k)=(\mathscr{E}_t(A^k))_{t\geqq 0}$ *is defined by* $\mathscr{E}_0(A^k):=1$ *and*

$$\mathscr{E}_t(A^k):=\exp\{A_t^k\cdot\prod_{s\leqq t}(1+\Delta A_s^k)\exp\{-\Delta A_s^k\}\qquad (t>0).$$

Proof of Lemma 3.4. (a) Because of (3.4) and Lemma 3.1 we know that $g(k,l_k)$ tends to zero as $k\uparrow\infty$. Therefore, (3.11) and the definition of τ_k imply

$$u^2\langle M^k\rangle_{\tau_k}\leqq T_k^2\langle M^k\rangle_{\tau_k}\leqq T_k^2[1+g(k,l_k)]\leqq 2T_k^2<\infty\quad \text{a.s.}$$

Moreover, using (3.8) and Lemma 3.1

$$\sup_{t\geqq 0}|u|\,|\Delta M_t^{k,\tau_k}|\leqq 2T_k\,g(k,l_k)^{\frac12}=2T_k h_k^{\frac12} f(t_k)^{\frac12} t_k^{-\rho/2}\ll k^{49/48}\,k^{\frac12-\rho\alpha/2}$$

$$\ll k^{-6}\leqq\tfrac14\quad \text{a.s.}$$

(b) Since the local characteristics of M^{k,τ_k} are given by $(0,\langle X^{k,c,\tau_k}\rangle,\eta^k)$, the predictability and the properties of the paths of A^k follow in the same way as in [18], Sec. II.§2d, 1., p. 85–86.

Now, let $t\geqq 0$ be arbitrary. Using the elementary estimate $|e^{iux}-1-iux|\leqq\frac12 u^2x^2$ we get

$$|\Delta A_t^k|\leqq\tfrac12 u^2\int x^2\,\eta^k(\{t\},dx)\leqq\tfrac12 u^2\Delta\langle M^{k,\tau_k}\rangle_t$$

and

$$\Delta\langle M^{k,\tau_k}\rangle_t\leqq\int_{\{x^2\leqq g(k,t)\}} x^2\,\nu^k(\{t\},dx)\leqq g(k,t)\tag{3.29}$$

from (3.11). Thus we obtain

$$|\Delta A_t^k|\leqq\tfrac12 T_k^2\,g(k,l_k)\ll k^{-12}\leqq\tfrac14\quad \text{a.s.}$$

Since we know from [18], Theorems I.4.61 and II.2.47(a), that $\mathscr{E}^{-1}(A^k)$ is well defined and Z^k is a local $\mathscr{F}^k$-martingale, it suffices to show for the remaining assertions of the lemma that the process $|Z^k|=|\mathscr{E}^{-1}(A^k)|$ is uniformly bounded by a deterministic constant. But this is shown in analogy to the proof of [21], Lemma 2, p. 675.

Applying Lemma 3.4 we get from (3.27) and (3.28), recalling the definition of $\mathscr{E}(A^k)$

$$\begin{aligned}\lambda_k^{(2)}(u)&=E[|E[\exp\{iuM_{\tau_k}^k\}|\mathscr{F}_0^k]-\exp\{-\tfrac12u^2\}\cdot E[Z_{l_k}^k|\mathscr{F}_0^k]|]\\&=E[|E[\exp\{iuM_{\tau_k}^k\}\cdot(1-\exp\{-\tfrac12u^2\}\cdot\mathscr{E}_{l_k}^{-1}(A^k))|\mathscr{F}_0^k]|]\\&\leqq E\left[\left|1-\exp\left\{-\tfrac12u^2+\tfrac12u^2\langle X^{k,c}\rangle_{\tau_k}-\int_0^{l_k}\!\!\int(e^{iux}-1-iux)\,\eta^k(ds,dx)\right\}\right.\right.\\&\qquad\left.\left.\cdot\prod_{s\leqq l_k}(1+\Delta A_s^k)^{-1}\exp(\Delta A_s^k)\right|\right].\end{aligned}$$

Since η^k is the compensator of the random measure associated to the jumps of M^{k,τ_k} we have

$$\langle M^{k,\tau_k}\rangle_t = \langle X^{k,c}\rangle_{t\wedge\tau_k} + \int_0^{l_k}\int x^2\,\eta^k(ds,dx)$$

for all $t\in\mathbb{R}^+$ and therefore

$$\lambda_k^{(2)}(u) \leqq E\left[\left|1-\exp\left\{\tfrac{1}{2}u^2(\langle M^{k,\tau_k}\rangle_{l_k}-1) - \int_0^{l_k}\int (e^{iux}-1-iux+\tfrac{1}{2}u^2x^2)\,\eta^k(ds,dx) + \sum_{s\leqq l_k}\{\Delta A_s^k - \log(1+\Delta A_s^k)\}\right\}\right|\right]. \tag{3.30}$$

In the following lemma we estimate the absolute values of the stochastic integral and the sum from the right hand side:

Lemma 3.5. *With probability* 1

$$\left|\int_0^{l_k}\int (e^{iux}-1-iux+\tfrac{1}{2}u^2x^2)\,\eta^k(ds,dx)\right| \leqq \tfrac{2}{3}T_k^3\,g(k,l_k)^{\frac{1}{2}}$$

and

$$\left|\sum_{s\leqq l_k}\{\Delta A_s^k - \log(1+\Delta A_s^k)\}\right| \leqq 2T_k^4\,g(k,l_k)$$

hold.

Proof of Lemma 3.5. Since the jumps of M^{k,τ_k} are bounded by $2g(k,l_k)^{\frac{1}{2}}$ (see proof of Lemma 3.4, (3.25)), we have $\eta^k(A\cap(\mathbb{R}^+\times\{x\in\mathbb{R}: |x|>2g(k,l_k)^{\frac{1}{2}}\}))=0$ for all $A\in\mathscr{B}^+\otimes\mathscr{B}$ and with (3.24)

$$\begin{aligned}
&\left|\int_0^{l_k}\int (e^{iux}-1-iux+\tfrac{1}{2}u^2x^2)\,\eta^k(ds,dx)\right| \\
&\leqq \int_0^{l_k}\int_{\{|x|\leqq 2g(k,l_k)^{\frac{1}{2}}\}} \tfrac{1}{6}|ux|^3\,\eta^k(ds,dx) \\
&\leqq \tfrac{1}{3}|u|^3 g(k,l_k)^{\frac{1}{2}} \int_0^{l_k}\int x^2\,\eta^k(ds,dx) \\
&\leqq \tfrac{1}{3}|u|\,g(k,l_k)^{\frac{1}{2}}\,u^2\langle M^{k,\tau_k}\rangle_{l_k} \\
&\leqq \tfrac{2}{3}T_k^3\,g(k,l_k)^{\frac{1}{2}} \quad \text{a.s.}
\end{aligned}$$

Observing that

$$|\Delta A_s^k| \leqq \tfrac{1}{2}u^2 \int_{\{|x|\leqq 2g(k,l_k)^{\frac{1}{2}}\}} x^2\,\eta^k(\{s\},dx) \leqq 2T_k^2\,g(k,l_k) \leqq \tfrac{1}{2}$$

for all $s \geqq 0$, with the estimate $|z-\log(1+z)| \leqq |z|^2 (|z| \leqq \frac{1}{2})$ and (3.24) we finally get

$$\begin{aligned} &|\sum_{s \leqq l_k} \{\Delta A_s^k - \log(1+\Delta A_s^k)\}| \\ &\quad \leqq \sum_{s \leqq l_k} |\Delta A_s^k|^2 \leqq 2 T_k^2\, g(k, l_k) \sum_{s \leqq l_k} |\Delta A_s^k| \\ &\quad \leqq T_k^2\, g(k, l_k) \sum_{s \leqq l_k} u^2 \int x^2\, \eta^k(\{s\}, dx) \\ &\quad \leqq T_k^2\, g(k, l_k)\, u^2 \langle M^{k,\tau_k} \rangle_{l_k} \leqq 2 T_k^4\, g(k, l_k) \quad \text{a.s.} \end{aligned}$$

Now we consider the difference $|\langle M^k \rangle_{\tau_k} - 1|$.

Since on the set $\{\langle M^k \rangle_{l_k} \geqq 1\}$ the inequality $\langle M^k \rangle_{\tau_k} \geqq 1$ holds, (3.29) and the definition of τ_k imply

$$|\langle M^k \rangle_{\tau_k} - 1|\, 1_{\{\langle M^k \rangle_{l_k} \geqq 1\}} \leqq (\Delta \langle M^k \rangle_{\tau_k})\, 1_{\{\langle M^k \rangle_{l_k} \geqq 1\}} \leqq g(k, l_k).$$

On the other hand the equality $\tau_k = l_k$ is valid on $\{\langle M^k \rangle_{l_k} < 1\}$; hence follows

$$\begin{aligned} &|\langle M^k \rangle_{\tau_k} - 1|\, 1_{\{\langle M^k \rangle_{l_k} < 1\}} \\ &\quad = (1 - \langle M^k \rangle_{l_k}) \{1_{\{0 \leqq \langle M^k \rangle_{l_k} < 1 - \varepsilon_k^3\}} + 1_{\{1 - \varepsilon_k^3 \leqq \langle M^k \rangle_{l_k} < 1\}}\} \\ &\quad \leqq 1_{\{0 \leqq \langle M^k \rangle_{l_k} < 1 - \varepsilon_k^3\}} + \varepsilon_k^3 \leqq 1_{\{|\langle M^k \rangle_{l_k} - 1| > \varepsilon_k^3\}} + \varepsilon_k^3. \end{aligned}$$

Putting these two inequalities together we obtain

$$E[|\langle M^k \rangle_{\tau_k} - 1|] \leqq \varepsilon_k^3 + P(|\langle M^k \rangle_{l_k} - 1| > \varepsilon_k^3) + g(k, l_k).$$

Because of (3.18) this means

$$E[|\langle M^k \rangle_{\tau_k} - 1|] \ll \varepsilon_k^3 + t_{k-1}^{-\gamma} + g(k, l_k). \tag{3.31}$$

Therefore, using (3.30), (3.31) and Lemma 3.5 we get

$$\begin{aligned} \lambda_k^{(2)}(u) \ll E\Bigg[& |\tfrac{1}{2} u^2 (\langle M^{k,\tau_k} \rangle_{l_k} - 1)| \\ & + \left| \int_0^{l_k} \int (e^{iux} - 1 - iux + \tfrac{1}{2} u^2 x^2)\, \eta^k(ds, dx) \right| \\ & + |\sum_{s \leqq l_k} \{\Delta A_s^k - \log(1 + \Delta A_s^k)\}| \Bigg] \\ \ll {} & T_k^2 \{\varepsilon_k^3 + t_{k-1}^{-\gamma} + g(k, l_k)\} + T_k^3\, g(k, l_k)^{\frac{1}{2}} + T_k^4\, g(k, l_k) \end{aligned}$$

for all $|u| \leqq T_k$. Now, (3.23) and the definitions of λ_k, $\lambda_k^{(1)}$ and $\lambda_k^{(2)}(u)$ yield

$$\lambda_k \ll \varepsilon_k T_k + T_k^2 \{\varepsilon_k^3 + t_{k-1}^{-\gamma} + g(k, l_k)\} + T_k^3\, g(k, l_k)^{\frac{1}{2}} + T_k^4\, g(k, l_k). \tag{3.32}$$

Since $t_k^{-\gamma} \leqq t_{k-1}^{-\gamma} \ll k^{-\alpha\gamma} \ll k^{-6,5}$ and, by Lemma 3.1 and (3.4)

$$g(k, l_k) = h_k^{-1} f(t_k)\, t_k^{-\rho} \ll k^{1-\alpha\rho} \ll k^{-1690/117} \ll k^{-14},$$

the estimate (3.32) results in

$$\lambda_k \ll T_k^3\, g(k, l_k)^{\frac{1}{2}} \ll k^{-599/144}.$$

By [1], Theorem 1, without loss of generality on this probability space a sequence of independent standard-normal distributed random variables $(Y_k)_{k\geq 1}$ exists such that

$$P(|X_{l_k}^k - W_{l_k}^k - Y_k| \geqq \alpha_k) \leqq \alpha_k,$$

where

$$\alpha_k \ll T_k^{-1} \log T_k + \lambda_k^{\frac{1}{4}}\, T_k + P(|N(0,1)| \geqq \tfrac{1}{4} T_k) \ll k^{-97/96}.$$

Thus the Borel-Cantelli lemma implies

$$|X_{l_k}^k - W_{l_k}^k - Y_k| \ll k^{-97/96} \quad \text{a.s.}$$

Taking [1], Lemma A1, into account we can assume that on this probability space there is a continuous process $Z = (Z_t)_{t\geqq 0}$ with independent increments, drift W and variance function f satisfying

$$Y_k + W_{l_k}^k = h_k^{-\frac{1}{2}}(Z_{t_k} - Z_{t_{k-1}})$$

for all $k \in \mathbb{N}$. Consequently, for $N \geqq 1$ we obtain

$$\begin{aligned} X_{t_N} - Z_{t_N} &= \sum_{k\leqq N} h_k^{\frac{1}{2}} (X_{l_k}^k - W_{l_k}^k - Y_k) \ll \sum_{k\leqq N} h_k^{\frac{1}{2}}\, k^{-97/96} \\ &= \sum_{k\leqq N} f(t_k)^{\frac{1}{2}} (h_k f(t_k)^{-1})^{\frac{1}{2}}\, k^{-97/96} \\ &\ll \sum_{k\leqq N} f(t_k)^{\frac{1}{2}}\, t_k^{-1/4\alpha}\, k^{-97/96} \quad \text{a.s.} \end{aligned}$$

by Lemma 3.1. Since $(4\alpha)^{-1} \leqq \frac{1}{2}\lambda_0$, the mapping $t \to f(t)^{\frac{1}{2}}\, t^{-1/(4\alpha)}$ is nondecreasing and therefore we conclude

(3.33) $$X_{t_N} - Z_{t_N} \ll f(t_N)^{\frac{1}{2}}\, t_N^{-1/4\alpha} \quad \text{a.s.}$$

Our proof is finished if we show that

(3.34) $$\sup_{0\leqq t\leqq l_k} h_k^{\frac{1}{2}}\, |X_t^k - Z_t^k| \ll f(t_k)^{\frac{1}{2}}\, t_k^{-\lambda}$$

with probability 1 holds for all $\lambda \in (0, (4\alpha)^{-1})$, where

$$Z_t^k := h_k^{-\frac{1}{2}} \{Z_{(t_{k-1}+t)\wedge t_k} - Z_{t_{k-1}}\}, \qquad t \geqq 0.$$

For this purpose let $k \in \mathbb{N}$ be sufficiently large and $\lambda \in (0, (4\alpha)^{-1})$ be arbitrary.

Defining $G_t^k := Z_t^k - W_t^k$, $t \geqq 0$, and splitting X^k according to (3.9) we get

$$(3.35)\qquad P(\sup_{0 \leqq t \leqq l_k} h_k^{\frac{1}{2}} |X_t^k - Z_t^k| \geqq 4 f(t_k)^{\frac{1}{2}} t_k^{-\lambda})$$

$$\leqq P(\sup_{0 \leqq t \leqq l_k} |V_t^k - W_t^k| \geqq h_k^{-\frac{1}{2}} f(t_k)^{\frac{1}{2}} t_k^{-\lambda})$$

$$+ P(\sup_{0 \leqq t \leqq l_k} |M_t^k| \geqq h_k^{-\frac{1}{2}} f(t_k)^{\frac{1}{2}} t_k^{-\lambda})$$

$$+ P(\sup_{0 \leqq t \leqq l_k} |G_t^k| \geqq h_k^{-\frac{1}{2}} f(t_k)^{\frac{1}{2}} t_k^{-\lambda})$$

$$+ P\left(\int_0^{l_k} \int_{\{x^2 > g(k,s)\}} |x| \, \mu^k(ds, dx) \geqq h_k^{-\frac{1}{2}} f(t_k)^{\frac{1}{2}} t_k^{-\lambda}\right).$$

In view of the Borel-Cantelli lemma it suffices to show for (3.34) that the expressions on the right hand side of (3.35) are at most of order k^{-2}. Now, we consider these terms separately. Since $0 < \lambda < (4\alpha)^{-1} \leqq \delta/26 \leqq \delta$ and

$$\int_0^{l_k} \int_{\{x^2 > g(k,s)\}} |x| \, \mu^k(ds, dx) \leqq h_k^{-\frac{1}{2}} \int_1^{t_k} \int_{\{x^2 > f(s) s^{-\rho}\}} |x| \, \mu(ds, dx) \quad \text{a.s.},$$

from assumption (1.1) we find

$$P\left(\int_0^{l_k} \int_{\{x^2 > g(k,s)\}} |x| \, \mu^k(ds, dx) \geqq h_k^{-\frac{1}{2}} f(t_k)^{\frac{1}{2}} t_k^{-\lambda}\right) \ll t_k^{-\gamma} \ll k^{-2}.$$

Further, since $0 < \lambda < (4\alpha)^{-1} \leqq \vartheta/26 \leqq \vartheta$ and therefore $\frac{1}{2} t_k^{-\lambda} \geqq t_{k-1}^{-\vartheta} \geqq t_k^{-\vartheta}$, we obtain

$$P(\sup_{0 \leqq t \leqq l_k} |V_t^k - W_t^k| \geqq h_k^{-\frac{1}{2}} f(t_k)^{\frac{1}{2}} t_k^{-\lambda})$$

$$\leqq P\left(\sup_{0 < t \leqq t_k} \left| B_t + \int_0^t \int_{\{1 < x^2 \leqq f(s) s^{-\rho}\}} x \, \nu(ds, dx) - W_t \right| \geqq f(t_k)^{\frac{1}{2}} t_k^{-\vartheta}\right)$$

$$+ P\left(\sup_{0 < t \leqq t_{k-1}} \left| B_t + \int_0^t \int_{\{1 < x^2 \leqq f(s) s^{-\rho}\}} x \, \nu(ds, dx) - W_t \right| \geqq f(t_{k-1})^{\frac{1}{2}} t_{k-1}^{-\vartheta}\right)$$

$$\ll t_{k-1}^{-\gamma} \ll k^{-2}$$

from definition (3.7) and assumption (1.3) using the triangle inequality. Moreover, the process $G^k = (G_t^k)_{t \geqq 0}$ is a continuous martingale satisfying $\langle G^k \rangle_{l_k} = 1$. By [29], Remark II.2.5, Ineq. (1), p. 26 (see also [23], Chap. 6, Example 1), we get

$$P(\sup_{0 \leqq t \leqq l_k} |G_t^k| \geqq h_k^{-\frac{1}{2}} f(t_k)^{\frac{1}{2}} t_k^{-\lambda})$$

$$\ll P(\langle G^k \rangle_{l_k} \geqq (h_k^{-\frac{1}{2}} f(t_k)^{\frac{1}{2}} t_k^{-\lambda})^2 (4 \log k)^{-1}) + 2 \exp(-2 \log k)$$

$$= 1_{\{1 \geqq h_k^{-1} f(t_k) t_k^{-2\lambda} (4 \log k)^{-1}\}} + 2 k^{-2}.$$

Hence follows

$$P(\sup_{0\leq t\leq l_k}|G_t^k|\geq h_k^{-\frac{1}{2}}f(t_k)^{\frac{1}{2}}t_k^{-\lambda})\ll k^{-2},$$

since by Lemma 3.1 $h_k^{-1}f(t_k)t_k^{-2\lambda}\geq(6\log k)^2>4\log k$ for large k.

These last considerations and the inequality

$$P(\sup_{0\leq t\leq l_k}|M_t^k|\geq h_k^{-\frac{1}{2}}f(t_k)^{\frac{1}{2}}t_k^{-\lambda})\leq P(\sup_{0\leq t\leq l_k}|M_t^k|\geq 6\log k)$$
$$=P(\sup_{0\leq t\leq l_k}M_t^k\geq 6\log k)+P(\sup_{0\leq t\leq l_k}(-M_t^k)\geq 6\log k)$$

show that it is sufficient to prove

$$P(\sup_{0\leq t\leq l_k}(uM_t^k)\geq 6\log k)\ll k^{-2} \tag{3.36}$$

for $u\in\{+1,-1\}$. For this let $u\in\{+1,-1\}$ and k be a sufficiently large integer. We have

$$P(\sup_{0\leq t\leq l_k}(uM_t^k)\geq 6\log k)$$
$$\leq P(\sup_{0\leq t\leq l_k}(uM_t^k-\tfrac{3}{4}\langle M^k\rangle_t)\geq 3\log k)+P(\langle M^k\rangle_{l_k}\geq 4\log k)$$
$$\leq P(\sup_{0\leq t\leq l_k}\exp(uM_t^k-\tfrac{3}{4}\langle M^k\rangle_t)\geq k^3)+P(|\langle M^k\rangle_{l_k}-1|\geq 4\log k-1).$$

By (3.18) we know that

$$P(|\langle M^k\rangle_{l_k}-1|\geq 4\log k-1)\leq P(|\langle M^k\rangle_{l_k}-1|\geq\varepsilon_k^3)\ll t_{k-1}^{-\gamma}\ll k^{-2}.$$

From the inequality above we obtain

$$P(\sup_{0\leq t\leq l_k}(uM_t^k)\geq 6\log k)\ll k^{-2}+P(\sup_{0\leq t\leq l_k}\exp(uM_t^k-\tfrac{3}{4}\langle M^k\rangle_t)\geq k^3). \tag{3.37}$$

In order to estimate the second summand we prove:

Lemma 3.6. *Let $N=(N_t)_{t\geq 0}$ be a square-integrable local martingale with respect to a right-continuous filtration $\mathcal{G}=(\mathcal{G}_t)_{t\geq 0}$ with $N_0=0$ and*

$$\sup_{t\geq 0}|\Delta N_t|\leq\varphi\leq\tfrac{1}{3}\log(3/2).$$

Denote the quadratic characteristic of N with respect to $\mathcal{G}$ by $\langle N\rangle$; then for all finite $\mathcal{G}$-stopping times S

$$E[\exp\{N_S-\tfrac{3}{4}\langle N\rangle_S\}]\leq 1$$

holds.

Proof of Lemma 3.6. By assumption,

$$(3\varphi)^{-2}[\exp\{3\varphi\}-1-3\varphi]=(3\varphi)^{-2}\sum_{n\geqq 2}(3\varphi)^n(n!)^{-1}\leqq\tfrac{1}{2}\exp\{3\varphi\}\leqq\tfrac{3}{4}.$$

Applying [29], Lemma I.1.4, we find that the process $L:=\exp\{N-(3\varphi)^{-2}[\exp\{3\varphi\}-1-3\varphi]\langle N\rangle\}$ is a positive supermartingale with respect to $\mathscr{G}$ and $L_0=1$. Note that the completeness of the filtration $\mathscr{G}$ which is assumed in the quoted lemma is not needed for that proof. Hence for all finite $\mathscr{G}$-stopping times S we conclude

$$E[\exp\{N_S-\tfrac{3}{4}\langle N\rangle_S\}]\leqq E[\exp\{N_S-(3\varphi)^{-2}[\exp\{3\varphi\}-1-3\varphi]\langle N\rangle_S\}]\leqq 1.$$

Now, we apply Lemma 3.6 to the local martingale $N:=(uM^k)\in\mathscr{M}^2_{\mathrm{loc}}(\mathscr{F}^k)$ and the $\mathscr{F}^k$-stopping time

$$S_k:=\inf\{t\leqq l_k\colon \exp(uM^k_t-\tfrac{3}{4}\langle M^k\rangle_t)>k^2\}\wedge(l_k+1)\quad(\inf\emptyset:=\infty).$$

From (3.8) and Lemma 3.1 we see that

$$\begin{aligned}\sup_{t\geqq 0}|\Delta(uM^k)_t| &= \sup_{t\geqq 0}\Big|\int_{\{x^2\leqq g(k,t)\}} x(\mu^k-\nu^k)(\{t\},dx)\Big|\\ &\leqq 2g(k,l_k)^{\frac{1}{2}}=2h_k^{-\frac{1}{2}}f(t_k)^{\frac{1}{2}}t_k^{-\rho/2}\leqq\tfrac{1}{3}\log(3/2).\end{aligned}$$

Since $\langle uM^k\rangle=\langle M^k\rangle$, Lemma 3.6 implies

$$E[\exp(uM^k_{S_k}-\tfrac{3}{4}\langle M^k\rangle_{S_k})]\leqq 1.$$

Furthermore, the jumps of the process $(uM^k-\frac{3}{4}\langle M^k\rangle)$ are bounded by $2g(k,l_k)^{\frac{1}{2}}+\frac{3}{4}g(k,l_k)\leqq 3g(k,l_k)^{\frac{1}{2}}\ll k^{-4}$ for large k and thus the set $\{S_k\leqq l_k\}$ contains $\{\sup_{0\leqq t\leqq l_k}\exp(uM^k_t-\frac{3}{4}\langle M^k\rangle_t)\geqq k^3\}$. Consequently

$$\begin{aligned}&P(\sup_{0\leqq t\leqq l_k}\exp(uM^k_t-\tfrac{3}{4}\langle M^k\rangle_t)\geqq k^3)\\ &\leqq E[1_{\{S_k\leqq l_k\}}]\\ &\leqq E[1_{\{S_k\leqq l_k\}}\exp(uM^k_{S_k}-\tfrac{3}{4}\langle M^k\rangle_{S_k})]\,k^{-2}\\ &\leqq k^{-2}\,E[\exp(uM^k_{S_k}-\tfrac{3}{4}\langle M^k\rangle_{S_k})]\\ &\leqq k^{-2}.\end{aligned}$$

Finally introducing this estimate in (3.37), we obtain the validity of (3.36). This concludes the proof of Theorem 1.

4. Proof of Theorem 3

In this proof we will reuse some parts of the derivation of Theorem 1; but here the assumptions (2.1)–(2.3) take the places of (1.1)–(1.3). For the sake of brevity we only state the essential changes.

Without loss of generality we can assume that $\gamma \leqq 73\beta/12$, $X_0=0$, $t^*=1$ and $\Delta X_s=0$ for all $s\in[0,1]$. There is a substantial distinction in the definition of t_k. We set

$$\chi := \tfrac{1}{2}(1+6\beta/\gamma) \in [145/146, 1)$$

and define

(4.1) $$t_k := f^{-1}(\exp\{k^\chi\}) \quad (t_0 := 0),$$

(4.2) $$l_k := t_k - t_{k-1}$$

and

(4.3) $$h_k := f(t_k) - f(t_{k-1}).$$

for $k \in \mathbb{N}$. Here f^{-1} denotes the inverse function of f.

Lemma 4.1. *For large k we have*

(4.4) $$\exp\{t_k^\alpha\} \ll f(t_k) = \exp\{k^\chi\} \ll \exp\{t_k^\beta\},$$

(4.5) $$k^{\chi/\beta} \ll t_k \ll k^{\chi/\alpha}$$

and

(4.6) $$\tfrac{1}{2}\chi k^{\chi-1} \leqq h_k f(t_k)^{-1} \ll k^{\chi-1}.$$

Proof of Lemma 4.1. The exponential behavior of f immediately yields (4.4). Taking the logarithm shows $t_k^\alpha \ll k^\chi$ and $k^\chi \ll t_k^\beta$; consequently, by raising to the power $1/\alpha$ respectively $1/\beta$ we get (4.5).

Now, let k be a sufficiently large integer. Then by (4.1)

$$h_k f(t_k)^{-1} = 1 - f(t_{k-1}) f(t_k)^{-1} = 1 - \exp\{(k-1)^\chi - k^\chi\}$$
$$\leqq k^\chi - (k-1)^\chi \ll k^{\chi-1} \to 0.$$

Further, since $(1-t)^\chi \leqq 1-\chi t$ for small $t \geqq 0$, $k^\chi - (k-1)^\chi = k^\chi[1-(1-k^{-1})^\chi] \geqq \chi k^{\chi-1}$ follow and thus

$$h_k f(t_k)^{-1} = 1 - \exp\{-[k^\chi - (k-1)^\chi]\} \geqq 1 - \exp\{-\chi k^{\chi-1}\}.$$

Using the well known estimate $1-e^{-t} \geqq t - 3t^2$, $t \geqq 0$, we conclude

$$h_k f(t_k)^{-1} \geqq \chi k^{\chi-1} - 3(\chi k^{\chi-1})^2 \geqq \tfrac{1}{2}\chi k^{\chi-1}.$$

In the following we use the notations made in the proof of Theorem 1 for the quantities not explicitly defined.

Let k be a sufficiently large integer. Putting

(4.7) $$T_k := k^{1+3(\gamma-6\beta)/32\beta} = k^{(1+3\gamma/8\beta)/4},$$

we can verbatim repeat the proof of Theorem 1 from Lemma 3.2 to the estimate (3.15) of $P(F_k^c)$. At this point we argue somehow different:

Since $\vartheta \geqq 6\beta$, from (4.5) and (4.6) we see

$$\varepsilon_k h_k^{\frac{1}{2}} f(t_k)^{-\frac{1}{2}} t_k^{\vartheta} \gg k^{-5{,}25} k^{\frac{1}{2}(\chi-1)} k^{\vartheta\chi/\beta} = k^{-5{,}75+\chi(\frac{1}{2}+\vartheta/\beta)} \gg k^{-5{,}75+6{,}5\chi} \uparrow \infty$$

and consequently $\frac{1}{2}\varepsilon_k h_k^{\frac{1}{2}} \geqq f(t_k)^{\frac{1}{2}} t_k^{-\vartheta} \geqq f(t_{k-1})^{\frac{1}{2}} t_{k-1}^{-\vartheta}$. Thus we obtain (3.16) just as in the proof of Theorem 1. The verification of (3.18) can be carried out in the same way as in that proof, since the needed inequality $\frac{1}{2} h_k \varepsilon_k^3 \geqq f(t_k) t_k^{-\psi} \geqq f(t_{k-1}) t_{k-1}^{-\psi}$ follows from the relation $\psi \geqq 16\beta$, Lemma 4.1 and

$$h_k \varepsilon_k^3 f(t_k)^{-1} t_k^{\psi} \gg k^{\chi-1} k^{-15,75} k^{\psi\chi/\beta} = k^{-16,75+\chi(1+\psi/\beta)} \uparrow \infty.$$

An analogous argument shows

$$\varepsilon_k h_k^{\frac{1}{2}} f(t_k)^{-\frac{1}{2}} t_k^{\delta} \gg k^{-5,25} k^{\frac{1}{2}(\chi-1)} k^{\delta\chi/\beta} = k^{-5,75+\chi(\frac{1}{2}+\delta/\beta)} \uparrow \infty$$

and $h_k^{\frac{1}{2}} \varepsilon_k \geqq f(t_k)^{\frac{1}{2}} t_k^{-\delta}$; therefore, the estimate (3.21) is deducible just as in the proof of Theorem 1.

Because in this setup (see (4.7) and Lemma 4.1) we have $\rho \geqq 16\beta$, $\gamma \leqq 73\beta/12$ and

$$\begin{aligned} 2T_k g(k, l_k)^{\frac{1}{2}} &= 2T_k h_k^{-\frac{1}{2}} f(t_k)^{\frac{1}{2}} t_k^{-\rho/2} \ll k^{\frac{1}{4}+3\gamma/32\beta} k^{(1-\chi)/2} k^{-\rho\chi/2\beta} \\ &= k^{3/4+3\gamma/32\beta-\chi(1+\rho/\beta)/2} \ll k^{-7,6+3\gamma/32\beta} \ll k^{-7} \leqq \tfrac{1}{4}, \end{aligned} \tag{4.8}$$

we can copy the proof of Theorem 1 from (3.22) until (3.32). Thus (see (3.32)) we get

$$\lambda_k \ll \varepsilon_k T_k + T_k^2 \{\varepsilon_k^3 + t_{k-1}^{-\gamma} + g(k, l_k)\} + T_k^3 g(k, l_k)^{\frac{1}{2}} + T_k^4 g(k, l_k).$$

The definitions of $g(k, l_k)$, t_k, ε_k and T_k and the relation $\rho \geqq 16\beta$ together with Lemma (4.1) yield

$$\begin{aligned} \lambda_k &\ll k^{-5+3\gamma/32\beta} + k^{-15,25+3\gamma/16\beta} \\ &\quad + k^{\frac{1}{2}+3\gamma/16\beta-\chi\gamma/\beta} + k^{3/2+3\gamma/16\beta-\chi(1+\rho/\beta)} \\ &\quad + k^{5/4+9\gamma/32\beta-\frac{1}{2}\chi(1+\rho/\beta)} + k^{2+3\gamma/8\beta-\chi(1+\rho/\beta)} \\ &\ll k^{-5+3\gamma/32\beta} + k^{-15,25+3\gamma/16\beta} \\ &\quad + k^{\frac{1}{2}-\gamma(\chi-3/16)/\beta} + k^{3/2+3\gamma/16\beta-17\chi} \\ &\quad + k^{5/4+9\gamma/32\beta-17\chi/2} + k^{2+3\gamma/8\beta-17\chi}. \end{aligned}$$

Since $6\beta < \gamma \leqq 73\beta/12$, $1 > \chi \geqq 145/146$ follows. Therefore, by elementary computations we obtain

$$\lambda_k \ll k^{\frac{1}{2}-\gamma(\chi-3/16)/\beta} = k^{-(2,5+5\gamma/16\beta)}.$$

Now, if we continue following the proof of Theorem 1 we conclude

$$\begin{aligned} \alpha_k &\ll T_k^{-1} \log T_k + \lambda_k^{\frac{1}{2}} T_k + P(|N(0,1)| \geqq \tfrac{1}{4} T_k) \\ &\ll k^{-\{1+3(\gamma-6\beta)/32\beta\}} \log k + k^{-(2,5+5\gamma/16\beta)/2} k^{1+3(\gamma-6\beta)/32\beta} \\ &\ll k^{-\{1+(\gamma-6\beta)/16\beta\}} + k^{-(13+\gamma/\beta)/16} \\ &\ll k^{-(5/8+\gamma/16\beta)} \end{aligned}$$

and finally from (4.5) and (4.6)

$$(4.9)\qquad \begin{aligned} X_{t_N}-Z_{t_N} &\ll \sum_{k\leqq N} h_k^{\frac{1}{2}} k^{-(5/8+\gamma/16\beta)} \\ &\ll \sum_{k\leqq N} f(t_k)^{\frac{1}{2}} k^{\frac{1}{2}(\chi-1)-5/8-\gamma/16\beta} \\ &= \sum_{k\leqq N} f(t_k)^{\frac{1}{2}} k^{3/8-\gamma/16\beta} k^{-[1+\frac{1}{2}(1-\chi)]} \\ &\ll f(t_N)^{\frac{1}{2}} t_N^{-\alpha(\gamma/16\beta-3/8)/\chi} \sum_{k\leqq N} k^{-[1+\frac{1}{2}(1-\chi)]} \\ &\ll f(t_N)^{\frac{1}{2}} t_N^{-\alpha(\gamma/16\beta-3/8)/\chi} \quad \text{a.s.} \end{aligned}$$

for large $N\in\mathbb{N}$. By our choice of χ we have $1-\chi\leqq\gamma/8\beta-3/4$ and thus $\frac{1}{2}(\chi^{-1}-1)\alpha\leqq\alpha(\gamma/16\beta-3/8)/\chi$. Therefore, it is sufficient to show that with probability 1

$$(4.10)\qquad \sup_{0\leqq t\leqq l_k} h_k^{\frac{1}{2}}|X_t^k-Z_t^k| \ll f(t_k)^{\frac{1}{2}} t_k^{-\lambda}$$

holds for $0<\lambda<\frac{1}{2}(\chi^{-1}-1)\alpha=\frac{1}{2}\frac{\gamma-6\beta}{\gamma+6\beta}\alpha$.

For this purpose, let $k\in\mathbb{N}$ be large enough and $\lambda\in(0,\frac{1}{2}(\chi^{-1}-1)\alpha)$.

From the relations $6\beta<\gamma\leqq 73\beta/12$ and $145/146\leqq\chi<1$ we get

$$\lambda<\frac{1}{2}\frac{\gamma-6\beta}{\gamma+6\beta}\alpha\leqq\alpha/288\leqq\alpha\leqq 6\beta\leqq\delta\wedge\vartheta$$

and

$$h_k^{-1} f(t_k) t_k^{-2\lambda}(\log k)^{-2} \gg k^{1-\chi(1+2\lambda/\alpha)}(\log k)^{-2}\uparrow\infty.$$

Especially, this yields $6\log k\leqq h_k^{-\frac{1}{2}} f(t_k)^{\frac{1}{2}} t_k^{-\lambda}$.

Furthermore, by Lemma 4.1 we obtain $t_k^{-\gamma}\leqq t_{k-1}^{-\gamma}\ll k^{-\gamma\chi/\beta}\ll k^{-2}$ and (see also (4.8))

$$g(k,t)^{\frac{1}{2}}=h_k^{-\frac{1}{2}} f(t_k)^{\frac{1}{2}} t_k^{-\rho/2}\ll k^{(1-\chi)/2-\rho\chi/2\beta}\ll k^{-4}\leqq\tfrac{1}{3}\log(3/2).$$

Now, using these estimates we conclude (4.10) in the same way as we deduced (3.34) in the proof of Theorem 1. So, Theorem 3 is completely proved.

5. Application to partial sum processes

In this section we apply our strong approximation theorem for polynomial time scales to the special case of discrete time partial sum processes with nonlinear growth of variance and a standard Brownian motion as reference process. The resulting invariance principle is a substantial improvement of the invariance principle for martingales of Morrow and Philipp [24].

Theorem 4. *Let $(\zeta_n)_{n\geqq 1}$ be a sequence of real random variables and $(\mathcal{L}_n)_{n\geqq 1}$ an increasing sequence of σ-fields such that ζ_n is $\mathcal{L}_n$-measurable and* $0<E[\zeta_n^2]<\infty$

for all $n \in \mathbb{N}$. *Denote the trivial* σ*-field by* $\mathscr{L}_0$ *and the conditional expectation with respect to* $\mathscr{L}_{i-1}$ *by* $E_i[\cdot]$ *and assume that*

$$V_n := \sum_{i=1}^{n} E_i[\zeta_i^2] \to \infty \quad a.s.$$

for $n \to \infty$. *Further, suppose* $\rho \in (0, 1)$ *exists such that*

$$\sum_{n \geq 1} E[\zeta_n^2 1_{\{\zeta_n^2 > V_n^{1-\rho}\}} V_n^{-(1-\rho)}] < \infty \tag{5.1}$$

and

$$\sum_{n \geq 1} E[|E_n[\zeta_n 1_{\{\zeta_n^2 \leq V_n^{1-\rho}\}}] V_n^{-(1-\rho)/2}|] < \infty. \tag{5.2}$$

Then without changing its joint distribution the sequence $(\zeta_n)_{n \geq 1}$ *can be redefined on a richer probability space on which a standard Brownian motion* $B = (B_t)_{t \geq 0}$ *exists such that with probability* 1

$$\sum_{n \geq 1} \zeta_n 1_{\{V_n \leq t\}} - B_t \ll t^{\frac{1}{2} - \lambda} \tag{5.3}$$

for $\lambda \in (0, \rho/104)$.

Remark. (a) If (5.1) holds, (5.2) is equivalent to the condition

$$\sum_{n \geq 1} E[|E_n[\zeta_n]| V_n^{-(1-\rho)/2}] < \infty. \tag{5.4}$$

(b) Since martingale difference sequences trivially satisfy (5.4), we obtain the strong invariance principle of Morrow and Philipp (see [24], Theorem 1) as a corollary of Theorem 4.

Proof of Remark (a). The stated equivalence follows immediately since by (5.1)

$$\begin{aligned} E[|E_n[\zeta_n 1_{\{\zeta_n^2 > V_n^{1-\rho}\}}]| V_n^{-(1-\rho)/2}] &\leq E[|\zeta_n| V_n^{-(1-\rho)/2} 1_{\{\zeta_n^2 > V_n^{1-\rho}\}}] \\ &\leq E[\zeta_n^2 V_n^{-(1-\rho)} 1_{\{\zeta_n^2 > V_n^{1-\rho}\}}] \end{aligned}$$

holds for all $n \in \mathbb{N}$.

Proof of Theorem 4. Let $t \in \mathbb{R}^+$ be arbitrary but fixed and define

$$\sigma_t := \inf\{n \geq 0 \colon V_{n+1} > t\}.$$

Then, σ_t is a stopping time with respect to $\mathscr{L} = (\mathscr{L}_n)_{n \geq 0}$ and for $k \geq 1$ we have

$$V_k = \inf\{s \geq 0 \colon \sigma_s = k\}.$$

Putting $\mathscr{F}_t := \mathscr{L}_{\sigma_t}$ and $X_t := \sum_{n \leq \sigma_t} \zeta_n$, we get $X_t = \sum_{n \geq 1} \zeta_n 1_{\{V_n \leq t\}}$.

From [16], Proposition III.1.45, we know that $\mathscr{F} = (\mathscr{F}_t)_{t \geq 0}$ is a right-continuous filtration and the continuous time process $X = (X_t)_{t \geq 0}$ is a semimartingale adapted to $\mathscr{F}$. Moreover, its local characteristics $(B, \langle X^c\rangle, \nu)$ and its random measure associated to the jumps μ are given by

$$B_t = \sum_{n\leqq\sigma_t} E_n[\zeta_n 1_{\{|\zeta_n|\leqq 1\}}], \tag{5.5}$$

$$\langle X^c\rangle = 0, \tag{5.6}$$

$$\nu((0,t]\times A) = \sum_{n\leqq\sigma_t} P(0 \neq \zeta_n \in A \mid \mathscr{L}_{n-1}) \tag{5.7}$$

and

$$\mu((0,t]\times A) = \sum_{n\leqq\sigma_t} 1_{\{0\neq\zeta_n\in A\}} \tag{5.8}$$

for $t\in\mathbb{R}^+$ and $A\in\mathscr{B}$.

In the following we verify the assumptions (1.1)–(1.3) of Theorem 1 with $f(t):=t$. For this purpose let $t>0$ be sufficiently large and set $D := \sum_{n\geqq 1} E[\zeta_n^2 1_{\{\zeta_n^2 > V_n^{1-\rho}\}} V_n^{-(1-\rho)}]$.

(i) Since X has jumps at the times $\{V_k: k\in\mathbb{N}\}$ only and $\Delta X_{V_k}=\zeta_k$, from (5.1) we get

$$E\left[\int_1^\infty \int_{\{x^2>s^{1-\rho}\}} x^2 s^{-(1-\rho)} \mu(ds,dx)\right] = E[\sum_{n\geqq 1} \zeta_n^2 1_{\{\zeta_n^2>V_n^{1-\rho}\}} V_n^{-(1-\rho)}] = D < \infty,$$

i.e. (1.7) is fulfilled. Just as in the proof of Proposition 1.2(i) we see that (1.1) holds for $\delta = \frac{\rho}{4} = \gamma$.

(ii) Define for $t\in\mathbb{R}^+$ $B_t' := B_t + \int_0^t \int_{\{x^2>1\}} x\,\nu(ds,dx)$.

Using (5.5), (5.7) and the Markov inequality, we obtain

$$\begin{aligned} P(\sup_{s\leqq t} |B_s'| \geqq t^{\frac12-\rho/4}) &\leqq t^{-(\frac12-\rho/4)} E[\sum_{n\leqq\sigma_t} |E_n[\zeta_n]|] \\ &\leqq t^{-\rho/4} E[\sum_{n\geqq 1} |E_n[\zeta_n]|\, V_n^{-(1-\rho)/2}] \\ &\ll t^{-\rho/4} \end{aligned}$$

from (5.4), since $V_n \leqq t$ holds on the set $\{n\leqq\sigma_t\}$. So, conditions (1.9) and especially (1.3) are satisfied for $\gamma = \vartheta = \rho/4$.

(iii) Because of (5.1) and (5.2) we have

$$\begin{aligned} &P(\sum_{s\leqq t} (\Delta B_s')^2 > \tfrac12 t^{1-7\rho/10}) \\ &\leqq P(\sum_{n\leqq\sigma_t} V_n^{-(1-\rho)} \{|E_n[\zeta_n 1_{\{\zeta_n^2\leqq V_n^{1-\rho}\}}]|^2 \\ &\quad + E_n[\zeta_n^2 1_{\{\zeta_n^2>V_n^{1-\rho}\}}]\} \geqq \tfrac12 t^{-3\rho/10}) \\ &\leqq 2t^{-3\rho/10} \sum_{n\geqq 1} \{E[|E_n[\zeta_n 1_{\{\zeta_n^2\leqq V_n^{1-\rho}\}}]|\, V_n^{-(1-\rho)/2}] \\ &\quad + E[\zeta_n^2 1_{\{\zeta_n^2>V_n^{1-\rho}\}} V_n^{-(1-\rho)}]\} \\ &\ll t^{-3\rho/10}. \end{aligned} \tag{5.9}$$

Moreover, by (5.7) the definitions of V_k and σ_t yield

$$P\left(\left|\int_0^t\int x^2\,\nu(ds,dx)-t\right|\geqq\tfrac{1}{2}t^{1-7\rho/10}\right)=P(t-V_{\sigma_t}\geqq\tfrac{1}{2}t^{1-7\rho/10}). \tag{5.10}$$

To evaluate this last probability we need:

Lemma 5.1. *We have*

$$P\Big(\sum_{n\geqq 0}E_{n+1}[\zeta_{n+1}^2]\,1_{\{\sigma_t=n\}}\geqq V_{\sigma_t+1}^{1-\rho}t^{\rho/4}\Big)\leqq(1+D)\,t^{-\rho/4}.$$

Proof of Lemma 5.1. Since with probability 1

$$\begin{aligned}
&\sum_{n\geqq 0}E_{n+1}[\zeta_{n+1}^2]\,1_{\{\sigma_t=n\}}\\
&\quad\leqq\sum_{n\geqq 0}1_{\{\sigma_t=n\}}\{V_{n+1}^{1-\rho}+E_{n+1}[\zeta_{n+1}^2\,1_{\{\zeta_{n+1}^2>V_{n+1}^{1-\rho}\}}V_{n+1}^{-(1-\rho)}]\,V_{n+1}^{1-\rho}\}\\
&\quad=V_{\sigma_t+1}^{1-\rho}\sum_{n\geqq 0}1_{\{\sigma_t=n\}}\{1+E_{n+1}[\zeta_{n+1}^2\,V_{n+1}^{-(1-\rho)}\,1_{\{\zeta_{n+1}^2>V_{n+1}^{1-\rho}\}}]\}\\
&\quad\leqq V_{\sigma_t+1}^{1-\rho}\{1+\sum_{n\geqq 1}E_n[\zeta_n^2\,V_n^{-(1-\rho)}\,1_{\{\zeta_n^2>V_n^{1-\rho}\}}]\},
\end{aligned}$$

we conclude

$$\begin{aligned}
&P\Big(\sum_{n\geqq 0}E_{n+1}[\zeta_{n+1}^2]\,1_{\{\sigma_t=n\}}\geqq V_{\sigma_t+1}^{1-\rho}t^{\rho/4}\Big)\\
&\quad\leqq P\Big(1+\sum_{n\geqq 1}E_n[\zeta_n^2\,V_n^{-(1-\rho)}\,1_{\{\zeta_n^2>V_n^{1-\rho}\}}]\geqq t^{\rho/4}\Big)\\
&\quad\leqq(1+D)\,t^{-\rho/4}.
\end{aligned}$$

with the Markov inequality. Thus the lemma is proved.

Applying Lemma 5.1, outside a set with probability $(1+D)\,t^{-\rho/4}$ we obtain

$$V_{\sigma_t}=V_{\sigma_t+1}-\sum_{n\geqq 0}E_{n+1}[\zeta_{n+1}^2]\,1_{\{\sigma_t=n\}}\geqq V_{\sigma_t+1}(1-t^{\rho/4}V_{\sigma_t+1}^{-\rho}).$$

Since $V_{\sigma_t}\leqq t<V_{\sigma_t+1}$, aside this small set we have

$$t\geqq V_{\sigma_t}\geqq V_{\sigma_t+1}(1-t^{\rho/4}V_{\sigma_t+1}^{-\rho})>t(1-t^{-3\rho/4})$$

and therefore $t-V_{\sigma_t}<t^{1-3\rho/4}$. Hence, from (5.10) follows

$$P\left(\left|\int_0^t\int x^2\,\nu(ds,dx)-t\right|\geqq\tfrac{1}{2}t^{1-7\rho/10}\right)\leqq(1+D)\,t^{-\rho/4}$$

for large t. Together with (5.6) and (5.9) this implies

$$P(|V(X)_t - t| \geqq t^{1-7\rho/10})$$
$$= P\left(\left|\int_0^t \int x^2 \, \nu(ds, dx) - \sum_{s \leqq t} (\Delta B'_s)^2 - t\right| \geqq t^{1-7\rho/10}\right)$$
$$\ll t^{-\rho/4}.$$

Thus conditions (1.8) and in particular (1.2) are met with $\psi = 7\rho/10$ and $\gamma = \rho/4$.

Now, Theorem 1 implies that there is a standard Brownian motion $B = (B_t)_{t \geq 0}$ without loss of generality on the given probability space such that

$$X_t - B_t \ll t^{\frac{1}{2}-\lambda} \quad \text{a.s.}$$

for $\lambda \in (0, \rho/104)$.

Recalling the definition of X we conclude that

$$\sum_{n \geqq 1} \zeta_n 1_{\{V_n \leqq t\}} - B_t \ll t^{\frac{1}{2}-\lambda} \quad \text{a.s.}$$

for $\lambda \in (0, \rho/104)$. Thus Theorem 4 is completely proved.

6. Strong invariance principles for Markov processes

In this chapter an almost sure invariance principle for integrated functions of Markov processes is deduced from Theorem 2. The assumptions are phrased in terms of the infinitesimal conditions. Therefore, we shortly introduce the basic terms in the following. For this, mainly the notations of [2, 3] and [9] are used.

Let $(\Omega, \mathfrak{A})$ be a measurable space, $\mathscr{F} = (\mathscr{F}_t)_{t \geq 0}$ a right-continuous increasing family of sub-σ-fields of $\mathfrak{A}$, S a Polish space and $\mathscr{B} = \mathscr{B}(S)$ the Borel-σ-field on S.

Moreover, let $X = (\Omega, \mathfrak{A}, \mathscr{F}, (P_x)_{x \in S}, (X_t)_{t \geqq 0})$ be a progressively measurable Markov process with state space $(S, \mathscr{B})$ and stationary transition probability function $p = \{p(t, x, B) \colon t \in \mathbb{R}^+, x \in S, B \in \mathscr{B}\}$ having an invariant distribution m on $(S, \mathscr{B})$, i.e. $m(B) = \int p(t, x, B)\, m(dx)$ for all $B \in \mathscr{B}$.

Define the distribution P_m on $(\Omega, \mathfrak{A})$ by $P_m(A) := \int P_x(A)\, m(dx)\, (A \in \mathfrak{A})$ and denote the expectation with respect to $P_x(P_m)$ by $E_x[\cdot]\,(E_m[\cdot])$.

In view of the Chapman-Kolmogorov relation the transition operators $(T_t)_{t \geqq 0}$, defined by

$$(T_t f)(x) := \int f(y)\, p(t, x, dy), \qquad t \geqq 0,\ x \in S,$$

form a semigroup of positive contractions on $L^2(S, m)$. Let $B_0 := \{f \in L^2(S, m) \colon \|T_t f - f\|_2 \to 0 \text{ as } t \downarrow 0\}$ be the center of this semigroup. The *infinitesimal operator* A *of* $(T_t)_{t \geqq 0}$ is defined on the domain $D(A) := \{f \in B_0 \colon \|t^{-1}(T_t f - f) - g\|_2 \to 0 \text{ for some } g \in B_0, \text{ as } t \downarrow 0\}$ by

$$Af := \lim_{t \downarrow 0} t^{-1}(T_t f - f).$$

Denoting the range of A by $R(A)$, i.e. $R(A) := \{Ag \colon g \in D(A)\}$, we have the following almost sure invariance principle for Markov processes:

Theorem 5. *Let $X=(\Omega, \mathfrak{A}, \mathscr{F}, (P_x)_{x\in S}, (X_t)_{t\geqq 0})$ be a progressively measurable Markov process with state space $(S, \mathscr{B})$, stationary transition probability function p having invariant distribution m and infinitesimal operator A, and $f\in R(A)$.*

Suppose there is an m-almost surely bounded $g\in D(A)$ such that $f=Ag$,

$$g^2\in D(A) \quad and \quad \sigma^2 := -2\int f(x)\, g(x)\, m(dx) > 0.$$

Further, assume that

$$P_m\left(\left|\frac{1}{t}\int_0^t [A g^2 - 2 f g](X_s)\, ds - \sigma^2\right| \geqq t^{-\psi}\right) \ll t^{-\gamma} \tag{6.1}$$

for some $\gamma, \psi>0$, then the process $X=(X_t)_{t\geqq 0}$ can be redefined on a richer probability space together with a Brownian motion $B=(B_t)_{t\geqq 0}$ with variance σ^2 such that

$$\int_0^t f(X_s)\, ds - B_t \ll t^{\frac{1}{2}-\lambda} \quad P_m\text{-a.s.}$$

for some $\lambda>0$.

Remark. If the Markov process X satisfies the assumptions made in Theorem 5 with the exception of (6.1) and, in addition, if it is ergodic, the ergodic theorem yields the convergence

$$\frac{1}{t}\int_0^t [A g^2 - 2 f g](X_s)\, ds \to \sigma^2$$

P_m-almost sure and in $L^1(\Omega, \mathfrak{A}, P_m)$ as $t\to\infty$ (see, e.g., proof of [18], Theorem VIII.3.65).

Condition (6.1) guarantees the necessary speed of convergence for the considered approximation rate.

Proof of Theorem 5. Define the process $Y=(Y_t)_{t\geqq 0}$ by $Y_t := \int_0^t f(X_s)\, ds$, $t\geqq 0$.

Thus, Y is a continuous process on $(\Omega, \mathfrak{A}, P_m)$ with $Y\in\mathscr{A}_{\text{loc}}(P_m, \mathscr{F})$.

Now, Y is linked with a martingale suitable for our approximation:

Lemma 6.1. *There is an $M\in\mathscr{M}^2(P_m, \mathscr{F})$ such that*

$$M_t = Y_t - g(X_t) + g(X_0) \tag{6.2}$$

and

$$\langle M\rangle_t = \int_0^t [A g^2 - 2 f g](X_s)\, ds \tag{6.3}$$

P_m-almost sure for all $t\in\mathbb{R}^+$.

The proof of (6.2) is a simple application of Fubini's theorem and Dynkin's formula (cf. [9], Chap. I, §2, 1.3.C, Eq. (1.5)). Since the verification of (6.3) can be carried out in the same way as in the proof of [18], Lemma VIII.3.68, part (b), we omit this proof.

In order to continue the proof of Theorem 5, we apply Theorem 2 to the martingale $M=(M_t)_{t\geqq 0}\in\mathscr{M}^2(\mathscr{F}, P_m)$ given by (6.2) and the functions $W_t=0$ and

$f(t)=\sigma^2 t$, $t \geqq 0$. Therefore, we check the assumptions of the theorem in this case: Since by (6.2)

$$|M_t - Y_t| = |g(X_t) - g(X_0)| \leqq 2C \quad P_m\text{-a.s.,} \tag{6.4}$$

the jumps of M are P_m-almost surely bounded by a deterministic constant. Thus, condition (1.7) is fulfilled. Furthermore, by (6.1) and (6.3)

$$P_m(|\langle M\rangle_t - \sigma^2 t| \geqq t^{1-\psi}) = P_m\left(\left|\frac{1}{t}\int_0^t [A g^2 - 2 f g](X_s)\, ds - \sigma^2\right| \geqq t^{-\psi}\right)$$
$$\ll t^{-\gamma}.$$

Hence, condition (1.12) is satisfied.

By Theorem 2 a Brownian motion $B=(B_t)_{t\geqq 0}$ with variance σ^2 exists on the given probability space without loss of generality such that

$$M_t - B_t \ll t^{\frac{1}{2}-\lambda} \quad P_m\text{-a.s.}$$

for some $\lambda > 0$.

Because the process $M - Y$ has right-continuous paths having limits from the left from (6.4)

$$Y_t - B_t \ll t^{\frac{1}{2}-\lambda} \quad P_m\text{-a.s.}$$

follows for some $\lambda > 0$. This concludes the proof of Theorem 5.

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