Probab. Th. Rel. Fields 87, 469–487 (1991)

Probability Theory and Related Fields

The lifetime of conditioned Brownian motion in planar domains of infinite area

Jianming Xu

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA

Received January 3, 1990; in revised form July 5, 1990

Summary. In this paper, it is shown that all expected lifetimes of h-processes in D are finite if and only if the area of D is finite if $D=\{(x, y)\colon \phi_-(x)<y<\phi_+(x), -\infty<x<\infty\}$, where $\phi_-<\phi_+$ are two Lipschitz functions. We show that if Ω is a bounded convex region in the plane, there is an h-process in Ω with expected lifetime at least c area(Ω). We also give an example of a planar domain D of infinite area such that the expected lifetime of each h-process in D is finite.

1. Introduction

This paper studies the lifetime of conditioned Brownian motion in simply connected planar domains, often domains of infinite area. If D is a domain in R^2 and Z_t, $t\geqq 0$ is a stochastic process, we define $\tau_D=\inf\{t>0\colon Z_t\notin D\}$. If g is a positive harmonic function in D, we use P_x^g, E_x^g, to denote probability and expectation for the h-process associated with g, started at x. This process will be discussed in more detail later. For now, we observe that for $g=K(\cdot, w)$, $w\in\partial D$, the Martin kernel function, the process is a standard two dimensional Brownian motion started at x and conditioned to exit D at w. We denote by P_x^w the above h-process and also by $P_x^w(Q)$ the probability that the h-process hits Q before leaving domain D for any subset Q of D. The lifetimes of h-processes have been studied by many authors. In 1983, Cranston and McConnell [3] proved that there is a universal constant $c>0$ such that for any domain D in R^2,

$$\sup E_x^h\tau_D\leqq c\,|D|, \tag{1.1}$$

where $x\in D$, $|D|$ is the area of D, and the sup is taken over all positive harmonic functions in D. From now on, we will use sup to denote the supremum over all variables in the expression. We also always assume h is a general positive harmonic function in the related domain. Other related results on the lifetime of h-process may be found in [1], [2] and [4]. These papers show that sup $E_x^h\tau_D$

$<\infty$ for domains in various classes. In this paper, we investigate whether some kind of converse inequality to (1.1) holds. For a class of domains with some regularity, as defined below, it does. We prove

Theorem 1. *Let $\Omega \subset R^2$ be the domain $\{(u, v): \phi_-(u) < v < \phi_+(u), u \in R\}$, where the functions $\phi_-(u)$ and $\phi_+(u)$ from R to R satisfy*

(1) $\phi_-(u) < \phi_+(u), \quad -\infty < u < \infty,$

(2) *there exists $M > 0$ such that*

$$|\phi_-(u_1) - \phi_-(u_2)| \leqq M|u_1 - u_2|$$

and

$$|\phi_+(u_1) - \phi_+(u_2)| \leqq M|u_1 - u_2|$$

for all $u_1, u_2 \in R$.

Then, there is $c > 0$, which depends only on M, a positive harmonic function h in Ω such that

$$\sup_{x \in \Omega} E_x^h \tau_\Omega \geqq c|\Omega|.$$

In the case that $|\Omega|$ is infinite, Theorem 1 is to be interpreted as saying $\sup_{x \in \Omega} E_x^h \tau_\Omega = \infty$.

In Theorem 1, the constant c depends on Lipschitz constant of the domain. For convex domains, we have a universal constant for the converse inequality to (1.1). In fact, we have

Theorem 2. *Let Ω be a convex planar domain. Then, there is a universal constant $c > 0$, a positive harmonic function h in Ω such that*

$$\sup_{x \in \Omega} E_x^h \tau_\Omega \geqq c|\Omega|.$$

In general, this kind of converse to (1.1) does not hold.

Theorem 3. *There is a simply connected domain Ω of infinite area such that*

$$\sup E_x^h \tau_\Omega < \infty.$$

We now briefly sketch the ideas involved in the proofs of Theorems 1 and 2. A square Q in D is called a Whitney square if Q satisfies that $\operatorname{diam}(Q) \leqq \operatorname{dist}(Q, \partial D) \leqq 4\, diam(Q)$, where ∂D is the ordinary Euclidean boundary of D. Any domain is a union of a countable number of Whitney squares with disjoint interiors (see [11]). In Davis [5], the local occupation time of h-process in a Whitney square has been studied. He proved the following theorem:

Theorem A. *Let D be a simply connected domain in R^2. Then, there are two universal constants c, C such that for any Whitney square Q in D, $x \in D$, $y \in \partial D$,*

$$(*) \qquad c P_x^y(Q)|Q| \leqq E_x^y T_Q \leqq C P_x^y(Q)|Q|.$$

In the proof of Theorem 2 we pick x and y at "the opposite ends" of Ω. We find, with the aid of a theorem we prove in Sect. 4 to the effect that Whitney squares are almost preserved under conformal mapping, a collection

$Q_1, Q_2, \ldots, Q_n$ of Whitney squares in the middle of Ω such that $P_x^y(Q_i) > p$ and $\Sigma|Q_i| > c|\Omega|$, where p and c are absolute positive constants. Theorem 2 now follows immediately from the left side of (*). The proof of Theorem 1 is similar.

2. Notation and preliminaries

All notation in Sect. 1 is retained. We usually identify R^2 with the complex plane $\mathbb{C}$, and denote by R_+^2 the upper half plane. We will use $c, c_0, c_1, c_2, \ldots,$ for absolute positive constants which may be different from one line to another and even in the same line, and $c(\mu)$ means that the constant depends on μ. We denote by M_δ the δ-neighborhood of domain M, that is, $M_\delta = \{y \in R^2$: dist $(y, x) < \delta$ for some $x \in M\}$, where $\operatorname{dist}(x, y)$ is the Euclidean distance between x and y. For any $x \in R^2$, we denote by $\delta(x, D)$ the Euclidean distance from x to the Euclidean boundary of D, usually abbreviate it by $\delta(x)$. Let W be the underlying sample space of all functions on $[0, \infty)$ which take value in $\mathbb{C} \cup \Delta$ and are right continuous and have left limits in Martin topology, where Δ is a trap state; the distance of Δ to each point in $\mathbb{C}$ is defined to be 1 and Δ is an absorbing state to which a killed process is sent upon its death (see [6]). Let $Z_t(\omega) = \omega_t$, $\forall \omega \in W$. If $Z_t(\omega) \neq \Delta$, let $X_t(\omega) = \operatorname{Re}(Z_1(\omega))$ and $Y_t(\omega) = \operatorname{Im}(Z_t(\omega))$. Let P_z or P_μ be the probability on W which makes Z_t a standard Brownian motion respectively starting at z or with initial measure μ on $\mathbb{C}$. Let D be a simply connected domain in R^2, and let Z_t, $t \geqq 0$ be a two dimensional Brownian motion. If $P_t^D(x, y)$ is the transition density of Z_t killed when it leaves D and h is a positive harmonic function in D, then the h-process is determined by the following transition density functions:

$$P_t^h(x, y) = P_t^D(x, y)\, h(y)/h(x), \qquad 0 \leqq t < \infty, x,\ y \in D.$$

Let $\mathscr{F}_t$, $t \geqq 0$ be the usual completed *σ-fields* of Z_t. All stopping times in this article will be stopping times with respect to $\mathscr{F}_t$, $t \geqq 0$. The shift operator θ_t from W to W is defined by $\theta_t(\omega)(s) = \omega(t+s)$, $s \geqq 0$. In this paper, we always assume the decompositions of Whitney squares $\{Q_i: i \in I\}$ of the related domains are fixed. Let Q_i, Q_j be two Whitney squares. We say that

$$Q_i = Q_0 \to Q_1 \to Q_2 \to \ldots \to Q_n = Q_j$$

is a chain of length n between Q_i and Q_j if all Q_k, $k = 0, 1, 2 \ldots n$ are Whitney squares and Q_k and Q_{k+1} have touching edges, $k = 0, 1, 2 \ldots n-1$.

Define

$$d_1(Q_i, Q_j) = \inf\{n: \text{there is a chain of Whitney squares of length } n \text{ connecting } Q_i \text{ and } Q_j\},$$

$$d_2(x, y) = \inf_\gamma \left\{ \int_\gamma \frac{1}{\delta(z)} |dz| \right\},$$

where the infimum is taken over all rectifiable curves γ in D which join x and y. It can be checked that d_1 and d_2 are equivalent on the space of Whitney squares [8; p. 44]. If f is a nonnegative function on I which is measurable w.r.t. the Borel field on I and $0<\int_I f(t)\,dt<\infty$, we use $N(f)$ to denote the normalization of f, i.e., $N(f)=f/\int_I f(t)\,dt$. For a positive harmonic function h, let Z^h be the h-process corresponding to h, $\mathscr{F}_t^h$ be the usual completed σ-field of Z^h. Let P_z^h or P_μ^h be the probability on W which makes Z^h an h-process started at z or with initial measure μ on $\mathbb{C}$. If η is a stopping time and $\Lambda\in\mathscr{F}(\eta)$, then

$$P_z^{D,h}(\Lambda\cap\{\eta<\tau_D\})=\int_{\Lambda\cap\{\eta<\tau_D\}} h(Z_\eta)/h(z)\,dP_z^D. \tag{2.1}$$

See Doob [6, p. 672] for a proof.

3. Some lemmas

Let $S_2=\{z\in\mathbb{C}:\ 0<\mathrm{Re}(z)<2,\ 0<\mathrm{Im}(z)<1\}$. Let $L_0=\{z\in\mathbb{C}:\ \mathrm{Re}(z)=0\}\cap\bar{S}_2$, where $\bar{S}_2$ is the closure of S_2 in the Euclidean topology. Define $H_{L_0}=\inf\{t>0:\ Z_t\in L_0\}$; if $\{\ldots\}=\emptyset$, $H_{L_a}=\infty$. Let I and J be two intervals. If $\Psi_s(t)$, $s\in I$, $t\in J$ is a nonnegative Borel-measurable function of (s,t) on $I\times J$ and v is a measure on the Borel subsets of I, then we say $g(t)=\int_I \Psi_s(t)\,dv(s)$ a mixture of the function Ψ_s. The following two lemmas come from Davis [5], pp. 405–406.

Lemma 3.1. *Let Ψ_s, $s\in I$ and g be as above and suppose all these functions have positive and finite integrals. If α and β are two functions on J such that*

$$\alpha\leqq N(\Psi_s)\leqq\beta,\qquad s\in I,$$

then,

$$\alpha\leqq N(g)\leqq\beta.$$

Lemma 3.2. *Let f_z be the density of $\{Z_{H_{L_0}};\ H_{L_0}\leqq\tau_{S_2}\}$ under $P_z^{S_2}$ with respect to linear Lebesgue measure. Then, there is $c\in(0,1)$ such that*

$$(1-c)\,N(f_z)\leqq N(f_w)\leqq(1+c)\,N(f_z)$$

for any z, $w\in L_1$.

Let $S\equiv\{z\in\mathbb{C}:\ -\infty<\mathrm{Re}(z)<\infty,\ 0<\mathrm{Im}(z)<1\}$. Let $Q_k\equiv\left\{z\in\mathbb{C}:\ \frac{k-1}{3}\leqq\mathrm{Re}(z)\leqq\frac{k}{3},\ \frac{1}{3}\leqq\mathrm{Im}(z)\leqq\frac{2}{3}\right\}$, $k=0,\ \pm1\pm2\ldots$. Define $\varphi: S\to R_+^2$ by $\varphi(z)=e^{\pi z}$, $z\in S$. It is obvious that φ is a univalent conformal mapping. Let

$$a_k=\left\{z\in\mathbb{C}:\ \mathrm{Re}(z)=\frac{k}{3},\ \mathrm{Im}(z)\in(0,\tfrac{1}{3})\right\}$$

$$b_k=\left\{z\in\mathbb{C}:\ \mathrm{Re}(z)=\frac{k}{3},\ \mathrm{Im}(z)\in(\tfrac{2}{3},1)\right\}$$

$$M_k=\left\{z\in\mathbb{C}:\ \mathrm{Re}(z)=\frac{k}{3},\ \mathrm{Im}(z)\in(0,1)\right\}$$

$$C_k=M_k\backslash(a_k\cup b_k),\qquad k=0\pm1\pm2\ldots$$

Then, we have

Lemma 3.3. *Suppose $K(x, y)$, $x \in S$, $y \in \partial S$ are Martin kernels for S. Then*

$$\sup_{x \in a_k \cup b_k} K(x, y) < c \cdot \inf_{x \in C_k} K(x, y)$$

for all $k \in Z$ and all $y \in \partial S$ satisfying that $\left|\mathrm{Re}(y) - \frac{k}{3}\right| \geqq 1$, where c is an absolute constant.

Proof. The proof is easy. By the ratio invariance of Martin kernels under conformal mapping, if $\mathrm{Re}(y) > \frac{k}{3} + 1$,

$$\begin{aligned}
&\sup\left\{\frac{K(u, y)}{K(v, y)}: u \in a_k \cup b_k,\ v \in C_k\right\} \\
&= \sup\left\{\frac{u_2}{(u_1-1)^2 + u_2^2} \Big/ \frac{v_2}{(v_1-1)^2 + v_2^2}: u_1^2 + u_2^2 = v_1^2 + v_2^2 = e^{2\left(\mathrm{Re}(y) - \frac{k}{3}\right)\pi}\right. \\
&\qquad \left. \mathrm{Arg}(u_1 + i u_2) \in \left(0, \frac{\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \pi\right), \quad \mathrm{Arg}(v_1 + i v_2) \in \left(\frac{\pi}{3}, \frac{2\pi}{3}\right)\right\} \\
&\leqq \sup\left\{\frac{(v_1-1)^2 + v_2^2}{(u_1-1)^2 + u_2^2}: u_1^2 + u_2^2 = v_1^2 + v_2^2 = e^{2\left(\mathrm{Re}(y) - \frac{k}{3}\right)\pi},\ |u_1| \geqq \tfrac{1}{2} e^{2\pi}\right\} \leqq 8.
\end{aligned}$$

Similarly, if $\mathrm{Re}(y) < \frac{k}{3} - 1$, it still holds that

$$\sup\left\{\frac{K(u, y)}{K(v, y)}: u \in a_k \cup b_k,\ v \in C_k\right\} \leqq 8.$$

Therefore, Lemma 3.3 holds. $\square$

Lemma 3.4. *Let $H_k = \inf\{t > 0: Z_t \in M_k\}$. Then, there is $\delta > 0$ such that*

$$P_x^y(Z_{H_k} \in C_k) \geqq \delta$$

for all $k \in Z$ and all x and y such that $x \in S$ and $y \in \partial S$ are on opposite sides of M_k and $\mathrm{dist}(x, M_k) \geqq 1$ *and* $\mathrm{dist}(y, M_k) \geqq 1$.

Proof. By the translation invariance of Brownian motion, it suffices to show the case $k = 6$. For simplicity, we may assume that $\mathrm{Re}(x) \leqq 1$ and $\mathrm{Re}(y) \geqq 3$, and $x \in S$ and $y \in \partial S$. Let g_z, $z \in M_3$ be the same density as before. There is a measure μ_x on M_3 such that

$$P_x^S(Z_{H_6}; H_6 < \tau_S) = \int_{M_3} g_z\, \mu_x(dz). \tag{3.5}$$

We have by Lemma 3.2 and symmetry of BM that there is $0 < c < 1$ such that

$$P_x^S(H_6 < \tau_S, Z_{H_6} \in C_6) \geqq c\, P_x^S(H_6 < \tau_S). \tag{3.6}$$

Since $P_x^y(H_6<\tau_S)=1$, by (2.1),

$$\begin{aligned}1&=\int_{H_6<\tau_S}\frac{K(Z_{H_6},y)}{K(x,y)}dP_x^S\\&=\int_{H_6<\tau_S,\,Z_{H_6}\in C_6}\frac{K(Z_{H_6},y)}{K(x,y)}dP_x^S+\int_{H_6<\tau_S,\,Z_{H_6}\in a_6\cup b_6}\frac{K(Z_{H_6},y)}{K(x,y)}dP_x^S\\&\equiv I+II.\end{aligned}$$

Now,

$$\begin{aligned}II&\leqq c\inf_{w\in C_6}\frac{K(w,y)}{K(x,y)}\cdot P_x^S(H_6<\tau_S,Z_{H_6}\in a_6\cup b_6)\\&\leqq cI.\end{aligned}$$

The first inequality comes from Lemma 3.3. The second is from (3.6). Therefore,

$$\int_{H_6<\tau_S,\,Z_{H_6}\in C_6}\frac{K(Z_{H_6},y)}{K(x,y)}dP_x^S\geqq\frac{1}{1+c}.$$

Thus

$$P_x^y(Z_{H_6}\in C_6)\geqq\frac{1}{1+c}\equiv\delta>0.\quad\square$$

The following corollary follows immediately.

Corollary 3.7. *There is $\delta>0$ such that for any Q_k,*

$$P_x^y(Q_k)\geqq\delta$$

if $x\in S$, $y\in\partial S$, and x and y are on the opposite side of Q_k and $\min(d(x,Q_k),d(y,Q_k))\geqq 1$.

Now, from Corollary 3.7 and Theorem A, we obtain the special case of Theorem 1.

Corollary 3.8. *Let S be as defined. Then,*

$$\sup E_x^h\tau_S=\infty.$$

Proof. Let $y\in\partial S$, $x\in S$ be very far apart such that there are at least N Whitney squares Q_i of sidelength $\frac{1}{3}$ in the middle of S between them. By Corollary 3.7 and Theorem A, the total time of $K(\cdot,y)$-process started at x in these squares is greater than a constant times the area of $N-6$ squares of the N squares. Therefore

$$\sup E_x^h\tau_S=\infty.\quad\square$$

Lemma 3.9. *Let Ω be a proper convex domain. Then, there are two functions ϕ_+ and ϕ_- from some interval I to $R\cup\{-\infty\}$ such that*

$$\Omega=\{(x,y):\phi_-(x)<y<\phi_+(x),\ x\in I\},\tag{**}$$

where $I=(0, \operatorname{diam}(\Omega))$ *if* Ω *is bounded;* $I=(0, \infty)$ *or* $I=(-\infty, \infty)$ *if* Ω *is unbounded and* $\phi_+(x)$ *is real-valued and concave and* $\phi_-(x)$ *is convex on* I *or identical to* $-\infty$.

Proof. Since Ω is a proper subdomain of R^2, $\partial\Omega$ is not empty. First, we claim that for any point T in $\partial\Omega$ and C in Ω the line segment from C to T completely lies in Ω except T.

Since Ω is convex, there exists r_0 such that $B(C, r_0)$ is in Ω. Then the convex hull generated by $B(C, r_0)$ and T must be in Ω except T. So, the claim holds.

Now, fix a point O_1 in Ω. Let $\delta(T)$ be the distance from O_1 to T, $\alpha(T)$ be the direction of O_1T, for all $T\in\partial\Omega$. There are several possible cases.

Case 1. If $\delta(T)$ is bounded on $\partial\Omega$, then $\partial\Omega$ is compact. So, there are two points $A, B\in\partial\Omega$ which satisfy $d(A, B)=\operatorname{diam}(\Omega)$. Choose rectangular coordinates such that A is at the origin and B is on the positive x-axis. Under this coordinate, it is clear that there are two functions ϕ_+ and ϕ_- on $I=(0, \operatorname{diam}(\Omega))$ such that $\Omega=\{(x, y)\colon \phi_-(x)<y<\phi_+(x), x\in I\}$.

Case 2. If $\delta(T)$ is unbounded on $\partial\Omega$, then there is a sequence of points $\{T_n\}$ on $\partial\Omega$ such that $\delta(T_n)\to\infty$ and $\alpha(T_n)\to\alpha_0$ for some $\alpha_0\in[0, 2\pi)$ as $n\to\infty$. The ray at O_1 with direction α_0 completely lies in Ω. Choose a rectangular coordinate system with origin at O_1 and the positive x-axis coincided with the ray. There are several possible cases.

(1) Assume that the x-axis completely lies in Ω.

(i) If $\partial\Omega$ is above the x-axis, then $\partial\Omega$ is a line which is parallel to the x-axis. If to the contrary there were two points $A(1)$ and $A(2)$ on $\partial\Omega$ such that the line passing through these two points intersects the x-axis at $A(3)$. This contradicts to the first claim. By the simply connectedness of Ω, Ω can be given by two functions as in the Lemma and in this case, ϕ_+ is a positive constant function on R and ϕ_- is identical to $-\infty$.

(ii) If $\partial\Omega$ has some points above the x-axis as well as below. By the same argument as in (i), there are two lines which are parallel to the x-axis above and below the axis respectively. So, ϕ_+ and ϕ_- are two constant functions on R.

(iii) If $\partial\Omega$ lies below the x-axis, choose a proper rectangular coordinate. Then Ω must be given the same as in (i).

(2) Assume that $\partial\Omega$ has some common point with the negative x-axis at $(a, 0)$.

(i) If $\partial\Omega$ has one common point with the y-axis, say $(0, b)$, $b>0$, choose a new system with the origin at $(a, 0)$ and the y-axis coincided with the ray from $(a, 0)$ to $(0, b)$. Let B be the farthest point in $\partial\Omega$ from the y-axis. Translate the system to the point B. Then the Lemma holds in this coordinate system.

(ii) If $\partial\Omega$ intersects with the y-axis on both sides, let $(0, a)$, $(0, b)$, $a<0<b$ be the two points. Then for each $y\in(a, b)$, there is unique x such that $(x, y)\in\partial\Omega$. By the convexity of Ω, there exists $(x_0, y_0)\in\partial\Omega$ such that $x_0=\inf\{x\colon (x, y)\in\partial\Omega, y\in(a, b)\}$. Translate the coordinate at O_1 to the point (x_0, y_0). Under this new system, since $\partial\Omega$ has two common points with the y-axis, there are two intersection points of $\partial\Omega$ with the line $x=c$ for any $c>0$. By the convexity of Ω, there are two functions ϕ_+ and ϕ_- on $(0, \infty)$ such that the Lemma holds in this case. □

Let Ω_I denote the convex domain corresponding to I as in Lemma 3.9 for I finite or $(0, \infty)$. Then we have

Lemma 3.10. *Let Ω and I be as in Lemma 3.9.*

(i) *Assume I is finite. Let $\eta<\frac{1}{4}$ be fixed. Let $\Omega_\eta=\{(x, y)\colon \phi_-(x)<y<\phi_+(x), d\eta<x<(1-\eta)d\}$, where $d=\operatorname{diam}(\Omega)$. Then,*

$$|\Omega_\eta|\geqq c(\eta)\,|\Omega|,$$

and $c(\eta)\to 1$ as $\eta\to 0$.

(ii) *Assume $I=(0, \infty)$. Let $\alpha=\min\left\{\frac{\pi}{4}, \tan^{-1}\phi_-(1), \tan^{-1}\phi_+(1)\right\}$. Let $\Omega_\alpha=\{(x, y)\colon \phi_-(x)<y<\phi_+(x), |y|<\alpha x, x\in(0, \infty)\}$. Then, $|\Omega_\alpha|=\infty$.*

Proof of (i). Let $P_\eta=(d\eta, \phi_+(d\eta))$ and $Q_\eta=(d\eta, \phi_-(d\eta))$. *Let* R be the intersection of the line through B and P_η with the y-axis, and S be the intersection of the line through B and Q_η with the y-axis, where B is as in the proof of Lemma 3.9. Then,

$$\frac{\operatorname{Area}(\Delta BP_\eta Q_\eta}{\operatorname{Area}(\Delta BRS)}=\frac{(1-\eta)^2}{1}.$$

So, by the convexity of Ω,

$$|\{(x, y)\colon \phi_-(x)<y<\phi_+(x), 0<x<d\eta\}|\leqq\frac{1-(1-\eta)^2}{(1-\eta)^2}|\Omega|.$$

By the same argument, we have

$$|\{(x, y)\colon \phi_-(x)<y<\phi_+(x), (1-\eta)d<x<d\}|\leqq\frac{1-(1-\eta)^2}{(1-\eta)^2}|\Omega|.$$

Therefore,

$$|\Omega_\eta|\geqq\frac{1-6\eta+3\eta^2}{(1-\eta)^2}|\Omega|.$$

The second part of the lemma is obvious because the area of Ω is infinite. □

4. The proofs of Theorems 1 and 2

Let us introduce some more notation. Let

$$u(y)=\{(x, y)\colon -\infty<x<\infty\}$$

$$u(y_1, y_2)=\{(x, y)\colon y_1<y<y_2, -\infty<x<\infty\}$$

$$v(x_1, x_2)=\{(x, y)\colon 0<y<1, x_1<x<x_2\}$$

for all $0\leqq y\leqq 1$, $0<y_1<y_2<1$ and $x_1<x_2$. Let a be a point $\mathbb{C}$ such that dist $(a, \Omega)>0$. Applying the conformal mapping $\phi(z)=\left(\frac{1}{z-a}-1\right)^{\frac{1}{2}}$ for certain branch to Ω, where Ω is defined as in Theorem 1, we can have that $\phi(\Omega)$ is a bounded simply connected Jordan domain. By Caratheodory's theorem [9] there is a univalent conformal mapping ψ from $\phi(\Omega)$ to the unit disk D_1 at 0 such that

ψ continuously $1-1$ maps the closure of $\phi(\Omega)$ to the closure of D_1. Therefore, there is a univalent conformal mapping F from S to Ω such that

$$F(u(1)) = \{(v, \phi_+(v)) \colon v \in (-\infty, \infty)\} \equiv U(1) \tag{4.0.0}$$

and

$$F(u(0)) = \{(v, \phi_-(v)) \colon v \in (-\infty, \infty)\} \equiv U(0). \tag{4.0.1}$$

If Ω is a convex domain, by Caratheodory's theorem there exists a conformal mapping from S to Ω such that (4.0.0) and (4.0.1) hold with $(-\infty, \infty)$ replaced by one of the I's as in Lemma 3.9. So, whether Ω is defined as in Theorem 1 or a convex domain, there exists a conformal mapping F satisfying (4.0.0) and (4.0.1).

Let $\theta \equiv \theta(v) = \phi_+(v) - \phi_-(v)$, $\Psi \equiv \Psi(v) = \frac{1}{2}(\phi_+(v) + \phi_-(v))$, $v \in I_0$, where $I_0 = R$ if Ω is as in Theorem 1 or I_0 equals to one of the I's as in Lemma 3.9 if Ω is convex.

Definition. A set $H \subset \Omega$ is called a Harnack region with bound $c > 1$ in Ω if H is connected and for any positive harmonic function h in Ω,

$$c^{-1} < \frac{h(z)}{h(w)} < c$$

for any $z, w \in H$.

The Harnack region is invariant under conformal mapping. Let Ω_1 and Ω_2 be two domains in $\mathbb{C}$ and ϕ be a conformal mapping from Ω_1 to Ω_2. If H is a Harnack region with bound c in Ω_1, then $\phi(H)$ is also a Harnack region with the same bound c in Ω_2. Any compact set $\Omega_3 \subset \Omega$ is in some Harnack region with some bound which depends on the relationship between Ω_3 and Ω. Conversely, if H is a Harnack region with bound c in Ω, then H has compact closure in Ω. Furthermore, we have

Theorem 4.0. *Assume Ω is a simply connected domain in R^2. Suppose $H_{c_0} \subset \Omega$ is a Harnack region of Ω with bound c_0. Then H_{c_0} can be covered by K Whitney squares $\{Q_i\}_{i=1}^K$, where K depends only on c_0.*

Proof. Step 1: We claim that for any $x_0 \in H_{c_0}$, there exists $M_0 > 0$, which is independent of x_0 and Ω such that

$$P_{x_0}^{\Omega}(Z_{\tau_\Omega} \in \partial\Omega \cap B(x_0, M_0\,\delta(x_0))) > \frac{c_0}{1+c_0},$$

where $\delta(x_0) \equiv \delta_0 = \operatorname{dist}(x_0, \Omega^c)$ and $B(x_0, M\delta(x_0))$ is the ball with center at x_0 and radius $M\delta(x_0)$.

Let $\Delta(M, x_0) = \partial\Omega \cap B(x_0, M\delta(x_0))$. We say that the Brownian motion makes a complete loop around a bounded subset S of R^2 in time t if the Brownian path Z_s; $0 < s \leqq t$ separates S from ∞. For any $y \in S(0, 1)$, let

$$p = P_y(B \text{ makes a complete loop between } S(0,1) \text{ and } S(0,2) \textit{ before leaving } B(0,2))$$

By the rotation invariance of Brownian motion, p is independent of y. It is clear that p is positive. So, by the scaling of Brownian motion,

$$P_{x_0}(Z_{\tau_{B(x_0,2\delta(x_0))}}\in\Omega)$$
$$\leqq 1-\inf_{z\in B(x_0,\delta(x_0))} P_z(Z \text{ makes a complete loop between}$$
$$S(x_0,\delta(x_0)) \text{ and } S(x_0,2\delta(x_0)) \text{ before leaving } B(x_0,2\delta(x_0)))=1-p.$$

Thus,

$$P_{x_0}(Z_{\tau_{B(x_0,2^N\delta(x_0))}}\in\Omega)\leqq(1-p)^N.$$

Therefore

$$P_{x_0}(Z_{\tau_\Omega}\in\Delta(M,x_0))\geqq 1-(1-p)^N.$$

So, there exists M_0 such that the claim holds.

Step 2: We claim that there exists $N_0>0$, independent of $x_0\in H_{c_0}$, such that $\operatorname{dist}(x_0,y)<N_0 M_0\,\delta(x_0)$ for all $y\in H_{c_0}$.

Let S_1 and S_2 be two concentric circles with radii 1 and 2 respectively.

For any $y\in S_2$, let

$$q=P_y(Z \text{ makes a complete loop around } S_1 \text{ before hitting } S_1).$$

Again, q is independent of y, and clearly $q>0$.

If $|y-x_0|\geqq 2^N M_0\,\delta_0$, we have

$$\begin{aligned}
&P_y^\Omega(Z_{\tau_\Omega}\in\Delta(M_0,x_0))\\
&\leqq E_y^\Omega(P^\Omega_{Z_{H_{S(x_0,2^{N-1}M_0\delta_0)}}}(Z_{\tau_\Omega}\in\Delta(M_0,\chi_0))\,I_{\{H_{S(x_0,2^{N-1}M_0\delta_0)}<\tau\Omega\}})\\
&\leqq(1-q)\sup_{w\in S(x_0,2^{N-1}M_0\delta_0)} P_\omega^\Omega(Z_{\tau_\Omega}\in\Delta(M_0,x_0)).
\end{aligned}$$

Therefore,

$$P_y^\Omega(Z_{\tau_\Omega}\in\Delta(M_0,x_0))\leqq(1-q)^N.$$

Note that $P_y^\Omega(Z_{\tau_\Omega}\in\Delta(M_0,x_0))$ is a positive harmonic function. So,

$$\frac{P_{x_0}^\Omega(Z_{\tau_\Omega}\in\Delta(M_0,x_0))}{P_y^\Omega(Z_{\tau_\Omega}\in\Delta(M_0,x_0))}\geqq\frac{1+c_0}{c_0}\,\frac{1}{(1-q)^N}.$$

Since $y\in H_{c_0}$, then there exists $N_0>0$ such that claim 2 holds.

Step 3: We claim that for any $y\in H_{c_0}$

$$\delta(y)\geqq\frac{1}{2N_0M_0}\,\delta(x_0).$$

If to the contrary, there is $y_0\in H_0$ such that $\delta(y_0)<\dfrac{1}{2N_0M_0}\,\delta(x_0)$, by claim 2,

$$\operatorname{dist}(y_0,x_0)<N_0M_0\,\delta(y_0)<\tfrac{1}{2}\,\delta(x_0).$$

So,

$$\delta(x_0) \leqq \operatorname{dist}(x_0, y_0) + \delta(y_0) < \frac{1}{2N_0 M_0} \delta(x_0) + \tfrac{1}{2}\delta(x_0) < \delta(x_0).$$

This is a contradiction. So, claim 3 holds.

Step 4: Assume $\delta(x_0)=1$, $c=\dfrac{1}{2N_0 M_0}$, $R_0=N_0 M_0$. We claim that there exists $M>0$ depending only on c_0 such that for any $y \in H_{c_0}$, there is a rectifiable curve γ in H_2 defined below connecting y and x_0 and

$$\operatorname{Arc}(\gamma) \leqq M\delta(x_0).$$

Let $H_1=\{y\in\Omega\colon \delta(y)\geqq c,\ \operatorname{dist}(x_0, y)\leqq R_0\}$ and $H_2=\left\{y\in\Omega\colon \delta(y)\geqq\frac{c}{4},\ \operatorname{dist}(x_0, y) \leqq R_0\right\}$. There are N uniform balls $\{B_i\}_{i=1}^N$ with radii $\frac{c}{4}$ which cover the closure of $B(x_0, R_0)$ and N depends only on c and R_0. Let $\mathscr{F}=\{2B_i\colon 2B_i\cap\Omega^c=\emptyset$ $i=1, 2, \ldots N\}$, where $2B_i$ is the ball with radius two times the one of B_i and concentric to B_i. Therefore,

$$H_{c_0} \subset H_1 \subset \bigcup_{2B_i\in\mathscr{F}} \mathscr{B}_i \subset H_2.$$

Since H_{c_0} is connected, there is a path in H_{c_0} from y to x_0. So, there are finitely many balls in $\mathscr{F}$ covering the path. Then, there is a rectifiable curve γ in H_2 from x_0 to y such that $\operatorname{Arc}(\gamma)\leqq c_0 N\pi$. By scaling, step 4 and the equivalence of distance d_1 and d_2, it follows that

$$d_1(x_0, y) \leqq \frac{N}{2}$$

for some constant N which depends only on c_0. Therefore, Theorem 4.0 holds. □

For fixed $\alpha_0>0$, let $D_1=\{(x, y)\colon |x|\leqq\alpha_0 y\}$, $D_2=\{(x, y)\colon |x|<\alpha_0(y-1)\}$, $D=D_1\backslash D_2$, $I=\{(x, y)\colon x=0, 0<y<1\}$.

Define for each $0<\mu<\frac{1}{2}$,

$$c_1(\mu)=\sup\{\operatorname{Im}(z)\colon P_z(Z_{\tau_D}\in\partial D_1)>1-\mu,\ z\in I\}.$$

Obviously,

$$c_1(\mu)\to 0, \qquad \text{as } \mu\to 0.$$

So, there is $\psi>0$ such that $c_1(\mu)<\frac{1}{4}$ for all $0<\mu<\psi$.

Lemma 4.1. (i) *Let Ω be as in Theorem 1, and let F be as described at the beginning of this section. Then, there is a constant β such that*

$$0<\beta<\left|\frac{\operatorname{Im}(F(z))-\Psi(\operatorname{Re}(F(z)))}{\theta(\operatorname{Re}(F(z)))}\right| \tag{4.1.1}$$

for $z \in u(\mu)$ *or* $z \in u(1-\mu)$ *when* $0<\mu<\psi$, *where* β *depends only on* μ *and* M. *In fact,* $\beta \to \frac{1}{2}$ *as* $\mu \to 0$.

(ii) *Suppose* Ω *is a described in Lemma 3.9 with* I *finite, and again let* F *be as described and* $\eta>0$. *Then there is* $\psi>0$ *and* $\beta(\mu)$ *defined for* $0<\mu<\psi$, *where* β *depends only on* μ *and* η, *such that* (4.1.1) *holds whenever* $\mathrm{Re}(F(z)) \in (d\mu, d(1-\mu))$.

(iii) *Suppose* Ω *is as described in Lemma 3.9 with* $I=(0,\infty)$. *Let* α *be as defined and* F *as described before. Then there is* $\psi>0$ *and* $\beta(\mu)$ *defined for* $0<\mu<\psi$ *such that* (4.1.1) *holds whenever* $\mathrm{Arg}(F(z)) \in (-\alpha, \alpha)$, *where* ψ *and* β *depend on* α.

Proof of (i). Let Ω be as in Theorem 1. Let $\alpha_0 = \tan^{-1} M$. Fix $\mu \in (0, \psi)$. We will prove the case that z belongs to $u(\mu)$. The other case follows similarly. For simplicity, let $v=\mathrm{Re}(F(z))$, $u=\mathrm{Im}(F(z))$ and $z \in u(\mu)$. Assume Z is a Brownian motion starting at z in S. With probability $1-\mu$, Z exits from S at some point on $u(0)$ and with probability μ on $u(1)$. By the invariance of Brownian motion under conformal mapping up to time-scaling, Brownian motion starting at $F(z)$ will exit Ω on $U(0)$ with probability $1-\mu$ and on $U(1)$ with probability μ. Let $\Gamma_1(v, \alpha)$ and $\Gamma_2(v, \alpha)$ denote the cones which open up and have aperture α at points $(v, \phi_-(v))$ and $(v, \phi_+(v))$, respectively. Let $\alpha = \alpha_0$. Then, the Brownian motion started at $F(z)$ will exit $\Gamma_1 \backslash \Gamma_2$ on $\partial \Gamma_1$ with probability at least $1-\mu$. By scale-invariance of BM, it follows that

$$\tfrac{1}{2} - c_1(\mu) < \frac{\Psi(v)-u}{\theta(v)}.$$

Similarly, if $z \in u(1-\mu)$,

$$\frac{u-\psi(v)}{\theta(v)} \geqq \tfrac{1}{2} - c_1(\mu).$$

Let $\beta = \frac{1}{2} - c_1(\mu)$. Therefore, part (i) holds.

Proof of (ii). Let Ω be as in Lemma 3.9. Assume $z \in u(\mu)$ and $v \in (\eta d, (1-\eta)d)$. For any P on $\partial\Omega$, let P_0 be the projection of P on the x-axis.

Since

$$\tan \angle APP_0 = \frac{|AP_0|}{|PP_0|} \geqq \frac{\eta a}{PP_0} \geqq \eta,$$

$$\angle APP_0 \geqq \tan^{-1} \eta,$$

where A is as in the proof of Lemma 3.9. Let $\alpha = \tan^{-1}\eta$. Then, $\Gamma_-(v, \alpha)$ intersects with $\partial\Omega$ only on $\{(v, u)\colon u=\phi_+(v), v \in (0, d)\}$ and $\Gamma_+(v, \alpha)$ only on $\{(v, u)\colon u=\phi_-(v), v \in (0, d)\}$. By the same argument as in the Proof of (i), the second part of the lemma holds.

Proof of (iii). The proof of (iii) is similar to the proof of (ii). □

Lemma 4.2. (1) *Let* Ω *be as in Theorem 1. There is* $c>0$ *and* $\psi>0$ *such that for all* $0<\mu<\psi$,

$$|F(u(\mu, 1-\mu))| \geqq c \cdot |\Omega|, \tag{4.2.1}$$

where c depends on μ and M. In fact, $c \to 1$ as $\mu \to 0$.

(2) *Let Ω be a bounded convex domain. Then, there exists a universal constant c such that* (4.2.1) *holds.*

(3) *Let Ω be as in Lemma 3.9 with $I=(0,\infty)$. Then, there is $\mu>0$ such that*

$$|F(u(\mu, 1-\mu))| = \infty.$$

Proof. We just prove part (1) of the lemma, since the rest parts easily follow from Lemmas 3.10 and 4.1 by a similar argument. By Lemma 4.1,

$$F(u(\mu, 1-\mu)) \supseteq \{(u,v)\colon \Psi(v) - \beta\theta(v) < u < \Psi(v) + \beta\theta(v)\}.$$

So,

$$|F(u(\mu, 1-\mu))| \geqq 2\beta \int_{-\infty}^{\infty} \theta(v)\,dv = 2\beta|\Omega|.$$

Therefore, the first part of the lemma follows from Lemma 4.1. □

In the rest of this section, Ω can be either as in Theorem 1 or in Theorem 2.

Now, fix μ sufficiently small. Because $u(\mu, 1-\mu) \cap v(x, x+1-2\mu)$ is a Harnack region of S with bound c_3 which is independent of x, $F(u(\mu, 1-\mu) \cap v(x, x+1-2\mu))$ is also a Harnack region in Ω with the same bound c_3. For convenience, let $W(x,\mu) = F(u(\mu, 1-\mu) \cap v(x, x+1-2\mu))$, $x \in R$. By Theorem 4.0, there are K Whitney squares $\{Q_i\}_{i=1}^K$ such that $W(x,\mu) \subseteq \bigcup_{j=1}^{K} Q_i$ and K depends only on μ. Let $\mathscr{L}(x) = \{Q_i\colon Q_i \in \{Q_j\}_{j=1}^K$ such that $W(x,\mu) \cap Q_i \neq \emptyset\}$. For each $Q_i \in \mathscr{L}(x)$, $|Q_i|$ cannot be very small when compared to $|W(x,\mu)|$. In fact, we obtain

Lemma 4.3. *There exists c which depends only on μ such that*

$$|Q_i| \geqq c \cdot |W(x,\mu)|$$

for any $Q_i \in \mathscr{L}(x)$.

Proof. We claim that for any two Whitney squares Q_1 and Q_2 if $Q_1 \cap Q_2 \neq \emptyset$, there is $c>0$ such that

$$|Q_1|/|Q_2| < c.$$

Since if $Q_1 \cap Q_2 \neq \emptyset$,

$$\begin{aligned}\operatorname{diam}(Q_1) &\leqq \operatorname{dist}(Q_1, \partial\Omega) \leqq \operatorname{diam}(Q_2) + \operatorname{dist}(Q_2, \partial\Omega)\\ &\leqq \tfrac{5}{4}\operatorname{diam}(Q_2).\end{aligned}$$

So,

$$\frac{|Q_1|}{|Q_2|} \leqq \left(\frac{5}{4}\right)^2.$$

Since there are at most K Whitney squares which cover the connected region $W(x,\mu)$, the ratio of the area of the largest Whitney square to the smallest in $\mathscr{L}(x)$ in fact is less than $(\frac{5}{4})^{2K}$. Therefore, there is $c>0$, which depends on μ, such that $|Q_i| \geqq c \cdot |W(x,\mu)|$. □

Let $S(x)=F^{-1}(\bigcup_{Q\in\mathscr{L}(x)} Q)$. Since $\bigcup_{Q\in\mathscr{L}(x)} Q$ is a Harnack region in Ω with bound c which depends only on μ, $S(x)$ is a Harnack region in S with the same bound c. Therefore there are at most L Whitney squares which cover $S(x)$ and L depends only on μ. So, $S(x)$ is contained in the L-neighborhood of $u(\mu, 1-\mu)\cap v(x, x+1-2\mu)$.

The Proofs of Theorems 1 and 2. If $\Omega=R^2$, Theorem 2 obviously holds because in this case, $h=constant$ and $E_x^h\tau_D=E_x\tau_{R^2}=\infty$. If Ω is as in Lemma 3.9 with $I=(-\infty,\infty)$, then, Ω is identical either to a half plane or an infinite strip. If Ω is a half plane, by Stegenga [10] we have $\sup_{x\in\Omega} E_x\tau_\Omega=\infty$. So, Theorem 2 holds. If Ω is an infinite strip, by Corollary 3.9, Theorem 2 still holds. So, we just need to show Theorems 1 and 2 for the convex domains as in Lemma 3.9 with I finite or $(0,\infty)$.

By Lemma 4.2, there is a sequence of squares $\{S_i\}_{i=1}^{\infty}$ in $u(\mu, 1-\mu)$ with sidelength $1-2\mu$ such that

(1) S_{i+1} is on the right side of S_i,

(2) $\operatorname{dist}(S_i, S_{i+1})=2L+2$, $i=1, 2, 3, \ldots$ and

(3) there is $c>0$ such that

$$\left|F\left(\bigcup_{i=1}^{\infty} S_i\right)\right|\geqq c\cdot|\Omega|,$$

where c depends only on μ and M if Ω is as in Theorem 1 or only on μ if Ω is as in Lemma 3.9 with I finite, and in the case that Ω is as in Lemma 3.9 with $I=(0,\infty)$, $\left|F\left(\bigcup_{i=1}^{\infty} S_i\right)\right|=\infty$.

Then, we have

$$\begin{aligned}(4.6)\qquad & P_{F(z)}^{\Omega, F(w)}(F(Z_t)\ \textit{hits}\ F(S_i)\ \textit{before leaving}\ \Omega)\\ &=P_z^{S,w}(Z_t\ \textit{hits}\ S_i\ \textit{before leaving}\ S)\end{aligned}$$

for any $z\in S$, $w\in\partial S$.

Let w be fixed on ∂S so that w is on the left side of S_1 and at least a distance 2 away from S_1. Then, by Corollary 3.7 and (4.6),

$$P_{F(z)}^{\Omega, F(w)}(F(Z_t)\ \textit{hits}\ F(S_i)\ \textit{before leaving}\ \Omega)\geqq\delta$$

if z is on the right side of S_i and a distance 2 away from S_i.

Let $\{Q_{i,j}\}_{j=1}^{K}$ be the collection of Whitney squares of Ω which cover $F(S_i)$. By Theorem A and Lemma 4.3,

$$\begin{aligned} E_{F(z)}^{\Omega, F(w)} T_{Q_{i,j}} &\geqq c P_{F(z)}^{\Omega, F(w)}(Q_{i,j})\,|Q_{i,j}|\\ &\geqq c\,|F(S_i)|\,P_{F(z)}^{\Omega, F(w)}(Q_{i,j}) \qquad \forall Q_{i,j}\cap F(S_i)\neq\emptyset.\end{aligned}$$

Since $F^{-1}(Q_{i,j})$ is in the L-neighborhood of S_i, $\{Q_{i,j}: Q_{i,j} \cap F(S_i) \neq \emptyset\}$ are disjoint in the sense that they have disjoint interiors. Thus,

$$\begin{aligned}
E_{F(z)}^{\Omega, F(w)} \tau_\Omega &\geqq \sum_{1 \leqq j \leqq K,\, 1 \leqq i \leqq m} E_{F(z)}^{\Omega, F(w)} T_{Q_{i,j}} \\
&\geqq c \sum_{1 \leqq j \leqq K,\, 1 \leqq i \leqq m} |F(S_i)|\, P_{F(z)}^{\Omega, F(w)}(Q_{i,j}) \\
&\geqq c \sum_{i=1}^{m} |F(S_i)|\, P_{F(z)}^{\Omega, F(w)}(F(S_i)) \\
&\geqq c\delta \sum_{i=1}^{m} |F(S^i)|,
\end{aligned}$$

if z is on the right side of L-neighborhood of S_m and at least distance 2 away from it. Therefore,

$$\sup E_{F(z)}^{\Omega, F(w)} \tau_\Omega \geqq c\delta \sum_{i=1}^{\infty} |F(S^i)| \geqq c\delta |\Omega|.$$

Thus, Theorems 1 and 2 hold. □

Remark 1. If the two curves $u = \phi_+(v)$, $u = \phi_-(v)$ in Theorem 1 satisfy condition (2) when $|v|$ is large enough and the area of the domain is infinite, then Theorem 1 still holds.

5. The proof of Theorem 3

Let $S_M = \{z \in \mathbb{C}: 0 < \mathrm{Re}(z) < M,\ 0 < \mathrm{Im}(z) < 1\}$ for any positive integer M. Let $L_a = \{z \in \mathbb{C}: \mathrm{Re}(z) = a\} \cap \bar{S}_M$ for any real number a, where $\bar{S}_M$ is the closure of S_M in the Euclidean topology. Define $H_{L_a} = \inf\{t > 0: Z_t \in L_a\}$; if $\{\ldots\} = \emptyset$, $H_{L_a} = \infty$. Let $c^* = \dfrac{1+c}{1-c}$, where c is a fixed constant for which Lemma 3.2 holds. From now on, all density functions will be density functions with respect to linear Lebesgue measure on various line segments. Let g_z be the density function of $\{Z_{H_{L_2}}, H_{L_2} \leqq \tau_{S_2}\}$ under $P_z^{S_2}$. By the symmetry of S_2 and Brownian motion Z, it follows that $N(g_z)(0+iy) = N(f_z)(2+iy)$ for $y \in (0, 1)$ and $z \in L_1$.

Lemma 5.1. *Let $M > 4$ and $4 \leqq n < M$. Let $g_{n,z}^M$ be the density of*

$$\{Z_{H_{L_2} \circ \theta_{H_{L_n}} + H_{L_n}}, H_{L_2} \circ \theta_{H_{L_n}} + H_{L_n} < \tau_{S_M}\}$$

under $P_z^{S_M}$. Then

$$(c^*)^{-1} N(g_w) \leqq N(g_{n,z}^M) \leqq c^* N(g_w), \qquad z, w \in L_1.$$

Proof. Define

$$\begin{aligned}
\tau_0 &= \inf\{t > H_{L_n}: Z_t \in L_3\}, \\
\tau_1 &= \inf\{t > \tau_0: Z_t \in L_2 \cup L_4\}, \\
\tau_{2i} &= \tau_0(\theta_{\tau_{2i-1}}) + \tau_{2i-1}, \\
\tau_{2i+1} &= \tau_1(\theta_{\tau_{2i}}) + \tau_{2i}, \qquad i = 1, 2, 3, \ldots.
\end{aligned}$$

Let $v_j = P_z^{S_M}(Y_{\tau_{2j}};\ \tau_{2j} < \tau_{S_M}\ Y_{\tau_{2i+1}} \in L_4\ \forall i < j)\ j = 0, 1, 2, \ldots$. Since $g_{n,z}^M$ is the density of

$$\sum_0^\infty P_z^{S_M}(Y_{\tau_{2j+1}}:\ \tau_{2j} < \tau_{S_M},\ Y_{\tau_{2j+1}} \in L_2,\ Y_{\tau_{2i+1}} \in L_4, \forall i < j)$$

and the jth term in the series is a mixture of f with v_j over L_3 by the strong Markov property of Z, it follows from Lemma 3.1, Lemma 3.2 and the translation invariance of Brownian motion that

$$(c^*)^{-1} N(g_w) \leqq N(g_{n,z}^M) \leqq c^* N(g_w), \qquad \forall z, w \in L_1. \quad \square$$

Then we have

Lemma 5.2. *Let* $\Omega = \{z \in \mathbb{C}:\ 0 < \mathrm{Re}(z) < 1,\ 0 < \mathrm{Im}(z) < 1\} \cup \{z \in \mathbb{C}:\ 0 < \mathrm{Re}(z) < 2^{-M},\ -2^M < \mathrm{Im}(z) \leqq 0\}$. *Let* $l_k = \{z \in \mathbb{C}:\ 0 < \mathrm{Re}(z) < 2^{-M},\ \mathrm{Im}(z) = k \cdot 2^{-M}\}$, $k = 0, 1, 2, 3, \ldots 4^M$. Let $\lambda_{2,z}^\Omega$ *be the density of* $Z_{H_{l_2}}$, $H_{l_2} < \tau_\Omega$ *and* $\lambda_{n,z}^\Omega$ *the density of* $\{Z_{H_{l_2} \circ \theta_{H_{l_n}} + H_{l_n}},\ H_{l_2} \circ \theta_{H_{l_n}} + H_{l_n} < \tau_\Omega\}$ *over* l_2 *under* P_z^Ω, $z \in \Omega$. *Then*

$$\lambda_{n,z}^\Omega \leqq c^* (1/2)^{n-1} \lambda_{2,z}^\Omega$$

for all $4 \leqq n < 4^M$ *and* z *satisfying* $\mathrm{Re}(z) \in (0, 1)$ *and* $\mathrm{Im}(z) \in (0, 1)$.

Proof. Let g_z^c and $g_{n,z}^{M,c}$ be defined as g_{cz} and $g_{n,cz}^M$, respectively, on $c \cdot S$ with L_k replaced by $c \cdot L_k$ for $k \in Z$. By scale invariance of Brownian motion, $g_{cz}^c = g_z$ for all $z \in S_2$, and $g_{n,cz}^{M,c} = g_{n,z}^M$ for all $z \in S_M$.

Let $c = 2^{-M}$. As in Lemma 5.1, there are two measures $\mu_{1,z}$ and $\mu_{n,z}$ on l_1 such that $\lambda_{2,z}^\Omega$ is a mixture of g^c with $\mu_{1,z}$ and $\lambda_{n,z}^\Omega$ a mixture of $g_{n,\cdot}^{4^M,c}$ with $\mu_{n,z}$. By Lemmas 3.1 and 5.1, it similarly follows that

$$c^* N(\lambda_{2,z}^\Omega) \leqq N(\lambda_{n,z}^\Omega) \leqq c^* N(\lambda_{2,z}^\Omega).$$

Then,

$$\lambda_{n,z}^\Omega \leqq c^* \lambda_{2,z}^\Omega \frac{P_z^\Omega(H_{l_2} \circ \theta_{H_{l_n}} + H_{l_n} < \tau_\Omega)}{P_z^\Omega(H_{l_2} < \tau_\Omega)}.$$

For any z such that $\mathrm{Re}(z) \in (0, 1)$ and $\mathrm{Im}(z) \in (0, 1)$,

$$\begin{aligned} P_z^\Omega(H_{l_2} \circ \theta_{H_{l_n}} + H_{l_n} < \tau_\Omega) &= P_z^\Omega(H_{l_2} < H_{l_n},\ H_{l_2} \circ \theta_{H_{l_n}} + H_{l_n} < \tau_\Omega) \\ &= E_z^\Omega(P_{Z_{H_{l_2}}}^\Omega(H_{l_2} \circ \theta_{H_{l_n}} + H_{l_n} < \tau_\Omega) I_{(H_{l_2} < \tau_\Omega)}) \\ &= E_z^\Omega(E_{Z_{H_{l_2}}}^\Omega(P_{Z_{H_{l_n}}}^\Omega(H_{l_2} < \tau_\Omega) I(H_{l_n} < \tau_\Omega)) I_{(H_{l_2} < \tau_\Omega)}) \\ &\leqq P_z^\Omega(H_{l_2} < \tau_\Omega) \sup_{w \in l_n} P_w^\Omega(H_{l_2} < \tau_\Omega). \end{aligned}$$

The second and third equations come from the strong Markov property of killed Brownian motion. Thus, to prove Lemma 5.2, it suffices to prove that

$$\sup_{w \in l_n} P_w^\Omega(H_{l_2} < \tau_\Omega) \leqq (\tfrac{1}{2})^{n-1}.$$

For any $w \in l_n$,

$$
\begin{aligned}
P_w^{\Omega}(H_{l_2} < \tau_{\Omega}) &= P_w^{\Omega}(H_{l_{n-1}} < H_{l_2} < \tau_{\Omega}) \\
&= E_w^{\Omega}(P_{Z_{H_{l_{n-1}}}}^{\Omega}(H_{l_2} < \tau_{\Omega}) I_{(H_{l_{n-1}} < \tau_{\Omega})}) \\
&\leqq \sup_{z \in l_{n-1}} P_z^{\Omega}(H_{l_2} < \tau_{\Omega}) \cdot P_w^{\Omega}(H_{l_{n-1}} < \tau_{\Omega}) \\
&\leqq \tfrac{1}{2} \sup_{z \in l_{n-1}} P_z^{\Omega}(H_{l_2} < \tau_{\Omega}).
\end{aligned}
$$

So,

$$
\sup_{w \in l_n} P_w^{\Omega}(H_{l_2} < H_{\Omega}) \leqq \tfrac{1}{2} \sup_{z \in l_{n-1}} P_z^{\Omega}(H_{l_2} < \tau_{\Omega}).
$$

Therefore, Lemma 5.2 holds. □

Remark. In the proof of Lemma 5.2, we never used any property of the square $\{z \in \mathbb{C}: \operatorname{Re}(z) \in (0, 1), \operatorname{Im}(z) \in (0, 1)\}$. So, Lemma 5.2 still holds if we attach an arbitrary subdomain of R^2 to the corridor $\{z \in \mathbb{C}: 0 < \operatorname{Re}(z) < 2^{-M}, -2^M < \operatorname{Im}(z) \leqq 0\}$ so that the portion $\{z: 0 < \operatorname{Re}(z) < 2^{-M}, \operatorname{Im}(z) = 0\}$ is in the resulting domain.

Now, we describe the domain Ω that will be used to prove Theorem 3.
Let

$$\Omega_0 \equiv \{z \in \mathbb{C}ß\ 0 < \operatorname{Re}(z) < 1, 0 < \operatorname{Im}(z) < 1\},$$

$$S_1 = \{z \in \mathbb{C}: 0 < \operatorname{Re}(z) < \tfrac{1}{2}, -2 < \operatorname{Im}(z) \leqq 0\},$$

$$S_2 = \left\{z \in \mathbb{C}: \frac{1}{2} < \operatorname{Re}(z) < \frac{1}{2} + \frac{1}{2^2}, -2^2 < \operatorname{Im}(z) \leqq 0\right\},$$

$$\ldots\ldots$$

$$S_k = \left\{z \in \mathbb{C}: \sum_{i=1}^{k-1} \frac{1}{2^i} < \operatorname{Re}(z) < \sum_{i=1}^{k} \frac{1}{2^i}, -2^k < \operatorname{Im}(z) \leqq 0\right\}$$

$$\ldots\ldots.$$

Finally, let $\Omega = \Omega_0 \cup \left(\bigcup_{k=1}^{\infty} S_k\right)$. Then, Ω is a simply connected domain in $\mathbb{C}$ and $|\Omega| = |\Omega_0| + \sum_{k=1}^{\infty} |S_k| = \infty$.

From Lemma 5.2, the Brownian motion Z, started from some point outside the corridor has a small chance of walking very far along the corridor and coming back to the main region. It will be seen that the same phenomenon arises when the Brownian motion is replaced by an h-process.

Now we show $\sup E_x^h \tau < \infty$, completing the proof of Theorem 3.

For each $k \geqq 1$, let

$$l_{k,j} \equiv \left\{z \in \mathbb{C}: \operatorname{Im}(z) = \frac{j}{2^k}\right\} \cap S_k, \quad j = 1, 2, \ldots$$

By the remark above and (2.1), if $z \in \Omega_0$,

$$(5.3)\qquad \begin{aligned} &P_z^{\Omega,h}(H_{l_{k,2}} \circ \theta_{H_{l_{k,n}}} + H_{l_{k,n}} < \tau_\Omega) \\ &= \frac{1}{h(z)} \int_{\{H_{l_{k,2}} \circ \theta_{H_{l_{k,n}}} + H_{l_{k,n}} < \tau_\Omega\}} h(Z_{H_{l_{k,2}} \circ \theta_{H_{l_{k,n}}} + H_{l_{k,n}}})\, dP_z^\Omega \\ &\leqq \frac{1}{h(z)} \int_{\{H_{l_{k,2}} < \tau_\Omega\}} c^* (\tfrac{1}{2})^{n-1} h(Z_{H_{l_{k,2}}})\, dP_z^\Omega \\ &\leqq c^* (\tfrac{1}{2})^{n-1} \end{aligned}$$

for all $4 \leqq n < 4^k$ and $k > 1$. Let M_0 be an positive integer s.t. $c^* < 2^{M_0}$. Let $n_k = 2k + 1 + M_0$, $M_1 = 4 + \dfrac{M_0}{2}$. Then

$$(5.4)\qquad c^* (\tfrac{1}{2})^{n_k - 1} < \frac{1}{4^k} \quad \text{and} \quad \frac{n_k}{2^k} \leqq M_1.$$

So, all $l_j^{(k)} j \leqq n_k$, $k = 1, 2, 3, \ldots$ are contained in the M_1-neighborhood of Ω_0, denoted by Ω_1. Let $\Omega_2 = \Omega_1 \cap \Omega$ and $\Omega(n) = \Omega_0 \cup \left(\bigcup_{k=1}^n S_k \right)$, $n = 1, 2, \ldots$. If $z \in \Omega_0$, then

$$\begin{aligned} E_z^h \tau_{\Omega(n)} &= E_z^h(\tau_{\Omega(n)}; \tau_{\Omega(n)} \leqq \tau_{\Omega_2}) + E_z^h(\tau_{\Omega(n)}; \tau_{\Omega(n)} > \tau_{\Omega_2}) \\ &\leqq E_z^h(\tau_{\Omega_2}) + E_z^h(E^h_{Z_{\tau_{\Omega_2}}}(\tau_{\Omega(n)}) I_{(\tau_{\Omega_2} < \tau_{\Omega(n)})}) \end{aligned}$$

by the strong Markov property of h-processes. Since h is a positive harmonic function in Ω, it is positive harmonic in Ω_2. By Cranston-McConnell's result, the first term on the R.H.S of the above inequality is less than $c|\Omega_2| < M_1^2 c$. So, it suffices to control the second term on the R.H.S. Let i be the index of S_j which contains $Z_{\tau_{\Omega_2}}$. Let d_j be the semicircle in Ω_0 over $\partial S_j \cap \partial \Omega_0$ and H_{d_j} be the hitting time of Z on d_j. Then,

$$\begin{aligned} E_z^h(E^h_{Z_{\tau_{\Omega_2}}}(\tau_{\Omega(n)}) I_{(\tau_{\Omega_2} < \tau_{\Omega(n)})}) &\leqq E_z^h(E^h_{Z_{\tau_{\Omega_2}}}(H_{d_i}) I_{(\tau_{\Omega_2} < \tau_{\Omega(n)})}) \\ &\quad + E_z^h(E^h_{Z_{\tau_{\Omega_2}}}(E^h_{Z_{\tau H_{d_i}}}(\tau_{\Omega(n)}) I_{\{H_{d_i} < \tau_{\Omega(n)}\}}) I_{\{\tau_{\Omega_2} < \tau_{\Omega(n)}\}}) \\ &\leqq 2c + \sup_{w \in \Omega_0} E_w^h(\tau_{\Omega(n)}) \cdot E_z^h(P^h_{Z_{\tau_{\Omega_2}}}(H_{d_i} < \tau_{\Omega(n)}) I_{\{\tau_{\Omega_2} < \tau_{\Omega(n)}\}}) \\ &\leqq 2c + \sup_{w \in \Omega_0} E_w^h(\tau_{\Omega(n)}) \cdot \sum_{k=1}^\infty \left(\frac{1}{4} \right)^k \\ &= 2c + \tfrac{1}{3} \sup_{w \in \Omega_0} E_w^h \tau_{\Omega(n)}. \end{aligned}$$

The first inequality comes from the strong Markov property of h-processes. The second is from (1.1) and the last is from Lemma 5.2, (5.3) and (5.4).

So,

$$\sup_{w \in \Omega_0} E_w^h \tau_{\Omega(n)} \leqq 3(M_1^2 + 2)c.$$

Since $\Omega(n) \nearrow \Omega$ as $n \nearrow \infty$, $E_w^h(\tau_{\Omega(n)}) \nearrow E_w^h(\tau_\Omega)$. So, $\sup_{w \in \Omega_0} E_w^h \tau_\Omega \leqq 3(M_1^2+2)c$. Now, if $z \in S_k$ for some $k \geqq 1$,

$$\begin{aligned} E_z^h \tau_\Omega &\leqq E_z^h H_{d_k} + E_z^h(E_{Z_{H_{d_k}}}^h \tau_\Omega I(H_{d_k} < \tau_\Omega) \\ &\leqq 2c + \sup_{w \in \Omega_0} E_w^h \tau_\Omega \\ &\leqq 3(M_1^2+2)c + 2c. \end{aligned}$$

Therefore

$$\sup E_z^h \tau_\Omega < \infty.$$

Acknowledgement. I would like to thank Prof. B. Davis, my academic advisor, for his guidance, help and encouragement, and I would also like to thank Prof. R. Bañuelos and Prof. D. Drasin for their interest and helpful comments.

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