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Probability Theory and Related Fields

Exact Convergence Rates in Strong Approximation Laws for Large Increments of Partial Sums

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Summary. Consider partial sums S_n of an i.i.d. sequence $X_1 X_2, \ldots$, of centered random variables having a finite moment generating function ϕ in a neighborhood of zero. The asymptotic behaviour of $U_n = \max_{0 \leq k \leq n-b_n} (S_{k+b_n} - S_k)$ is investigated, where $1 \leq b_n \leq n$ denotes an integer sequence such that $b_n/\log n \to \infty$ as $n \to \infty$. In particular, if $b_n = o(\log^p n)$ as $n \to \infty$ for some $p > 1$, the exact convergence rate of $U_n/b_n \alpha_n = 1 + o(1)$ is determined, where α_n depends upon b_n and the distribution of X_1. In addition, a weak limit law for U_n is derived. Finally, it is shown how strong invariance takes over if $\lim_{n \to \infty} b_n (\log\log n)^2/\log^3 n = \infty$.

1. Introduction

Consider a sequence $X_1, X_2, \ldots$ of independent, identically distributed random variables with moment generating function $\phi(t) = E(\exp(tX_1))$ and satisfying the conditions

(A) $E(X_1) = 0; \quad \sigma^2 = E(X_1^2) < \infty;$

(B) X_1 is nondegenerate, i.e. $P(X_1 = x) < 1 \quad$ for all x;

(C) $t_0 = \sup\{t : \phi(t) < \infty\} > 0, t_1 = \inf\{t : \phi(t) < \infty\} < 0.$

Let

$$\rho(\alpha) = \inf_t \{\phi(t)\, e^{-t\alpha}\}, \quad \text{and}$$

$$S_0 = 0, S_n = X_1 + \ldots + X_n (n = 1, 2, \ldots).$$

We are interested in the asymptotic behavior of

$$U_n = \max_{0 \leq k \leq n-b_n} (S_{k+b_n} - S_k),$$

where $\{b_n, n \geqq 1\}$ denotes an integer sequence satisfying $1 \leqq b_n \leqq n$ and other assumptions specified below.

In 1970, Erdös and Rényi proved that, for any $\alpha \in \{\phi'(t)/\phi(t): 0 < t < t_0\}$,

$$\lim_{n \to \infty} (U_n/b_n \alpha) = 1 \text{ a.s.,} \tag{1.1}$$

if $b_n = [c \log n]$ and $c = c(\alpha)$ is related to α via the equation

$$\exp(-1/c) = \rho(\alpha). \tag{1.2}$$

This fundamental result was followed by a number of extensions and refinements (see, e.g., S. Csörgö (1979); M. Csörgö and Steinebach (1981), Deheuvels, Devroye and Lynch (1986); Deheuvels and Devroye (1987); Steinebach (1985).

Let

$$m(t) = \phi'(t)/\phi(t), \qquad t_1 < t < t_0,$$

$$A = \lim_{t \uparrow t_0} m(t) \quad \text{and} \quad c_0 = 1/\int_0^{t_0} t m'(t)\, dt.$$

Deheuvels et al. (1986) showed that (1.1) holds for any $c_0 < c < \infty$ and $0 < \alpha < A$ related via (1.2). Furthermore, if $t = t^* = t(\alpha)$ is the positive solution of

$$m(t) = \alpha, \tag{1.3}$$

they proved that

$$\limsup_{n \to \infty} \frac{t^*(U_n - b_n \alpha)}{\operatorname{loglog} n} = \frac{1}{2}, \quad \liminf_{n \to \infty} \frac{t^*(U_n - b_n \alpha)}{\operatorname{loglog} n} = -\frac{1}{2} \text{ a.s.,} \tag{1.4}$$

and

$$\lim_{n \to \infty} \frac{t^*(U_n - b_n \alpha)}{\operatorname{loglog} n} = -\frac{1}{2} \quad \text{in probability,} \tag{1.5}$$

where $b_n = [c \log n]$ and $c = c(\alpha)$.

The distributions for which $c_0 > 0$, together with the limiting behaviour of U_n when $b_n = [c \log n]$, $0 < c \leqq c_0$ are treated in Deheuvels et al. (1986), Deheuvels and Devroye (1987) and Deheuvels (1985). It appears that the case $b_n = [c \log n]$ is critical under (A), (B) and (C), and that one needs further assumptions on the distribution of X_1 to characterize the limiting behaviour of U_n when $b_n = o(\log n)$.

For the reasons above, we will specialize in the sequel in the case where $b_n/\log n \to \infty$. Csörgö and Steinebach (1981) have proved that if $b_n = [\tilde{b}_n]$, where $\tilde{b}_n$ is a sequence such that

$$\tilde{b}_n/\log n \uparrow \infty; \quad \tilde{b}_n/\log^p n \to 0 \quad \text{for some } p > 1; \quad 0 < \tilde{b}_n \leqq n; \quad \tilde{b}_n/n \downarrow, \tag{1.6}$$

then

$$U_n - b_n a_n = o(b_n^{1/2}) \quad \text{a.s.} \quad \text{as } n \to \infty, \tag{1.7}$$

where a_n is the unique positive solution of the equation

$$c(a_n)=\frac{b_n}{\log(n/b_n)}. \tag{1.8}$$

They also showed by the use of strong invariance principles that if $b_n=[\tilde{b}_n]$, where $\tilde{b}_n$ is a sequence such that

$$\tilde{b}_n/\log^2 n\to\infty;\quad \lim_{n\to\infty}\frac{\log(n/\tilde{b}_n)}{\log\log n}=\infty;\quad 0<\tilde{b}_n\leqq n;\quad \tilde{b}_n/n\downarrow, \tag{1.9}$$

then

$$U_n-\sigma\sqrt{2b_n\log(n/b_n)}=o(b_n^{1/2})\quad \text{a.s.}\quad \text{as } n\to\infty. \tag{1.10}$$

In this paper we obtain the exact rates of convergence of (1.7) and (1.10) together with a weak law for U_n after a suitable normalization. Our results extend also those of Deheuvels, Devroye and Lynch (1986) to the case where b_n is a general sequence satisfying $b_n=O(\log n)$.

Note that it is possible to prove by asymptotic expansions that (1.7) coincides with (1.10) in the range $b_n/\log^2 n\to\infty$, $b_n/\log^p n\to 0$ for some $p>2$. In this case, strong invariance takes over by the Komlós-Major-Tusnády (1976) approximation of partial sums by Wiener processes. On the other hand, for $[c\log n]\leqq b_n=O(\log^2 n)$, there exists a grey zone between strong invariance and strong noninvariance, with respect to the precision of the results given above.

In the sequel, we shall give expansions similar to (1.7) and (1.10) with the $o(b_n^{1/2})$ replaced by $O(b_n^{1/2}(\log\log n)(\log n)^{-1/2})$ together with precise lim inf and lim sup results including the best constants. Because of this increased precision, the range uncovered by strong invariance principles will be extended to $[c\log n]\leqq b_n=O((\log^3 n)/(\log\log n)^2)$.

Before stating our results in detail, we need to discuss asymptotic expansions related to the moment generating function ϕ and some auxiliary large deviation results for the sequence $\{S_n\}$. This will be done in Sects. 2 and 3, respectively. Section 4 presents the above mentioned weak limit law for $\{U_n\}$, and Sect. 5 contains the almost sure results. Finally, Sect. 6 discusses what strong invariance principles give in our present situation.

2. Expansions Related to the Moment Generating Function

Let X_1 be a random variable satisfying conditions (A), (B) and (C) of Sect. 1. In the sequel, we shall make use of the following notations. Let

$$\phi(t)=E(\exp(tX_1)),\qquad \psi(t)=\log\phi(t),\qquad m(t)=\frac{d}{dt}(\log\phi(t))=\psi'(t)=\phi'(t)/\phi(t),$$

$$\sigma^2(t)=m'(t)=\psi''(t).$$

Let also

$$A=\lim_{t\uparrow t_0} m(t)\quad \text{and}\quad c_0=1/\int_0^{t_0} t m'(t)\,dt,\qquad \text{with the convention that } 1/\infty=0.$$

It is well known (see, e.g., Deheuvels et al. (1986)) that ϕ, ψ, m and $\sigma^2(.)$ are infinitely differentiable on (t_1, t_0). Furthermore $\sigma^2(t)>0$ and $m(t)$ is strictly increasing on $[0, t_0)$ with $m(0)=0$. It follows that, for any $0<\alpha<A$, the equation $m(t)=\alpha$ has a unique positive solution $t=t^*=t(\alpha)$ such that

$$\rho=\rho(\alpha)=\inf_t\{\phi(t)\,e^{-t\alpha}\}=\phi(t^*)\,e^{-t^*\alpha}.$$

Let $c=c(\alpha)=-1/\log\rho(\alpha)$. It may be proved (see Deheuvels, Devroye and Lynch (1986)) that $c(\alpha)$ is a decreasing differentiable function of $\alpha\in(0, A)$ with

$$\lim_{\alpha\downarrow 0} c(\alpha)=\infty \quad \text{and} \quad \lim_{\alpha\uparrow A} c(\alpha)=c_0.$$

Hence $c(\alpha)$ takes all values in (c_0, ∞) as α decreases from A to 0. Likewise $t^*=t(\alpha)$ increases from 0 to t_0 as α increases from 0 to A.

Next, the finiteness of $\phi(t)$ for $|t|<\min(t_0, -t_1)$ implies the existence of all moments of X_1 and the fact that the expansion

$$\varphi(t)=\tfrac{1}{2!}t^2 M_2+\tfrac{1}{3!}t^3 M_3+\ \dots,\qquad \text{where } M_k=E(X_1^k),$$

holds for t in a neighborhood of 0 (see e.g., Lukacs, (1970), Chap. 7). The same is true for the cumulant generating function

$$\psi(t)=\log\phi(t)=\tfrac{1}{2!}t^2\kappa_2+\tfrac{1}{3!}t^3\kappa_3+\ \dots, \tag{2.1}$$

where $\kappa_1=0$, $\kappa_2=M_2$, $\kappa_3=M_3$, $\kappa_4=M_4-3M_2^2$, ... are the cumulants of X_1.

From there follow the expansions

$$m(t)=tM_2+\tfrac{1}{2}t^2 M_3+O(t^3),$$

and

$$\sigma^2(t)=M_2+tM_3+O(t^2)\quad as\ \ t\to 0.$$

In the sequel, it will be convenient to set $\sigma^2=\sigma^2(0)=M_2$ and $\gamma=M_3\,M_2^{-3/2}$. Straightforward computations show that, when $\alpha\to 0$,

$$t^*=t(\alpha)=\frac{\alpha}{M_2}-\frac{M_3}{2M_2^3}\alpha^2+O(\alpha^3), \tag{2.2}$$

and

$$-\frac{1}{c(\alpha)}=\log\varphi(t^*)-t^*\alpha=-\frac{\alpha^2}{2M_2}+\frac{M_3}{6M_2^3}\alpha^3+O(\alpha^4). \tag{2.3}$$

Let $\{b_n, n\geqq 1\}$ be a sequence such that

(*D*) $b_n/\log n\to\infty$ as $n\to\infty$.

By the arguments above, there exists always an n_0 such that $b_n/\log n>c_0$ for $n\geqq n_0$, and hence for which the equation in α

$$c(\alpha)=\frac{b_n}{\log n}\Leftrightarrow\exp\left(-\frac{b_n}{c(\alpha)}\right)=\frac{1}{n}\Leftrightarrow\rho^{b_n}(\alpha)=\frac{1}{n}, \tag{2.4}$$

has a unique solution $\alpha = \alpha_n \in (0, A)$. For sake of simplicity, we shall always assume in the sequel that $n \geqq n_0$.

Clearly $(D) \Leftrightarrow c(\alpha_n) \to \infty$, which implies that $\alpha_n \to 0$ as $n \to \infty$. Hence (2.3) guarantees that

$$\alpha_n = \sigma \left\{\frac{2 \log n}{b_n}\right\}^{1/2} + \frac{\sigma\gamma}{3}\left\{\frac{\log n}{b_n}\right\} + O\left\{\frac{\log n}{b_n}\right\}^{3/2} \quad \text{as } n \to \infty. \tag{2.5}$$

Put, in the sequel, $t_n = t(\alpha_n)$. By (2.2) and (2.5), we have

$$t_n = \sigma^{-1}\left\{\frac{2 \log n}{b_n}\right\}^{1/2} - \frac{2\gamma}{3\sigma}\left\{\frac{\log n}{b_n}\right\} + O\left\{\frac{\log n}{b_n}\right\}^{3/2} \quad \text{as } n \to \infty. \tag{2.6}$$

Define likewise for n large enough $a = a_n$ as the unique positive solution of the equation

$$c(a) = \frac{b_n}{\log(n/b_n)} \Leftrightarrow \exp\left(-\frac{b_n}{c(a)}\right) = \frac{b_n}{n} \Leftrightarrow \rho^{b_n}(a) = \frac{b_n}{n}, \tag{2.7}$$

where we assume, in addition to (D):

(E) There exists a $p > 1$ such that $b_n/\log^p n \to 0$ as $n \to \infty$.

In the sequel, we shall make use of the following expansions, whose proofs are similar to those of (2.5) and (2.6) and will therefore be omitted. We have

$$a_n = \sigma\left\{\frac{2 \log(n/b_n)}{b_n}\right\}^{1/2} + \frac{\sigma\gamma}{3}\left\{\frac{\log(n/b_n)}{b_n}\right\} + O\left\{\frac{\log(n/b_n)}{b_n}\right\}^{3/2}, \tag{2.8}$$

and

$$b_n \alpha_n - t_n^{-1} \log b_n = b_n a_n + o(t_n^{-1} \operatorname{loglog} n) \quad \text{as } n \to \infty, \tag{2.9}$$

if $\{b_n\}$ satisfies (D) and (E).

3. Auxiliary Large Deviation Results

In this section, we use the hypotheses and notations of Sects. 1 and 2. Moreover, define $\mu = P_{X_1}$ (resp. $\mu_k = P_{S_k}$) as the probability distribution of X_1 (resp. S_k), and let $G(\mu)$ denote the smallest additive subgroup of $\mathbb{R}$ containing all differences of numbers in $\operatorname{supp}(\mu)$. Note that, since μ is nondegenerate, either $G(\mu) = \mathbb{R}$ or $G(\mu) = l\mathbb{Z}$ for some $l > 0$. Let further

$$u(x) = \frac{e^{x/2}}{\sqrt{2\pi}} \int_{\sqrt{x}}^{\infty} e^{-t^2/2}\, dt, \qquad x > 0,$$

and

$$v(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\exp\{x(e^{i\xi} - 1 - i\xi)\}}{1 - y e^{i\xi}}\, d\xi, \qquad x > 0, \qquad y > 0.$$

Some applications of the following large deviation result will play a key role in the proof of our main theorems. It is an extension due to Höglund (1979) of an earlier result of Petrov (1965).

Theorem A. *a) If* $G(\mu)=\mathbb{R}$, *and if* $0<a<A$ *is fixed, then uniformly in* $\alpha\in(0,a)$, *as* $k\to\infty$,

$$P(S_k\geqq k\alpha)=\rho^k(\alpha)\,u(kt^2(\alpha)\,\sigma^2(t(\alpha)))\,\{1+o(1)\}. \tag{3.1}$$

b) If $G(\mu)=l\mathbb{Z}$, *and if* $0<a<A$ *is fixed, then uniformly in* $\alpha\in(0,a)$ *such that* $k\alpha\in\operatorname{supp}(\mu_k)$, *we have, as* $k\to\infty$,

$$P(S_k\geqq k\alpha)=\rho^k(\alpha)\,v\left(\frac{k\sigma^2(t(\alpha))}{2},e^{-lt(\alpha)}\right)\{1+O/(k^{-1/2})\}. \tag{3.2}$$

In the sequel, we shall apply Theorem A to the special case where $k=b_n$ and

$$\alpha=\alpha_n-b_n^{-1}\,t_n^{-1}\{\log b_n+O(\log_2 n)\}. \tag{3.3}$$

By (2.5) and (2.6) we have $\alpha_n b_n t_n\sim 2\log n$ as $n\to\infty$, whenever (D) holds. In order to have $\alpha\sim\alpha_n$ in (3.1), we shall make the complementary assumption that

(F) $(\log b_n)/\log n\to 0$ as $n\to\infty$.

In the remainder of this section, we assume that the assumptions (D) and (E) are satisfied and investigate the behaviour of the corresponding coefficients of $\rho^k(\alpha)$ in (3.1) and (3.2) for $\alpha=\alpha_n$, $k=b_n$ and $n\to\infty$. Making use of the facts

$$u(x)\sim 1/\sqrt{2\pi x} \quad \text{as } x\to\infty, \tag{3.4}$$

and

$$|v(x,e^{-b})-u(xb^2)|\leqq K\left(b+\frac{1}{\sqrt{x}}\right)u(xb^2), \quad x>0, \quad b>0, \tag{3.5}$$

where K is a constant (see, e.g., Höglund (1979), pp. 106–107), we immediately obtain the following expansions.

Lemma 1. *Let* $\{c_n\}$ *be a positive sequence such that* $c_n^2/\log n\to 0$ *as* $n\to\infty$. *Let* $k=b_n$ *and assume that* $\alpha=\alpha_n+b_n^{-1}\,t_n^{-1}\,z_n$. *Then, there exists a constant C depending upon the distribution of* X_1 *only, such that, uniformly over all sequences* $\{z_n\}$ *such that* $|z_n|\leqq c_n$, *we have*

$$P(S_k\geqq k\alpha)\sim C(\log n)^{-1/2}\rho^k(\alpha) \quad \text{as } n\to\infty.$$

Proof. Note that

$$kt^2(\alpha)\,\sigma^2(t(\alpha))\sim 2\log n\to\infty \quad \text{as } n\to\infty, \tag{3.6}$$

and combine (3.1)–(3.6).

Observe here that the result of Lemma 1 holds without restrictions on α such as those required in Theorem A (b).

Lemma 2. *Let $\{c_n\}$ be a positive sequence such that $c_n^2/\log n \to 0$ as $n \to \infty$. Let $k = b_n$ and assume that $\alpha = \alpha_n + b_n^{-1} t_n^{-1} z_n$. Then, there exists a constant C, depending upon the distribution of X_1 only, such that, uniformly over all sequences $\{z_n\}$ such that $|z_n| \leqq c_n$, we have*

$$P(S_k \geqq k\alpha) \sim \frac{C}{n} (\log n)^{-1/2} \exp(-z_n).$$

Proof. Observe that $\rho^k(\alpha_n) = 1/n$. Also, by the Taylor expansion of $-\log \rho(\alpha)$, we have

$$-\log \rho(\alpha) = -\log \rho(\alpha_n) + t(\tilde{\alpha})(\alpha - \alpha_n),$$

where $\tilde{\alpha}$ is between α_n and α. By our assumptions and (2.2), it can be seen that

$$\frac{t(\tilde{\alpha}) - t_n}{t_n} = \frac{t(\tilde{\alpha}) - t(\alpha_n)}{t(\alpha_n)} \sim \frac{\tilde{\alpha} - \alpha_n}{\alpha_n} = O\{z_n/(b_n \alpha_n t_n)\} = O\{z_n/\log n\},$$

where we have used (2.5) and (2.6). It follows that

$$\rho(\alpha) = \exp\left(-\frac{\log n}{k} - \frac{1}{k} z_n + \frac{1}{k} O\{z_n^2/\log n\}\right),$$

which, by Lemma 1, gives the result we seek.

4. A Weak Law of the Erdös-Rényi Type

In this section, we prove the following theorem.

Theorem 1. *Assume that $1 \leqq b_n < n$ is an integer sequence such that*

(i) $b_n/\log n \to \infty$ *as* $n \to \infty$;

(ii) $b_n/\log^p n \to 0$ *as* $n \to \infty$ *for some* $p > 1$.

Then

$$\lim_{n\to\infty} \frac{(U_n - b_n \alpha_n) t_n + \log b_n}{\log\log n} = \frac{1}{2} \quad \textit{in probability,} \tag{4.1}$$

where α_n is the unique positive solution of the Eq. (2.4) *and t_n is as in* (2.5). *Furthermore*

$$\lim_{n\to\infty} \frac{(U_n - b_n a_n) t_n}{\log\log n} = \lim_{n\to\infty} \frac{(2\log n)^{1/2}}{\sigma b_n^{1/2} \log\log n} (U_n - b_n a_n) = \frac{1}{2} \quad \textit{in probability} \tag{4.2}$$

where a_n is the unique positive solution of the Eq. (2.7).

Remark 1. Deheuvels et al. (1986) have proved that (4.1) holds for $b_n = [c \log n]$, where $c \in (c_0, \infty)$ is fixed. A close look to their proofs shows that they remain valid without modification whenever

$$c_1 \log n < b_n < c_2 \log n,$$

where $c_0 < c_1 < c_2 < \infty$ are fixed. It follows that (4.1) holds for any sequence b_n such that $b_n/\log n$ converges to a limit as $n \to \infty$ and satisfying

$$\lim_{n\to\infty} b_n/\log n \in (c_0, \infty], \quad \text{and} \quad b_n/\log^p n \to 0 \quad \text{as } n\to\infty, \quad \text{for some } p>1.$$

Remark 2. A direct application of Theorem 1 shows, without any further regularity assumption on the sequence $\{b_n\}$, that

$$\liminf_{n\to\infty} \frac{(U_n - b_n \alpha_n)\, t_n + \log b_n}{\log\log n} \leqq \frac{1}{2} \quad \text{a.s.,} \tag{4.3}$$

and

$$\limsup_{n\to\infty} \frac{(U_n - b_n \alpha_n)\, t_n + \log b_n}{\log\log n} \geqq \frac{1}{2} \quad \text{a.s.} \tag{4.4}$$

The proof of Theorem 1 is captured in the following sequence of lemmas.

Lemma 3. *Let $1 \leqq i \leqq k$ and let $S_i = X_1 + \ldots + X_i$, $S'_{k-i} = X_{i+1} + \ldots X_k$, $S''_i = X_{k+1} + \ldots X_{k+i}$. Then, for any x and y and for any $t \in (0, t_n)$, we have, if $k = b_n$*

$$P(S_i + S'_{k-i} \geqq x, \quad S'_{k-i} + S''_k \geqq x) \leqq (\phi(t_n))^{k-i} e^{-t_n q} + P(S_k \geqq x)\, (\phi(t))^i e^{-t(x-q)}.$$

Proof. See Deheuvels et al. (1986), Lemma 4.

We shall apply Lemma 3 with the following choices of k, x and q:

$$k = b_n, \quad \alpha = \alpha_n, \quad t = t_n/2, \quad q = \alpha k - (i/t_n) \log \phi(t_n) + (2/t_n) \log k,$$
$$x = \alpha k + ((\tfrac{1}{2} - \varepsilon) \log\log n - \log k)/t_n.$$

Note that, by Lemma 2, we have

$$P(S_k \geqq x) \sim \frac{C}{n} (k/\log N)(\log n)^{\varepsilon} \quad \text{as } n \to \infty. \tag{4.5}$$

Consider the events

$$A_i = \{S_{i+k} - S_i \geqq x\}, \quad 0 \leqq i \leqq n-k. \tag{4.6}$$

By Lemma 3, we have, for $0 \leqq i \leqq k$ and $0 < \varepsilon \leqq 1/2$,

$$\begin{aligned} P(A_0 \cap A_i) &\leqq (\phi(t_n))^{k-i} e^{-t_n q} + P(A_0)\, (\phi(t))^i e^{-t(x-q)} \\ &\leqq n^{-1} k^{-2} + P(A_0)\, k^{3/2} e^{-\theta i}, \end{aligned}$$

where, by (2.1) and (2.6),

$$\theta = t((1/t_n) \log \phi(t_n) - (1/t) \log \phi(t)) \sim \frac{\sigma^2}{2} t(t_n - t) = \frac{\sigma^2}{8} t_n^2 \sim \frac{1}{4} \frac{\log n}{k} \quad \text{as } n \to \infty.$$

Let $l \in \mathbb{R}$ be such that $1 \leqq l \leqq k$. We have evidently, for $0 < \varepsilon \leqq 1/2$ and n sufficiently large, uniformly in l,

$$\begin{aligned} \sum_{i=1}^{k} P(A_0 \cap A_i) &= \sum_{i=1}^{[l]} P(A_0 \cap A_i) + \sum_{i=[l]+1}^{k} P(A_0 \cap A_i) \\ &\leqq n^{-1} k^{-1} + P(A_0)\, l + P(A_0)\, k^{5/2} e^{-l\theta}. \end{aligned}$$

Let us now choose $l=(k/\log n)(\log n)^{\varepsilon/2}$. Using (4.5) and assumption (ii) of Theorem 1, we have

$$n\sum_{i=1}^{k} P(A_0\cap A_i)\leqq\{1+o(1)\}\, n l P(A_0)=\{1+o(1)\}\, C(k/\log n)^2(\log n)^{3\varepsilon/2} \quad (4.7)$$

$$=o\{(nP(A_0))^2\} \qquad \text{as } n\to\infty.$$

We will now make use of the following lemma, due to Chung and Erdös (1952).

Lemma 4. *For arbitrary of events* $A_0, \ldots, A_n$, *we have*

$$P\left(\bigcup_{i=0}^{N} A_i\right)\geqq\left(\sum_{i=0}^{N} P(A_i)\right)^2\Big/\left(\sum_{i=0}^{N} P(A_i)+\sum_{1\leqq i\neq j\leqq N} P(A_i\cap A_j)\right). \qquad (4.8)$$

Lemma 5. *Under the assumptions of Theorem* 1, *for any* $\varepsilon>0$,

$$\lim_{n\to\infty} P\left(\frac{(U_n-b_n\alpha_n)\,t_n+\log b_n}{\operatorname{loglog} n}\geqq\frac{1}{2}-\varepsilon\right)=1.$$

Proof. Let $A_0, \ldots, A_N$, $N=n-k$, $k=b_n$, be as in (4.5). We have $P(A_0)\to 0$ and

$$\sum_{i=0}^{N} P(A_i)=NP(A_0)\sim C(k/\log n)(\log n)^{\varepsilon}\to\infty \qquad \text{as } n\to\infty.$$

Likewise

$$\sum_{i=0}^{N} P(A_i)+\sum_{1\leqq i\neq j\leqq N} P(A_i\cap A_j)=\left(\sum_{i=0}^{N} P(A_i)\right)^2+\sum_{i=0}^{N}(P(A_i)-P^2(A_i))$$
$$+\sum_{1\leqq|i-j|\leqq k}(P(A_i\cap A_j)-P(A_i)\,P(A_j))\sim(nP(A_0))^2,$$

by (4.7).

Making use of Lemma 4 renders the proof complete.

Remark 3. In view of the proof of Lemma 7, we see that a critical step occurs in the proof of (4.7), where we have to show that, for an arbitrary $\varepsilon>0$,

$$\frac{k^{5/2}}{l}e^{-l\theta}=k^{3/2}(\log n)^{1-\varepsilon/2}\exp\left(-\frac{1+o(1)}{4}\log^{\varepsilon} n\right)\to 0 \qquad \text{as } n\to\infty.$$

It follows that the conclusion of Lemma 5 remains true if $k=b_n$ satisfies

$$b_n/\log n\to\infty \quad \text{and} \quad \frac{\operatorname{loglog} b_n}{\operatorname{loglog} n}\to 0 \qquad \text{as } n\to\infty. \qquad (4.9)$$

Lemma 6. *Let* $W_N=\max\{|S_i|,\ 1\leqq i\leqq N\}$. *There exists constants* $\beta>0$ *and* $\delta>0$ *such that, for any* $s>0$ *and* $N\geqq 1$, *we have*

$$P(W_N\geqq s\sqrt{N})\leqq 2\{\exp(-\beta s^2)+\exp(-\delta s\sqrt{N})\}.$$

Proof. By (2.1), for any $0<T<t_0$, there exists a constant $0<D<\infty$ such that

$$\sup_{0<t\leqq T} \frac{\log \phi(t)}{t^2} \leqq \frac{1}{2D}. \tag{4.10}$$

Let $V'_N=\max\{S_i,\ 1\leqq i\leqq N\}$. Consider the function $h_t^\omega(x)=e^{t(x-\omega)}$ which is convex and nondecreasing in x for any fixed $t>0$ and $\omega\in\mathbb{R}$. It is straightforward (see, e.g., Hall and Heyde (1980), pp. 13–14) that $\{h_t^\omega(S_n), n\geqq 1\}$ is a submartingale whenever $0<t<t_0$, from where, by the submartingale inequality, we get

$$P(h_t^\omega(V'_N)>1)=P(V'_N>\alpha)\leqq E(h_t^\omega(S_N))=\phi^N(t)\,e^{-t\omega}.$$

By taking $\omega=s\sqrt{N}$ and using the remark that the right hand side of the inequality above is continuous in ω, it follows that

$$P(V'_N\geqq s\sqrt{N})\leqq \phi^N(t)\exp(-ts\sqrt{N}), \qquad \text{for } 0<t\leqq T \quad \text{and arbitrary } s>0.$$

If $s\leqq T\sqrt{N}/D$, choose $t=Ds/\sqrt{N}<T$, so that the inequality above, jointly with (4.10) yields

$$P(V'_N\geqq s\sqrt{N})\leqq \phi^N(Ds/\sqrt{N})\exp(-Ds^2)\leqq \exp(-Ds^2/2).$$

If $s>T\sqrt{N}/D$, choose $t=T$. We get likewise

$$P(V'_N\geqq s\sqrt{N})\leqq \phi^N(T)\exp(-Ts\sqrt{N})\leqq \exp\left(\frac{NT^2}{2D}-Ts\sqrt{N}\right).$$

Since in this case $N\leqq (sD/T)\sqrt{N}$, we have $\dfrac{NT^2}{2D}\leqq \dfrac{Ts\sqrt{N}}{2}$. Hence we get

$$P(V''_N\geqq s\sqrt{N})\leqq \exp(-Ds^2/2)+\exp\left(-\frac{T}{2}s\sqrt{N}\right), \qquad s>0.$$

The same arguments used for $V''_n=\max\{-S_i,\ 1\leqq i\leqq N\}$ yield

$$P(V''_N\geqq s\sqrt{N})\leqq \exp(-D's^2/2)+\exp\left(-\frac{T'}{2}s\sqrt{N}\right), \qquad s>0,$$

where $D'>0$ and $T'>0$ are constants. The result follows by the inequality

$$P(W_N\geqq s\sqrt{N})\leqq P(V'_N\leqq s\sqrt{N})+P(V''_N\geqq s\sqrt{N}),$$

and by taking $\beta=\min(D,D')/2$ and $\delta=\min(T,T')/2$.

Lemma 7. *Let $0<R<\infty$ be a constant, and let $\{c_n\}$ be a positive sequence such that $c_n^2/\log n\to 0$ as $n\to\infty$. There exists constants $0<C_1\leqq C_2<\infty$ and $n_0<\infty$, such that, for all $n\geqq n_0$, $|k-b_n|\leqq Rb_n/\log n$ and $|z_n|\leqq c_n$, we have*

$$\frac{C_1}{n}(\log n)^{-1/2}\exp(-z_n)\leqq P(S_k\geqq b_n\alpha_n+t_n^{-1}z_n)\leqq \frac{C_2}{n}(\log n)^{-1/2}\exp(-z_n).$$

Proof. Let $\alpha=\alpha(k)$ be defined as in (2.4), to be the unique positive solution of $c(\alpha)=k/\log n$. Since $c(\alpha)\sim b_n/\log n$ as $n\to\infty$, the expansions given in Sect. 2 yield, as $n\to\infty$,

$$\alpha\sim\alpha_n\sim\sigma\left\{\frac{2\log n}{b_n}\right\}^{1/2},$$

$$t=t(\alpha)\sim\alpha/\sigma^2\sim\alpha_n/\sigma^2\sim t_n,$$

and

$$(\alpha_n-\alpha)/\alpha_n\sim(k-b_n)/2b_n.$$

Hence, by the condition imposed on k, it follows that there exists an $n_0<\infty$ such that, for $n\geqq n_0$ and uniformly in k,

$$|k\alpha-b_n\alpha_n|\leqq|k-b_n|\alpha+b_n|\alpha_n-\alpha|=O\{(b_n/\log n)^{1/2}\}=O(t^{-1})\quad\text{as}\quad n\to\infty.$$

Likewise, we obtain that $t-t_n\sim(\alpha-\alpha_n)/\sigma^2$, and hence

$$t^{-1}z_n-t_n^{-1}z_n^{-1}=t^{-1}z_n\frac{t_n-t}{t_n}=t^{-1}z_n O\left(\frac{k-b_n}{b_n}\right)=t^{-1}z_n O\left(\frac{1}{\log n}\right)\quad\text{as}\quad n\to\infty.$$

By all this, we see that

$$P(S_k\geqq b_n\alpha_n+t_n^{-1}z_n)=P(S_k\geqq k\alpha+t^{-1}(z_n+O(1))),$$

from where the result follows by Lemma 1 and 2.

Remark 4. It is immediate from the proof of Lemma 7 that if $|k-b_n|\leqq r_n$, where $r_n=o(b_n/\log n)$ as $n\to\infty$, then the constants C_1 and C_2 can be replaced by $C(1+o(1))$, where C is as in Lemmas 1 and 2.

Lemma 8. *Let $0<A, B<\infty$ be arbitrary constants. Let $N=[Ab_n/\log n]$ and let k be an integer such that $|k-b_n|\leqq BN$. Define, for all n sufficiently large, $T_n=S'_k+W_N+W'_N$, where S'_k, W_N and W'_N are independent, S'_k following the same distribution as S_k, while W_N is defined as in Lemma 6 and following the same distribution as W'_N.*

Then, for any $\varepsilon\in\mathbb{R}$ and $v>0$, we have, as $n\to\infty$, uniformly in k,

$$nP(T_n\geqq b_n\alpha_n+t_n^{-1}(-\log b_n+(\tfrac{1}{2}+\varepsilon)\operatorname{loglog} n))=\frac{b_n}{\log n}O\{(\log n)^{v-\varepsilon}\}.$$

Proof. Let $\log_2 n=\operatorname{loglog} n$. By Lemmas 2 and 7, we have

$$P_l=P(S'_k\geqq b_n\alpha_n+t_n^{-1}(-\log b_n+(\tfrac{1}{2}+\varepsilon-lv)\log_2 n))=O\left\{\frac{1}{n}\left(\frac{b_n}{\log n}\right)(\log n)^{lv-\varepsilon}\right\},$$

the result being uniform in $|l|\leqq d_n$, where $d=d_n$ is any sequence such that $d_n=o(\sqrt{\log n}/\log_2 n)$. Take in the sequel $d_n=[(\log n)^{1/4}/\log_2 n]$, and consider

$$\Sigma_1=P_0+\sum_{l=1}^{d}(P_l-P_{l-1})P(W_N+W'_N\geqq t_n^{-1}(l-1)v\log_2 n)\leqq 2P_1+\sum_{l=2}^{d}P_lQ_l,$$

where by Lemma 6, $Q_l = P(W_N + W'_N \geqq t_n^{-1}(l-1)\nu \log_2 n)$ is such that

$$Q_l \leqq 2P(W_N \geqq \tfrac{1}{2} t_n^{-1}(l-1)\nu \log_2 n) \leqq 4\left\{\exp(-B(l-1)^2 \log_2^2 n) + \exp\left(-E(l-1)\left(\frac{b_n}{\log n}\right)^{1/2} \log_2 n\right)\right\},$$

where B and E are appropriate positive constants. It is now straightforward that

$$\Sigma_1 = O(P_1) = O\left\{\frac{1}{n}\left(\frac{b_n}{\log n}\right)(\log n)^{\nu-\varepsilon}\right\} \quad \text{as } n \to \infty.$$

Consider now the sequence $c = c_n = [K(\log n)/\log_2 n]$, where $0 < K < \infty$ is fixed. By Lemma 6

$$\Sigma_2 = \sum_{l=c+1}^{\infty} (P_l - P_{l-1}) Q_l \leqq Q_c \leqq 2P(W_N \geqq \tfrac{1}{2} t_n^{-1}(c-1)\nu \log_2 n) = o(1/n^2) = o(P_1).$$

Here, we have used the fact that $b_n/\log n \to \infty$.

Finally, we need consider only

$$\Sigma_3 = \sum_{l=d+1}^{c} (P_l - P_{l-1}) Q_l \leqq \sum_{l=d+1}^{c} P_l Q_l.$$

We note that, in the range of interest for Σ_3, P_l is of the form $P(S_k \geqq k\alpha + t^{-1} z_n)$, where α and t are as in Lemma 7, and where $|z_n| = O(\log n)$. By the same arguments as used in the proofs of Lemmas 1 and 2, we see that for such z_n's, there exists constants $\gamma > 0$ and $E > 0$, such that $P_l \leqq \frac{E}{n} \exp(-\gamma z_n)$. This last result follows from the fact that, whenever $0 < \alpha$, $\alpha + h < A$, $\rho^k(\alpha + h) = \rho^k(\alpha) \exp(-kht(\tilde{\alpha}))$, where $\tilde{\alpha}$ lies between α and $\alpha + h$.

In the present case, $h = k^{-1} t^{-1} z_n = O\left\{\left(\frac{\log n}{b_n}\right)^{1/2}\right\}$, so that, for some appropriate constants $0 < G < H < \infty$, $0 < G\alpha < \tilde{\alpha} < H\alpha$. Finally, we use (2.5) and (2.6) to show that $t(\tilde{\alpha}) \sim \tilde{\alpha}/\sigma^2$ as $n \to \infty$. We have used here the fact that $0 < K < \infty$ may be chosen as small as desired, which enables us to have $G > 0$.

By all this, in Σ_3,

$$P_l Q_l \leqq \frac{4E}{n} \exp\left(\gamma \nu \{1 + o(1)\}(l-1) \log_2 n - B(l-1)^2 \log_2^2 n - E(l-1)\left(\frac{b_n}{\log n}\right)^{1/2} \log_2 n\right),$$

so that, as $n \to \infty$,

$$\Sigma_3 = \frac{1}{n} O\left\{\exp\left(-\frac{1}{2} B d_n^2 \log_2^2 n\right)\right\} = \frac{1}{n} O\left\{\exp\left(-\frac{1}{4} B(\log n)^{1/4}\right)\right\} = o(P_1).$$

The proof of the lemma follows from the fact that the probability we seek is bounded above by $\Sigma_1+\Sigma_2+\Sigma_3$.

Lemma 9. *For any* $\varepsilon>0$,

$$\lim_{n\to\infty} P\left(\frac{(U_n-b_n\alpha_n)\,t_n+\log b_n}{\log\log n}\leqq\frac{1}{2}+\varepsilon\right)=1.$$

Proof. In the proof, we will use the remark that $U_n\leqq V_n$, where

$$V_n=\max\{(S_{j+b_n}-S_{j+N})+(S_{j+N}-S_{j+i})+(S_{j+b_n+l}-S_{j+b_n}),\\ 0\leqq i,l\leqq N,\ j=0,N,2N,\ldots,j\leqq n\},$$

where we choose $N=[b_n/\log n]$.

Note that $V_n\leqq\max\{V_{ni},\ 1\leqq i\leqq 1+(n/N)\}$, where

$$V_{ni}=(S_{(i-1)N+b_n}-S_{iN})+\max\{S_{iN}-S_{(i-1)N+l},0\leqq l\leqq N\}\\ +\max\{S_{(i-1)N+b_n+l}-S_{(i-1)N+b_n},0\leqq l\leqq N\}.$$

We now apply Lemma 8, with the following choices of k, A and B. Let $k=b_n-N$, $A=B=1$. If

$$d=b_n\alpha_n+t_n^{-1}(-\log b_n+(\tfrac{1}{2}+\varepsilon)\log\log n),$$

then, for any fixed $v>0$, as $n\to\infty$,

$$P(V_n\geqq d)\leqq\left(1+\frac{n}{N}\right)P(V_{n1}\geqq d)=O\left\{\frac{n}{N}\left(\frac{b_n}{n\log n}\right)(\log n)^{v-\varepsilon}\right\}=O\{(\log n)^{v-\varepsilon}\}.$$

Take now $\varepsilon>0$ and $v=\varepsilon/2$. It is straightforward that $P(V_n\geqq d)\to 0$ as $n\to\infty$. Since $\varepsilon>0$ may be chosen as small as desired, we have the proof of Lemma 9.

Proof of Theorem 1. (4.1) follows from Lemmas 5 and 9. The equivalence between (4.1) and (4.2) follows from (2.9).

Remark 5. The proof of Theorem 1 shows that the results can be extended to the case where $\phi(t)<\infty$ for $0\leqq t<t_0$ only (without assuming that $\phi(t)<\infty$ for some $t<0$). However, in the expansions of the moment generating function, we have used the existence of $\sigma^2=E(X_1^2)$ and we would need to assume (say) that $E(|X_1|^{2+\delta})<\infty$ for some $\delta>0$. On the other hand, in view of Sect. 6, such an assumption looses interest because of the lack of sharpness in the Komlós-Major-Tusnády strong approximation of partial sums if one does not assume the finiteness of $\phi(t)$ in a neighborhood of zero. This explains why we assume throughout that (*C*) holds.

5. Strong Laws

In this section, we shall prove the following theorem.

Theorem 2. *Assume that* $b_n\uparrow$ *is an integer sequence such that* $1\leqq b_n\leqq n$, $b_n/\log n\to\infty$, *and* $b_n/\log^p n\to 0$ *for some* $p>1$. *Assume further that there exists a real* *valued sequence* $\tilde{b}_n$ *such that*

$$(i)\ b_n-\tilde{b}_n=O\left(\frac{b_n}{\log n}\right), \quad and \quad (ii)\ \tilde{b}_{n+1}-\tilde{b}_n=O\left(\frac{\tilde{b}_n}{n\log n}\right), \qquad as\ n\to\infty.$$

Then

$$\limsup_{n\to\infty}\frac{(U_n-b_n\alpha_n)\,t_n+\log b_n}{\log\log n}=\frac{3}{2} \quad a.s., \tag{5.1}$$

and

$$\liminf_{n\to\infty}\frac{(U_n-b_n\alpha_n)\,t_n+\log b_n}{\log\log n}=\frac{1}{2} \quad a.s., \tag{5.2}$$

where α_n *is the unique positive solution of the Eq.* (2.4) *and* t_n *is as in* (2.5). *Furthermore*

$$\limsup_{n\to\infty}\frac{(U_n-b_n a_n)\,t_n}{\log\log n}=\limsup_{n\to\infty}\frac{(2\log n)^{1/2}}{\sigma b_n^{1/2}\log\log n}(U_n-b_n a_n)=\frac{3}{2} \quad a.s., \tag{5.3}$$

and

$$\liminf_{n\to\infty}\frac{(U_n-b_n a_b)\,t_n}{\log\log n}=\liminf_{n\to\infty}\frac{(2\log n)^{1/2}}{\sigma b_n^{1/2}\log\log n}(U_n-b_n a_n)=\frac{1}{2} \quad a.s., \tag{5.4}$$

where a_n *is the unique positive solution of the Eq.* (2.7).

Remark 6. (5.1) and (5.2) are in agreement with (1.4), proved by Deheuvels et al. (1986), when $b_n=[c\log n]$, $c>c_0$. By using the same arguments as in their proofs, one may extend the validity of (5.1) and (5.2) to the case where $b_n/\log n \to c\in(c_0,\infty)$, where $\{b_n\}$ is a sequence satisfying assumptions (i) and (ii) in Theorem 1.

Remark 7. Some regularity assumptions on the sequence b_n are used in our proofs. We will now discuss these conditions. Let $a>1$ be a fixed number, and consider the integer sequence $n_j=[a^j]$, $j=1,2,\ldots$; let $B_j=b_{n_j}$, $A_j=\alpha_{n_j}$ and $T_j=t_{n_j}$. We shall denote by (Ga) the assumption that (recall that $\log_2 t=\log\log t$)

$$(Ga)\ b_{n_{j+1}}-b_{n_j}=o\left(\frac{b_{n_j}\log_2 n_j}{\log n_j}\right)\Leftrightarrow B_{j+1}-B_j=o\left(\frac{B_j\log j}{j}\right) \qquad \text{as } j\to\infty$$

Observe that the sequence $B_j=[\exp(\log^\delta j)]$ satisfies (Ga) for all $a>1$ iff $1\leqq\delta<2$. It is also straightforward that the range of interest covered by (Ga) includes all sequences of the form

$$b_n=[\exp(\log_2^\delta n)], \qquad 1\leqq\delta<2,$$
$$b_n=[\log^p n], \qquad p\geqq 1,$$

among others.

The reason for assuming (Ga) is that (by the same arguments as used in the proof of Lemma 7) we have, as $j\to\infty$,

$$B_{j+1}A_{j+1}-B_jA_j\sim\sigma\left\{\frac{\log n_j}{2B_j}\right\}^{1/2}(B_{j+1}-B_j)$$

$$\sim\frac{1}{T_j}\left\{\frac{\log n_j}{B_j}\right\}(B_{j+1}-B_j)=\frac{1}{T_j}o(\log n_j).$$

Simple sets of conditions which imply $(G a)$ for all $a>0$ are as follows:
(G_2) There exists a real sequence $\{\tilde{b}_n\}$ such that

(i) $\tilde{b}_{n+1}-\tilde{b}_n=o\left(\frac{\tilde{b}_n\log_2 n}{n\log n}\right)$ as $n\to\infty$;

(ii) $b_n-\tilde{b}_n=o\left(\frac{b_n\log_2 n}{\log n}\right)$ as $n\to\infty$;

(G_1) There exists a real sequence $\{\tilde{b}_n\}$ such that

(i) $\tilde{b}_{n+1}-\tilde{b}_n=O\left(\frac{\tilde{b}_n}{n\log n}\right)$ as $n\to\infty$;

(ii) $b_n-\tilde{b}_n=O\left(\frac{b_n}{\log n}\right)$ as $n\to\infty$.

Note that $(G_1)\Rightarrow(G_2)\Rightarrow(Ga)$ for all $a>0$. (G_1), in turn is satisfied under the following assumption:

(G) There exists a real sequence $\{\tilde{b}_n\}$ such that

(i) $\tilde{b}_n/\log n\uparrow$;
(ii) $\tilde{b}_n/\log^p n\downarrow$ for some $p>1$;
(iii) $b_n=[\tilde{b}_n]$.

In the remainder of this section, and unless otherwise specified, we shall assume that (G_1) holds, even though most of our arguments remain valid under weaker assumptions.

Lemma 10. *Under (G_1), there exists a constant $0<D<\infty$ and a $J<\infty$ such that, for all $j\geqq J$ and $n_{j-1}\leqq n\leqq n_j$,*

$$|b_n-b_{n_j}|\leqq D(b_{n_j}/\log n_j),$$

and

$$|\alpha_n b_n-\alpha_{n_j}b_{n_j}|\leqq D(b_{n_j}/\log n_j).$$

Proof. We use the same arguments as in the proof of Lemma 7. Details are omitted.

Let us now define, as in the proof of Lemma 9,

$$V_n=\max\{(S_{j+b_n}-S_{j+N})+(S_{j+N}-S_{j+i})+(S_{j+b_n+l}-S_{j+b_n}),$$
$$0\leqq i,l\leqq N,\ j=0,N,2N,\ldots,j\leqq N\},$$

where we choose here $N=[2Db_n/\log n]$, D being as in Lemma 10.

By such a definition of V_n, we see evidently that, for all $n_{j-1} \leqq n \leqq n_j$, we have $U_n \leqq V_{n_j}$. By Lemma 8, using the same arguments as in the proof of Lemma 9, it follows that

$$P\Big(\bigcup_{n_{j-1} \leqq n \leqq n_j} \{U_n \geqq \alpha_n b_n + t_n^{-1}(-\log b_n + (\tfrac{3}{2}+\varepsilon)\log_2 n)\}\Big)$$
$$\leqq P\left(V_{n_j} \geqq \alpha_{n_j} b_{n_j} + t_{n_j}^{-1}\left(-\log b_{n_j} + \left(\frac{3}{2}+\frac{\varepsilon}{2}\right)\log_2 n_j\right)\right) = O((\log n_j)^{-1-\varepsilon/4})$$

as $j \to \infty$.

Observe that $\sum_j (\log n_j)^{-1-\varepsilon/4} < \infty$. By Borel-Cantelli, we have just proved:

Lemma 11. *Under* (G_1), *if* $b_n/\log n \to \infty$, *we have*

$$\limsup_{n \to \infty} \frac{(U_n - b_n \alpha_n) t_n + \log b_n}{\log\log n} \leqq \frac{3}{2} \quad a.s.$$

In order to prove that the upper bound given in Lemma 11 is sharp, one needs only show that

$$\limsup_{j \to \infty} \frac{(U_{n_j} - b_{n_j} \alpha_{n_j}) t_{n_j} + \log b_{n_j}}{\log\log n_j} \geqq \frac{3}{2} \quad \text{a.s.}$$

This, in turn, will follow from

$$\limsup_{j \to \infty} \frac{(\xi_{n_j} - b_{n_j} \alpha_{n_j}) t_{n_j} + \log b_{n_j}}{\log\log n_j} \geqq \frac{3}{2} \quad \text{a.s.},$$

where we define here ξ_{n_j} by $\xi_{n_j} = \max\{S_{i+b_{n_j}} - S_i,\ n_{j-1} \leqq i \leqq n_j - b_{n_j}\}$. It is noteworthy that the ξ_{n_j} are independent for different values of j. Hence, by Borel-Cantelli, all we need is to show that, for any $\varepsilon > 0$,

$$\sum_j P_j = \sum_j P(\xi_{n_j} - b_{n_j} \alpha_{n_j} \geqq t_{n_j}^{-1}(-\log b_{n_j} + (\tfrac{3}{2}-\varepsilon)\log_2 n_j)) = \infty,$$

The latter relation will be obtained by deriving a lower bound for P_j. Let $k = b_{n_j}$ and $N = [b_{n_j}/\log n_j]$. We have evidently $\xi_{n_j} \geqq \zeta_{n_j}$, where

$$\zeta_{n_j} = \max\{S_{n_{j-1}+i+k} - S_{n_{j-1}+i},\ i = 0, N, 2N, \ldots, i < n_j - n_{j-1} - k\}.$$

Let $x = b_{n_j} \alpha_{n_j} + t_{n_j}^{-1}(-\log b_{n_j} + (\tfrac{3}{2}-\varepsilon)\log_2 n_j)$ and $R = \left[\dfrac{n_j - n_{j-1} - k}{N}\right]$. We have

$$P_j \geqq P(\zeta_{n_j} \geqq x) = P\Big(\bigcup_{1 \leqq l \leqq R} \{S_{Nl+k} - S_{Nl} \geqq x\}\Big) = P\Big(\bigcup_{1 \leqq l \leqq R} A_l\Big).$$

By Lemma 2, we have

$$P(A_l) = P(A_1) \sim \frac{C}{n_j}\left(\frac{b_{n_j}}{\log n_j}\right)(\log n_j)^{-1+\varepsilon} \quad \text{as } j \to \infty.$$

Hence

$$\sum_{l=1}^{R} P(A_l)=RP(A_1)\sim C\left(\frac{a-1}{a}\right)(\log a)^{-1+\varepsilon} j^{-1+\varepsilon}=cj^{-1+\varepsilon}, \quad \text{as } j\to\infty.$$

Let us now use the Chung-Erdös evaluation (i.e. Lemma 4). We have

$$P\Big(\bigcup_{1\leqq l\leqq R} A_l\Big)\geqq (RP(A_1))^2 \Big/ \left(RP(A_1)+(RP(A_1))^2+2R\sum_{l=1}^{m} P(A_0\cap A_l)\right),$$

where $A_0=\{S_k\geqq x\}$ and $m=[K/N]+1\sim\log n_j\sim j\log a$ as $j\to\infty$ (we use the fact that A_i and A_l are independent for $|i-l|>m$). Observe that, for $0<\varepsilon<1$,

$$RP(A_1)+(RP(A_1))^2\sim RP(A_1)\sim cj^{-1+\varepsilon} \quad \text{as } j\to\infty.$$

In the sequel, we will show that, as $j\to\infty$,

$$R\sum_{l=1}^{m} P(A_0\cap A_l)=O(j^{-1+3\varepsilon/2}), \tag{5.5}$$

which will in turn imply that, for a suitable constant $d>0$,

$$P_j\geqq P\Big(\bigcup_{1\leqq l\leqq R} A_l\Big)\geqq dj^{-1+\varepsilon/2}.$$

This, in turn, will imply that $\sum_j P_j=\infty$, which suffices for our needs. By Lemma 3, we have, for any $0<t<t_{n_j}$, x and q,

$$P(A_0\cap A_l)\leqq(\phi(t_{n_j}))^{k-Nl}\exp(-t_{n_j}q)+P(A_0)(\phi(t))^{Nl}\exp(-t(x-q))=Q_l'+Q_l'',$$

where $k=b_{n_j}$. We shall now make the following choices for x and q. Let

$$q=b_{n_j}\alpha_{n_j}-(Nl/t_{n_j})\log\phi(t_{n_j})+K\frac{\log l}{t_{n_j}},$$

where $K=K_{n_j}=(\log n_j)^{\varepsilon/4}$, and

$$x=b_{n_j}\alpha_{n_j}+t_{n_j}^{-1}(-\log b_{n_j}+(\tfrac{3}{2}-\varepsilon)\log_2 n_j).$$

For these choices, we have $Q_l'=n_j^{-1}\,l^{-K}$, while

$$Q_l''=P(A_0)\exp\left\{-Nlt\left(\frac{1}{t_{n_j}}\log\phi(t_{n_j})-\frac{1}{t}\log\phi(t)\right)\right.$$
$$\left.-\frac{t}{t_{n_j}}\left(-\log b_{n_j}+(\tfrac{3}{2}-\varepsilon)\log_2 n_j-K\log l\right)\right\}.$$

Take now $t=t_{n_j}/2$. By (2.1) and (2.6), and using the fact that $N t_{n_j}^2 \sim 2/\sigma^2$, we get

$$Q_l''=P(A_0)\exp\left(-\frac{l}{4}\{1+o(1)\}\right) l^{K/2} b_{n_j}^{1/2}(\log n_j)^{-\frac{3}{4}+\frac{\varepsilon}{2}}$$

$$=o\left(\frac{1}{n_j} l^{K/2} e^{-l/8} b_{n_j}^{3/2}(\log n_j)^{-\frac{11}{4}+\frac{3\varepsilon}{2}}\right) \quad \text{as } j\to\infty.$$

Let $r=[(\log n_j)^{\varepsilon/2}]\sim m^{\varepsilon/2}$. We have

$$\sum_{l=r}^{m} Q_l'=\frac{1}{n_j} O(r^{1-K})=\frac{1}{n_j} O\{(\log n_j)^{\varepsilon(1-K)/2}\},$$

and

$$\sum_{l=r}^{m} Q_l''=\frac{1}{n_j} O(e^{-r/16})=o(P(A_0)),$$

where we have used the assumption (E) that $b_n=O(\log^p n)$ for some $p>1$, or the weaker assumption $(\log b_n)/\log n\to 0$.

Using the trivial upper bound $P(A_0\cap A_l)\leqq P(A_0)$, we have

$$\sum_{l=1}^{r} P(A_0\cap A_l)\leqq rP(A_0)\sim(\log a)^{\varepsilon/2} j^{\varepsilon/2} P(A_0)=R^{-1} O\{j^{-1+3\varepsilon/2}\}.$$

Recall that $P(A_0)\sim R^{-1} c j^{-1+\varepsilon}$ as $j\to\infty$.

In order to complete the proof of (5.5), only one piece of the puzzle is missing, namely to show that

$$R\sum_{l=r}^{m} Q_l'=\frac{R}{n_j} O\{j^{\varepsilon(1-K)/2}\}=O\{j^{-1+3\varepsilon/2}\} \quad \text{as } j\to\infty.$$

This last statement is in turn equivalent to

$$j^{1-\frac{3\varepsilon}{2}+\frac{\varepsilon}{2}(1-K)}=O\left\{\frac{b_{n_j}}{\log n_j}\right\} \quad \text{as } j\to\infty,$$

which evidently holds since $K=K_{n_j}\to\infty$ and $b_n/\log n\to\infty$.

By all this, we have completed the proof of (5.5). Observe that we did not make use in our arguments of regularity conditions on b_n. We have just proved:

Lemma 12. *Let $1\leqq b_n\leqq n$ be an arbitrary integer sequence such that $b_n/\log n\to\infty$ and $b_n/\log^p n\to 0$ as $n\to\infty$ for some $p>1$. Then*

$$\limsup_{n\to\infty}\frac{(U_n-b_n\alpha_n)t_n+\log b_n}{\log\log n}\geqq\frac{3}{2} \quad \text{a.s.}$$

All we need to conclude the proof of Theorem 2 is to show that

$$\liminf_{n\to\infty} \frac{(U_n - b_n \alpha_n) t_n + \log b_n}{\log\log n} \geqq \frac{1}{2} \quad \text{a.s.}$$

For this, we will first evaluate, for $M \geqq 1$ integer to be precised later on,

$$P\Big(\max\{S_{i+b_n} - S_i, i=0, M, 2M, \ldots, i \leqq b_n\}$$

$$\geqq b_n \alpha_n + \frac{1}{t_n}(-\log b_n + (\tfrac{1}{2} - \varepsilon)\log_2 n)\Big) = P\Big(\bigcup_{i=1}^{R} A_i\Big),$$

where $A_i = \{S_{(i-1)M+b_n} - S_{(i-1)M} \geqq b_n \alpha_n + \frac{1}{t_n}\;(-\log b_n + (\frac{1}{2} - \varepsilon)\;\log_2 n\}$, and $R = [b_n/M] + 1$.

Bonferroni's inequality yields

$$P\Big(\bigcup_{i=1}^{R} A_i\Big) \geqq RP(A_0) - 2R \sum_{l=1}^{R} P(A_0 \cap A_l). \tag{5.6}$$

By a proper choice of $M = o(b_n)$, we want to ensure that, as $n \to \infty$,

$$\sum_{l=1}^{R} P(A_0 \cap A_l) = o(P(A_0)). \tag{5.7}$$

Note that by Lemma 2,

$$RP(A_0) \sim \frac{C}{n}(b_n/M)(b_n/\log n)(\log n)^{\varepsilon} \qquad \text{as } n \to \infty.$$

For the proof of (5.7), we use again Lemma 3 with $k = b_n$, $t = t_n/2$, $i = Ml$,

$$x = b_n \alpha_n + t_n^{-1}(-\log b_n + (\tfrac{1}{2} - \varepsilon)\log_2 n),$$

and

$$q = b_n \alpha_n - t_n^{-1}\, Ml \log \phi(t_n) + 2 t_n^{-1} \log(Ml).$$

Observe by (2.1) and (2.6) that, as $n \to \infty$,

$$\log \phi(t_n) \sim \frac{\log n}{b_n} \quad \text{and} \quad \log \phi(t) \sim \frac{\log n}{4 b_n}.$$

It follows from Lemma 3 that

$$P(A_0 \cap A_l) \leqq \frac{1}{nMl^2} + P(A_0)\, Ml\, b_n^{1/2} (\log n)^{\frac{\varepsilon}{2} - \frac{1}{4}} \exp\left(-\frac{Ml}{4}\{1 + o(1)\}\frac{\log n}{b_n}\right)$$

$$\leqq \frac{1}{nMl^2} + P(A_0)\, Ml\, b_n^{1/2} (\log n)^{\frac{\varepsilon}{2} - \frac{1}{4}} \exp\left(-\frac{Ml}{8}\frac{\log n}{b_n}\right),$$

for n large enough.

Let us now choose $M \sim (b_n/\log n)\; b_n^{\delta}$. If we assume that, for some $p > 1$,

$b_n/\log^p n \to 0$ as $n \to \infty$, the condition $\delta < 1/p$ ensures that $M = o(b_n)$. Furthermore,

$$\sum_{l=1}^{R} \frac{1}{nMl^2} \leq \frac{\pi^2}{6n} M^{-1} = o(P(A_0)) \quad \text{as } n \to \infty,$$

whenever $\delta > -\varepsilon$. Using the fact that $b_n/\log n \to \infty$ for the remainding terms, we see that, for any $0 < \delta < 1/p$, (5.7) holds with the above choice of M.

By (5.6), this, in turn, implies that

$$P\left(\bigcup_{i=1}^{R} A_i\right) \sim RP(A_0) \sim \frac{C}{n} b_n^{1-\delta} (\log n)^{\varepsilon} \quad \text{as } n \to \infty. \tag{5.7}$$

Let now $Q_j = \max\{S_{j+i+b_n} - S_{j+i},\ i = 0, M, 2M, \ldots, i \leq b_n\}$. We see that

$$U_n \geq \max\{Q_j, j = 0, 2b_n, 4b_n, \ldots, j \leq n - b_n\}.$$

Hence, noting that the Q_j's are independent and identically distributed, we have

$$\begin{aligned} P(U_n < x) &\leq \prod_{0 \leq l < [(n-b_n)/2b_n]} (1 - P(Q_{lb_n} \geq x)) \\ &\leq \exp\left(-\frac{n}{2b_n} \frac{C}{n} b_n^{1-\delta} (\log n)^{\varepsilon} \{1 + o(1)\}\right). \end{aligned} \tag{5.8}$$

From there, we deduce the lemma:

Lemma 13. *Assume that* $b_n/\log n \to \infty$, *and that, for some* $p > 1$, $b_n/\log^p n \to 0$. *Then, for any* $0 < \varepsilon < 1$, *if* C *is as in Lemma* 1,

$$\begin{aligned} &P\left(U_n < b_n \alpha_n + \frac{1}{t_n}(-\log b_n + (\tfrac{1}{2} - \varepsilon) \operatorname{loglog} n)\right) \\ &\quad = O\left\{\exp\left(-\frac{C}{4} (\log n)^{\varepsilon/2}\right)\right\} \quad \text{as } n \to \infty. \end{aligned}$$

Proof. In (5.8), take $\delta = \varepsilon/(2p) < 1/p$.

In the sequel, we will use the remark that if $b_n = j$ is constant for all $N_1 \leq n \leq N_2$, then $U_{N_1} = \min\{U_n, N_1 \leq n \leq N_2\}$. Since $c(\alpha)$ is decreasing in α and α_n satisfies $c(\alpha_n) = j/\log n \downarrow$, it follows that α_n as well as $t_n = t(\alpha_n)$ is increasing on $[N_1, N_2]$. Let

$$x = x(n, \varepsilon) = b_n \alpha_n + \frac{1}{t_n}(-\log b_n + (\tfrac{1}{2} - \varepsilon) \operatorname{loglog} n)).$$

We see evidently that, for any $\varepsilon > 0$ and all $N_1 \leq n \leq N_2$,

$$x(n, \varepsilon) \leq x(N_1, \varepsilon) + (b_{N_2} \alpha_{N_2} - b_{N_1} \alpha_{N_1})(1 + o(1)) \quad \text{as } N_1 \to \infty. \tag{5.9}$$

Assume now that $N_2/N_1 \leq a < \infty$. Then, by (2.5) and (2.6), we have

$$b_{N_2} \alpha_{N_2} - b_{N_1} \alpha_{n_1} = O\{\alpha_{N_1}^{-1} \log(N_2/N_1)\} = o\{t_{N_1}^{-1} \operatorname{loglog} N_1\}, \quad \text{as } N_1 \to \infty.$$

It follows that, if $N_2/N_1 \leqq a$, we have, for large enough N_1, and a fixed $\varepsilon > 0$,

$$x(n, \varepsilon) \leqq x(N_1, \varepsilon/2), \qquad N_1 \leqq n \leqq N_2 \tag{5.10}$$

Hence, if (5.10) holds,

$$\bigcup_{N_1 \leqq n \leqq N_2} \{U_n \leqq x(n, \varepsilon)\} \subset \{U_{N_1} \leqq x(N_1, \varepsilon/2)\}. \tag{5.11}$$

Consider now the sequence $n_j = [a^j]$, where $a > 1$ is fixed, and the sequence m_j defined recursively by

$$m_1 = 1, \qquad m_j = \min\{m > m_{j-1}, b_m > b_{m-1}\}, \qquad j = 2, 3, \ldots.$$

Because of (5.10), all we need is to show that, for all $0 < \varepsilon < 1$,

$$\sum_j P(U_{n_j} \leqq x(n_j, \varepsilon/2)) < \infty, \tag{5.12}$$

and

$$\sum_j P(U_{m_j} \leqq x(m_j, \varepsilon/2)) < \infty. \tag{5.13}$$

By Lemma 13, (5.12) is straightforward. Next, observe that (5.13) is equivalent to

$$\sum_n P(U_n \leqq x(n, \varepsilon/2))\, 1_{\{b_n > b_{n-1}\}} < \infty, \qquad \text{all } 0 < \varepsilon < 1,$$

where 1_A denotes the indicator function of A. This, in turn, follows from

$$\sum_n 1_{\{b_n > b_{n-1}\}} \exp\left(-\frac{C}{4} (\log n)^{\varepsilon/2}\right) < \infty, \qquad \text{all } 0 < \varepsilon < 1,$$

Evidently, when b_n is nondecreasing,

$$\sum_{i=1}^{n} 1_{\{b_i > b_{i-1}\}} \leqq b_n = O(\log^p n) \qquad \text{as } n \to \infty.$$

On the other hand, for $0 < \varepsilon < 1$,

$$\exp\left(-\frac{C}{4} (\log(n+1))^{\varepsilon/2}\right) - \exp\left(-\frac{C}{4} (\log n)^{\varepsilon/2}\right) = O(n^{-1} \log^{-p-2} n) \qquad \text{as } n \to \infty.$$

But this suffices for our needs by Abel's lemma. By all this, we have just proved:

Lemma 14. *Assume that $b_n/\log n \to \infty$, $b_n \uparrow$ and for some $p > 1$, $b_n/\log^p n \to 0$. Then*

$$\liminf_{n \to \infty} \frac{(U_n - b_n \alpha_n)\, t_n + \log b_n}{\log\log n} \geqq \frac{1}{2} \quad \text{a.s.}$$

Proof of Theorem 2. It follows from (4.3) and Lemmas 11, 12 and 14. The equivalence between (5.1) (resp. (5.2)) and (5.3) (resp. (5.4)) follows from (2.9).

Remark 8. It is noteworthy that the results

$$\liminf_{n\to\infty}\frac{(U_n-b_n\alpha_n)+\log b_n}{\operatorname{loglog} n}=\frac{1}{2} \quad\text{and}\quad \limsup_{n\to\infty}\frac{(U_n-b_n\alpha_n)+\log b_n}{\operatorname{loglog} n}\geqq\frac{3}{2} \quad\text{a.s.}$$

are valid under the sole assumptions that $b_n/\log n\to\infty$, $b_n/\log^p n\to 0$ for some $p>1$, and $b_n\uparrow$.

6. What Strong Invariance Principle Give

Consider U_n as defined in Sect. 1, and assume conditions (A), (B) and (C). Then, by the Komlós-Major-Tusnády (1976) approximation, there exists a probability space which carries a Wiener process $\{W(t),\ t\geqq 0\}$ and a sequence with the same distribution as $X_1, X_2, \ldots$ (and which will be assumed to be our original sequence, without loss of generality), such that, almost surely,

$$S_n-\sigma W(n)=O(\log n) \qquad \text{as } n\to\infty. \tag{6.1}$$

For $0<h<t$, denote by

$$R(t,h)=\max_{0\leqq s\leqq t-h}(W(s+h)-W(s)),$$

the maximal increment of size h of the Wiener process in $(0,t)$. Let also $\{\tilde{b}_t, t>0\}$ be a function of t such that

$$0<\tilde{b}_t<t, \quad\text{and}\quad b_n-\tilde{b}_n=O(\log n) \qquad \text{as } n\to\infty. \tag{6.2}$$

We have then (see, e.g., M. Csörgö and Révész (1981), Theorem 1.2.1):

$$U_n-\sigma R(n,\tilde{b}_n)=O(\log n) \quad \text{a.s.} \qquad \text{as } n\to\infty. \tag{6.3}$$

It follows from (6.2) that we can characterize the limiting behaviour of U_n up to the precision $O(\log n)$ by the corresponding results known for $R(t,h)$. In the sequel, we shall use the results of Révész (1982) and Ortega and Wschebor (1984). As in Révész (1982), we assume that

$$(H)\quad t^{-1}\tilde{b}_t\downarrow, \qquad \lim_{t\to\infty}(\log(t/\tilde{b}_t))/\operatorname{loglog} t=\infty, \quad\text{and}\quad \tilde{b}_t\uparrow.$$

From Theorem 2.1 of Révész (1982), by expanding the functions $a_3(t)$ and $a_4(t)$ give there, we deduce that

$$a_0(t)=(2\log(t/\tilde{b}_t))^{1/2}\left(1+\frac{(1+\varepsilon)\log_2 t}{2(2\log(t/\tilde{b}_t))}\right)\in\begin{pmatrix}LLC, & \varepsilon<0,\\ LUC, & \varepsilon>0,\end{pmatrix}$$

provided $\operatorname{loglog}(t/\tilde{b}_t)\sim\operatorname{loglog} t$. This gives:

Lemma 15. *Under the assumptions above,*

$$\liminf_{t\to\infty}\frac{R(t,\tilde{b}_t)-\{2\tilde{b}_t\log(t/\tilde{b}_t)\}^{1/2}}{\{2\tilde{b}_t/\log(t/\tilde{b}_t)\}^{1/2}\operatorname{loglog} t}=\frac{1}{2} \quad \text{a.s.}$$

Ortega and Wschebor (1984) consider the upper classes of $\tilde{b}_t^{-1/2} R(t, \tilde{b}_t)$. They introduce an additional assumption which can be stated as follows in the case we consider.

(I) $\tilde{b}_t$ has a continuous first derivative $\tilde{b}'_t$ such that

$$\tilde{b}'_t/\tilde{b}_t = O\left(\frac{1}{t \log(t/\tilde{b}_t)}\right), \quad \text{as } t \to \infty.$$

It is noteworthy that (I) is the continuous version of condition (G_1) (i), namely

$$\tilde{b}_{n+1} - \tilde{b}_n = O\left(\frac{\tilde{b}_n}{n \log n}\right), \quad \text{as } n \to \infty.$$

If (H) and (I) hold, an immediate consequence of Theorems 2 and 4 in Ortega and Wschebor is that

$$a_1(t) = (2 \log(t/\tilde{b}_t))^{1/2}\left(1 + \frac{(3+\varepsilon)\log_2 t}{2(2\log(t/\tilde{b}_t))}\right) \in \begin{pmatrix} UUC, & \varepsilon > 0, \\ ULC, & \varepsilon < 0, \end{pmatrix}$$

provided $\operatorname{loglog}(t/\tilde{b}_t) \sim \operatorname{loglog} t$. This gives:

Lemma 16. *Under the assumptions above,*

$$\limsup_{t\to\infty} \frac{R(t, \tilde{b}_t) - \{2\tilde{b}_t \log(t/\tilde{b}_t)\}^{1/2}}{\{2\tilde{b}_t/\log(t/\tilde{b}_t)\}^{1/2} \operatorname{loglog} t} = \frac{3}{2} \quad \text{a.s.}$$

Now, in order to obtain a precise rate in the limiting behaviour of U_n by the above strong approximation approach, we need assume that

$$\lim_{t\to\infty} \frac{\log t}{\{\tilde{b}_t/\log(t/\tilde{b}_t)\}^{1/2} \operatorname{loglog} t} = 0,$$

which amounts to

$$\lim_{t\to\infty} \tilde{b}_t (\operatorname{loglog} t)^2/\log^3 t = \infty. \tag{6.4}$$

By Lemmas 15 and 16, we prove easily the following result.

Theorem 3. *Assume that b_n is an integer sequence such that $1 \leqq b_n \leqq n$, $b_n(\operatorname{loglog} n)^2/\log^3 n \to \infty$ and $(\operatorname{loglog}(n/b_n))/\operatorname{loglog} n \to 1$ as $n \to \infty$. Assume further that there exists a real valued sequence $\tilde{b}_n \uparrow$, such that*

(i) $b_n - \tilde{b}_n = O(\log n)$,

(ii) $\tilde{b}_{n+1} - \tilde{b}_n = O\left(\frac{\tilde{b}_n}{n \log n}\right)$ as $n \to \infty$,

(iii) $n^{-1}\tilde{b}_n \downarrow$.

Then

$$\limsup_{n\to\infty} \frac{(2\log(n/b_n))^{1/2}}{\sigma b_n^{1/2}\operatorname{loglog} n}(U_n-\sigma(2b_n\log(n/b_n))^{1/2})=\frac{3}{2}\quad \text{a.s.,} \tag{6.5}$$

and

$$\liminf_{n\to\infty} \frac{(2\log(n/b_n))^{1/2}}{\sigma b_n^{1/2}\operatorname{loglog} n}(U_n-\sigma(2b_n\log(n/b_n))^{1/2})=\frac{1}{2}\quad \text{a.s.} \tag{6.6}$$

Remark 9. In the range where $b_n(\operatorname{loglog} n)^2/\log^2 n\to\infty$, $b_n/\log^p n\to 0$ for some $p>3$, the results of Theorems 2 and 3 coincide almost exactly, as can be seen from the expansions (2.5) and (2.6). In fact, Theorem 2 is there slightly stronger since it does not require that $\tilde{b}_n/n$ to be nonincreasing, and allows $b_n-\tilde{b}_n$ to be as large as $O(b_n/\log n)$.

On the other hand, it can be verified from the same expansions that the results of Theorem 3 are invalid in general if one drops the assumption that $b_n(\operatorname{loglog} n)^2/\log^3 n\to\infty$.

This can be seen from the following corollary of Theorem 2:

Corollary 1. *Under the assumptions of Theorem* 2, *if* $b_n/\log^2 n\to\infty$ *and* $b_n/\log^p n\to 0$ *for some* $p>2$, *we have*

$$\limsup_{n\to\infty} \frac{(2\log(n/b_n))^{1/2}}{\sigma b_n^{1/2}\operatorname{loglog} n}\left(U_n-\sigma(2b_n\log(n/b_n))^{1/2}-\frac{\sigma\gamma}{3}\log n\right)=\frac{3}{2}\quad a.s., \tag{6.7}$$

and

$$\liminf_{n\to\infty} \frac{(2\log(n/b_n))^{1/2}}{\sigma b_n^{1/2}\operatorname{loglog} n}\left(U_n-\sigma(2b_n\log(n/b_n))^{1/2}-\frac{\sigma\gamma}{3}\log n\right)=\frac{1}{2}\quad a.s., \tag{6.8}$$

where $\gamma=E(X_1^3)/E(X_1^2)^{3/2}$.

In the particular case where $b_n(\operatorname{loglog} n)^2/\log^3 n\to d\in(0,\infty)$, *we have*

$$\limsup_{n\to\infty} \frac{(2\log(n/b_n))^{1/2}}{\sigma b_n^{1/2}\operatorname{loglog} n}(U_n-\sigma(2b_n\log(n/b_n))^{1/2})=\frac{3}{2}+\frac{\gamma\sqrt{2}}{3\sqrt{d}}\quad a.s., \tag{6.9}$$

and

$$\limsup_{n\to\infty} \frac{(2\log(n/b_n))^{1/2}}{\sigma b_n^{1/2}\operatorname{loglog} n}(U_n-\sigma(2b_n\log(n/b_n))^{1/2})=\frac{1}{2}+\frac{\gamma\sqrt{2}}{3\sqrt{d}}\quad a.s. \tag{6.10}$$

Proof. Straightforward by the expansions in Sect. 1.

We will discuss now weak laws. In the case, the following result holds:

Theorem 4. *Assume that* b_n *is an integer sequence such that* $1\leqq b_n\leqq n$, $b_n(\operatorname{loglog} n)^2/\log^3 n\to\infty$ *and* $b_n/n\to 0$ *as* $n\to\infty$. *Then*

$$\lim_{n\to\infty} \frac{(2\log(n/b_n))^{1/2}}{\sigma b_n^{1/2}\operatorname{loglog}(n/b_n)}(U_n-\sigma(2b_n\log(n/b_n))^{1/2})=\frac{1}{2}\quad \text{in probability.} \tag{6.10}$$

Proof. A proof of this result, together with additional weak laws for U_n in the range covered by Theorem 4, is given in Deheuvels and Révész (1986). It can be seen that (6.10) is a simple consequence of Theorem (1.5.5) in M. Csörgö and Révész (1981). We omit details.

Remark 10. In the case where $X_1, X_2, \ldots$ are i.i.d. Gaussian $N(0, \sigma^2)$ random variables, we get exactly $a_n b_n = \sigma(2b_n \log(n/b_n))^{1/2}$, so that the results in Theorems 2 and 3 (resp. 1 and 4) coincide for all sequences b_n in study.

In general, the range of validity of Theorems 3 and 4 depends of the index m of the first non-zero cumulant κ_r, $r \geqq 3$, in the expansion (2.1). The case where $\kappa_r = 0$ for all $r \geqq 3$ coincides with the normal distribution, and corresponds to the most general situation where these theorems can be applied.

Remark 11. Several extensions of our results are possible, in particular for Shepp's statistic, i.e. $\max_{0 \leqq k \leqq n} (S_{k+b_k} - S_k)$, or for the maximal modulus of continuity $\max_{0 \leqq k \leqq n-b_n} \max_{0 \leqq i \leqq b_n} (S_{k+i} - S_k)$. The corresponding theorems will be published elsewhere.

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