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Multiple Points in the Sample Paths of a Lévy Process

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Summary. We obtain a sufficient condition for the sample paths of a Lévy process to contain multiple points. Our condition is close to one conjectured by Hendricks and Taylor.

1. Introduction

Suppose that Y is a Lévy process with transition function $Q_t(x, B)$ and potential function

$$V^\alpha(x, B) = \int_0^\infty \exp(-at)\, Q_t(x, B)\, dt.$$

A problem that has aroused a considerable degree of interest is to determine necessary and sufficient conditions on Q which will ensure for some integer $k \geqq 2$ that almost surely the sample paths of Y possess k-tuple points (we say that a point x is a k-tuple point of Y if the cardinality of set $Y^{-1}(\{x\})$ is at least k).

The case of Brownian motion was dealt with by Dvoretzky et al. [2, 3], and Dvoretzky et al. [4], and that of stable processes by Taylor [15, 16] and Fristedt [7]. Fairly complete results were obtained in Hendricks [11, 12] for processes with independent stable components and in Hawkes [8] for spherically symmetric processes which possess a suitably behaved family of transition densities.

All of the above work deals with processes for which

$$Q_t(x, dy) = q_t(y-x)\, dy \tag{1.1}$$

for some measurable function q_t, $t > 0$, and hence

$$V^\alpha(x, dy) = v^\alpha(y-x)\, dy \tag{1.2}$$

for some measurable function v^α. In all cases k-tuple points are present whenever

$$\int_{|x| \leqq 1} (v^1(x))^k \, dx < \infty \tag{1.3}$$

and it has been conjectured by Hendricks and Taylor [13] that the condition (1.3) coupled with some form of 'non-singularity' condition to rule out cases such as processes with subordinator projections should be necessary and sufficient for the existence of k-tuple points.

Work of the author in Evans [5] shows that for symmetric processes (1.1) (which in this case is equivalent to (1.2)) and (1.3) are indeed sufficient for the existence of k-tuple points. The most general sufficient condition known to us is found in LeGall et al. [14]. There the authors work with processes which possess a strong-Feller semigroup (these are precisely the processes for which (1.1) holds) and they prove a theorem which implies that (1.3) and the non-singularity condition

$$v^1(0) > 0 \tag{1.4}$$

are together sufficient for the existence of k-tuple points.

In this paper we shall improve this last result by replacing the assumption (1.1) by the weaker assumption (1.2), which is equivalent to the assumption that the resolvent of Y is strong-Feller. Furthermore, we provide another approach to this type of result in that our proofs are potential theoretic and Fourier analytic in spirit, as opposed to the local time methods of LeGall et al. [14].

The main tool which we use is a theorem from Evans [5] which provides a sufficient condition for a multiparameter process formed from several independent Lévy processes to 'hit' a given set. We recall this result in Sect. 2 along with some relevant notation. The precise statement and proof of our result are given in Sect. 3 along with an example of a process which satisfies the conditions of our theorem but does not posses a strong-Feller semigroup.

2. A Multiparameter Result

Suppose that $X^i = (\Omega^i, \mathscr{M}^i, \mathscr{M}^i_{t^i}, X^i_{t^i}, \theta^i_{t^i}, P^i_{x^i})$, $1 \leqq i \leqq k$, are standard Markov processes (see, e.g., Sect. I–9 of Blumenthal and Getoor [1]) with state spaces $(E^i, \mathscr{B}^i)$ augmented by Δ^i. As usual, we set $E^i_{\Delta^i} = E^i \cup \{\Delta^i\}$ and let $\mathscr{B}^i_{\Delta^i}$ be the σ-field on $E^i_{\Delta^i}$ generated by $\mathscr{B}^i$.

Set $E = \prod_i E_i$ and $\mathscr{B} = \prod_i \mathscr{B}_i$. Define E_Δ and $\mathscr{B}_\Delta$ similarly. We will adopt the convention that when the domain of a function is not expressly stated it will be assumed to be E. We extend such functions to E_Δ by setting them to be 0 on $E_\Delta \backslash E$. We also adopt the analogous convention for measures.

Define a measurable space $(\Omega, \mathscr{M})$ by setting $\Omega = \prod_i \Omega^i$ and $\mathscr{M} = \prod_i \mathscr{M}^i$. For $t = (t^1, \ldots, t^k) \in [0, \infty[^k$, $\omega = (\omega^1, \ldots, \omega^k) \in \Omega$ and $x = (x^1, \ldots, x^k) \in E_\Delta$ set $X_t(\omega) = (X^i_{t^i}(\omega^i))$ and $P_x = \prod_i P^i_{x^i}$.

If Z (resp. Z^i) is a random variable defined on $(\Omega, \mathcal{M})$ (resp. $(\Omega^i, \mathcal{M}^i)$) we will denote the expectation of Z (resp. Z^i) with respect to P_x (resp. $P^i_{x^i}$) by $E_x Z$ (resp. $E^i_{x^i} Z^i$).

If μ is a σ-finite measure on $(E_\Delta, \mathcal{B}_\Delta)$ we may define a measure P_μ on $(\Omega, \mathcal{M})$ by $P_\mu(\cdot)=\int \mu(dx)\, P_x(\cdot)$.

Consider now the special case of this general construction that obtains when each of the X^i are Lévy processes on $E^i=\mathbb{R}^{d^i}$. We say that a set $B\in\mathcal{B}$ is *essentially polar* if $P_\lambda(\exists t\in]0,\infty[^k : X_t\in B)=0$, where $\lambda=\prod_i \lambda^i$ is Lebesgue measure on E. Theorem (7.4) in Evans [5] provides a sufficient condition for a set not to be essentially polar. We restate this result below as Theorem (2.2). Since it has been pointed out to the author that it is not clear from the proof in Evans [5] how one deals with certain difficulties presented by the existence of discontinuities in the paths of the process, we also provide a more complete proof of this result. First, however, we will require some more notation and a Fourier analytic lemma which appears as part of Theorem 1 in Hawkes [9].

Notation. If μ is a measure we denote the Fourier-Stieltjes transform of μ by $\hat{\mu}$. If C is a measurable set we denote the Lebesgue measure of C by $|C|$.

Lemma (2.1). *Let A and B be compact subsets of $\mathbb{R}^n$, $n\geqq 1$. If there exist non-trivial finite measures μ and v supported on A and B respectively and such that*

$$\int |\hat{\mu}(z)|^2\, |\hat{v}(z)|^2\, dz<\infty$$

then

$$|\{x:(x+A)\cap B\neq\phi\}|>0.$$

Notation. As usual, define the exponent of X^j to be the function ψ^j such that

$$\exp(-t^j\psi^j(z^j))=E^j_0 \exp(i X^j(t^j)\cdot z^j).$$

If μ is a measure on E and $\beta>0$ set

$$I(\beta;\psi;\mu)=\int \Big[\prod_i \operatorname{Re}((\beta+\psi^i(z^i))^{-1})\Big]\, |\hat{\mu}(z)|^2\, dz.$$

Theorem (2.2). *If K is a compact subset of E then a sufficient condition for K to be not essentially polar is that there exists $\beta>0$ and a finite measure μ supported on K such that $I(\beta;\psi;\mu)<\infty$.*

Proof. Let $\tilde{X}^i$ denote the standard process obtained by killing X^i at constant rate β(see, e.g., III.3 of Blumenthal and Getoor [1]). Denote the lifetime of $\tilde{X}^i$ by ζ^i. Let $\tilde{X}$ be the multiparameter process formed from the family $\{\tilde{X}^i: 1\leqq i\leqq k\}$ using the construction described above and consider the measures

$$\tau(A)=\int I_A(\tilde{X}(t))\, dt, \qquad A\in\mathcal{B}$$
$$\tau^i(A^i)=\int I_{A^i}(\tilde{X}^i(t^i))\, dt^i, \qquad A^i\in\mathcal{B}^i.$$

In an obvious notation we have

$$\begin{aligned}
\tilde{E}_0^j |\tilde{\tau}^j(u)|^2 &= \tilde{E}_0 \Big|\int_0^\infty \exp(iu\cdot \tilde{X}_j(t))\,dt\Big|^2\\
&= \tilde{E}_0 \int_0^\infty \int_0^\infty \exp(iu\cdot(\tilde{X}^j(t)-\tilde{X}^j(s)))\,dt\,ds\\
&= \int_0^\infty \int_0^\infty \exp(-\beta(s\vee t))\,E_0 \exp(iu\cdot(X^j(t)-X^j(s)))\,dt\,ds\\
&= 2\,\mathrm{Re}\int\int_{\{(s,t):t\geqq s\}} \exp(-\beta t)\exp(-(t-s)\,\psi^j(u))\,dt\,ds\\
&= 2\,\mathrm{Re}\int_0^\infty \exp(-\beta s)\int_s^\infty \exp(-(t-s)(\beta+\psi^j(u)))\,dt\,ds\\
&= 2\beta^{-1}\,\mathrm{Re}((\beta+\psi^j(u))^{-1}),
\end{aligned}$$

and so

$$\begin{aligned}
\tilde{E}_0 |\hat{\tau}(z)|^2 &= \prod_i \tilde{E}_0^i |\hat{\tau}^i(z^i)|^2\\
&= 2^k \beta^{-k} \prod_i \mathrm{Re}((\beta+\psi^i(z^i))^{-1}).
\end{aligned}$$

Now ζ^i is almost surely finite and τ^i is supported on the compact set

$$C^i = \{\tilde{X}^i(s^i), \tilde{X}^i(s^i-): 0 \leqq s^i < \zeta^i\} \cup \{X^i(\zeta^i-)\}.$$

Applying Lemma (2.1) gives

$$(\tilde{P}_0 \times \lambda)(\{(\omega, x): \prod_i C^i(w) \cap (K-x) \neq \emptyset\}) > 0 \qquad (*).$$

Let $\pi^i: E \to E^i$ denote the usual projection map and set

$$D^i = \{\tilde{X}^i(s^i): 0 \leqq s^i < \zeta^i\}.$$

Observe that the statement of Proposition I.10.20 in Blumenthal and Getoor [1] still holds if the σ-field of Borel sets is replaced by the family of analytic sets so that if $F \subset E^i$ is analytic we have

$$\tilde{P}_0^i(C^i \cap F \neq \emptyset) = \tilde{P}_0^i(D^i \cap F \neq \emptyset).$$

Thus for any x we see from Fubini's Theorem that

$$\begin{aligned}
\tilde{P}_0(\prod_i C^i \cap (K-x) \neq 0) &= \tilde{P}_0(C^1 \cap \pi^1((E^1 \times \prod_{i>1} C^i) \cap (K-x)) \neq \emptyset)\\
&= \tilde{P}_0(D^1 \cap \pi^1((E^1 \times \prod_{i>1} C^i) \cap (K-x)) \neq \emptyset)\\
&= \tilde{P}_0(D^1 \times \prod_{i>1} C^i \cap (K-x) \neq \emptyset)\\
&\dots\\
&= \tilde{P}_0\Big(\prod_i D^i \cap (K-x) \neq \emptyset\Big).
\end{aligned}$$

Combining this with $(*)$ we have

$$(\tilde{P}_0 \times \lambda)(\{(\omega, x): \prod_i D^i(\omega) \cap (K-x) \neq \emptyset\}) > 0$$

and from the relationship between $\tilde{X}^i$ and X^i this certainly implies

$$\begin{aligned} 0 &< (P_0 \times \lambda)(\{(\omega, x): (x + X([0, \infty[^k)) \cap K \neq \emptyset\}) \\ &= \int \lambda(dx)\, P_x(X([0, \infty[^k) \cap K \neq \emptyset) \end{aligned}$$

and so

$$\lambda(\{x: P_x(X([0, \infty[^k) \cap K \neq 0\}) > 0.$$

If

$$\lambda(\{x: P_x(X([0, \infty[^k \setminus]0, \infty[^k) \cap K \neq \emptyset) > 0\}) = 0$$

then we have nothing more to prove, so we assume that this is not the case. By renumbering the X^i if necessary, we may suppose for some j, $1 \leqq j \leqq k$, that

$$\lambda(\{x: P_x(\{(X^1, \ldots, X_j)(0, \ldots, 0)\} \times (X^{j+1}, \ldots, X^k)(]0, \infty[^{k-j}[) \cap K \neq \emptyset) > 0\}) > 0.$$

Set $X' = (X^1, \ldots, X^j)$, $X'' = (X^{j+1}, \ldots, X^k)$ and decompose λ and P_x similarly. By assumption we have

$$(P''_0 \times \lambda' \times \lambda'')(\{x: \{x'\} \times (X''(]0, \infty[^{k-j}) - x'') \cap K \neq \emptyset\}) > 0$$

and so

$$\begin{aligned} 0 &< (P'_0 \times P''_0 \times \lambda' \times \lambda'')(\{x: \{X'(1, \ldots, 1) - x'\} \times (X''(]0, \infty[^{k-j}) - x'') \cap K = \emptyset\}) \\ &= \int \lambda(dx)\, P_x(X(\{(1, \ldots, 1)\} \times]0, \infty[^{k-j}) \cap K \neq \emptyset) \end{aligned}$$

which certainly implies

$$\lambda(\{x: P_x(X(]0, \infty[^k) \cap K \neq \emptyset) > 0\}) > 0.$$

3. A Sufficient Condition for Multiple Points

Notation. Suppose that Y is a Lévy process on $\mathbb{R}^d$ with transition function

$$Q_\cdot(\cdot, \cdot): [0, \infty[\times \mathbb{R}^d \times \mathscr{B}(\mathbb{R}^d) \to [0, 1].$$

Set

$$V^\alpha(z, B) = \int_0^\infty \exp(-\alpha s)\, Q_s(z, B)\, ds$$

for $\alpha > 0$, $z \in \mathbb{R}^d$ and $B \in \mathscr{B}(\mathbb{R}^d)$. Assume that Y has a strong-Feller resolvent. We have from Propositions 1 and 3 in Hawkes [10] that this will be the case if and only if there exists for each $\alpha > 0$ a unique measurable function v^α such that

i) for each y, $z \to v^\alpha(y-z)$ is α-excessive (and hence lower semicontinuous);

ii) $V^\alpha(z, B)) = \int_B v^\alpha(y-z)\,dy$;

iii) $v^\alpha - v^\beta = (\beta-\alpha)\, v^\beta * v^\alpha$.

We call $\{v^\alpha : \alpha > 0\}$ the family of canonical resolvent densities.

Theorem (3.1). *Let Y be a Lévy process on $\mathbb{R}^d$ with canonical resolvent densities $\{v^\alpha\}$. If for some $k \in \{2, 3, \ldots\}$ there exists $\beta, \varepsilon > 0$ such that*

i) $v^\beta(0) > 0$

and

ii) $\int_{|z| \leq \varepsilon} (v^\beta(z))^k\,dz < \infty$

then the sample paths of Y have k-tuple points almost surely.

Proof. Let $X^1, \ldots, X^k$ be k copies of Y, and form a process X from these in the manner of Sect. 2. We will begin by showing that the set

$$\partial = \{x \in (\mathbb{R}^d)^k : x^1 = \ldots = x^k\}$$

is not essentially polar for X.

For $x = (x^1, \ldots, x^k) \in (\mathbb{R}^d)^k$ put

$$u(x) = v^\beta(x^1) \ldots v^\beta(x^k)$$

and let μ be Lebesgue measure on ∂ restricted to $\{y \in \partial : |y^i| \leq \varepsilon/3,\ 1 \leq j \leq k\}$. Note that if $x \in (\mathbb{R}^d)^k$ with $|x^j| \leq \varepsilon/3$, $1 \leq j \leq k$, then for some constant c

$$\begin{aligned}(u * \mu * \mu)(x) &\leq c \int_{|z| \leq 2\varepsilon/3} v^\beta(z-x^1) \ldots v^\beta(z-x^k)\,dz \\ &\leq c \prod_{j=1}^{k} \Big(\int_{|z| \leq 2\varepsilon/3} (v^\beta(z-x^j))^k\,dz\Big)^{1/k} \\ &\leq c \int_{|z| \leq \varepsilon} (v^\beta(z))^k\,dz.\end{aligned}$$

Thus if we put

$$w(x) = 2^{-k} \prod_{j=1}^{k} (v^\beta(x^j) + v^\beta(-x^j))$$

and

$$f(x) = (w * \mu * \mu)(x)$$

then

$$\sup\{f(x) : x \in (\mathbb{R}^d)^k, \quad |x^i| \leq \varepsilon/3, \quad 1 \leq j \leq k\} = c' < \infty.$$

Note also that, since v^β is integrable, w and hence f are also integrable.

Observe that if we set $g=\hat{f}$ then

$$g(\xi)=\prod_j \operatorname{Re}((\beta+\varphi(\xi^j))^{-1})\,|\hat{\mu}(\xi)|^2$$
$$\geqq 0, \qquad \xi=(\xi^1, \ldots, \xi^k)\in(\mathbb{R}^d)^k,$$

where φ is the exponent of Y. From the obvious multidimensional analogue to Eq. (3.4) in Chap. XV of Feller [6] we have, for $a>0$, that

$$(2\pi)^{-dk}\int_{(\mathbb{R}^d)^k} g(\xi)\exp\left(\frac{-a^2|\xi|^2}{2}\right)d\xi=(2\pi)^{-dk/2}\,a^{-dk}\int_{(\mathbb{R}^d)^k}\exp\left(\frac{-|x|^2}{2a^2}\right)f(x)\,dx$$
$$\leqq c'+(2\pi)^{-dk/2}\,a^{-dk}\exp\left(\frac{-|\varepsilon|^2}{18a^2}\right)\int_{(\mathbb{R}^d)^k} f(x)\,dx.$$

Letting $a\downarrow 0$, it is clear from the above that

$$\int_{(\mathbb{R}^d)^k} g(\xi)\,d\xi<\infty$$

and so Theorem (2.2) gives that ∂ is not essentially polar for X.

Consider the following construction. By possibly enlarging the probability space on which Y is defined, construct exponential random variables $\rho_1, \rho_2, \ldots$ with means β^{-1} which are mutually independent of Y for all initial distributions. Define $T_0=0$ and $T_j=\rho_1+\ldots+\rho_j$, $j=1, 2, \ldots$. Suppose that

$$\bigcap_{j=0}^{k-1}\{Y(t):t\in[T_{2j}, T_{2j+1}[\}=\emptyset \text{ a.s.,} \tag{**}$$

that is

$$\bigcap_{j=0}^{k-2}\{Y(t):t\in[T_{2j}, T_{2j+1}[\}$$
$$\cap(Y(T_{2k-3}+(Y(T_{2k-2})-Y(T_{2k-3})))+\{Y(t)-Y(T_{2k-2}):$$
$$t\in[T_{2k-2}, T_{2k-1}[\}=\emptyset$$

almost surely. Since the σ-fields

$$\sigma\{Y(t)-Y(T_j): t\in[T_j, T_{j+1}[\}, \qquad j=0, 1, \ldots$$

are mutually independent and $Y(T_{2k-2})-Y(T_{2k-3})$ has density βv^β which, by the following Lemma (3.2) is every where positive, we see that (**) implies that for all z_{j-1} not belonging to a set of zero Lebesgue measure we have

$$\bigcap_{j=0}^{k-2}\{Y(t):t\in[T_{2j}, T_{2j+1}[\}$$
$$\cap\{z_{j-1}+Y(t)-Y(T_{2k-2}):t\in[T_{2k-2}, T_{2k-1}[\}=\emptyset$$

almost surely. Continuing in this fashion we find that (**) implies that for all $(z_1, z_2, \dots, z_{j-1})$ not belonging to a set of zero Lebesgue measure we have

$$\{Y(t): t\in[T_0, T_1[\} \cap \bigcap_{j=1}^{k-1} \{z_j + Y(t) - Y(T_{2j}): t\in[T_{2j}, T_{2j+1}[\} = \emptyset$$

almost surely. This is, however, not possible since ∂ is not essentially polar for X.

We therefore have that

$$P_0(\exists x: \mathrm{card}(Y^{-1}(\{x\}) \cap [T_0, T_{2k-1}[) \geqq k) = P > 0;$$

and hence

$$\begin{aligned} &P_0(\exists x: \mathrm{card}\, Y^{-1}(\{x\}) \geqq k) \\ &\quad \geqq P_0(\exists n, \exists x: \mathrm{card}((Y(\cdot) - Y(T_{n(2k-1)}))^{-1}(\{x\}) \cap [T_{n(2k-1)}, T_{(n+1)(2k-1)}[) \geqq k) \\ &\quad \lim_{m\to\infty} 1-(1-p)^m = 1 \end{aligned}$$

as claimed. □

Lemma (3.2). *Under the conditions of Theorem (3.1), $v^\beta(z) > 0$, $z\in\mathbb{R}^d$.*

Proof. Observe that

$$\begin{aligned} V^\beta(0,\cdot) * V^\beta(0,\cdot) &= \int\int \exp(-\beta s)\exp(-\beta t)\, Q_s(0,\cdot) * Q_t(0,\cdot)\, ds\, dt \\ &= \int\int \exp(-\beta(s+t))\, Q_{s+t}(0,\cdot)\, ds\, dt \\ &= \int r \exp(-\beta r)\, Q_r(0,\cdot)\, dr \end{aligned}$$

and so it is clear that

$$\begin{aligned} \mathrm{supp}\, V^\beta(0,\cdot) &= \mathrm{supp}\, V^\beta(0,\cdot) * V^\beta(0,\cdot) \\ &= \mathrm{supp}\, V^\beta(0,\cdot) + \mathrm{supp}\, V^\beta(0,\cdot). \end{aligned}$$

From the lower semicontinuity of v^β it is clear that supp $V^\beta(0,\cdot)$ contains a neighbourhood of 0 and hence from the above supp $V^\beta(0,\cdot) = \mathbb{R}^d$. It is shown in Hawkes [10] that v^β is positive on int supp $V^\beta(0,\cdot)$ and so the lemma follows. □

The following example shows that there are processes satisfying the conditions of Theorem (3.2) which do not posses a strong-Feller semigroup.

Example. Let Z be a simple Poisson process with parameter 1. Consider the process Y given by $Y(t) = Z(t) - t$. Since the distribution of $Y(t) - Y(0)$ is atomic, it is clear that (1.1) does not hold for this process. However, a straightforward calculation shows that Y has canonical resolvent densities given by

$$v^\alpha(x) = \begin{cases} \displaystyle\sum_{k=\lceil x\rceil}^{\infty} \exp(-(1+\alpha)(k - x + \lceil x\rceil - 1))(k - x + \lceil x\rceil - 1)^k/k!, & \text{if } x > 0; \\ \displaystyle\sum_{k=0}^{\infty} \exp(-(1+\alpha)(k-x))(k-x)^k/k!, & \text{otherwise.} \end{cases}$$

It is clear that (1.4) holds and the integral in (1.3) is finite for all $k \geqq 2$. Applying Theorem (3.1) gives that for all $k \geqq 2$ the sample paths of Y possess k-tuple points almost surely. This conclusion is of course also obvious from totally elementary considerations.

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