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# On the almost sure central limit theorem and domains of attraction

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**Summary.** We give necessary and sufficient criteria for a sequence $(X_n)$ of i.i.d. r.v.'s to satisfy the a.s. central limit theorem, i.e.,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k\le N} \frac{1}{k} I\left\{\frac{S_k}{a_k} - b_k < x\right\} = \phi(x) \quad \text{a.s. for all } x$$

for some numerical sequences $(a_n)$, $(b_n)$ where $S_n = X_1 + \cdots + X_n$ and $I$ denotes indicator function. Our method leads also to new results on the limit distributional behavior of $S_n/a_n - b_n$ along subsequences ("partial attraction"), as well as to necessary and sufficient criteria for averaged versions of the central limit theorem such as

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k\le N} P\left(\frac{S_k}{a_k} - b_k < x\right) = \phi(x) \qquad \text{for all } x\,.$$

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## 1 Introduction

Let us say that a sequence $(X_n)$ of r.v's satisfies the a.s. central limit theorem (ASCLT) if there exist numerical sequences $(a_n), (b_n)$ such that setting $S_n = X_1 + \ldots + X_n$ we have

$$\text{(1.1)} \qquad \lim_{N\to\infty} \frac{1}{\log N} \sum_{k\le N} \frac{1}{k}\, I\left\{\frac{S_k}{a_k} - b_k < x\right\} = \phi(x) \quad \text{a.s. for all } x$$

where $I$ denotes indicator function. The purpose of this paper is to give necessary and sufficient criteria for an i.i.d. sequence $(X_n)$ to satisfy the ASCLT.

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The first to prove a.s. central limit theorems were Brosamler (1988) and Schatte (1988) who proved independently that if $(X_n)$ are i.i.d. with $EX_1 = 0$, $EX_1^2 = 1, E|X_1|^{2+\delta} < +\infty$ for some $\delta > 0$ ($\delta = 1$ for Schatte) then

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k\le N} \frac{1}{k} I\left\{\frac{S_k}{\sqrt{k}} < x\right\} = \phi(x) \quad \text{a.s. for all } x \ . \tag{1.2}$$

(Actually, in [9, p. 270] Lévy formulated a result very similar to (1.2) but he gave no proof). Lacey and Philipp [8] showed that (1.2) remains valid assuming only $EX_1 = 0, EX_1^2 = 1$ and in [2] we proved that the converse is also valid: if an i.i.d. sequence $(X_n)$ satisfies (1.2) then $EX_1 = 0, EX_1^2 = 1$. Thus in the special case $a_n = \sqrt{n}, b_n = 0$ the ASCLT (1.1) is equivalent to the ordinary CLT

$$S_n/a_n - b_n \xrightarrow{\mathscr{D}} N(0,1) \ . \tag{1.3}$$

For general $(a_n), (b_n)$ the situation is different and more delicate. Let us first note, as observed in [2], [3], that for general $(a_n), (b_n)$ the ASCLT (1.1) can hold in a curious (but degenerate) situation when the distribution of $S_n/a_n - b_n$ is near degenerate for all $n$ and thus (1.3) is not valid. Indeed, the ASCLT can even hold for nonrandom sequences, i.e., there exists a numerical sequence $(c_n)$ such that

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k\le N} \frac{1}{k} I\{c_k < x\} = \phi(x) \quad \text{for all } x \ . \tag{1.4}$$

To get such a sequence let e.g. $(\alpha_n)$ be a sequence in (0,1) unformly distributed in the Weyl sense, i.e.,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k\le N} I\{\alpha_k < x\} = x \quad \text{for all} \quad 0 \le x \le 1 \ .$$

(For example, we can choose $\alpha_n = \{n\alpha\}$ where $\alpha$ is any irrational number and $\{\ \}$ means fractional part.) Then letting $c_n = \phi^{-1}(\alpha_n)$ we clearly have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k\le N} I\{c_k < x\} = \phi(x) \quad \text{for all } x$$

which immediately implies (1.4) (cf. Lemma 7). Now if $(X_n)$ is a sequence of r.v's with $S_n = X_1 + \ldots + X_n$ and $(a_n), (b_n), (d_n)$ are numerical sequences such that

$$\frac{S_n}{a_n} - d_n \xrightarrow{P} 0 \tag{1.5}$$

and the sequence $c_n = d_n - b_n$ satisfies (1.4) then clearly $(X_n)$ satisfies the ASCLT (1.1) but the validity of (1.1) in this case is due not to the random fluctuations of $S_n$ but the fluctuations of the numerical sequence $d_n - b_n$. To give further degenerate examples for (1.1) let $\mu_L^*$ denote the upper log density of sets $H \subseteq N$, i.e.,

$$\mu_L^*(H) = \limsup_{N\to\infty} \frac{1}{\log N} \sum_{k\le N, k\in H} \frac{1}{k} \tag{1.6}$$

If in (1.6) actually the limit exists then $\mu_L^*(H)$ reduces to the log density of H which will be denoted by $\mu_L(H)$. Let now $N = H_1 \cup H_2$ where $H_1, H_2$ are disjoint sets of positive integers with positive upper log density and let $(X_n)$ be a sequence of i.i.d. r.v's such that for some numerical sequences $(a_n), (b_n), (c_n), (d_n)$ we have

$$\frac{S_n}{a_n} - d_n \xrightarrow{P} 0 \qquad \text{as} \quad n \to \infty, \quad n \in H_1 \tag{1.7}$$

$$\frac{S_n}{a_n} - b_n \xrightarrow{\mathscr{D}} N(0,1) \qquad \text{as} \quad n \to \infty, \quad n \in H_2 \tag{1.8}$$

and $c_k = d_k - b_k$ satisfies

$$\sum_{k\le N, k\in H_1} \frac{1}{k} I\{c_k < x\} \sim \left( \sum_{k\le N, k\in H_1} \frac{1}{k} \right) \phi(x) \quad \text{as} \quad N \to \infty\,. \tag{1.9}$$

(Again, sequences satisfying (1.9) are easy to construct; for example, if $(\alpha_n)$ is uniformly distributed as above then

$$c_k = \begin{cases} 0 & \text{if} \quad k \notin H_1 \\ \phi^{-1}(\alpha_n) & \text{if} \quad k \text{ is the } n\text{-th element of } H_1 \end{cases}$$

satisfies (1.9) by Lemma 7.) From Lemma 5 it follows that the sequence $(X_n)$ satisfies the ASCLT (1.1) but the validity of (1.1) is again due partly to a non-random effect, namely the fluctuations of the numerical sequence $d_n - b_n$ on $H_1$. To avoid such degenerate situations, let us say that a sequence $(X_n)$ of r.v's satisfies the ASCLT *nontrivially* if (1.1) holds for some $(a_n), (b_n)$ but there is no numerical sequence $(d_n)$ such that (1.5) holds on a set $H \subseteq N$ with positive upper log density. In [2] the following result was proved.

**Theorem.** *Let $X_1, X_2, \dots$ be i.i.d. r.v's with distribution function $F$. If*

$$\lim_{x\to\infty} \frac{x^2(1 - F(x) + F(-x))}{\int_{|t|\le x} t^2 dF(t)} = 0 \tag{1.10}$$

*then there exist numerical sequences $(a_n), (b_n)$ such that the ASCLT (1.1) holds nontrivially. On the other hand, if the ASCLT (1.1) holds nontrivially for some $(a_n), (b_n)$ then*

$$\liminf_{x\to\infty} \frac{x^2(1 - F(x) + F(-x))}{\int_{|t|\le x} t^2 dF(t)} = 0\,. \tag{1.11}$$

*Neither implication can be reversed: (1.10) is not necessary and (1.11) is not sufficient for the ASCLT (1.1) to hold nontrivially for some $(a_n), (b_n)$.*

Note that (1.10) is the classical necessary and sufficient condition for an i.i.d. sequence $(X_n)$ with distribution function $F$ to satisfy the ordinary CLT (1.3) for some $(a_n), (b_n)$ while (1.11) is the necessary and sufficient condition for (1.3) to hold along an infinite sequence of $n$'s tending to $+\infty$ (see e.g. [9] p. 113). Hence the above theorem shows that the ASCLT (1.1) is a strictly weaker result than the distributional CLT (1.3), a rather surprising conclusion. Conditions (1.10) and (1.11) are very similar but in reality (1.11) is essentially weaker than (1.10): while (1.10) implies (1.3) and thus completely determines the weak limit behavior of $S_n$, (1.11) gives information on $S_n$ only on a (possibly thin) sequence of $n$'s, leaving its behavior undetermined for all other $n$'s. (For example, in [2] we constructed an i.i.d. sequence $(X_n)$ such that (1.11) holds but except a thin set of $n$'s where (1.3) is valid, $S_n/a_n - b_n$ has a limiting Cauchy distribution). The purpose of this paper is to close the gap between (1.10) and (1.11) and to find a necessary and sufficient condition for the ASCLT (1.1). To formulate our results, we need some notation. Given a nondegerate distribution function $F$, define for any $0 < \varepsilon < 1$ and $n \geq 1$

$$
\begin{aligned}
c_n^{(\varepsilon)} &= \inf\{x > 0 : n(1 - F(x) + F(-x)) \leq \varepsilon\} \\
d_n^{(\varepsilon)} &= \int_{|x| \leq c_n^{(\varepsilon)}} x dF(x) \\
\sigma_n^{(\varepsilon)} &= \sqrt{n} \left[ \int_{|x| \leq c_n^{(\varepsilon)}} x^2 dF(x) - \left( \int_{|x| \leq c_n^{(\varepsilon)}} x dF(x) \right)^2 \right]^{1/2} \\
b_n^{(\varepsilon)} &= \left[ \frac{x^2(1 - F(x) + F(-x))}{\int_{|t| \leq x} t^2 dF(t)} \right]_{x = c_n^{(\varepsilon)}}
\end{aligned}
\tag{1.12}
$$

Then we have

**Theorem 1.** *Let $X_1, X_2, \ldots$ be i.i.d. r.v's with continuous distribution function $F$ centered at median and satisfying $\int x^2 dF(x) = +\infty$. Then the following statements are equivalent:*

*(A) $X_n$ satisfies the ASCLT (1.1) nontrivially for some $(a_n), (b_n)$.*
*(B) There exist numerical sequences $(a_n), (b_n)$ and a set $H \subseteq N$ with $\mu_L(H) = 1$ such that the CLT (1.3) holds as $n \to \infty, n \in H$.*
*(C) For each $0 < \varepsilon < 1$ $\mu_L\{n : b_n^{(\varepsilon)} \geq \varepsilon^2\} = 0$.*
*(D) There exists a set $H \subseteq N$ with $\mu_L(H) = 1$ such that for each $0 < \varepsilon < 1$ the inequality $b_n^{(\varepsilon)} \geq \varepsilon^2$ can hold only for finitely many $n \in H$.*

Clearly, centering the $X_n$ at medians in Theorem 1 is no restriction of generality and neither is $\int x^2 dF(x) = +\infty$ since the case $\int x^2 dF(x) < +\infty$ is covered by the theorem of Lacey and Philipp [8]. The assumption of the continuity of $F$ is also unessential and serves only to make the formulation of Theorem 1 simpler; the theorem remains valid for discontinuous $F$ with a slightly modified definition of $b_n^{(\varepsilon)}$. For general $F$ define the 'jump factor' $\rho_n^{(\varepsilon)}$ by

$$\rho_n^{(\varepsilon)} = \varepsilon/n(1 - F(c_n^{(\varepsilon)}) + F(-c_n^{(\varepsilon)}))_+$$

and change the definition of $b_n^{(\varepsilon)}$ in (1.12) to

$$b_n^{(\varepsilon)} = \rho_n^{(\varepsilon)} \left[ \frac{x^2(1 - F(x) + F(-x))_+}{\int_{|t|\le x} t^2 dF(t)} \right]_{x=c_n^{(\varepsilon)}} \tag{1.12a}$$

where $f(x)_+$ denotes the right limit of $f$ at $x$. Clearly $n(1-F(c_n^{(\varepsilon)})+F(-c_n^{(\varepsilon)}))_+ \le \varepsilon$ and thus $\rho_n^{(\varepsilon)} \ge 1$; also, $\rho_n^{(\varepsilon)} = 1$ if $F$ is continuous and hence in the continuous case (1.12a) reduces to the original definition of $b_n^{(\varepsilon)}$ in (1.12). If, however, $c_n^{(\varepsilon)}$ is a point of discontinuity of $1-F(x)+F(-x)$, the new $b_n^{(\varepsilon)}$ can be essentially larger than the original one in (1.12). As we shall see, with the modified definition of $b_n^{(\varepsilon)}$, Theorem 1 will be valid for arbitrary distribution functions $F$. The same remark applies for Theorems 2 and 3 below.

Theorem 1 shows that the validity of the ASCLT depends on the behavior of $b_n^{(\varepsilon)}$, i.e., the behaviour of the fraction

$$\frac{x^2(1 - F(x) + F(-x))}{\int_{|t|\le x} t^2 dF(t)} \tag{1.13}$$

along the values $x = c_n^{(\varepsilon)}$. Note that the ASCLT does not require that $b_n^{(\varepsilon)} \to 0$ as $n \to \infty$ (which holds if (1.10) is valid), only that $b_n^{(\varepsilon)}$ is small for small $\varepsilon$ and most $n$'s. Theorem 1 also shows that even though (1.1) and (1.3) are not equivalent, (1.1) implies (1.3) for 'almost all $n$', the exceptional set having log density zero. Note however, that permitting an exceptional set of log density zero in (1.3) changes the nature of the CLT (1.3) radically: for example, while the validity of (1.3) for all $n$ implies that $a_n = \sqrt{n}L(n)$ with a slowly varying function $L$, in [2] we constructed an example where (1.3) holds with the exception of a log zero set of $n$'s but $\limsup_{n\to\infty} a_n/n^2 > 0$.

As we observed earlier, the ASCLT (1.1) can hold in a trivial way if $S_n/a_n$ becomes asymptotically degenerate on a set $H \subseteq N$ with positive upper log density and $S_n$ satisfies the ordinary CLT on the complement $H^c$ of $H$. Such cases were excluded in Theorem 1 by considering only the nondegenerate case, i.e., assuming that (1.5) cannot hold on a set of $n$'s with positive upper log density. As the proof of Theorem 1 will show (cf. Lemma 8) there are no other cases: if $(X_n)$ satisfies (1.1) then either it does it nontrivially or it belongs to the type described above.

It is worth noting that the proof of Theorem 1 leads to new information even in the classical central limit theorem. As we mentioned above, (1.10) is the necessary and sufficient condition for the CLT (1.3) to hold for some $(a_n)$,$(b_n)$ ("$F$ belongs to the domain of attraction of the normal law"), while (1.11) is the necessary and sufficient condition for the CLT (1.3) to hold, with some $(a_n)$, $(b_n)$, along a suitable infinite sequence of $n$'s ("$F$ belongs to the domain of partial attraction of the normal law"). No condition seems to be known, however, for (1.3) to hold along a *specified* sequence of $n$'s. (Clearly, the condition for this lies between (1.10) and (1.11).) The following theorem answers this question, showing the significance of the inequality $b_n^{(\varepsilon)} \geq \varepsilon^2$ in the asymptotic behavior of $S_n$ and in the fine structure of the domain of partial attraction of the normal law.

**Theorem 2.** *Let $X_1, X_2, \ldots$ be i.i.d. r.v's satisfying the conditions of Theorem 1 and let $H \subseteq N$ be an arbitrary set of positive integers. Then the following statements are equivalent:*

*(A) There exist sequences $(a_n), (b_n)$ such that the CLT (1.3) holds along $H$.*
*(B) For each $0 < \varepsilon < 1$ the inequality $b_n^{(\varepsilon)} \geq \varepsilon^2$ can hold only for finitely many $n \in H$.*

In conclusion, we formulate one more theorem concerning certain 'average' forms of the CLT. Let $q = (q_1, q_2, \ldots)$ be a weight vector where the $q_n$ are positive numbers with $\Sigma q_n = +\infty$; let $Q_n = \sum_{i \leq n} q_i$. We say that $(X_n)$ satisfies the CLT in $q$-average if there exist numerical sequences $(a_n), (b_n)$ such that setting $S_n = X_1 + \ldots + X_n$ we have

$$\lim_{N \to \infty} \frac{1}{Q_N} \sum_{k \leq N} q_k P\left(\frac{S_k}{a_k} - b_k < x\right) = \phi(x) \quad \text{for all } x . \tag{1.14}$$

Clearly, (1.14) holds if $(X_n)$ satisfies the ordinary CLT (1.3) but the converse is false: (1.14) can hold as the result of an averaging effect even if (1.3) fails. To formulate a necessary and sufficient condition for (1.14) let us define, analogously to the log density,

$$\mu_q^*(H) = \limsup_{N \to \infty} \frac{1}{Q_N} \sum_{k \leq N, k \in H} q_k \tag{1.15}$$

for any $H \subseteq N$. If in (1.15) actually the lim exists then we shall write $\mu_q$ instead of $\mu_q^*$ and we call it the $q$-density of $H$. We then have

**Theorem 3.** *Let $q = (q_1, q_2, \ldots)$ be a fixed weight vector and $(X_n)$ an i.i.d. sequence of r.v's satisfying the conditions of Theorem 1. Then the following statements are equivalent:*

*(A) There exist numerical sequences $(a_n), (b_n)$ such that the average CLT (1.14) holds nontrivially.*
*(B) There exist numerical sequences $(a_n)$, $(b_n)$ and a set $H \subseteq N$ with $\mu_q(H) = 1$ such that the CLT (1.3) holds as $n \to +\infty, n \in H$.*
*(C) For each $0 < \varepsilon < 1$ $\quad \mu_q\{n : b_n^{(\varepsilon)} \geq \varepsilon^2\} = 0$.*

The nontriviality of (1.14) is defined similarly as in the case of the ASCLT (1.1): we require that (1.5) cannot hold for some $(d_n)$ on a set $H \subseteq N$ with $\mu_q^*(H) > 0$.

Note that, in contrast to Theorem 1, the weights in Theorem 3 are arbitrary, subject only to $\Sigma q_n = +\infty$. Thus, there is an essential difference between the situations when we have $I$ or $P$ in (1.1). Theorem 1 itself is not valid for arbitrary weights $q_k$ i.e.,

$$\lim_{N\to\infty} \frac{1}{Q_N} \sum_{k\le N} q_k I \left\{ \frac{S_k}{a_k} - b_k < x \right\} = \phi(x) \quad \text{a.s. for all } x \tag{1.16}$$

is generally not equivalent to the condition

$$\mu_q\{n : b_n^{(\varepsilon)} \ge \varepsilon^2\} = 0 \quad \text{for each } 0 < \varepsilon < 1 . \tag{1.17}$$

Since (1.16) clearly implies (1.14) by the bounded convergence theorem, Theorem 3 shows that (1.17) is a necessary condition for (1.16). However, the implication (1.17) $\Rightarrow$ (1.16) is false in general as one can see in the case $q_k = 1, EX_1 = 0$, $EX_1^2 = 1$, $a_k = \sqrt{k}$, $b_k = 0$ when (1.10) holds and thus $\lim_{n\to\infty} b_n^{(\varepsilon)} = 0$ for any $\varepsilon > 0$ i.e., (1.17) is valid but (1.16) fails for $x = 0$ by the arc sine law (see [8]).

## 2 Proof of the theorems

In what follows, the continuity of the distribution function $F$ of the r.v.'s $X_n$ will not be assumed and $b_n^{(\varepsilon)}$ will be defined by (1.12a).

**Lemma 1.** *Let $X_1, X_2, \dots$ be i.i.d. r.v's centered at medians. Then letting $S_n = X_1 + \dots + X_n$ we have for any $0 < \varepsilon < 1$, all real $d$, $x \ge 2$ and $n \ge 2x$*

$$P\left\{ \left| \frac{S_n - d}{\sigma_n^{(\varepsilon)}} \right| \ge \frac{1}{16} x \sqrt{\frac{b_n^{(\varepsilon)}}{\varepsilon}} \right\} \ge \frac{1}{4} \left( \frac{\varepsilon}{32x} \right)^x . \tag{2.1}$$

*Proof.* Throughout this proof, $[t]$ will denote the integral part of $t$. Let $F^*$ denote the symmetrized distribution function of $F$, let $X_1^*, X_2^*, \dots$ be i.i.d. r.v's (on some probability space) with distribution function $F^*$ and set $S_n^* = X_1^* + \dots + X_n^*$. Letting $G(x) = 1 - F(x) + F(-x)$ we have by the definition of $\sigma_n^{(\varepsilon)}$, $\rho_n^{(\varepsilon)}$ and (1.12a)

$$\left( \frac{c_n^{(\varepsilon)}}{\sigma_n^{(\varepsilon)}} \right)^2 \ge \frac{(c_n^{(\varepsilon)})^2 G(c_n^{(\varepsilon)})_+}{nG(c_n^{(\varepsilon)})_+ \int_{|x|\le c_n^{(\varepsilon)}} x^2 dF(x)} = \rho_n^{(\varepsilon)} \frac{(c_n^{(\varepsilon)})^2 G(c_n^{(\varepsilon)})_+}{\varepsilon \int_{|x|\le c_n^{(\varepsilon)}} x^2 dF(x)} = \frac{b_n^{(\varepsilon)}}{\varepsilon} \tag{2.2}$$

and thus using Lévy's inequality (see e.g. [5, p. 149, Lemma 2]) and the symmetrization inequalities in [10, p. 245] it follows that the left hand side of (2.1) is

$$\begin{aligned} &\ge \frac{1}{2} P \left\{ \left| \frac{S_n^*}{\sigma_n^{(\varepsilon)}} \right| \ge \frac{1}{8} x \sqrt{\frac{b_n^{(\varepsilon)}}{\varepsilon}} \right\} \\ &\ge \frac{1}{2} P\left\{ |S_n^*| \ge x c_n^{(\varepsilon)}/8 \right\} \ge \frac{1}{2} P\left\{ |S_n^*| \ge [x] c_n^{(\varepsilon)}/4 \right\} \end{aligned}$$

$$
\begin{aligned}
(2.3)\qquad &\geq \frac{1}{2}P\left\{\sum_{i=l[n/x]+1}^{(l+1)[n/x]} X_i^* \geq c_n^{(\varepsilon)}/4,\ 0\leq l\leq [x]-1,\ \sum_{i=[x][n/x]+1}^{n} X_i^*\geq 0\right\}\\
&\geq \frac{1}{4}P\left\{S_{[n/x]}^*\geq c_n^{(\varepsilon)}/4\right\}^{[x]}\geq \frac{1}{4}\left(\frac{1}{2}P\{|S_{[n/x]}^*|\geq c_n^{(\varepsilon)}/4\}\right)^x\\
&\geq \frac{1}{4}\left(\frac{1}{4}P\{\max_{k\leq[n/x]}|X_k^*|>c_n^{(\varepsilon)}/4\}\right)^x\\
&= \left(\frac{1}{4}\right)^{x+1}(1-\{1-2(1-F^*(c_n^{(\varepsilon)}/4))\}^{[n/x]})^x
\end{aligned}
$$

By the definition of $c_n^{(\varepsilon)}$ and a further application of the symmetrization inequalities in [10, p. 245] we see that

$$n(1-F^*(c_n^{(\varepsilon)}/4))\geq \frac{1}{4}nG(c_n^{(\varepsilon)}/2)\geq \varepsilon/4$$

and thus the last expression of (2.3) is

$$
\begin{aligned}
&\geq \left(\frac{1}{4}\right)^{x+1}\left(1-\left(1-\frac{2\varepsilon}{4n}\right)^{[n/x]}\right)^x\geq\left(\frac{1}{4}\right)^{x+1}\left(1-\exp\left(-\frac{2\varepsilon[n/x]}{4n}\right)\right)^x\\
(2.4)\qquad &\geq \left(\frac{1}{4}\right)^{x+1}\left(\frac{\varepsilon}{8x}\right)^x\geq\frac{1}{4}\left(\frac{\varepsilon}{32x}\right)^x
\end{aligned}
$$

using the fact that $t/2\leq 1-\exp(-t)\leq t$ for $0\leq t\leq 1/2$. This completes the proof of Lemma 1.

**Lemma 2.** *Let $X_1, X_2, \ldots$ be i.i.d. r.v's with distribution function $F$ satisfying $\int x^2dF(x)=+\infty$. Then setting $S_n=X_1+\ldots+X_n$ we have for any $0<\varepsilon<1$, $n\geq n_0$ and all real $x$*

$$\left|P\left(\frac{S_n-nd_n^{(\varepsilon)}}{\sigma_n^{(\varepsilon)}}<x\right)-\phi(x)\right|\leq 96\left(\varepsilon+\sqrt{\frac{b_n^{(\varepsilon)}}{\varepsilon}}\right). \tag{2.5}$$

*Proof.* Let $X_k^*=X_kI(|X_k|\leq c_n^{(\varepsilon)}), 1\leq k\leq n, Y_k^*=X_k^*-EX_k^*=X_k^*-d_n^{(\varepsilon)}, S_n^*=X_1^*+\ldots+X_n^*$. Choosing $x_0$ so large that $\int_{|x|\geq x_0}dF(x)\leq 1/6$ we get, using the Cauchy-Schwarz inequality, $\int x^2dF(x)=+\infty$ and observing that $c_n^{(\varepsilon)}\to+\infty$ as $n\to\infty$, uniformly in $\varepsilon$,

$$
\begin{aligned}
\left|\int_{|x|\leq c_n^{(\varepsilon)}}xdF(x)\right| &\leq \left|\int_{|x|\leq x_0}xdF(x)\right|\\
&+\left(\int_{x_0\leq|x|\leq c_n^{(\varepsilon)}}dF(x)\right)^{1/2}\left(\int_{|x|\leq c_n^{(\varepsilon)}}x^2dF(x)\right)^{1/2}
\end{aligned}
$$

$$\leq \frac{1}{2}\left(\int_{|x|\leq c_n^{(\varepsilon)}} x^2 dF(x)\right)^{1/2} \qquad \text{for } n \geq n_0$$

and thus

$$\sigma_n^{(\varepsilon)} \geq \frac{1}{2}\sqrt{n}\left(\int_{|x|\leq c_n^{(\varepsilon)}} x^2 dF(x)\right)^{1/2} \qquad (n \geq n_0)\,. \tag{2.6}$$

Hence for $n \geq n_0$ we have a converse inequality to (2.3):

$$\left(\frac{c_n^{(\varepsilon)}}{\sigma_n^{(\varepsilon)}}\right)^2 \leq \frac{4(c_n^{(\varepsilon)})^2 G(c_n^{(\varepsilon)})_+}{nG(c_n^{(\varepsilon)})_+ \int_{|x|\leq c_n^{(\varepsilon)}} x^2 dF(x)} = \frac{4b_n^{(\varepsilon)}}{\varepsilon}\,. \tag{2.7}$$

Also $|Y_k^*| \leq 2c_n^{(\varepsilon)}$, $E|Y_k^*|^2 \leq E|X_k^*|^2$ and thus using (2.6) we get for $n \geq n_0$

$$\sum_{k\leq n} E|Y_k^*|^3 \leq 2c_n^{(\varepsilon)} \sum_{k\leq n} E|Y_k^*|^2 \leq 2c_n^{(\varepsilon)} \sum_{k\leq n} E|X_k^*|^2 \leq 8c_n^{(\varepsilon)}(\sigma_n^{(\varepsilon)})^2$$

whence by (2.7)

$$(\sigma_n^{(\varepsilon)})^{-3} \sum_{k\leq n} E|Y_k^*|^3 \leq 8c_n^{(\varepsilon)}/\sigma_n^{(\varepsilon)} \leq 16\sqrt{b_n^{(\varepsilon)}/\varepsilon}\,.$$

The last relation and the Berry-Esseen theorem (see e.g. [5, p. 544]) show that replacing $S_n$ by $S_n^*$, the left side of (2.5) will be $\leq 96\sqrt{b_n^{(\varepsilon)}/\varepsilon}$. Since we have $P(S_n \neq S_n^*) \leq nG(c_n^{(\varepsilon)})_+ \leq \varepsilon$, Lemma 2 is proved.

*Remark.* For later reference we note the obvious fact that for any i.i.d. sequence $(X_n)$ we have (even without $\int x^2 dF(x) = +\infty$)

$$P\left(\left|\frac{S_n - nd_n^{(\varepsilon)}}{\sigma_n^{(\varepsilon)}}\right| \geq t\right) \leq \varepsilon + t^{-2} \qquad \text{for any } t > 0\,. \tag{2.8}$$

Indeed, $P(S_n \neq S_n^*) \leq \varepsilon$ as noted above so (2.8) is immediate from the Chebisev inequality.

**Lemma 3.** *Let $X_1, X_2 \ldots$ be i.i.d. r.v's with distribution function $F$ centered at median and set $S_n = X_1 + \ldots + X_n$. Assume that for some infinite set $H \subseteq N$ and numerical sequences $(a_n), (b_n)$ we have*

$$\frac{S_n}{a_n} - b_n \xrightarrow{\mathscr{D}} N(0,1) \qquad \text{as } n \to \infty,\ n \in H\,. \tag{2.9}$$

*Then for any $0 < \varepsilon < 1$ the inequality $b_n^{(\varepsilon)} \geq \varepsilon^2$ can hold only for finitely many $n \in H$.*

*Proof.* Set $K = \varepsilon^{-1}, x = \varepsilon^{-4}$ and

$$(2.10)\qquad \begin{aligned} H^* &= \{n \in N : b_n^{(\varepsilon)} \ge \varepsilon^2, \quad a_n \le K\sigma_n^{(\varepsilon)}\} \\ H^{**} &= \{n \in N : b_n^{(\varepsilon)} \ge \varepsilon^2, \quad a_n > K\sigma_n^{(\varepsilon)}\} \end{aligned}$$

It suffices to prove that both $H \cap H^*$ and $H \cap H^{**}$ are finite. Clearly

$$\frac{xa_n}{K}\sqrt{\varepsilon} \le x\sqrt{\frac{b_n^{(\varepsilon)}}{\varepsilon}}\sigma_n^{(\varepsilon)} \qquad \text{for} \quad n \in H^*$$

and thus we get by Lemma 1 for $0 < \varepsilon < 1/32$

$$\begin{aligned} P\left\{\left|\frac{S_n - a_n b_n}{a_n}\right| \ge \frac{x}{16K}\sqrt{\varepsilon}\right\} &\ge P\left\{\left|\frac{S_n - a_n b_n}{\sigma_n^{(\varepsilon)}}\right| \ge \frac{x}{16}\sqrt{\frac{b_n^{(\varepsilon)}}{\varepsilon}}\right\} \\ &\ge \frac{1}{4}\left(\frac{\varepsilon}{32x}\right)^x \qquad n \in H^* \end{aligned}$$

i.e.,

$$(2.11)\qquad P\left\{\left|\frac{S_n}{a_n} - b_n\right| \ge \frac{1}{16}\varepsilon^{-5/2}\right\} \ge \frac{1}{4}(\varepsilon^5/32)^{\varepsilon^{-4}} \ge \frac{1}{4}\varepsilon^{6\varepsilon^{-4}} \qquad n \in H^* .$$

If the set $H \cap H^*$ were infinite then we could choose an infinite sequence of $n$'s along which both (2.9) and (2.11) would hold, but then letting $n \to \infty$ in (2.11) we get, in view of (2.9),

$$(2.12)\qquad 2\left(1 - \phi\left(\frac{1}{16}\varepsilon^{-5/2}\right)\right) \ge \frac{1}{4}\varepsilon^{6\varepsilon^{-4}}$$

which is a contradiction for small enough $\varepsilon > 0$ since $1 - \phi(x) \le \exp(-x^2/2)$ for $x \ge 1$ and thus the ratio of the left and right side of (2.12) is

$$\le 8\exp\left(-\frac{1}{512}\varepsilon^{-5} - 6\varepsilon^{-4}\ln\varepsilon\right) \le 8\exp\left(-\varepsilon^{-5}\left(\frac{1}{512} + 6\varepsilon\ln\varepsilon\right)\right) = O(\varepsilon^5)$$

which tends to 0 if $\varepsilon \to 0$. Hence for $\varepsilon$ small enough, the set $H \cap H^*$ is finite. To prove the finiteness of $H \cap H^{**}$ let us note that by the Remark preceding Lemma 3 we have

$$(2.13)\qquad \begin{aligned} &P\left\{\left|\frac{S_n - nd_n^{(\varepsilon)}}{a_n}\right| \ge \varepsilon^{1/2}\right\} \\ &\le P\left\{\left|\frac{S_n - nd_n^{(\varepsilon)}}{\sigma_n^{(\varepsilon)}}\right| \ge K\varepsilon^{1/2}\right\} \le 2\varepsilon \qquad n \in H^{**} . \end{aligned}$$

The last relation shows that for $n \in H^{**}$ the distribution of $S_n/a_n$, and thus also the distribution of $S_n/a_n - b_n$, attach probability $\ge 1 - 2\varepsilon$ to an interval of length $\le 2\varepsilon^{1/2}$. Since for $0 < \varepsilon \le \varepsilon_0$ such a sequence $S_n/a_n - b_n$ obviously cannot converge to the standard normal distribution, the set $H \cap H^{**}$ cannot be infinite. Thus we proved Lemma 3 for $0 < \varepsilon \le \varepsilon_0$; changing the definition of $K$ and $x$ to $K = c_1\varepsilon^{-1}$, $x = c_2\varepsilon^{-4}$ and replacing $\varepsilon^{1/2}$ in the first probability of (2.13) by $c_3\varepsilon^{1/2}$ for suitably chosen $c_1$, $c_2$ $c_3$, we easily get the statement of the lemma for all $0 < \varepsilon < 1$.

**Lemma 4.** *Let $H \subseteq N$ be a set of positive integers with $\Sigma_{k\in H} 1/k = +\infty$ and $\{\xi_i, i \in H\}$ a uniformly bounded sequence of r.v's with $E\xi_i = 0$ and*

$$|E(\xi_k \xi_l)| \leq const\,(k/l)^\alpha \qquad k \leq l, \quad k,l \in H \tag{2.14}$$

*for some constant $\alpha > 0$. Then letting $\lambda_N = \sum_{k\leq N, k\in H} 1/k$ we have*

$$\lim_{N\to\infty} \frac{1}{\lambda_N} \sum_{i\leq N, i\in H} \frac{1}{i}\xi_i = 0 \qquad a.s.$$

For $H = N$ this lemma is a key ingredient in the proof of most a.s. central limit theorems (see e.g. [1], [4], [8], [11]). To prove the general case assume, without loss of generality, that $\alpha < 1$ and let $C$ denote a uniform upper bound for the $|\xi_i|$. Then we get, using (2.14),

$$\begin{aligned} E\left(\frac{1}{\lambda_N}\sum_{i\leq N,\, i\in H}\frac{1}{i}\xi_i\right)^2 &\leq \frac{\text{const}}{\lambda_N^2}\sum_{\substack{i,j\in H,\\ 1\leq i\leq j\leq N}}\frac{1}{ij}\left(\frac{i}{j}\right)^\alpha \\ &\leq \frac{\text{const}}{\lambda_N^2}\sum_{i\leq N,\, i\in H}\frac{1}{i^{1-\alpha}}\sum_{j\geq i}\frac{1}{j^{1+\alpha}} \leq \frac{\text{const}}{\lambda_N^2}\sum_{i\leq N, i\in H}\frac{1}{i} = \frac{\text{const}}{\lambda_N}\,. \end{aligned} \tag{2.15}$$

Since $\lambda_N \to +\infty$ and $\lambda_{N+1} - \lambda_N \to 0$, there exists an increasing sequence $(N_k)$ of positive integers such that $\lambda_{N_k} \sim k^2$. Hence letting $T_N = \lambda_N^{-1}\sum_{i\leq N, i\in H} i^{-1}\xi_i$ we get by (2.15) $E(T_{N_k}^2) \leq$ const $k^{-2}$ whence by the Beppo Levi theorem $\sum_{k\geq 1} T_{N_k}^2 < +\infty$ a.s. and thus $T_{N_k} \to 0$ a.s. Now for $N_k \leq N \leq N_{k+1}$ we have

$$|T_N| \leq |T_{N_k}| + \frac{C}{\lambda_N}\sum_{\substack{N_k < i \leq N\\ i\in H}}\frac{1}{i} \leq |T_{N_k}| + \frac{C(\lambda_{N_{k+1}} - \lambda_{N_k})}{\lambda_{N_k}}\,.$$

Since $\lambda_{N_{k+1}}/\lambda_{N_k} \to 1$, it follows that $T_N \to 0$ a.s., as stated.

**Lemma 5.** *Let $H \subseteq N$ be a set of positive integers with $\Sigma_{k\in H} 1/k = +\infty$ and let $X_1, X_2, \ldots$ be i.i.d. r.v's such that, setting $S_n = X_1 + \ldots X_n$, we have for some numerical sequences $(a_n), (b_n)$,*

$$\frac{S_n}{a_n} - b_n \xrightarrow{\mathscr{D}} N(0,1) \qquad as \quad n \to \infty, \; n \in H\,. \tag{2.16}$$

*Then letting $\lambda_N = \Sigma_{k\leq N, k\in H} 1/k$ we have*

$$\lim_{N\to\infty}\frac{1}{\lambda_N}\sum_{k\leq N, k\in H}\frac{1}{k} I\left\{\frac{S_k}{a_k} - b_k < x\right\} = \phi(x) \qquad \text{a.s. for all } x\,. \tag{2.17}$$

Under additional moment conditions such as

$$E\left|\frac{S_n}{a_n} - b_n\right|^p = O(1) \quad \text{for some} \quad p > 0$$

the statement of Lemma 5 can be proved essentially in the same way as the ASCLT's in our earlier paper [1]. However, to prove the precise characterization results in this paper we need to show the implication (2.16) $\Rightarrow$ (2.17) without any additional conditions on $(X_n)$.

*Proof of Lemma 5.* Let $d_n = a_n b_n/n$, $Y_k^{(n)} = X_k - d_n$, $1 \le k \le n$, $S_n' = \Sigma_{k\le n} Y_k^{(n)}$, $S_n^* = \Sigma_{k\le n} Y_k^{(n)} I(|Y_k^{(n)}| < a_n)$, $M_n = \max_{1\le k\le n} |Y_k^{(n)}|$. (2.16) and (2.17) can then be written equivalently

$$S_n'/a_n \xrightarrow{\mathscr{D}} N(0,1) \quad \text{as} \quad n \to \infty, \quad n \in H \tag{2.16a}$$

and

$$\lim_{N\to\infty} \frac{1}{\lambda_N} \sum_{k\le N, k\in H} \frac{1}{k} I\left\{\frac{S_k'}{a_k} < x\right\} = \phi(x) \quad \text{a.s. for all} \quad x \tag{2.17a}$$

respectively. Clearly

$$|I\{S_n'/a_n < x\} - I\{S_n^*/a_n < x\}| \le I\{M_n \ge a_n\} \qquad (n \ge 1)$$

and thus (2.17a) will follow if we show that

$$\lim_{N\to\infty} \frac{1}{\lambda_N} \sum_{k\le N, k\in H} \frac{1}{k} I\left\{\frac{S_k^*}{a_k} < x\right\} = \phi(x) \quad \text{a.s. for all} \quad x \tag{2.18}$$

and

$$\lim_{N\to\infty} \frac{1}{\lambda_N} \sum_{k\le N, k\in H} \frac{1}{k} I\{M_k \ge a_k\} = 0 \qquad \text{a.s.} \tag{2.19}$$

To prove (2.18) let us note that (2.16a) and the standard normal convergence criterion (see e.g., [10, p. 316]) imply

$$n \int_{|x|\ge \varepsilon a_n} dF_n(x) \to 0 \ \text{ for any } \varepsilon > 0 \text{ as } n \to \infty, \quad n \in H \tag{2.20}$$

$$\frac{n}{a_n^2}\left[\int_{|x|<a_n} x^2 dF_n(x) - \left(\int_{|x|<a_n} x dF_n(x)\right)^2\right] \to 1 \tag{2.21}$$

as $n \to \infty, \quad n \in H$

$$\frac{n}{a_n} \int_{|x|<a_n} x dF_n(x) \to 0 \quad \text{as} \quad n \to \infty, \quad n \in H \tag{2.22}$$

where $F_n$ is the common distribution function of the r.v.'s $Y_k^{(n)}$, $1 \leq k \leq n$. By a standard observation in the theory of the ASCLT (see e.g., [8]) (2.18) will follow if we show that

$$\lim_{N\to\infty} \frac{1}{\lambda_N} \sum_{k\leq N, k\in H} \frac{1}{k} f\left(\frac{S_k^*}{a_k}\right) = \int_{-\infty}^{+\infty} f(x)d\phi(x) \quad \text{a.s.} \tag{2.23}$$

for any $f : R \to R$ belonging to the bounded Lipschitz class BL of functions satisfying

$$|f(x) - f(y)| \leq K|x - y|, \quad |f(x)| \leq K, \quad x, y \in R \tag{2.24}$$

for some $K > 0$. Since $P(S_n' \neq S_n^*) \leq n \int_{|x|\geq a_n} dF_n(x) \to 0$ for $n \to \infty, n \in H$ by (2.20), relation (2.16a) remains valid if $S_n'$ is replaced by $S_n^*$ and thus

$$Ef\left(\frac{S_n^*}{a_n}\right) \to \int_{-\infty}^{+\infty} f(x)d\phi(x) \qquad \text{as } n \to \infty, \quad n \in H$$

whence

$$\lim_{N\to\infty} \frac{1}{\lambda_N} \sum_{k\leq N, k\in H} \frac{1}{k} Ef\left(\frac{S_k^*}{a_k}\right) = \int_{-\infty}^{+\infty} f(x)d\phi(x)$$

Hence setting

$$\xi_k = f\left(\frac{S_k^*}{a_k}\right) - Ef\left(\frac{S_k^*}{a_k}\right)$$

relation (2.23) is equivalent to

$$\lim_{N\to\infty} \frac{1}{\lambda_N} \sum_{k\leq N, k\in H} \frac{1}{k} \xi_k = 0 \qquad \text{a.s.} \tag{2.25}$$

which, in view of Lemma 4, will be proved if we show that

$$|E(\xi_k \xi_l)| \leq \text{const } (k/l)^\alpha \qquad k \leq l, \qquad k, l \in H \tag{2.26}$$

for some $\alpha > 0$. Setting $S_{k,l}^* = \sum_{k<i\leq l} Y_i^{(l)} I(|Y_i^{(l)}| < a_l)$ we get by using (2.21), (2.22), (2.24) and observing that $S_k^*$ and $S_{k,l}^*$ are independent,

$$\begin{aligned}
|E(\xi_k \xi_l)| &= \left|\text{Cov}\left(f\left(\frac{S_k^*}{a_k}\right), f\left(\frac{S_l^*}{a_l}\right)\right)\right| \\
&= \left|\text{Cov}\left(f\left(\frac{S_k^*}{a_k}\right), f\left(\frac{S_l^*}{a_l}\right) - f\left(\frac{S_{k,l}^*}{a_l}\right)\right)\right| \\
&\leq 4K^2 E\left|\frac{1}{a_l} \sum_{i\leq k} Y_i^{(l)} I(|Y_i^{(l)}| < a_l)\right| \\
&\leq 4K^2 E\left|\frac{1}{a_l} \sum_{i\leq k} \{Y_i^{(l)} I(|Y_i^{(l)}| < a_l) - \int_{|x|<a_l} x dF_l(x)\}\right|
\end{aligned}$$

$$+4K^2\frac{k}{a_l}\left|\int_{|x|<a_l} xdF_l(x)\right|$$

$$\leq 4K^2\left\{\frac{1}{a_l^2}k\left[\int_{|x|<a_l} x^2dF_l(x)-\left(\int_{|x|<a_l} xdF_l(x)\right)^2\right]\right\}^{1/2}$$

$$+4K^2\frac{k}{a_l}\left|\int_{|x|<a_l} xdF_l(x)\right| \leq \text{const}\left(\frac{k}{l}\right)^{1/2}+\text{const}\left(\frac{k}{l}\right) \qquad k,l\in H$$

proving (2.26) and thus (2.18). To prove (2.19) it will be again sufficient to verify

$$\lim_{N\to\infty}\frac{1}{\lambda_N}\sum_{k\leq N, k\in H}\frac{1}{k}f\left(\frac{M_k}{a_k}\right)=f(0)\quad \text{a.s.} \tag{2.27}$$

for any function $f$ satisfying (2.24). Now $M_k/a_k \xrightarrow{P} 0$ as $k\to\infty, k\in H$ by (2.20) and thus $Ef(M_k/a_k)\to f(0)$ as $k\to\infty, k\in H$. Hence (2.27) is again equivalent to (2.25) where now

$$\xi_k=f(M_k/a_k)-Ef(M_k/a_k)$$

and in view of Lemma 4 it remains to show that (2.26) holds. Set $M_{k,l}=\max_{k<i\leq l}|Y_i^{(l)}|$, $M_{k,l}^*=\max_{i\leq k}|Y_i^{(l)}|$ for $k\leq l$; clearly $M_k$ and $M_{k,l}$ are independent and $|M_l-M_{k,l}|\leq M_{k,l}^*$. Also, (2.16a) and a well known lemma from central limit theory (see e.g. [10], p.307) imply

$$nE(|Y_1^{(n)}/a_n|\wedge 2)^2=O(1) \qquad (n\in H)$$

and thus by Chebishev's inequality

$$P(|Y_1^{(n)}|\geq ta_n)\leq P(|Y_1^{(n)}/a_n|\wedge 2\geq t)\leq C/(nt^2) \qquad (n\in H,\ 0\leq t\leq 2)$$

for some constant $C>0$. Thus

$$|E(\xi_k\xi_l)|=\left|\text{Cov}\left(f\left(\frac{M_k}{a_k}\right), f\left(\frac{M_l}{a_l}\right)\right)\right|$$

$$=\left|\text{Cov}\left(f\left(\frac{M_k}{a_k}\right), f\left(\frac{M_l}{a_l}\right)-f\left(\frac{M_{k,l}}{a_l}\right)\right)\right|$$

$$\leq 4KE\left|K\frac{M_{k,l}^*}{a_l}\wedge 2K\right|=4K^2E\left|\frac{M_{k,l}^*}{a_l}\wedge 2\right|\leq 4K^2\int_0^2 P(M_{k,l}^*\geq ta_l)dt$$

$$\leq 4K^2\left[T+\int_T^2 kP(|Y_1^{(l)}|\geq ta_l)dt\right]$$

$$\leq 4K^2\left[T+\frac{Ck}{l}\int_T^2 t^{-2}dt\right]\leq 4K^2\left(T+\frac{Ck}{l}T^{-1}\right), \qquad k,l\in H$$

for any $0<T\leq 2$. Choosing $T=(k/l)^{1/2}$ we get again (2.26), completing the proof of Lemma 5.

The following two lemmas are well known and easily proved; we formulate them here for purposes of reference.

**Lemma 6.** (see [1], [6].) *Let $(x_n)$ be a numerical sequence. Then the following statements are equivalent:*

*(i) There exists a set $H \subseteq N$ of log density 1 such that $x_n \to 0$ as $n \to \infty, n \in H$*
*(ii) For all $\varepsilon > 0$ the set $\{n \in N : |x_n| \geq \varepsilon\}$ has log density 0, i.e.,*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k\leq N} \frac{1}{k} I\{|x_k| \geq \varepsilon\} = 0 .$$

**Lemma 7.** (see e.g. [7], p. 63) *Let $(c_n)$ be a numerical sequence satisfying $\lim_{n\to\infty}(c_1 + \ldots + c_n)/n = c$ for some finite $c$. Then for any positive decreasing sequence $(\lambda_n)$ with $\sum \lambda_n = +\infty$ we have*

$$\lim_{n\to\infty} \frac{\lambda_1 c_1 + \ldots + \lambda_n c_n}{\lambda_1 + \ldots + \lambda_n} = c .$$

*Proof of Theorem 1.* Assume first that (A) holds i.e., (1.1) is valid nontrivially for some $(a_n), (b_n)$. Since the expression $(\log N)^{-1} \sum_{k\leq N}\{\,\}$ on the left hand side of (1.1) is uniformly bounded, (1.1) can be integrated to give

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k\leq N} \frac{1}{k} P\left(\frac{S_k}{a_k} - b_k < x\right) = \phi(x) \quad \text{for all } x . \tag{2.28}$$

Set now $K = \varepsilon^{-1}, x = \varepsilon^{-4}$ and define $H^*, H^{**}$ by (2.10); let further

$$H^{***} = \{n \in N : b_n^{(\varepsilon)} < \varepsilon^2\} .$$

Define also, for any r.v. $X$,

$$\begin{aligned} \rho(X) &= \inf_{a\in R} \mathscr{L}(\text{dist}(X), \delta_a) \\ \tau(X) &= \inf_{a,b\in R, a>0} \mathscr{L}(\text{dist}(\frac{X}{a} - b), N(0,1)) \end{aligned}$$

where dist ( ) denotes the distribution of the r.v. in the brackets, $\delta_a$ is the probability distribution concentrated at $a$ and $\mathscr{L}$ denotes the Lévy distance of probability distributions. By (2.28) and (2.11) we get

$$\begin{aligned} 2\left(1 - \phi\left(\frac{1}{16}\varepsilon^{-5/2}\right)\right) &= \lim_{N\to\infty} \frac{1}{\log N} \sum_{k\leq N} \frac{1}{k} P\left\{\left|\frac{S_k}{a_k} - b_k\right| \geq \frac{1}{16}\varepsilon^{-5/2}\right\} \\ &\geq \limsup_{N\to\infty} \frac{1}{\log N} \sum_{k\leq N, k\in H^*} \frac{1}{k}\frac{1}{4}\varepsilon^{6\varepsilon^{-4}} = \frac{1}{4}\varepsilon^{6\varepsilon^{-4}} \mu_L^*(H^*) \end{aligned}$$

whence

$$\mu_L^*(H^*) \leq 8\varepsilon^{-6\varepsilon^{-4}} \left(1 - \phi\left(\frac{1}{16}\varepsilon^{-5/2}\right)\right) \leq \text{const} \cdot \varepsilon^5 \tag{2.29}$$

since the ratio of the left and right sides of (2.12) was shown to be $O(\varepsilon^5)$ in the proof of Lemma 3. (In (2.29), and in the relations below, the constants depend only on the sequence $(X_n)$ i.e., they are independent of $\varepsilon$.) On the other hand, (2.13) gives

$$\rho\left(\frac{S_n}{a_n}\right) \le 2\varepsilon^{1/2} \quad \text{for } n \in H^{**} . \tag{2.30}$$

Finally, on the set $H^{***}$ we can apply Lemma 2 to get

$$\mathscr{L}\left(\text{dist}\left(\frac{S_n - nd_n^{(\varepsilon)}}{\sigma_n^{(\varepsilon)}}\right), N(0,1)\right) \le \text{const}\sqrt{\varepsilon} \qquad n \in H^{***}$$

i.e.,

$$\tau(S_n) \le \text{const}\sqrt{\varepsilon} \qquad n \in H^{***} . \tag{2.31}$$

Let now

$$\pi_n = \min\left(\tau(S_n), \rho\left(\frac{S_n}{a_n}\right)\right) .$$

Clearly $N = H^* \cup H^{**} \cup H^{***}$ and thus (2.29)–(2.31) imply

$$\mu_L^*\{n \in N : \pi_n \ge \text{const}\ \varepsilon^{1/2}\} \le \text{const} \cdot \varepsilon \tag{2.32}$$

Since $\pi_n$ depends only on $n$ (but not on $\varepsilon$), the left side of (2.32) is independent of $\varepsilon$ and thus letting $\varepsilon \to 0$ in (2.32) we get

$$\mu_L^*\{n \in N : \pi_n \ge \delta\} = 0 \quad \text{for all } \ \delta > 0 .$$

Hence in view of Lemma 6 there exists a set $H \subseteq N$ with $\mu_L(H) = 1$ such that $\pi_n \to 0$ on $H$, i.e.,

$$\min\left(\tau(S_n), \rho\left(\frac{S_n}{a_n}\right)\right) \to 0 \quad \text{as } n \to \infty, \ \ n \in H .$$

Setting

$$H_1 = \left\{n \in H : \tau(S_n) \le \rho\left(\frac{S_n}{a_n}\right)\right\}, \quad H_2 = \left\{n \in H : \tau(S_n) > \rho\left(\frac{S_n}{a_n}\right)\right\}$$

it follows that $\mu_L(H_1 \cup H_2) = 1$ and $\tau(S_n) \to 0$ on $H_1, \rho(S_n/a_n) \to 0$ on $H_2$ provided that both $H_1$ and $H_2$ are infinite; if one of $H_1$ and $H_2$ is finite, only the convergence relation formulated for the other one holds. By the definition of $\rho$ and $\tau$ this means that

$$S_n/a_n' - b_n' \xrightarrow{\mathscr{D}} N(0,1) \qquad \text{on } H_1 \text{ for some } (a_n'), (b_n') \tag{2.33}$$

$$S_n/a_n - d_n \xrightarrow{P} 0 \ \text{ on } H_2 \text{ for some } (d_n) \tag{2.34}$$

in case $H_1$ resp. $H_2$ are infinite. If now $(X_n)$ satisfies (1.1) nontrivially then the set $H_2$ in (2.34) cannot have positive upper log density, i.e., $\mu_L(H_2) = 0$ and thus $\mu_L(H_1) = 1$ but then (2.33) shows that $(X_n)$ satisfies statement (B) of Theorem 1.

Let us note that in the just completed proof of $(A) \Rightarrow (B)$ the nontriviality of $(X_n)$ was used only at the very end and our argument actually yields the following more general statement.

**Lemma 8.** *Let $(X_n)$ be an i.i.d. sequence satisfying the conditions of Theorem 1. If $(X_n)$ satisfies the ASCLT (1.1) (nontrivially or not) then there exist disjoint sets $H_1, H_2 \subseteq N$ such that $\mu_L(H_1 \cup H_2) = 1$ and (2.33), (2.34) are valid (assumed again that $H_1$ resp. $H_2$ are infinite.)*

Continuing the proof of Theorem 1, the implication $(B) \Rightarrow (D)$ is contained in Lemma 3 while $(D) \Rightarrow (C)$ is obvious. Thus we proved $(A) \Rightarrow (B) \Rightarrow (D) \Rightarrow (C)$ and it remains to verify $(C) \Rightarrow (A)$. To this end assume that $(C)$ holds, then by Lemma 2 we get

$$\mu_L\{n \in N : \tau(S_n) \leq \text{const}\sqrt{\varepsilon}\} = 1 \qquad \text{for all } \varepsilon > 0 .$$

Since $\tau(S_n)$ depends only on $n$ but not on $\varepsilon$, the last relation and Lemma 6 imply that $\tau(S_n) \longrightarrow 0$ on a suitable set $H \subseteq N$ with log density 1 i.e., (2.16) holds with suitable $(a_n), (b_n)$. But then Lemma 5 implies the ASCLT (2.17) where now $\lambda_N \sim \log N$ and the restriction $k \in H$ in the sum (2.17) can be removed since $\mu_L(H) = 1$. Hence $(X_n)$ satisfies (1.1), moreover (2.16) and $\mu_L(H) = 1$ obviously imply the impossibility of (1.5) on a set $H_1$ of positive upper log density and thus (A) holds. This completes the proof of Theorem 1.

The proofs of Theorem 2 and Theorem 3 are immediate consequences of the proof of Theorem 1. In Theorem 2, the implication $(A) \Rightarrow (B)$ is contained in Lemma 3 while $(B) \Rightarrow (A)$ follows from the observation that $(B)$ and Lemma 2 together imply that for any $\varepsilon > 0$ we have $\tau(S_n) \leq \text{const}\sqrt{\varepsilon}$ for $n \in H, n \geq n_0(\varepsilon)$ i.e. $\tau(S_n) \longrightarrow 0$ on $H$ which is equivalent to $(A)$. To get Theorem 3 observe that the proof of Theorem 1 remains valid with arbitrary weights $q_n$ (satisfying $\Sigma q_n = +\infty$) with the exception of an application of Lemma 5 in the proof of the implication $(C) \Rightarrow (A)$ where the special nature of the weights $q_k = 1/k$ was used in an essential way. (Note that Lemma 6 remains valid for arbitrary $q_n$, see [6], Lemma (4.9).) However, replacing $I$ by $P$ in (2.17), Lemma 5 becomes obvious with arbitrary weights and Theorem 3 follows.

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