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Tsirel'son's equation in discrete time

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Summary. Motivated by Tsirel'son's equation in continuous time, a similar stochastic equation indexed by discrete negative time is discussed in full generality, in terms of the law of a discrete time noise. When uniqueness in law holds, the unique solution (in law) is not strong; moreover, when there exists a strong solution, there are several strong solutions. In general, for any time n, the σ-field generated by the past of a solution up to time n is shown to be equal, up to negligible sets, to the σ-field generated by the 3 following components: the infinitely remote past of the solution, the past of the noise up to time n, together with an adequate independent complement.

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1 Introduction

(1.1) Motivated by Tsirel'son's equation in continuous time [2, 5, 6], which we shall describe in the second part of this introduction, we are interested in the following equation in discrete negative time:

$$\text{(T)} \qquad \eta_k = \xi_k + \{\eta_{k-1}\} \quad (k \in -\mathbb{N}),$$

where $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$, and the ξ_k's are considered as given, whilst the η_k's are unknown.

In a more precise way, we assume that the laws of the ξ_k's, say μ_k, are given once and for all; we call $\mu = (\mu_k; k \in -\mathbb{N})$ the sequence of these laws, and we should like to give a precise description of the family $\mathscr{P}_\mu(T)$ of all probabilities P on $\mathbb{R}^{-\mathbb{N}}$ such that: if we denote $\eta_k(\omega) = \omega(k)$, $\xi_k = \eta_k - \{\eta_{k-1}\}$, $\mathscr{F}_k = \sigma(\eta_n, n \leqq k)$, and $\mathscr{E}_k = \sigma(\xi_n, n \leqq k)$, then, under P:

for every k, ξ_k is independent of $\mathscr{F}_{k-1}$, and has distribution μ_k.

Our paper is organized as follows:
we first give a characterization of the extremal points of $\mathscr{P}_\mu(T)$, and a description of the asymptotic σ-field $\mathscr{F}_{-\infty} \equiv \bigcap_k \mathscr{F}_k$; then, we show that, for any given sequence $\mu = (\mu_k;\, k \in -\mathbb{N})$, there exists at least a solution in $\mathscr{P}_\mu(T)$.

It would then be natural to look for a characterization, in terms of μ, of the uniqueness of the solutions in $\mathscr{P}_\mu(T)$. However, at this point, this is being postponed, because such a characterization shall appear in a clearer way, once we have obtained, for any given k, a general formula for the distribution of $\{\eta_k\}$ given the σ-field $\mathscr{F}_{-\infty} \vee \mathscr{E}_k$, in terms of μ.

Once this is done, then not only does a criterion for uniqueness follow naturally, but the general formula also illuminates the discussion of whether $\mathscr{F}_k$ equals $\mathscr{F}_{-\infty} \vee \mathscr{E}_k$, and, if this is not the case, an independent complement of $\mathscr{F}_{-\infty} \vee \mathscr{E}_k$ in $\mathscr{F}_k$ is found.

One of the questions which we have found to be of great interest in this general study of (T) is that it gives a particularly clear example of a situation where exchanging the order of taking the supremum and the intersection of σ-fields must be done with great care (see H. von Weizsäcker [8] for a general discussion and resolution of this problem, and D. Williams [9], Exercise (4.12), p. 48, for a particularly simple example, due to M. Barlow and E. Perkins, which, in fact, is quite close to Tsirel'son's equation).
Indeed, note that for any $n \geqq 0$, and any k, we have:

$$\mathscr{F}_k = \mathscr{F}_{k-n} \vee \mathscr{E}_k$$

and therefore: $\mathscr{F}_k = \bigcap_{(n \geqq 0)} (\mathscr{F}_{k-n} \vee \mathscr{E}_k)$, but the σ-field:

$$\mathscr{F}_{-\infty} \vee \mathscr{E}_k = \left(\bigcap_{n \geqq 0} \mathscr{F}_{k-n} \right) \vee \mathscr{E}_k$$

is, in most cases in our study, strictly contained in $\mathscr{F}_k$.

This is, in particular, the case when uniqueness holds, i.e.:
$\mathscr{P}_\mu(T)$ consists of only one solution, in which case:

$$\mathscr{F}_k = \mathscr{E}_k \vee \sigma(\{\eta_k\}), \quad \mathscr{F}_{-\infty} \text{ is trivial (so: } \mathscr{E}_k \equiv \mathscr{F}_{-\infty} \vee \mathscr{E}_k)$$

and $\{\eta_k\}$ is uniformly distributed on $[0, 1[$, and independent of $\mathscr{E}_k$.

In the above discussion, the equalities between σ-fields should be understood up to P-negligible sets, for $P \in \mathscr{P}_\mu(T)$.
(1.2) We now discuss Tsirel'son's equation in continuous time [6], and its close relationship with (T).

We first recall a striking result of Zvonkin [12]: let $(B_t, t \geqq 0)$ be a real-valued Brownian motion, and $b: \mathbb{R} \to \mathbb{R}$ a Borel bounded function. Zvonkin [12] showed that pathwise uniqueness holds for the stochastic differential equation:

$$dX_t = dB_t + b(X_t)\,dt\,. \tag{1.a}$$

As a consequence (of a general theorem of Yamada-Watanabe [10]; but here the following property may be proved directly), the unique solution of (1.a) is *strong*, i.e.: it is adapted to the natural filtration of B.

Note that this uniqueness result is remarkable, since, in order that the deterministic equation:

$$dx_t = b(x_t)dt$$

has a unique solution, some fairly strong continuity assumption on b, such as Lipschitz continuity, needs to be made.

An even more striking, and certainly much harder to prove, result in this vein is that of Veretennikov [7] who showed that, in the multidimensional case, the equation:

$$dX_t = dB_t + b(t, X_t)\,dt, \tag{1.a$'$}$$

where $b: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is a Borel bounded function, has a unique strong solution.

We now proceed with the discussion of some more general one-dimensional stochastic differential equations than (1.a). Once Zvonkin's result was established, it seemed plausible that the same strong measurability property would hold for a much larger class of bounded, Borel drifts $b: \mathbb{R}_+ \times \mathscr{W} \to \mathbb{R}$, where $\mathscr{W}$ denotes the space $C(\mathbb{R}_+, \mathbb{R})$ of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ and b is, more precisely, assumed to be predictable with respect to the filtration $\mathscr{G}_t = \sigma(x_s;\ s \leqq t)$, where $x_s(\omega) = \omega(s)$, for $\omega \in \mathscr{W}$.

To any such drift b, we associate the stochastic differential equation

$$dX_t = dB_t + b(t, X_.)\,dt, \tag{1.b}$$

which, at least, thanks to Girsanov's theorem, enjoys the uniqueness in law property. However, Tsirel'son [6] produced the following example of a drift b^* for which the unique solution (in law) of (1.b) – which we now denote as (1.b*) – is not strong:

if $(t_k, k \in -\mathbb{N})$ is an increasing sequence of reals such that: $\lim_{k\to-\infty} t_k = 0$, then b^* is defined as:

$$b^*(t, \omega) = \sum_{k\in-\mathbb{N}} \left\{ \frac{\omega(t_k) - \omega(t_{k-1})}{t_k - t_{k-1}} \right\} 1_{]t_k, t_{k+1}]}(t).$$

The original proof by Tsirel'son [6] was complemented by Stroock and Yor ([5], Proposition (6.13)) in the following form:

Theorem 0 *If* $(X_t, t \geqq 0)$ *is a solution of* (1.b*), *then*:

(i) *for any* $k \in -\mathbb{N}$, *and any pair* (s, t) *such that*: $t_{k-1} \leqq s < t \leqq t_k$, *the random variable* $\{(X_t - X_s)/(t - s)\}$ *is uniformly distributed, and independent of* B;

(ii) *the germ* σ*-field* $\mathscr{F}_{0+} \equiv \bigcap_{\varepsilon>0} \sigma(X_s, s \leqq \varepsilon)$ *is* P*-trivial.*

The proof of the first assertion of Theorem 0 is obtained essentially by considering the two sequences of r.v.'s:

$$\eta_k = \frac{X_{t_k} - X_{t_{k-1}}}{t_k - t_{k-1}} \quad \text{and} \quad \xi_k = \frac{B_{t_k} - B_{t_{k-1}}}{t_k - t_{k-1}},$$

which satisfy (T), hence the idea of considering (T) with more general assumptions about the laws of the ξ_k's. In particular, one of the original motivations of this paper was to understand the rôle played by the Gaussian distributions of the ξ_k's, in connection with the uniform distribution of the fractional parts $\{\eta_k\}$. It simply turns out that Tsirel'son's (Brownian) example, when considered in the present general study of (T) falls into the uniqueness in law subcase (see Sect. 5 below) which, in general, ensures that the fractional parts $\{\eta_k\}$ are uniformly distributed.

(1.3) A substantial part of the results contained in this paper has been announced without proof in the Comptes Rendus Note [11]. Here, full proofs are given, and the discussion in terms of the sequence $\mu = (\mu_k;\ k \in -\mathbb{N})$ is completed, as far as questions of uniqueness in law, and existence of strong solutions, for example, are concerned (we use here the same terms as for continuous time stochastic differential equations).

In conclusion of this Introduction, we should like to point out that the most interesting features of the results obtained in this work are their seemingly paradoxical character:

in particular, if $\mathscr{P}_\mu(T)$ enjoys the uniqueness in law property, then the unique solution (in law) P_μ^* is not strong (see Sect. 5); moreover, in the case where there exists a strong solution, then there are several strong solutions (see Sect. 6).

We hope that the detailed and elementary study made in this paper shall help the reader with such "paradoxes"; in fact, the author is much indebted to Rogers and Williams [4], p. 156, for their amazed comments following their Theorem (18.3) $\equiv$ Theorem 0 above, concerning the independence of any fractional part and the Brownian motion.

2 On the extremal points of $\mathscr{P}_\mu(T)$

We keep the notation introduced in (1.1). The following lemma is a first easy step in our study.

Lemma 1. *A probability P on $(\mathbb{R}^{-\mathbb{N}}, \mathscr{F}_0)$ belongs to $\mathscr{P}_\mu(T)$, if, and only if, under P, the sequence $(\eta_k;\ k \in -\mathbb{N})$ enjoys the following (inhomogeneous) Markov property: for any Borel, bounded function $f\colon \mathbb{R} \to \mathbb{R}$, we have:*

$$E[f(\eta_k) \mid \mathscr{F}_{k-1}] = \int d\mu_k(y) f(y + \{\eta_{k-1}\}) .$$

It is now clear that $\mathscr{P}_\mu(T)$ is a convex set of probabilities on $(\mathbb{R}^{-\mathbb{N}}, \mathscr{F}_0)$; the next theorem describes its extremal points, as well as the asymptotic σ-field $\mathscr{F}_{-\infty}$, up to negligible sets, for any $P \in \mathscr{P}_\mu(T)$.

Theorem 1. *Let $P \in \mathscr{P}_\mu(T)$. Then,*

1) $\mathscr{F}_{-\infty}$ coïncides, up to P negligible sets, with the σ-field generated by the sequence $(\theta_{k,p};\ k \in -\mathbb{N}, p \in \mathbb{Z}^*)$, defined by:

$$\theta_{k,p} = \lim_{n \to \infty} \exp(2i\pi p \eta_{k-n}) \prod_{j=k-n+1}^{k} \varphi_j(p)$$

(this limit exists P almost surely), where $\varphi_j(p) = \int d\mu_j(x) \exp(2i\pi p x)$.

2) P is an extremal point of $\mathscr{P}_\mu(T)$ if, and only if, $\mathscr{F}_{-\infty}$ is P-trivial.

Proof of Theorem 1.

1) $\mathscr{F}_{-\infty}$ coïncides (up to P-negligible sets) with the σ-field generated by

$$\lim_{n \to \infty} E(Y \mid \mathscr{F}_{-n}) ,$$

as Y varies among the family

$$F = (f_k(\eta_k, \eta_{k+1}, \ldots, \eta_0);\ k \in -\mathbb{N}, f_k \in b(\mathscr{B}(\mathbb{R}^{-k+1}))\ ,$$

since this family is total in $L^2(P)$.

Thanks to the Markov property satisfied by P (see Lemma 1 above), $\mathscr{F}_{-\infty}$ coïncides in fact with the σ-field generated by:

$$\lim_{n\to\infty} E[\varphi(\exp 2i\pi\,\eta_k)|\mathscr{F}_{k-n}]\,,$$

as k varies in $-\mathbb{N}$, and φ is a generic continuous function on the unit circle.

Next, approximating φ uniformly by a trigonometric polynomial, we obtain that $\mathscr{F}_{-\infty}$ coïncides with the σ-field generated by:

$$\lim_{n\to\infty} E[\exp(2i\pi p\,\eta_k)|\mathscr{F}_{k-n}]$$

as $k\in -\mathbb{N}$, and $p\in\mathbb{Z}^*$.
Finally, we deduce from (T) that:

$$E[\exp(2i\pi p\,\eta_k)\,|\,\mathscr{F}_{k-n}] = \exp(2i\pi p\,\eta_{k-n}) \prod_{j=k-n+1}^{k} \varphi_j(p)$$

and the first assertion is proved.

2) We now assume that P is an extremal point of $\mathscr{P}_\mu(T)$, and we show that $\mathscr{F}_{-\infty}$ is P-trivial. If not, there would exist $A\in\mathscr{F}_{-\infty}$ such that:

$0 < P(A) < 1$, and we would have:

$$P = P(A)P_A + P(A^c)P_{A^c}, \quad \text{where: } P_B \equiv \frac{P(\cdot\cap B)}{P(B)}\,.$$

But, it follows from Lemma 1 that, for any $B\in\mathscr{F}_{-\infty}$ such that: $P(B)>0$, the probability P_B belongs to $\mathscr{P}_\mu(T)$. Hence, there exists no non-trivial set $A\in\mathscr{F}_{-\infty}$.

3) Conversely, we assume that $\mathscr{F}_{-\infty}$ is P-trivial, and we show that P is an extremal point of $\mathscr{P}_\mu(T)$. It suffices to show that the only probabilities $Q\in\mathscr{P}_\mu(T)$ which are equivalent to ($\equiv$ mutually absolutely continuous with respect to) P on $\mathscr{F}_0$ are in fact equal to P. This follows immediately from the general statement of Lemma 2 below. □

Lemma 2. *Let $P\in\mathscr{P}_\mu(T)$, and let D be a strictly positive, $\mathscr{F}_0$-measurable r.v. such that: $E_P(D)=1$. Define $Q=D\cdot P$ on $\mathscr{F}_0$.*
Then, $Q\in\mathscr{P}_\mu(T)$ if, and only if, D is $\mathscr{F}_{-\infty}$ measurable.

Proof. 1) If D is $\mathscr{F}_{-\infty}$-measurable, it follows from Lemma 1 that Q belongs to $\mathscr{P}_\mu(T)$.

2) We now assume that $Q\in\mathscr{P}_\mu(T)$.
Using again Lemma 1, we know that, for any bounded r.v. G_k, which is $\mathscr{G}_k$ measurable, where: $\mathscr{G}_k\equiv\sigma(\eta_k,\eta_{k+1},\ldots,\eta_0)$, we have:

$$E_Q(G_k\,|\,\mathscr{F}_{k-1}) = E_P(G_k\,|\,\mathscr{F}_{k-1})\,,$$

hence, using the fact that: $\mathscr{F}_0=\mathscr{F}_{k-1}\vee\mathscr{G}_k$, we deduce from the monotone class theorem that:

$$E_Q(X\,|\,\mathscr{F}_{k-1}) = E_P(X\,|\,\mathscr{F}_{k-1})$$

for any $\mathbb{R}_+$-valued, $\mathscr{F}_0$-measurable r.v. X.

In terms of P and D, this equality is equivalent to:

$$E_P(XD \mid \mathscr{F}_{k-1}) = E_P(X \mid \mathscr{F}_{k-1}) E_P(D \mid \mathscr{F}_{k-1}) ,$$

from which we easily deduce that, for any k, $D \in L^1(\mathscr{F}_{k-1}, P)$, hence D belongs to $L^1(\mathscr{F}_{-\infty}, P)$. □

The identification of $\mathscr{F}_{-\infty}$, under $P \in \mathscr{P}_\mu(T)$, which is stated in Theorem 1, shall allow us now to write P as an integral of extremal points of $\mathscr{P}_\mu(T)$. We begin by defining $\hat{\mathscr{F}}_{-\infty}$ as the σ-field generated by the sequence

$$(\hat{\theta}_{k,p};\, k \in -\mathbb{N},\, p \in \mathbb{Z}^*) \quad \text{defined by:}$$

$$\hat{\theta}_{k,p} = \limsup_{n\to\infty} (\operatorname{Re} \theta^{(n)}_{k,p}) + i \limsup_{n\to\infty} (\operatorname{Im} \theta^{(n)}_{k,p}) ,$$

where

$$\theta^{(n)}_{k,p} = \exp(2i\pi p\, \eta_{k-n}) \prod_{j=k-n+1}^{k} \varphi_j(p) .$$

We have, obviously, $\hat{\mathscr{F}}_{-\infty} \subset \mathscr{F}_{-\infty}$, and we have seen, in Theorem 1, that if $P \in \mathscr{P}_\mu(T)$, then $\hat{\mathscr{F}}_{-\infty}$ coïncides with $\mathscr{F}_{-\infty}$ up to P-negligible sets.

If P is a probability measure on $(\mathbb{R}^{-\mathbb{N}}, \mathscr{F}_0)$, then there exists a regular conditional distribution $P_\omega(d\omega')$ of P, given $\hat{\mathscr{F}}_{-\infty}$, since $\hat{\mathscr{F}}_{-\infty}$ is separable. We have:

$$P = \int \hat{P}(\mathrm{d}\omega) P_\omega , \tag{2.a}$$

where $\hat{P}(\mathrm{d}\omega)$ is the restriction of P to $\hat{\mathscr{F}}_{-\infty}$. We may now prove

Theorem 1′. *If $P \in \mathscr{P}_\mu(T)$, the identity (2.a) also gives the integral representation of P as an integral of extremal points of $\mathscr{P}_\mu(T)$, that is:*

$$\hat{P}(\mathrm{d}\omega) \quad \text{a.s.,} \quad P_\omega \in \operatorname{ext}(\mathscr{P}_\mu(T)) .$$

Proof. 1) It suffices to show:

$$\hat{P}(\mathrm{d}\omega) \quad \text{a.s.,} \quad P_\omega \in \mathscr{P}_\mu(T) , \tag{2.b}$$

since, by construction, we know that $\hat{\mathscr{F}}_{-\infty}$ is trivial under P_ω, and, on the other hand, if (2.b) holds, we know, from both assertions in Theorem 1, that $\hat{P}(d\omega)$ a.s., $\mathscr{F}_{-\infty}$ is trivial under P_ω, hence $P_\omega \in \operatorname{ext}(\mathscr{P}_\mu(T))$.

2) We now prove (2.b). From Lemma 1, it suffices to show that, for a fixed $k \in -\mathbb{N}$, and any given Borel function $f: \mathbb{R} \to \mathbb{R}_+$, we have:

$$\hat{P}(\mathrm{d}\omega) \text{ a.s.,} \quad E_{P_\omega}[f(\eta_k) \mid \mathscr{F}_{k-1}] = \int \mathrm{d}\mu_k(y) f(y + \{\eta_{k-1}\}) \tag{2.c}$$

(indeed, the monotone class theorem allows to restrict ourselves to a countable set of such functions f, which generate the Borel σ-field). Now, (2.c) follows immediately from the identity:

$$E_P[1_A 1_{B_{k-1}} f(\eta_k)] = E_P[1_A 1_{B_{k-1}} \int \mathrm{d}\mu_k(y) f(y + \{\eta_{k-1}\})] ,$$

for any $A \in \hat{\mathscr{F}}_{-\infty}$, and $B_{k-1} \in \mathscr{F}_{k-1}$, and the proof is ended. □

Remark. It should be mentioned here that the work of Dynkin [1] is very relevant for this paragraph, but we prefered to derive the results from scratch, in order to keep with the simplicity of the present paper.

3 The set $\mathscr{P}_\mu(T)$ contains at least one element

We now prove that, no matter what the sequence $\mu = (\mu_k; k \in -\mathbb{N})$ may be, the set $\mathscr{P}_\mu(T)$ is never empty. More precisely, we have the following

Theorem 2. *There exists a unique probability P^*_μ in $\mathscr{P}_\mu(T)$ such that:*
(3.a) *under P^*_μ, for any $k \in -\mathbb{N}$, $\{\eta_k\}$ is uniformly distributed on* $[0, 1[$.
Moreover, under P^*_μ, for any $k \in -\mathbb{N}$, the variable $\{\eta_k\}$ is independent of $\sigma\{\xi_j; j \in -\mathbb{N}\}$, where $\xi_j = \eta_j - \{\eta_{j-1}\}$.

Finally, if the variables $\{\xi_j\}$ are all uniformly distributed, then the variables $(\{\eta_j\}, j \in -\mathbb{N})$ are independent.

We shall see, in the sequel, that P^*_μ plays a fundamental rôle in the study of $\mathscr{P}_\mu(T)$.

To begin the proof of Theorem 2, we first remark that the condition (3.a) and the equation (T) specify uniquely the distribution on $\mathbb{R}^{-k}$, which we denote by π_k, of $(\eta_{k+1}, \eta_{k+2}, \ldots, \eta_0)$ under P^*_μ, for any $k \in -\mathbb{N}$.

The first part of the next Proposition, which may in fact be considered as a key throughout all our study, shows obviously that the different probabilities π_k are compatible between themselves, and the first assertion of Theorem 2 now follows from the extension theorem of Kolmogorov (see, for example, Neveu [3], p. 78).

Proposition 1. *Let U be a random variable which takes its values in* $[0, 1[$, *and X be a real-valued random variable, which is independent of U.*

1) *If U is uniformly distributed on* $[0, 1[$, *then $\{U + X\}$ and X are independent, and $\{U + X\}$ is uniformly distributed on* $[0, 1[$;

2) *Conversely, if the law of X is diffuse, and if $\{U + X\}$ and X are independent, then U is uniformly distributed on* $[0, 1[$.

Proof of the Proposition. 1) In order to prove, at the same time, that $\{U + X\}$ and X are independent, and that $\{U + X\}$ is uniformly distributed, it suffices to show that:

for any $p \in \mathbb{Z}^*$, and any $\lambda \in \mathbb{R}$,

$$E[\exp(2ip\pi\{U + X\} + i\lambda X)] = 0\,. \tag{3.b}$$

However, the left-hand side of (3.b) is equal to:

$$\begin{aligned} E[\exp 2i\pi p(U + X)\exp i\lambda X] &= E[\exp(2i\pi pU)]E[\exp i(2\pi p + \lambda)X] \\ &= 0, \text{ since } U \text{ is uniformly distributed.} \end{aligned}$$

This proves (3.b).

2) Conversely, from our hypothesis, we have, on one hand:

$$E[\exp(2i\pi p(U + X) + i\lambda X)] = E[\exp 2i\pi p(U + X)]E[\exp i\lambda X]\,.$$

Hence, since U and X are independent, this equality may also be written:

$$E[\exp(2i\pi pU)]E[\exp(2i\pi p + i\lambda)X] = E[\exp(2i\pi pU)]E[\exp 2i\pi pX]E[\exp i\lambda X].$$

We denote $\varphi(p) = E[\exp(2i\pi pU)]$, and we take $\lambda = -2\pi p$.
We obtain:

$$\varphi(p)(1 - |E(e^{2i\pi pX})|^2) = 0\,.$$

However, the hypothesis made on X implies: $|E(\exp 2i\pi pX)| < 1$, for $p \neq 0$, from which we deduce:

$$\varphi(p) = 0, \quad \text{for any } p \in \mathbb{Z}^* ;$$

in other terms, U is uniformly distributed on $[0, 1[$. □

End of the proof of Theorem 2

a) To prove the second assertion of the Theorem, it suffices to show that, for any $p \in \mathbb{Z}^*$, and for any $(\lambda_k, \lambda_{k-1}, \ldots, \lambda_{k-n}) \in \mathbb{R}^{n+1}$, with $n \in \mathbb{N}$, the following quantity:

$$\varphi(p; \lambda_j(k-n \leqq j \leqq k)) = E[\exp(2i\pi p\eta_k)\exp i(\lambda_k\xi_k + \lambda_{k-1}\xi_{k-1} + \cdots + \lambda_{k-n}\xi_{k-n})]$$

equals 0.

However, from the identity (T): $\eta_k = \xi_k + \{\eta_{k-1}\}$, we deduce:

$$\begin{aligned}\varphi(p; \lambda_j(k-n \leqq j \leqq k)) &= E[\exp i((\lambda_k + 2\pi p)\xi_k + (\lambda_{k-1} + 2\pi p)\xi_{k-1} \\ &\quad + \cdots + (\lambda_{k-n} + 2\pi p)\xi_{k-n})]E[\exp 2i\pi p\eta_{k-n-1}] \\ &= 0, \text{ since } \{\eta_{k-n-1}\} \text{ is uniformly distributed.}\end{aligned}$$

b) To prove the last assertion of the Theorem, it suffices to show that for any $n \in \mathbb{N}$, and $(p_0, p_{-1}, \ldots, p_{-n+1}) \in \mathbb{Z}^n$, the equality:

$$E\left[\exp\left(2i\pi \sum_{j=-n+1}^{0} p_j\eta_j\right)\right] = 0 \tag{3.c}$$

holds as soon as one of the p_j's is not 0.

We denote $\varphi_j(p) = E[\exp(2i\pi p\xi_j)]$; then, we have:

$$\begin{aligned}&E\left[\exp\left(2i\pi \sum_{j=-n+1}^{0} p_j\eta_j\right)\right] \\ &= \varphi_0(p_0)E\left[\exp 2i\pi\left((p_0 + p_{-1})\eta_{-1} + \sum_{j=-n+1}^{-2} p_j\eta_j\right)\right] \\ &= \varphi_0(p_0)\varphi_{-1}(p_0 + p_{-1})\varphi_{-2}(p_0 + p_{-1} + p_{-2}) \ldots \varphi_{-n+1}(p_0 + \cdots + p_{-n+1}) \\ &\quad \times E\left[\exp\left(2i\pi \left(\sum_{j=-n+1}^{0} p_j\right)\eta_{-n}\right)\right].\end{aligned}$$

Consequently, if $p_0 \neq 0$, we have: $\varphi_0(p_0) = 0$; then, if $p_0 = 0$, but $p_{-1} \neq 0$, we have: $\varphi_{-1}(p_{-1}) = 0$, and so on; by iteration, we have shown (3.c). □

Remark. In the last part of the proof of Theorem 2, we did not use the hypothesis that $\{\eta_k\}$ is uniformly distributed. This shows therefore that, if for any $k \in -\mathbb{N}$, $\{\xi_k\}$ is uniformly distributed, then $\mathscr{P}_\mu(T)$ contains only one element, which is P_μ^*. A full characterization of the uniqueness case for $\mathscr{P}_\mu(T)$ is given in Sect. 5.

4 A general formula for a conditional distribution, and some consequences

(4.1) A key rôle shall be played in the sequel by the following

Proposition 2. *For any $p \in \mathbb{Z}$, and $k \in -\mathbb{N}$, we have, for any $P \in \mathscr{P}_\mu(T)$:*

$$E[\exp(2i\pi p\eta_k) \mid \mathscr{F}_{-\infty} \vee \mathscr{E}_0] = \delta_p \exp(2i\pi p\eta_k) \tag{4.a}$$

where:

$$\delta_p = \begin{cases} 0, \text{ if for any } k', & \lim_{n\to\infty} \prod_{j=k'-n}^{k'} \varphi_j(p) = 0 \\ 1, \text{ if not.} \end{cases}$$

Proof. Remark that, since for any j, ξ_j is independent of $\mathscr{F}_{j-1}$, the left-hand side of (4.a) is equal to:

$$E[\exp(2i\pi p\eta_k) | \mathscr{F}_{-\infty} \vee \mathscr{E}_k] .$$

1) We first assume that $\delta_p = 0$.
(i) We first prove

$$E[\exp(2i\pi p\eta_k) | \mathscr{F}_{-\infty}] = 0 . \tag{4.b}$$

Indeed, we have:

$$E[\exp(2i\pi p\eta_k) | \mathscr{F}_{-\infty}] = \prod_{j=k-n}^{k} E[\exp(2i\pi p\xi_j)] E[\exp(2i\pi p\eta_{k-n-1}) | \mathscr{F}_{-\infty}]$$

and, since $\delta_p = 0$, the product $\prod_{j=k-n}^{k}$ converges to 0, as $n \to \infty$. This proves (4.b).

(ii) Then, we remark that, for every $n \in \mathbb{N}$, we have:

$$\begin{aligned} &E[\exp(2i\pi p\eta_k) | \mathscr{F}_{-\infty} \vee \sigma(\xi_{k-n}, \ldots, \xi_k)] \\ &= \exp 2i\pi p(\xi_{k-n} + \cdots + \xi_k) E[\exp(2i\pi p\eta_{k-n-1}) | \mathscr{F}_{-\infty}] \\ &= 0, \text{ from (4.b).} \end{aligned}$$

Finally, letting $n \to \infty$, we then obtain:

$$E[\exp(2i\pi p\eta_k) | \mathscr{F}_{-\infty} \vee \mathscr{E}_k] = 0 ,$$

which is (4.a) in the case $\delta_p = 0$.
2) We now consider the case where $\delta_p = 1$, that is when there exists k' sufficiently small such that:

$$\prod_{j=-\infty}^{k'} |\varphi_j(p)| > 0 .$$

We may assume $k' < k$.
We then write $\exp(2i\pi p\eta_k)$ as:

$$\exp(2i\pi p\eta_k) = \exp(2i\pi p\eta_{k'}) \exp(2i\pi p(\xi_{k'+1} + \cdots + \xi_k)) ,$$

and it remains to show that $\exp(2i\pi p\eta_{k'})$ is measurable with respect to $\mathscr{F}_{-\infty} \vee \mathscr{E}_0$. However, we have: $\exp(2i\pi p\eta_{k'}) = \Phi_n \Psi_n$, where:

$$\Phi_n = \frac{\exp 2i\pi p(\xi_{k'} + \cdots + \xi_{k'-n})}{\prod_{j=k'-n}^{k'} \varphi_j(p)}$$

and $\Psi_n = \exp(2\pi i p\eta_{k'-n-1}) \left(\prod_{j=k'-n}^{k'} \varphi_j(p) \right)$.

(With the notation introduced in the proof of Lemma 2, we have: $\Psi_n = \theta_{k',p}^{(n+1)}$).

Now, from the convergence result for bounded martingales, Φ_n converges almost surely to a variable which is $\mathscr{E}_{k'}$-measurable; hence, Ψ_n also converges almost surely, to a variable which is $\mathscr{F}_{-\infty}$-measurable. □

Remark. We note here that the variables $\Phi_\infty \equiv \lim_{n\to\infty}\Phi_n$ and $\Psi_\infty = \lim_{n\to\infty}\Psi_n$ which appeared at the end of the proof of the Proposition are closely related respectively to:

$$E[\exp(2i\pi p\eta_k)|\mathscr{E}_0] \quad \text{and} \quad E[\exp(2i\pi p\eta_k)|\mathscr{F}_{-\infty}] .$$

To simplify the discussion, we shall assume that for every $j \leqq k$, $\varphi_j(p) \neq 0$, so that we have, with the above notation for Φ_n and Ψ_n, for which we take $k' = k$:

$$E[\exp(2i\pi p\eta_k)|\mathscr{F}_{k-n-1}] = \Psi_n$$

and therefore:

$$E[\exp(2i\pi p\eta_k)|\mathscr{F}_{-\infty}] = \Psi_\infty, \tag{4.c}$$

whilst:

$$E[\exp(2i\pi p\eta_k)|\xi_{k-n}, \dots, \xi_k] = \exp 2i\pi p(\xi_{k-n} + \cdots + \xi_k) E[\exp(2i\pi p\eta_{k-n-1})] . \tag{4.d}$$

Taking expectations on both sides of (4.d), we obtain:

$$E[\exp(2i\pi p\eta_k)] = \left(\prod_{j=k-n}^{k} \varphi_j(p)\right) E[\exp(2i\pi p\eta_{k-n-1})]$$

so that, putting this back in (4.d), we obtain:

$$E[\exp(2i\pi p\eta_k)|\xi_{k-n}, \dots, \xi_k] = \Phi_n E[\exp(2i\pi p\eta_k)]$$

and, in case $\delta_p = 1$:

$$E[\exp(2i\pi p\eta_k)|\mathscr{E}_k] = \Phi_\infty E[\exp(2i\pi p\eta_k)] , \tag{4.e}$$

so that:

(i) *In the case* $\delta_p = 1$, since $\exp(2i\pi p\eta_k) = \Phi_\infty \Psi_\infty$, we have:

$$\exp(2i\pi p\eta_k) = E[\exp(2i\pi p\eta_k)|\mathscr{F}_{-\infty}] \frac{E[\exp(2i\pi p\eta_k)|\mathscr{E}_0]}{E[\exp(2i\pi p\eta_k)]} \tag{4.f}$$

from which we can then recover (4.a).

(ii) On the other hand, *in the case* $\delta_p = 0$, since we have, by (4.a):

$$E[\exp(2i\pi p\eta_k)|\mathscr{F}_{-\infty} \vee \mathscr{E}_0] = 0 ,$$

this implies: $\quad E[\exp(2i\pi p\eta_k)|\mathscr{F}_{-\infty}] = E[\exp(2i\pi p\eta_k)|\mathscr{E}_0] = 0 .$

(4.2) Once formula (4.a) has been established, it is natural to look for an explicit description of the law of $\{\eta_k\}$ given $\mathscr{F}_{-\infty} \vee \mathscr{E}_0$. In order to do this, we need to know more about the structure of the set

$\mathbb{Z}_+(\mu) \overset{\text{def}}{=} \{p \in \mathbb{Z}: \delta_p = 1\}$. In fact, we have the

Proposition 3. $\mathbb{Z}_+(\mu) \overset{\text{def}}{=} \{p \in \mathbb{Z}: \delta_p = 1\}$ *is a subgroup of* $\mathbb{Z}$.

Hence, there exists $p_\mu \in \mathbb{N}$ *such that:* $\mathbb{Z}_+(\mu) = p_\mu \mathbb{Z}$.

Proof. To simplify notation, we shall assume here that the set of the indexes k of the *variables* ξ_k is $\mathbb{N}$, instead of $-\mathbb{N}$.

1) Since $\varphi_j(-p) = \overline{\varphi_j(p)}$, it is obvious that $p \in \mathbb{Z}_+(\mu)$ if and only if $(-p)$ does; moreover, it is obvious that $0 \in \mathbb{Z}_+(\mu)$.

2) It remains to verify that if p and q belong to $\mathbb{Z}_+(\mu)$, so does $p+q$. By definition of $\mathbb{Z}_+(\mu)$, there exists k sufficiently large such that:

$$\prod_{j=k}^{\infty} |\varphi_j(p)| > 0 \quad \text{and} \quad \prod_{j=k}^{\infty} |\varphi_j(q)| > 0 \,.$$

To simplify our notation again, we may assume, without loss of generality, that $k = 0$.

We already remarked, while proving (4.a), that the two $(\mathscr{F}_n,\, n \geqq 0)$-martingales:

$$\Phi_n(p) = \frac{\exp 2i\pi p(\xi_0 + \cdots + \xi_n)}{\prod_{j=0}^{n} \varphi_j(p)}$$

and $\Phi_n(q)$, defined similarly, converge a.s. towards two complex variables, which are a.s. different from 0.

Hence, so does the sequence:

$$H_n = \frac{\exp 2i\pi(p+q)(\xi_0 + \cdots + \xi_n)}{\left(\prod_{j=0}^{n} \varphi_j(p)\right)\left(\prod_{j=0}^{n} \varphi_j(q)\right)} .$$

Consequently, we have: $\lim\limits_{\substack{m<n \\ m,n\to\infty}} \dfrac{H_n}{H_m} = 1$, a.s. and, since the sequence $(H_n/H_m;\ 0 \leqq m \leqq n < \infty)$ is uniformly bounded, we deduce: $\lim\limits_{\substack{m<n \\ m,n\to\infty}} E\left(\dfrac{H_n}{H_m}\right) = 1$,

that is:

$$\lim_{\substack{m<n \\ m,n\to\infty}} \frac{\prod_{j=m+1}^{n} \varphi_j(p+q)}{\left(\prod_{j=m+1}^{n} \varphi_j(p)\right)\left(\prod_{j=m+1}^{n} \varphi_j(q)\right)} = 1 \,.$$

On the left-hand side, we may replace the function φ_j by $|\varphi_j|$, and since, by definition of $\mathbb{Z}_+(\mu)$, we have:

$$\lim_{\substack{m<n \\ m,n\to\infty}} \prod_{j=m+1}^{n} |\varphi_j(p)| = 1, \text{ and the same for } q,$$

we obtain:

$$\lim_{\substack{m<n \\ m,n\to\infty}} \prod_{j=m+1}^{n} |\varphi_j(p+q)| = 1 \,.$$

Consequently, for m sufficiently large, we have:

$$\prod_{j=m-1}^{\infty} |\varphi_j(p+q)| > 0 ,$$

which means that: $p + q \in \mathbb{Z}_+(\mu)$. □

Although it would now be natural to give some examples of sequences $\mu = (\mu_k; k \in -\mathbb{N})$ for which $p_\mu = 0$, 1, or any positive integer, we prefer to postpone this to the next paragraphs 5, 6, 7 which are respectively devoted to the cases $p_\mu = 0$, $p_\mu = 1$, and $p_\mu > 1$.

We end up this subparagraph (4.2) with the following remark: the previous arguments can also be applied, in fact more simply, to the study of the set $\mathbb{Z}_{+,c}(\mu)$ which we define as follows: $p \in \mathbb{Z}_{+,c}(\mu)$ if there exists k large enough such that $\prod_{j=k}^{k+n} \varphi_j(p)$ converges, as $n \to +\infty$, towards a complex number different from 0.

We then have the

Lemma 3. *$p \in \mathbb{Z}_{+,c}(\mu)$ if, and only if: $\exp 2i\pi p(\xi_0 + \xi_1 + \cdots + \xi_n)$ converges almost surely.*
Consequently, $\mathbb{Z}_{+,c}(\mu)$ is a subgroup of $\mathbb{Z}$ (in fact, it is a subgroup of $\mathbb{Z}_+(\mu)$) and, therefore, there exists an integer p_μ^c which is a multiple of p_μ, such that: $\mathbb{Z}_{+,c}(\mu) = p_\mu^c \mathbb{Z}$.

Proof. 1) If $p \in \mathbb{Z}_{+,c}(\mu)$, then, for some k large enough, and every $j \geqq k$, we have: $\varphi_j(p) \neq 0$, and

$$\frac{\exp 2i\pi p(\xi_k + \cdots + \xi_{k+n})}{\prod_{j=k}^{k+n} \varphi_j(p)}$$

converges almost surely, as $n \to \infty$, by the martingale convergence theorem. Consequently, $\exp 2i\pi p(\xi_0 + \cdots + \xi_n)$ converges almost surely.
2) Conversely, if $\exp 2i\pi p(\xi_0 + \cdots + \xi_n)$ converges almost surely as $n \to \infty$, then:

$$\lim_{\substack{m<n \\ m,n\to\infty}} \exp 2i\pi p(\xi_m + \cdots + \xi_n)$$

exists a.s. and is equal to 1.
Hence, by the dominated convergence theorem, we have:

$$\lim_{\substack{m<n \\ m,n\to\infty}} \prod_{j=m}^{n} \varphi_j(p) = 1$$

which easily implies that $p \in \mathbb{Z}_{+,c}(\mu)$. □

Example. A study of the case where the variables ξ_n are Gaussian, with mean m_n, and variances σ_n^2, quickly provides examples where $\mathbb{Z}_{+,c}(\mu)$ is strictly contained in $\mathbb{Z}_+(\mu)$.

In fact, in this case, we have:

$$\mathbb{Z}_+(\mu) = \begin{cases} \mathbb{Z}, & \text{if } \sum_n \sigma_n^2 < \infty \\ \{0\}, & \text{if } \sum_n \sigma_n^2 = \infty \end{cases}$$

while, in case: $\sum_n \sigma_n^2 < \infty$, we have: $\mathbb{Z}_{+,c}(\mu) = \{p \in \mathbb{Z} : \exp(2ip\pi M_n)$ converges, as $n \to \infty\}$ with $M_n = \sum_{k=0}^n m_k$.

(4.3) It is now possible to discuss, in terms of the values of p_μ, the law of $\{\eta_k\}$, given $\mathscr{F}_{-\infty} \vee \mathscr{E}_0$.

Proposition 4. 1) *If* $p_\mu = 0$, *then* $\{\eta_k\}$ *is uniformly distributed on* $[0, 1[$ *and independent of* $\mathscr{F}_{-\infty} \vee \mathscr{E}_0$.

2) *If* $p_\mu = 1$, *then* $\{\eta_k\}$ *is measurable with respect to* $\mathscr{F}_{-\infty} \vee \mathscr{E}_0$.

3) *If* $p_\mu \neq 0, 1$, *then* $\{p_\mu \eta_k\} \equiv \{p_\mu \{\eta_k\}\}$ *is measurable with respect to* $\mathscr{F}_{-\infty} \vee \mathscr{E}_0$, *while the integer part:* $[p_\mu\{\eta_k\}]$ *is uniformly distributed on* $(0, 1, \ldots, (p_\mu - 1))$, *and independent of* $\mathscr{F}_{-\infty} \vee \mathscr{E}_0$.

Proof. a) The two first points follow immediately from (4.a), since, in the first case, we have:

for any $p \neq 0$, $E[\exp(2i\pi p\eta_k) | \mathscr{F}_{-\infty} \vee \mathscr{E}_0] = 0$, and, in the second case, we have: for any $p \in \mathbb{Z}$, $E[\exp(2i\pi p\eta_k) | \mathscr{F}_{-\infty} \vee \mathscr{E}_0] = \exp(2i\pi p\eta_k)$.

b) In the last case, we first remark that, for any $p \in \mathbb{Z}$,

$$E[\exp(2i\pi p p_\mu \eta_k) | \mathscr{F}_{-\infty} \vee \mathscr{E}_0] = \exp(2i\pi p p_\mu \eta_k)$$

so that $\{p_\mu \eta_k\} \equiv \{p_\mu\{\eta_k\}\}$ is measurable with respect to $\mathscr{F}_{-\infty} \vee \mathscr{E}_0$.
Moreover, for any $p \in (1, 2, \ldots, (p_\mu - 1))$, we have

$$E[\exp(2i\pi p\eta_k) | \mathscr{F}_{-\infty} \vee \mathscr{E}_0] = 0 .$$

To simplify notation, we write $\bar{\eta}_k$ for $\{\eta_k\}$. Since $p_\mu \bar{\eta}_k$ is $\mathscr{F}_{-\infty} \vee \mathscr{E}_0$ measurable, we obtain:

$$E\left[\exp\left(2i\pi \frac{p}{p_\mu}[p_\mu \bar{\eta}_k]\right) \middle| \mathscr{F}_{-\infty} \vee \mathscr{E}_0 \right] = 0 ,$$

for any $p \in (1, 2, \ldots, p_\mu - 1)$, and, from the injectivity of the Fourier transform, we deduce that $[p_\mu \bar{\eta}_k]$ is uniformly distributed on $(0, 1, \ldots, (p_\mu - 1))$. □

In agreement with the first part of the Introduction, we now remark that the last Proposition 4 shows that, in any case, the σ-field $(\mathscr{F}_{-\infty} \vee \mathscr{E}_k)$ admits an independent complement in $\mathscr{F}_k$.

Proposition 5. 1) *If* $p_\mu = 0$, *then* $\mathscr{P}_\mu(T)$ *consists of only one element, i.e.:* P_μ^*; $\mathscr{F}_{-\infty}$ *is trivial,* $\mathscr{F}_k = \mathscr{E}_k \vee \sigma(\{\eta_k\})$, *and* $\{\eta_k\}$ *is independent from* $\mathscr{E}_k$.

2) *If* $p_\mu = 1$, *then* $\{\eta_k\}$ *is measurable with respect to* $\mathscr{F}_{-\infty} \vee \mathscr{E}_k$, *and:*

$$\mathscr{F}_k = \mathscr{F}_{-\infty} \vee \mathscr{E}_k .$$

3) *If* $p_\mu > 1$, *then* $[p_\mu\{\eta_k\}]$ *is uniformly distributed on* $(0, 1, \ldots, (p_\mu - 1))$; *it is independent of* $\mathscr{F}_{-\infty} \vee \mathscr{E}_k$, *and:*

$$\mathscr{F}_k = \mathscr{F}_{-\infty} \vee \mathscr{E}_k \vee \sigma([p_\mu\{\eta_k\}]) .$$

Remark. In the statement of either of the 3 points, we may replace $\{\eta_k\}$ by $\{\eta_j\}$, for any $j < k$. In particular, in cases: $p_\mu = 0$, and $p_\mu > 1$, this shows that there exist infinitely many independent complements of $(\mathscr{F}_{-\infty} \vee \mathscr{E}_k)$ in $\mathscr{F}_k$.

Proof of Proposition 5. a) The statements concerning the complementation of $(\mathscr{F}_{-\infty} \vee \mathscr{E}_k)$ in $\mathscr{F}_k$ follow immediately from the previous Proposition 4, once one has remarked that, in any case:

$$\mathscr{F}_k = \mathscr{E}_k \vee \sigma(\{\eta_k\}) = \mathscr{E}_k \vee \sigma(\{\eta_j\}), \quad \text{for any } j < k$$

as a consequence of the equation (T).

b) It remains to finish the proof of the first assertion.

If $p_\mu = 0$, then from Proposition 4, $\{\eta_k\}$ is uniformly distributed on $[0, 1[$. Hence, by Theorem 2, $P = P_\mu^*$, so that $\mathscr{P}_\mu(T)$ consists only of $\mathscr{P}_\mu^*$, and $\mathscr{F}_{-\infty}$ is P_μ^*-trivial, by Theorem 1. □

5 The condition $p_\mu = 0$ characterizes uniqueness in law

(5.1) The last Proposition 5 states that, if $p_\mu = 0$, then $\mathscr{P}_\mu(T)$ consists of only one element, i.e.: P_μ^*. The next Theorem shows that the converse is true.

Theorem 3. *The following assertions are equivalent*:

1) $p_\mu = 0$.
2) $\mathscr{P}_\mu(T)$ *consists of only one element, i.e.*: P_μ^*.
3) P_μ^* *is an extremal point of* $\mathscr{P}_\mu(T)$.
4) *Under* P_μ^*, *for any* $k \in -\mathbb{N}$, $\{\eta_k\}$ *is independent of* $\mathscr{F}_{-\infty}$.

Proof. The implication 1) ⇒ 2) has been proved in the last Proposition 5.
2) ⇒ 3) is obvious; so is 3) ⇒ 4) since, if P_μ^* is an extremal point, then, by Theorem 1, $\mathscr{F}_{-\infty}$ is trivial under P_μ^*.

It remains to prove: 4) ⇒ 1).
Remark that, for $p \in \mathbb{Z}$:

$$E_\mu^*[\exp(2i\pi p\eta_k) \mid \mathscr{F}_{k-n}] = \exp(2i\pi p\eta_{k-n}) \prod_{j=k-n+1}^{k} \varphi_j(p) . \tag{5.a}$$

If $p \neq 0$, then, from the independence hypothesis made in 4), we obtain:

$$E_\mu^*[\exp(2i\pi p\eta_k) \mid \mathscr{F}_{k-n}] \xrightarrow[n\to\infty]{\text{a.s.}} E_\mu^*[\exp(2i\pi p\eta_k)] = 0 .$$

This implies that the right-hand side of (5.a) converges to 0 as $n \to \infty$, which shows that $p_\mu = 0$. □

(5.2) We now give some important examples of sequences $\mu = (\mu_k; k \in -\mathbb{N})$ for which the condition: $p_\mu = 0$ is satisfied. It may be helpful to denote this condition by (C_0), and to recall that, from the definition of p_μ given in Proposition 3 in (4.2), we have:

$$(C_0) \qquad \text{for any } p \in \mathbb{Z}^*, \text{ and } k \in -\mathbb{N}, \lim_{n\to\infty} \prod_{j=k-n}^{k} \varphi_j(p) = 0 .$$

It will be easier in this paragraph to take the set of indexes to be $\mathbb{N}$ instead of $-\mathbb{N}$; hence, from now on in this paragraph, if $n \in \mathbb{N}$, we shall write ξ_n for ξ_{-n} and φ_n for φ_{-n}.

The analytical condition (C_0) may be written in a more probabilistic, but equivalent way, as recorded in the following easy statements, the equivalence of which is left to the reader.

Lemma 4. *The following conditions are equivalent:*

(i) *for any $k \in \mathbb{N}$, the sequence $\{\xi_k + \xi_{k+1} + \cdots + \xi_n\}$ converges in law, as $n \to \infty$, towards a uniformly distributed random variable on $[0, 1[$;*

(ii) *for any $k \in \mathbb{N}$, the sequence of $(k+1)$ dimensional random vectors:*

$$(\xi_0, \xi_1, \ldots, \xi_k, \{\xi_0 + \xi_1 + \cdots + \xi_n\})$$

converges in law, as $n \to \infty$, towards:

$$(\xi_0, \xi_1, \ldots, \xi_k, U),$$

where U is a random variable which is uniformly distributed on $[0, 1[$, and is independent of the vector $(\xi_0, \xi_1, \ldots, \xi_k)$.

(iii) *the condition (C_0) is satisfied.*

Remark. This concerns condition (ii) in the Lemma; in the statement of this condition (ii), it would not be sufficient to assume only that $\{\xi_0 + \cdots + \xi_n\}$ converges in law, as $n \to \infty$, towards a uniform random variable, since this condition is satisfied as soon as one of the $\{\xi_k\}$'s is uniformly distributed; in such a case, obviously, the independence statement which is a part of (ii) may not be satisfied. □

We now give several sufficient conditions on the sequence $\mu = (\mu_k; k \in \mathbb{N})$ which imply (C_0); it may be remarked that these conditions are quite varied and numerous.

Proposition 6. *We assume that there exists a subsequence (n_j) of $\mathbb{N}$, and a random variable ξ such that:*

$$\xi n_j \overset{\text{(law)}}{=} \varepsilon_j \xi, \quad \textit{for some} \quad \varepsilon_j \in \mathbb{R}\,. \tag{5.b}$$

Then, either one of the following conditions implies that (C_0) is satisfied:

(i) *the law of ξ is absolutely continuous, and $|\varepsilon_j| \xrightarrow[j \to \infty]{} \infty$;*

(ii) *the law of ξ is diffuse, and there exist α and A such that: $0 < \alpha < A < \infty$ and: for all j's, $\alpha \leqq |\varepsilon_j| \leqq A$;*

(iii) *ξ has a moment of order 2, and*

$$\varepsilon_j \xrightarrow[j \to \infty]{} 0; \quad \sum_j \varepsilon_j^2 = \infty\,.$$

Consequently, if the law of ξ is absolutely continuous, and has a moment of order 2, then (C_0) is satisfied if and only if:

$$\sum_i \varepsilon_j^2 = \infty\,.$$

Proof. We first remark that, for all $k_1 < k_2$, we have:

$$\left| \prod_{j=n_{k_1}}^{n_{k_2}} \varphi_j(p) \right| \leqq \prod_{l=k_1}^{k_2} |\varphi(2\pi p \varepsilon_l)| \,, \tag{5.c}$$

where φ denotes the characteristic function of ξ.
– In the case (i), we may majorize the right-hand side of (5.c) by $|\varphi(2\pi p \varepsilon_{k_2})|$ which converges to 0, as $k_2 \to \infty$, as a consequence of the Riemann-Lebesgue theorem;
– In the case (ii), if we define: $M = \sup_{\alpha \leqq |x| \leqq A} |\varphi(2\pi p x)|$,
we have: $M < 1$, since the law of ξ is diffuse, and the right-hand side of (5.c) is then majorized by: $M^{k_2 - k_1}$, which converges to 0, as $k_2 \to \infty$, with k_1 fixed.
– In the case (iii), we have:

$$1 - |\varphi(2\pi p \varepsilon_j)|^2 \underset{j \to \infty}{\sim} c p^2 \varepsilon_j^2 \,; \tag{5.d}$$

therefore, thanks to (5.c), the condition (C_0) is satisfied.
– Finally, to prove the last assertion of the Proposition, we remark that, either $\sum_j \varepsilon_j^2 < \infty$, and then, from (5.d), we deduce that we are in the case $p_\mu = 1$ which is being studied in the next Sect. 6, or $\sum_j \varepsilon_j^2 = \infty$, in which case there exists a subsequence of the ε_j's which satisfies one of the hypotheses (i), (ii), or (iii) so that then (C_0) is satisfied. □

Remark. Under the hypothesis (5.b), and if the law of ξ is absolutely continuous, then the criterion: $\sum_j \varepsilon_j^2 = \infty$, which ensures that (C_0) is satisfied, has a lot to do with the hypothesis: ξ admits a moment of order 2. Indeed, if, on the contrary, we take for ξ a symmetric stable variable of order α, $0 < \alpha < 2$, then, obviously, (C_0) is satisfied if, and only if:

$$\sum_j |\varepsilon_j|^\alpha = \infty \,.$$

6 The condition $p_\mu = 1$ characterizes existence of strong solutions

Following the usual terminology in stochastic differential equations, we say that $P(\in \mathscr{P}_\mu(T))$ is a strong solution (in law) of (T) if, under P, the equality:

$$\mathscr{F}_k = \mathscr{E}_k, \text{ for every } k, \text{ is satisfied} \,.$$

We then have

Theorem 4. *The following assertions are equivalent:*

1) $p_\mu = 1$
2) $\mathscr{P}_\mu(T)$ *admits at least one strong solution.*
2′) $\mathscr{P}_\mu(T)$ *admits several distinct strong solutions.*
3) *for every* $P \in \mathscr{P}_\mu(T)$, *and for every* $k \in -\mathbb{N}$, *the "exchange identity"* $\mathscr{F}_k = \mathscr{F}_{-\infty} \vee \mathscr{E}_k$ *is satisfied.*
4) *At least one of extremal points of* $\mathscr{P}_\mu(T)$ *is a strong solution.*
4′) *Any extremal point of* $\mathscr{P}_\mu(T)$ *is a strong solution.*
5) *The set of strong solutions of* $\mathscr{P}_\mu(T)$ *coïncides with* $\text{ext}(\mathscr{P}_\mu(T))$.

Proof. These equivalences are immediate, by inspection of the different possible cases enumerated in Proposition 5, and also thanks to the fact that every point in $\mathscr{P}_\mu(T)$ is an integral of extremal points, as stated in Theorem 1′. □

7 The condition $p_\mu > 1$; more case studies

(7.1) The following theorem is the companion of Theorem 3 and Theorem 4; again, its proof is immediate by inspection of the different possible cases enumerated in Proposition 5.

Theorem 5. *The following assertions are equivalent:*

1) $p_\mu > 1$;

2) $\mathscr{P}_\mu(T)$ *admits no strong solution, and does not enjoy the uniqueness in law property.*

(7.2) As an illustration of the different cases: $p_\mu = 0$, $p_\mu = 1$, $p_\mu > 1$, we now consider the case where all the ξ_j's are identically distributed. In this case, we denote by φ the common characteristic function:

$$\varphi(p) = E[\exp(2i\pi p\xi_1)] \quad (p \in \mathbb{Z}) .$$

Since, in this case, we have: $\prod_{j=k-n+1}^{k} \varphi_j(p) = (\varphi(p))^n$, it follows that $\mathbb{Z}_+(\mu) = \{q \in \mathbb{Z}: |\varphi(q)| = 1\}$.

Consequently, in the i.i.d. case, we have the following elementary

Lemma 5. *If the* ξ_j*'s are i.i.d., then:*

a) $p_\mu = 0$ *if, and only if, for any* $p \in \mathbb{Z}^*$, *and any* $x \in \mathbb{R}$, $P(\xi \in x + (1/p)\mathbb{Z}) < 1$;

b) $p_\mu = 1$ *if, and only if, there exists* $x \in \mathbb{R}$ *such that:* $P(\xi \in x + \mathbb{Z}) = 1$;

c) $p_\mu = p > 1$ *if, and only if, there exists* $x \in \mathbb{R}$ *such that:* $P(\xi \in x + (1/p)\mathbb{Z}) = 1$, *and, for any integer* q *such that:* $0 < q < p$, *and any* $y \in \mathbb{R}$, $P(\xi \in y + (1/q)\mathbb{Z}) < 1$.

We continue our study in the i.i.d. case by giving some particular examples of distributions $\mu_1(dx) \equiv P(\xi_1 \in dx)$ for which $p_\mu = p > 1$.

In the sequel, we shall write $\mathbb{Z}_p$ for the set $\left(0, \frac{1}{p}, \ldots, \frac{p-1}{p}\right)$.

Proposition 7. *Let* $p \in \mathbb{Z}^*$, $p > 1$.

1) *If* ξ *(or rather:* $\{\xi\}$*) is uniformly distributed on* $\mathbb{Z}_p$, *then:* $p_\mu = p$.

2) *If* ξ *takes its values in* $\mathbb{Z}_p$, *and if:* $m_k \equiv P(\xi = k/p) > 0$, *for any* k *with* $0 \leqq k \leqq p-1$, *then:* $p_\mu = p$.

Proof. 1) If ξ is uniformly distributed on $\mathbb{Z}_p$, we have, for any integer r such that: $0 \leqq r \leqq p$:

$$E[\exp(2i\pi r\xi)] = \frac{1}{p}\sum_{k=0}^{p-1} \exp\left(2i\pi r\frac{k}{p}\right) = \frac{\exp(2i\pi r) - 1}{\exp\left(2i\pi\frac{r}{p}\right) - 1} = 0 ,$$

whereas: $E[\exp(2i\pi p\xi)] = 1$, hence: $p_\mu = p$.

2) If $m_k > 0$ for any k with $0 \leqq k \leqq p-1$, then there exists $\alpha < 1$ such that: $m_k \geqq \alpha/p > 0$ (we may take: $\alpha = \beta/2$, where: $\beta = \inf m_k$).

Hence, if we define m'_k by: $m_k = \alpha/p + (1-\alpha)m'_k$, the sequence $(m'_k; 0 \leqq k \leqq p-1)$ is a probability measure on $\mathbb{Z}_p$, and we may write:

$$E[\exp(2i\pi r\xi)] = \alpha E[\exp(2i\pi r v)] + (1-\alpha)E[\exp(2i\pi r\xi')] \qquad (7.a)$$

where v is uniformly distributed on $\mathbb{Z}_p$, and ξ' is distributed on $\mathbb{Z}_p$ with $(m'_k; 0 \leqq k \leqq p-1)$.

As we saw in 1) above, we have: $E[\exp(2i\pi r v)] = 0$, for $0 < r < p$, hence, we have from (7.a) that, for $0 < r < p$:

$$|E[\exp(2i\pi r\xi)]| \leqq (1-\alpha) < 1\,. \qquad (7.b)$$

Consequently, we have: $p_\mu = p$, in this case. □

(7.3) Finally, leaving aside the i.i.d. case, we remark that the estimate (7.b) made in the second part of the proof of Proposition 7 allows us to obtain the following class of examples of sequences $\mu = (\mu_j; j \in -\mathbb{N})$ of probabilities on $\mathbb{Z}_p$ for which $p_\mu = p$.

Proposition 8. *Let* $\beta_j = \inf\{\mu_j(k); 0 \leqq k \leqq p-1\}$.
Then, if $\sum_{j=-\infty}^{0} \beta_j = \infty$, *we have*: $p_\mu = p$.

Proof. Since $\sum_{j=-\infty}^{0} \beta_j = \infty$, there exists at least a subsequence $(n_j, j \to -\infty)$ such that: $\beta_{n_j} > 0$, for any j. We may obviously assume this subsequence to be the whole set $-\mathbb{N}$. Now, we deduce from the estimate (7.b), that, for $0 < r < p$, we have:

$$\left|\prod_{j=k-n+1}^{k} \varphi_j(r)\right| \leqq \prod_{j=k-n+1}^{k} \left(1 - \frac{\beta_j}{2}\right),$$

and the divergence of the series $\sum_{j=-\infty} \beta_j = +\infty$ implies that the above infinite product converges to 0, as $n \to \infty$. This implies: $p_\mu = p$. □

References

1. Dynkin, E.B.: Sufficient statistics and extreme points. Ann. Probab. **6**, 705–730 (1978)
2. Lipcer, R.S., Shyriaev, A.N.: Statistics of random processes, I. General theory. Applications of mathematics, vol. 5. Berlin Heidelberg New York: Springer 1977
3. Neveu, J.: Bases mathématiques du calcul des probabilités, 2nd edn. Paris: Masson 1970
4. Rogers, L.C.G., Williams, D.: Diffusions, Markov processes and martingales, vol. 2. Itô calculus, New York: Wiley 1987
5. Stroock, D.W., Yor, M.: On extremal solutions of martingale problems. Ann. Sci. Ec. Norm. Super., IV. Ser. **13**, 95–164 (1980)
6. Tsirel'son, B.S.: An example of a stochastic equation having no strong solution. Teor. Veroyatn. Primen. **20** (2), 427–430 (1975)
7. Veretennikov, A.Y.: On strong solutions and explicit formulas for solutions of stochastic integral equations. Math. USSR Sb. **39**, 387–403 (1981)
8. Weizsäcker, H. von: Exchanging the order of taking suprema and countable intersection of sigma algebras. Ann. Inst. Henri Poincaré **19**, 91–100 (1983)
9. Williams, D.: **Probability with martingales.** Cambridge mathematical textbooks. Cambridge: Cambridge University Press 1991
10. Yamada, Y., Watanabe, S.: On the uniqueness of solutions of stochastic differential equations. J. Math. Kyoto Univ. **11**, 155–167 (1971)
11. Yor, M.: De nouveaux résultats sur l'équation de Tsirel'son. C.R. Acad. Sci., Paris, Sér. I. **309**, 511–514 (1989)
12. Zvonkin, A.K.: A transformation of the phase space of a diffusion process that removes the drift. Math. USSR, Sb. **22**, 129–149 (1974)