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**Probability Theory and Related Fields**



# Limiting Behaviour of the Occupation of Wedges by Complex Brownian Motion

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**Summary.** We prove a theorem which gives the lim inf behaviour as $t$ tends to 0 for the amount of time a complex Brownian motion spends in a wedge with apex at the origin. The result is then shown to hold uniformly for all wedges a.s..

## Introduction

It is well known that a planar Brownian motion $\{B(t): t \geqq 0\}$ spins infinitely about its starting point in any time interval containing zero (see, e.g., Itô and McKean (1965), Sect. 7.11). It is also known that there do exist random times $\tau$ at which this regularity breaks down so that for some $h$ and some non-trivial wedge $W$ with apex at the origin,

$$B(t+\tau) - B(\tau) \in W \quad \forall t \in [0, h)$$

(see Burdzy (1984); Evans (1986); Le Gall (1987)). At the fixed time point 0, however, the Brownian motion must enter and leave each wedge infinitely often.

The occupation at time $t$ for a wedge $W$ is the amount of time spent inside it by the Brownian motion up to time $t$:

$$T_{\mathrm{occ}}(t) = \int_0^t I_{\{B(u) \in W\}} du .$$

In this paper we attempt to describe how extreme the occupation of wedges of the plane can be.

A routine application of Blumenthal's 0–1 Law shows that for a fixed wedge $W$, $\limsup_{t \to 0} \frac{T_{\mathrm{occ}}(t)}{t} = 1$ a.s. and $\liminf_{t \to 0} \frac{T_{\mathrm{occ}}(t)}{t} = 0$ a.s.. We shall investigate further the lim inf properties of the occupation of wedges and prove:

**Theorem 1.** *Let $W_\alpha$ be the wedge $\{\mathrm{r}e^{i\theta}: r \geqq 0, \theta \in (-\alpha/2, \alpha/2)\}$ and let $\{B(t): t \geqq 0\}$ be a planar Brownian motion started at 0. Define $T^\alpha_{\mathrm{occ}}(t) = \int_0^t I_{\{B(u) \in W_\alpha\}} du$. Then*

$$\liminf_{t\to 0} \frac{T^\alpha_{\text{occ}}(t)}{t} \times \log^p 1/t = \begin{cases} \infty & \text{if} \quad p > \frac{2}{\pi}(2\pi - \alpha) \\ 0 & \text{if} \quad p < \frac{2}{\pi}(2\pi - \alpha) . \end{cases}$$

By the isotropy of planar Brownian motion, this Theorem holds for any wedge with apex at the origin and angle $\alpha$. We then show in the last section that the result of Theorem 1 holds a.s. for all wedges of angle $\alpha$ simultaneously.

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**1.** In this section we reformulate the result in terms of the occupation behaviour as time tends to infinity and then reduce the problem of the proof to looking at a countable sequence of random variables.

It is well known that if $\{B(u): t \geqq 0\}$ is a Brownian motion then so is

$$\begin{aligned} Y(u) &= uB(1/u) \quad \text{for} \quad u > 0 \\ &= 0 \quad \text{for} \quad u = 0 . \end{aligned}$$

Using this fact, we can see that Theorem 1 is equivalent to

**Proposition 1.1.** *Let $\{B(t): t \geqq 0\}$ be a planar Brownian motion and define*

$$V^\alpha(u) = \int_u^\infty \frac{1}{v^2} I_{\{B(v) \in W_\alpha\}} \, dv .$$

*For* $p > \frac{2}{\pi}(2\pi - \alpha)$, $\liminf_{t\to\infty} t \log^p(t) V^\alpha(t) = \infty$,

*while for* $p < \frac{2}{\pi}(2\pi - \alpha)$, $\liminf_{t\to\infty} t \log^p(t) V^\alpha(t) = 0$.

We now observe that (since $V^\alpha(t)$ is decreasing) for $2^n \leqq t \leqq 2^{n+1}$,

$$\begin{aligned} t \log^p(t) V^\alpha(t) &\geqq t \log^p(t) V^\alpha(2^{n+1}) \\ &\geqq 2^n \log^p(2^n) V^\alpha(2^{n+1}) . \end{aligned}$$

For $n$ large enough, the last term is greater than or equal to

$$\frac{1}{3} 2^{n+1} \log^p(2^{n+1}) V^\alpha(2^{n+1}) .$$

Similarly, $t \log^p(t) V^\alpha(t) \leqq 3 \cdot 2^n \log^p(2^n)$ for $n$ sufficiently large. Given this observation, it is easy to see that Proposition 1.1 is equivalent to

**Proposition 1.2.** *Let $V^\alpha$ be defined as above and let $t_n = 2^n$. Then*

(*i*) $\liminf_{n\to\infty} t_n \log^p(t_n) V^\alpha(t_n) = \infty$ *if* $p > \frac{2}{\pi}(2\pi - \alpha)$

(*ii*) $\liminf_{n\to\infty} t_n \log^p(t_n) V^\alpha(t_n) = 0$ *if* $p < \frac{2}{\pi}(2\pi - \alpha)$ .

We prove Proposition 1.2 in sections two and three, thereby establishing Theorem 1.

**2.** In this section we wish to prove the part of Proposition 1.2 which refers to the case $p > \frac{2}{\pi}(2\pi - \alpha)$. This is equivalent to proving

**Proposition 2.1.** *If* $p > \frac{2}{\pi}(2\pi - \alpha)$, *then* $\liminf_{n\to\infty} t_n \log^p(t_n) V^\alpha(t_n) \geqq 1/2$.

To see this equivalence, note that if Proposition 2.1 holds and $p$ is greater than $\frac{2}{\pi}(2\pi - \alpha)$, then for any $p_1$ in the interval $\left(\frac{2}{\pi}(2\pi - \alpha), p\right)$,

$$\liminf_{n\to\infty} t_n \log^{p_1}(t_n) V^\alpha(t_n) \geqq 1/2 .$$

Since $\log^{p-p_1}(t_n) \to \infty$ as $n \to \infty$, $\liminf_{n\to\infty} t_n \log^p(t_n) V^\alpha(t_n) = \infty$.

Rather than looking at $V^\alpha$ directly, we prove this proposition by finding a suitable stochastic interval $(T_n, S_n)$ with $T_n \geqq t_n$ and such that the Brownian motion $B$ is within $W_\alpha$ for the entire interval. Then

$$\begin{aligned}\liminf t_n \log^p(t_n) V^\alpha(t_n) &\geqq \liminf t_n \log^p(t_n) \int_{T_n}^{S_n} \frac{1}{u^2} du \\ &= \liminf t_n \log^p(t_n) \left(\frac{1}{T_n} - \frac{1}{S_n}\right).\end{aligned}$$

Below we define the stopping times $T_n$ and $S_n$ which will make this proof work. Throughout this section we consider $p$ and $\alpha \left(p > \frac{2(2\pi - \alpha)}{\pi}\right)$ to be fixed. We first choose $\alpha_1$, $p_1$ and $\varepsilon(>0)$ such that

$$p > p - 3\varepsilon > p_1 > p_1 - 3\varepsilon > \frac{2(2\pi - \alpha_1)}{\pi} > \frac{2(2\pi - \alpha)}{\pi} \quad \text{and} \quad \alpha_1 > \alpha - 3\varepsilon . \qquad (*)$$

We also choose $p_2$ to be in the interval $(p_1 - 3\varepsilon, p_1 - 2\varepsilon)$. We now define the stopping times in terms of these constants:

$U_n = \inf\{t > t_n : |B(t)| > t_n^{1/2}\}$
$T_n = \inf\{t > U_n : B(t) \in W_{\alpha_1}\}$ (recall the definition of $W_\beta$ given in the statement of Theorem 1)
$S_n = \inf\{t > T_n : B(t) \in W_\alpha^c\}$, where $W_\alpha^c$ is the complement of $W_\alpha$.

We now wish to obtain some bounds on the expression $\int_{T_n}^{S_n} \frac{1}{t^2} dt = \frac{S_n - T_n}{T_n \times S_n}$. A key result in this venture is the following lemma which first requires some notation.

*Notation.* $r(t) = |B(t)|$ and $R_n = r(T_n)$.

**Lemma 2.1.** *Let $P_n$ be the probability that after hitting $W_{\alpha_1}$ at time $T_n$, the Brownian motion $B(t)$ will leave $W_\alpha$ before additional time $\frac{R_n^2}{[\log\log t_n]^2}$. Then $\sum_{n=1}^{\infty} P_n < \infty$.*

*Proof.* There exists a constant $K$ (depending on $\alpha$ and $\alpha_1$) such that every non-zero point $z$ in $W_{\alpha_1}$ is the centre of a square of side $K|z|$ entirely contained in $W_\alpha$. Now it is well known (see, e.g., Itô and McKean (1965) p. 25) that the log of the probability that 1-dimensional Brownian motion leaves $[-1, 1]$ before time $t$ is of the order $-1/t$ for small $t$. We obtain the lemma using scaling and the fact that for standard planar Brownian motion to leave $[-1, 1]^2$ by time $t$, at least one of the component Brownian motions must have left $[-1, 1]$. □

So (by the first Borel-Cantelli Lemma and Lemma 2.1), a.s. for all $n$ large enough

$$
\begin{aligned}
V^\alpha(t_n) &\geqq \int_{T_n}^{S_n} \frac{1}{u^2}\,du \geqq \int_{T_n}^{T_n + \frac{R_n^2}{(\log\log t_n)^2}} \frac{1}{t^2}\,dt \\
&= \frac{1}{T_n} - \frac{1}{T_n + \frac{R_n^2}{(\log\log t_n)^2}} \\
&= \frac{\frac{R_n^2}{(\log\log t_n)^2}}{(T_n)\left(T_n + \frac{R_n^2}{(\log\log t_n)^2}\right)} \\
&\geqq \min\left\{\frac{1}{2T_n}, \frac{\left(\frac{R_n^2}{(\log\log t_n)^2}\right)}{2(T_n)^2}\right\}.
\end{aligned}
$$

So we have reduced the problem of proving Proposition 2.1 to showing the following:

**Proposition 2.2.** *The events*

$$
(C_n)\ \frac{1}{(T_n)} \leqq \frac{1}{t_n \log^p t_n} \quad \textit{and} \quad (D_n)\ \frac{\left(\frac{R_n^2}{(\log\log t_n)^2}\right)}{(T_n)} \leqq \frac{T_n}{t_n \log^p t_n}
$$

*satisfy* $P\left[\limsup_{n\to\infty} C_n \cup D_n\right] = 0$.

In proving that $C_n \cup D_n$ cannot happen infinitely often, we have to be aware of what could go wrong. If $|B(t_n)|$ is large, then naturally $T_n$ may be correspondingly large, so we treat large values of $|B(t_n)|$ separately from tamer values of $|B(t_n)|$. Also, if $R_n$ is too small then $D_n$ may occur, so we have to treat smaller values of $|B(t_n)|$ separately.

*Proof of Proposition 2.2.* In this proof our choices of $p_1$, $\alpha_1$, and $\varepsilon$ will still obey the inequalities $(*)$ and $p_2$ will still be in the interval $(p_1 - 3\varepsilon, p_1 - 2\varepsilon)$. The plan of proof is to split the events $C_n$ and $D_n$ into three separate cases, depending on the magnitude of $r_n (= r(t_n))$, and then to use the first Borel-Cantelli Lemma.

*Case 1.* $r_n \in (t_n^{1/2}, 3(t_n \log\log t_n)^{1/2})$
*Case 2.* $r_n \leqq t_n^{1/2}$
*Case 3.* $r_n \geqq 3(t_n \log\log t_n)^{1/2}$.

Recall that $T_n$ is the first hitting time of wedge $W_{\alpha_1}$ after time $U_n$, so that in cases 1 and 3 it is the first hitting time of $W_{\alpha_1}$ after time $t_n$.

We further subdivide case 1 into 3 cases:
(A) $T_n \geqq t_n \log^{p_2} t_n$
(B) $T_n \leqq t_n \log^{\varepsilon} t_n$
(C) Neither (A) nor (B) occurs.

We intend to show separately that for $i = 1, 2, 3$, case $i \cap (C_n \cup D_n)$ cannot occur infinitely often. It turns out that case 1 is the real problem and that part C of this case is the most difficult to prove. We now treat each of the above cases in turn.

*Case 1(A).* We wish to evaluate $P[T_n \geqq t_n \log^{p_2} t_n]$. We will need the following facts and Lemma 2.2 (below):

1. Let $\{X(t): t \geqq 0\}$ *be a 1-dimensional Brownian motion with* $X(0) \in (-\alpha/2, \alpha/2)$. *Then* $E_{\alpha/2} = \inf\{t: |X(t)| = \alpha/2\}$ *satisfies*

$$P[E_{\alpha/2} > t] \leqq K e^{-(\pi/\alpha)^2 t/2}$$

(see Itô and McKean (1965), p. 31).

2. *Recall that* $r(t)$ *is the magnitude of* $B(t)$. *Then*

$$E\left[\exp\left(-\frac{\alpha^2}{2}\int_0^t \frac{1}{r(u)^2}\,du\right)\middle|\, r(0) = a,\, r(t) = b\right] = \frac{I_{|\alpha|}\left(\frac{ab}{t}\right)}{I_0\left(\frac{ab}{t}\right)}$$

*where we write* $I_\beta(\cdot)$ *for the modified Bessel function of order* $\beta$ (see Pitman and Yor (1982)).

Let $T^X_{\text{exit}\,R}$ denote the first leaving time of a region $R$ by a planar Brownian motion $\{X(t): t \geqq 0\}$. (We suppress the superscript when dealing with the process $B$.)

**Lemma 2.2.** *For planar Brownian motion* $X(t)$ *starting within* $W^c_\alpha$, *with* $|X(0)| = 1$, *there exist constants* $K$ (*not depending on* $X(0)$) *and* $K'$(*depending on* $X(0)$) *such that for* $t$ *greater than one,*

$$K' t^{-\frac{\pi}{2(2\pi-\alpha)}} < P[T_{\text{exit}\,W^c_\alpha} \geqq t] < K t^{-\frac{\pi}{2(2\pi-\alpha)}}.$$

*Proof.* The skew product decomposition of planar Brownian motion (see Itô and McKean (1965), Sect. 7.15) tells us that planar Brownian motion can be written as $r(t)e^{i\theta(t)}$ where $\theta(t)$ is a Brownian motion run with clock $\int_0^t \frac{1}{r(u)^2}\,du$ but otherwise independent of the process $\{r(t): t \geqq 0\}$. Now the Brownian motion $X$ exiting $W_\alpha^c$ is equivalent to $\theta$ exiting an interval of length $2\pi - \alpha$. Consequently (by fact 1 above), given $\{r(u): 0 \leqq u \leqq t\}$,

$$P(T_{\text{exit } W_\alpha^c} \geqq t) \leqq K\, e^{-\frac{1}{2}\left(\frac{\pi}{2\pi-\alpha}\right)^2 \int_0^t \frac{1}{r(u)^2} du} .$$

If we then use fact 2 above ($a = |X(0)| = 1$ here), we obtain

$$P(T_{\text{exit } W_\alpha^c} \geqq t) \leqq \int_0^\infty \frac{I_{\frac{\pi}{2\pi-\alpha}}\left(\frac{b}{t}\right)}{I_0\left(\frac{b}{t}\right)} P[r(t) \in b, b + db] .$$

Now substituting the transition density of the two-dimensional Bessel process (see Itô and McKean (1965)), we see that the right-hand side equals

$$= \int_0^\infty \frac{2b}{2t} e^{\frac{-b^2}{2t}} e^{\frac{-a^2}{2t}} I_0\left(\frac{b}{t}\right) \frac{I_{\frac{\pi}{2\pi-\alpha}}\left(\frac{b}{t}\right)}{I_0\left(\frac{b}{t}\right)} db$$

$$\leqq C \int_0^\infty re^{-r^2/2} I_{\frac{\pi}{2\pi-\alpha}}\left(\frac{r}{t^{1/2}}\right) dr$$

$$\leqq kt^{-\frac{\pi}{2(2\pi-\alpha)}}$$

for some $C$, $K$. This calculation uses the estimates for Bessel functions found on p. 77 of Watson (1966). The left-hand inequality of the lemma follows from similar arguments. □

This is by no means an original calculation; see Le Gall (1987) for similar calculations.

We can now resume the examination of case 1(A):

$$P[T_n \geqq t_n \log^{p_2} t_n | B(t_n)] =$$

(by scaling)

$$P\left[ T_{\text{exit } W_{\alpha_1}^c} \geqq \frac{t_n \log^{p_2} t_n - t_n}{(r(t_n))^2} \,\middle|\, |X(0)| = 1, \arg(X(0)) = \arg(B(t_n)) \right]$$

(by Lemma 2.2 and the assumption that $r(t_n) \leqq 3(t_n \log\log t_n)^{1/2}$)

$$\leqq K \left[ \frac{\log^{p_2} t_n}{\log\log t_n} \right]^{\frac{-\pi}{2(2\pi-\alpha_1)}}$$

(from the definition of $p_1$ and $\alpha_1$)

$$\leqq \frac{K_1}{n^{1+\delta}}$$

for some $K_1$ and $\delta > 0$ independent of $n$. The first Borel-Cantelli Lemma shows that case 1(A) cannot happen infinitely often.

*Case 1(B).* We wish to show that neither case 1(B) and $C_n$ nor case 1(B) and $D_n$ can occur infinitely often. We see from the definitions of the events that case 1(B) and $C_n$ are incompatible, so we have only to prove that

$$\left\{R_n^2/(\log\log t_n)^2 \leqq \frac{2(T_n)^2}{t_n(\log^p t_n)}\right\}$$

cannot happen infinitely often when $T_n \leqq t_n(\log^\varepsilon t_n)$ and $r(t_n) \in (t_n^{1/2}, 3(t_n \log\log t_n)^{1/2})$. We now need another lemma:

**Lemma 2.3.** *Uniformly for Brownian motion started within wedge $W_{\alpha_1}^c$ at $r(0) = 1$,*

$$P[r(T_{\text{exit}\, W_{\alpha_1}^c}) \leqq \varepsilon] \leqq {}^{\mu}\varepsilon^{\frac{\pi}{(2\pi-\alpha_1)}} .$$

*Proof.* The image of Brownian motion by an analytic map is a time-changed Brownian motion which consequently has the same hitting distributions as Brownian motion. Consider the analytic function $z \to z^{\frac{\pi}{2\pi-\alpha_1}}$ which maps the wedge $W_{\alpha_1}^c$ into the left-half plane. This mapping takes Brownian motion starting at $|X(0)|$ equal to 1 to a time-change of Brownian motion starting with initial point possessing magnitude 1. The event $\{r(T_{\text{exit}\, W_{\alpha_1}^c}) \leqq \varepsilon\}$ is mapped into the event that the time-changed Brownian motion leaves the left half-plane in the interval $\operatorname{Im}(z) \in (-\varepsilon^{\frac{\pi}{2\pi-\alpha_1}}, +\varepsilon^{\frac{\pi}{2\pi-\alpha_1}})$. The result follows smoothly. □

The argument above is reproduced from Burdzy (1984), pp. 60–64.

We are now ready to complete case 1(B). The event

$$\left\{R_n^2/(\log\log t_n)^2 \leqq \frac{2(T_n)^2}{t_n \log^p t_n} \quad \text{and} \quad T_n \leqq t_n(\log^\varepsilon t_n)\right\}$$

is contained in the event

$$\left\{R_n \leqq \frac{\sqrt{2t_n}\log\log(t_n)}{\log^{p/2-\varepsilon} t_n}\right\}.$$

By scaling and Lemma 2.3, the event $\left\{R_n \leqq \frac{\sqrt{2t_n}\log\log(t_n)}{\log^{p/2-\varepsilon} t_n}\right\} \cap \{r(t_n) \geqq t_n^{1/2}\}$ has probability

$$\begin{aligned} &\leqq K\left[\frac{\log\log t_n}{\log^{(p/2-\varepsilon)} t_n}\right]^{\frac{\pi}{2\pi-\alpha_1}} \\ &\leqq K(\log t_n)^{-\frac{p_1\pi}{2(2\pi-\alpha_1)}} \\ &\leqq \frac{K}{n^{1+\delta}} \end{aligned}$$

for some $K$ and $\delta > 0$. The first Borel-Cantelli Lemma disposes of case 1(B).

*Case 1(C)*. If $T_n \in (t_n \log^{i\varepsilon} t_n, t_n \log^{(i+1)\varepsilon} t_n]$ (for $i = 1, 2, \ldots, [p_1/\varepsilon] - 2$), then automatically $C_n$ cannot occur and $D_n$ can only hold if

$$R_n^2 \leqq \frac{(\log\log t_n)^2 2(T_n)^2}{t_n \log^p t_n}$$

$$\leqq t_n \log^{-(p-(2i+3)\varepsilon)}(t_n)$$

for $n$ large enough; i.e., if

$$R_n \leqq t_n^{1/2}[\log t_n]^{-(p_1 - 2i\varepsilon)/2} .$$

So if we can show that for each $i$ in $\{1, 2, \ldots, [p_1/\varepsilon] - 2\}$,

$$P[T_n \geqq t_n[\log t_n]^{i\varepsilon} \quad \text{and} \quad R_n \leqq t_n^{1/2}[\log t_n]^{-(p_1 - 2i\varepsilon)/2}] \leqq \frac{K}{n^{1+\delta}}$$

for some $K$ and $\delta > 0$, then we will be done. Let $P_{b,\theta}\{t_n^{1/2}[\log t_n]^{-(p_1 - 2i\varepsilon)/2}\}$ be the probability that Brownian motion starting at $r(B(0)) = b$ and $\arg(B(0)) = \theta$ hits the wedge $W_{\alpha_1}$ with radial magnitude $\leqq t_n^{1/2}[\log t_n]^{-(p_1 - 2i\varepsilon)/2}$. Conditional upon $B(t_n)$ (of magnitude greater than $t_n^{1/2}$), the left hand side of the above inequality is exactly equal to:

$$\int_0^\infty P\{r(t_n \log^{i\varepsilon} t_n) \in db, \arg B(t_n \log^{i\varepsilon} t_n) \in d\theta,$$

$$T_n > (t_n \log^{i\varepsilon} t_n) | B(t_n)\} \cdot P_{b,\theta}\{t_n^{1/2}[\log t_n]^{-(p_1 - 2i\varepsilon)/2}\} .$$

Putting together Lemma 2.3 and the facts used to prove Lemma 2.2 and letting $t_n^i = t_n(\log(t_n))^{i\varepsilon}$, we majorize this integral by

$$K \int_0^\infty \frac{2b}{2t_n^i} e^{-\frac{b^2}{2t_n^i}} \cdot I_{\frac{\pi}{2\pi - \alpha_1}}\left(\frac{r(t_n)b}{t_n^i}\right)\left(1 \wedge \frac{t_n^{1/2}[\log t_n]^{-(p_1 - 2i\varepsilon)/2}}{b}\right)^{\frac{\pi}{2\pi - \alpha_1}} db .$$

Changing variables to $u = b/(t_n^i)^{1/2}$ reformulates the above integral as

$$K \int_0^\infty u e^{-\frac{u^2}{2}} I_{\frac{\pi}{2\pi - \alpha_1}}\left(\frac{r(t_n)u}{(t_n^i)^{1/2}}\right)\left(1 \wedge \frac{[\log t_n]^{-(p_1 - i\varepsilon)/2}}{u}\right)^{\frac{\pi}{2\pi - \alpha_1}} du .$$

Using the assumption $r(t_n) \leqq 3(t_n \log\log t_n)^{1/2}$ and the fact that for $x \in [0, 1]$, $[I_{\frac{\pi}{2\pi - \alpha_1}}(x)]/[x^{\frac{\pi}{2\pi - \alpha_1}}]$ is bounded (see Watson (1966), p. 79), we see that this last expression is less than or equal to

$$K \int_0^{[\log t_n]^{-(p_1 - i\varepsilon)/2}} u^{\frac{\pi}{2\pi - \alpha_1} + 1}\left[\frac{(\log\log t_n)^{1/2}}{\log^{i\varepsilon/2} t_n}\right]^{\frac{\pi}{2\pi - \alpha_1}} du .$$

$$+ \int_{[\log t_n]^{-(p_1 - i\varepsilon)/2}}^\infty u e^{-\frac{u^2}{2}}\left[I_{\frac{\pi}{2\pi - \alpha_1}}\left(\frac{r(t_n)u}{(t_n^i)^{1/2}}\right) \Big/ u^{\frac{\pi}{2\pi - \alpha_1}}\right]\{[\log t_n]^{-(p_1 - i\varepsilon)/2}\}^{\frac{\pi}{2\pi - \alpha_1}} du .$$

Using again the fact that for $x \in [0, 1]$, $[I_{\frac{\pi}{2\pi - \alpha_1}}(x)]/[x^{\frac{\pi}{2\pi - \alpha_1}}]$ is bounded and also the fact that the Bessel function is bounded by a multiple of the exponential (see

Watson (1966), p. 195), we find that the above sum is

$$\leqq K[\log t_n]^{-p_2 \frac{\pi}{2(2\pi-\alpha_1)}} \leqq K'/n^{1+\delta},$$

thereby completing case 1(C).

*Case 2.* In this case, we need the following:

**Lemma 2.4.** *Let $T_e$ be the time for Brownian motion starting at $\underset{\sim}{x}$ ($|\underset{\sim}{x}| \leqq 1$) to hit the unit circle. $T_e$ satisfies*

$$P_{\underset{\sim}{x}}[T_e \geqq (\log\log t_n)^2] \leqq \frac{K}{\log^2 t_n}$$

*for some $K > 0$.*

The proof of this lemma is much the same as that of Lemma 2.1.

We know from Lemma 2.4 and the first Borel-Cantelli Lemma that a.s. apart from a finite number of times, the stopping time

$$U_n = \inf\{t > t_n : |B(t)| > t_n^{1/2}\}$$

of the Brownian motion $B(u)$ will be less than $t_n(\log\log t_n)^2$. Reviewing the proofs of case 1, we see that even if $T_n$ is augmented by $t_n(\log\log(t_n))^2$, we can still prove that $P[C_n \cup D_n]$ is $O(1/n^{1+\delta})$. Hence, case 2 is completed.

*Case 3.* The Law of the Iterated Logarithm tells us that $r(t)$ will be less than $3\sqrt{t\log\log(t)}$ for all sufficiently large $t$, so we need not consider this case further.

Collecting the results of the 3 cases, we see that $C_n \cup D_n$ cannot happen infinitely often; that is, Proposition 2.2 is proven. □

Recalling the discussion at the beginning of this section (up to the statement of Proposition 2.2), we see that we have established part (i) of Proposition 1.2. In the next section we prove part (ii), and therefore complete the proof of Theorem 1.

**3.** In this section we complete the proof of Proposition 1.2 (and hence that of Theorem 1). We require the following lemma:

**Lemma 3.1.** *Consider planar Brownian motion $\{B(t): t \geqq 0\}$. Let $\delta > 0$ be such that $(1+\delta)p < \frac{2}{\pi}(2\pi - \beta)$ and let $s_n = 2^{[n^{1+\delta}]}$. Define the events*

$$B_n^\beta = \{|B(s_n)| \in (s_n^{1/2}, 2s_n^{1/2})\} \cap \{Tw_\beta\,\theta_{s_n} \geqq s_n \log^p(s_n)\}$$
$$\cap \{|B(s_n \log^p(s_n))| \leqq s_n^{1/2} \log^{p/2+2}(s_n)\}.$$

*Then with probability one the events $B_n^\beta$ occur infinitely often.*

*Proof.* By Lemma 2.2 of Sect. 2 and simple calculations,

$$\text{(i)}\ \ P[B_n^\beta] \geqq C\log(s_n)^{-\frac{p\pi}{2(2\pi-\beta)}} - P[|B(s_n(\log(s_n))^p)| > s_n^{1/2}\log^{p/2+2}(s_n)] \geqq \frac{K}{n}.$$

Now for $n < m$, the events $B_n^\beta$ and $B_m^\beta$ are conditionally independent given the values $|B(s_n(\log(s_n))^p)|$ and $|B(s_m)|$. Now the conditional density of $B(s_m)$ given $B(s_n)$

at point $\underset{\sim}{x}$ is

$$\frac{1}{2\pi(s_m - s_n)}\exp(-(|\underset{\sim}{x} - B(s_n)|^2/2(s_m - s_n)))$$

and so the ratio of this quantity with the unconditional density at $x$ is

$$\frac{s_m}{(s_m - s_n)}\exp(-(|\underset{\sim}{x}|^2 s_n)/(2s_m(s_m - s_n)))\exp(\underset{\sim}{x}\,.\,B(s_n)/(s_m - s_n))$$
$$\times\exp(-(|B(s_n)|^2/2(s_m - s_n)))$$

Now for all $|\underset{\sim}{x}| \leqq 2s_m^{1/2}$ and $|B(s_n)| \leqq s_n^{1/2}\log^{p/2+2}(s_n)$, this quantity will be arbitrarily close to 1 as $n$ and $m$ become large. This shows that

(ii) $\lim\limits_{n,\,m\to\infty}\dfrac{P[B_n^\beta]P[B_m^\beta]}{P[B_n^\beta \cap B_n^\beta]} = 1.$

Putting (i) and (ii) together, we conclude from Chung (1974) p. 77, that $B_n^\beta$ occurs infinitely often a.s. . □

We can now complete the proof of Proposition 1.2. Lemma 3.1 tells us that if $p < \frac{2}{\pi}(2\pi - \alpha)$ then $\liminf\limits_{m\to\infty} t_m \log^p(t_m)V^\alpha(t_m) \leqq 1$. Reasoning similar to that employed after the statement of Proposition 2.1 completes the proof.

**4.** We now extend the result of Theorem 1 by proving:

**Theorem 2.** *Theorem* 1 *holds for all wedges simultaneously.*

*Proof.* We give only the proof that part (ii) of Proposition 1.2 holds uniformly for all wedges, as the proof that part (i) also holds uniformly is very similar. We need only consider rational $\alpha$ and rational $p$, since given $p$ and $\alpha$ as in part (ii), we can find rational $p_1$ and $\alpha_1$ such that $\dfrac{2(2\pi - \alpha_1)}{\pi} < \dfrac{2(2\pi - \alpha)}{\pi}$. Thus, if the theorem holds for all rational $\alpha$ and $p$, it holds for all $\alpha$ and $p$. Let us now prove that for given rationals $\alpha_1$ and $p_1$, part (ii) holds uniformly for all wedges of angle $\alpha_1$ a.s. .

Given $p_1 < \dfrac{2(2\pi - \alpha_1)}{\pi}$, we choose finitely many wedges $W_{\alpha_2}^1, W_{\alpha_2}^2, \ldots, W_{\alpha_2}^n$ of angle $\alpha_2$, where

$$p_1 < \frac{2(2\pi - \alpha_2)}{\pi} < \frac{2(2\pi - \alpha_1)}{\pi}$$

and such that every wedge of angle $\alpha_1$ is strictly contained in one of the wedges. Then for $T_{\mathrm{occ}}^j(t) = \int_0^t I_{\{B(u)\in W_{\alpha_2}^j\}}\,du$, we have by Theorem 1 that with probability one,

$$\liminf_{t\to 0}\frac{\log^{p_1}(1/t)\,T_{\mathrm{occ}}^j(t)}{t} = 0\,.$$

But for every wedge of angle $\alpha_1$, $T_{\mathrm{occ}}(t) \leqq \max\limits_{1\leqq j\leqq n} T_{\mathrm{occ}}^j(t)$, and so the result follows. □

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