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Probability Theory and Related Fields

Convergence to equilibrium for classical and quantum spin systems

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Summary. The paper is devoted to stochastic equations describing the evolution of classical and quantum unbounded spin systems on discrete lattices and on Euclidean spaces. Existence and asymptotic properties of the corresponding transition semigroups are studied in a unified way using the theory of dissipative operators on weighted Hilbert and Banach spaces. This paper is an enlarged and rewritten version of the paper [7].

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1 Introduction

In this paper we are concerned with transition semigroups describing evolution of unbounded spin systems on the lattice $\mathbb{Z}^d$ and on $\mathbb{R}^d$. In particular, we study their asymptotic behaviour.

Systems on $\mathbb{Z}^d$ are determined, see e.g. [9, 14, 15, 23] by an infinite matrix $(a_{\gamma,j})_{\gamma,j\in\mathbb{Z}^d}$ and a real function $f:\mathbb{R}\to\mathbb{R}$, called respectively a *global interaction matrix* and a *local interaction function*. Let $\mathscr{H}_n$, $n\in\mathbb{N}$, $\mathscr{H}_\infty$, be spaces of sequences $\{x_\gamma\}_{\gamma\in\mathbb{Z}^d}$, vanishing for $|\gamma|>n$, $\gamma=\{\gamma_1,\dots,\gamma_d\}\in\mathbb{Z}^d$, and $|\gamma|=\Sigma_{k=1}^d|\gamma_k|$ or growing no faster than polynomials. Let $C_b(\mathscr{H}_n)$, $n\in\mathbb{N}$, $C_b(\mathscr{H}_\infty)$ be the space of all bounded continuous functions on $\mathscr{H}_n$, $n\in\mathbb{N}$, and $\mathscr{H}_\infty$.

Starting from properly defined semigroups P_t^n, $t\geqq 0$ on $C_b(\mathscr{H}_n)$ describing the dynamics of finite *configurations* on $\mathscr{H}_n$, one can construct, see e.g. [14, 26], under proper conditions on $\{a_{ij}\}$ and f, the so called *thermodynamic limit* P_t, $t\geqq 0$, acting on $\mathscr{H}_\infty$. Limit properties of P_t, $t\geqq 0$ as $t\to\infty$ are of

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great concern in applications. In particular, in his recent paper, [26], Zegarlinski proved that, in some cases, there exists a unique invariant probability measure μ for P_t, $t \geqq 0$ on $\mathscr{H}_\infty$, the so called, *Gibbs measure*, and that there exist $\omega > 0$ and a finite function $c(x) > 0$, $x \in \mathscr{H}_\infty$ such that

$$|P_t\varphi(x) - (\varphi, \mu)| \leqq c(x)e^{-\omega t}d(\varphi), \quad t \geqq 0, \ x \in \mathscr{H}_\infty . \tag{1.1}$$

In (1.1) φ is any bounded function on $\mathscr{H}_\infty$ depending on finite number of coordinates, of class C^1 and such that

$$d(\varphi) = \sum_\gamma \sup_x \left|\frac{\partial\varphi}{\partial x_\gamma}(x)\right| + \left(\int_{\mathscr{H}_\infty} (\varphi(x) - (\varphi,\mu))^2 \mu(dx)\right)^{1/2} < +\infty , \tag{1.2}$$

where

$$(\varphi, \mu) = \int_{\mathscr{H}_\infty} \varphi(y)\mu(dy) .$$

This paper presents a direct way of constructing P_t, $t \geqq 0$, and proving (1.1) using the theory of stochastic equations and properties of dissipative mappings. Under weaker conditions than in [26], the semigroup P_t, $t \geqq 0$, will be defined by a Markov process $X = \{X_\gamma\}_{\gamma\in\mathbb{Z}_d}$ satisfying an infinite system of Ito's equations

$$\begin{aligned} dX_\gamma(t) &= \left(\sum_j a_{\gamma j}X_j(t) + f(X_\gamma(t))\right) dt + dW_\gamma(t) , \\ X_\gamma(0) &= x_\gamma, \quad \gamma \in \mathbb{Z}^d, \ t \geqq 0 , \end{aligned} \tag{1.3}$$

in which W_γ, $\gamma \in \mathbb{Z}^d$, are real Brownian motions.

A typical equation to which our theory is applicable is of the form

$$\begin{aligned} dX_\gamma(t) &= ((\Delta_d - \alpha)X_\gamma(t)) + f(X_\gamma(t)))\,dt + dW_\gamma(t) , \\ X_\gamma(0) &= x_\gamma, \quad \gamma \in \mathbb{Z}^d, \ t \geqq 0 , \end{aligned} \tag{1.4}$$

where Δ_d is the discrete Laplacian and α is a constant.

Under additional assumptions the estimate (1.1) will be derived with an explicit formula for the function c and with the formula for d slightly different from that of [26]. The estimate will be a special case of a general exponential inequality obtained by considering solutions X on growing time intervals $[\lambda, +\infty[$, $\lambda \in \mathbb{R}$, and letting $\lambda \to -\infty$. For the special case of solutions to (1.4) the estimate will be valid without requiring that φ is smooth or depends on a finite number of coordinates, compare [26]. Our conditions on local interaction functions are neither weaker nor stronger than those in [26], see Sect. 3. Moreover the Wiener processes W_γ, $\gamma \in \mathbb{Z}^d$, can be correlated and in the extreme situation can be identically zero. This latter case is outside of the scope of the logarithmic Sobolev inequality approach and of the Bakry–Emery criterion [2], as the generators of P_t, $t \geqq 0$ and P_t^n, $t \geqq 0$, $n \in \mathbb{N}$ are degenerate.

In a similar manner as Eq. (1.3) we will treat spin systems on $\mathbb{R}^d$ restricting our considerations to continuous versions of (1.3) of the form

$$dX(t,\xi) = ((\Delta - \alpha)X(t,\xi) + f(X(t,\xi)))\,dt + dW(t,\xi)\,, \tag{1.5}$$

where $W(\,\cdot\,,\xi)$, $\xi \in \mathbb{R}^d$ are Wiener processes and $W(t,\,\cdot\,)$, $t \geqq 0$, are stationary Gaussian fields describing random environments, see [8].

The third class of systems discussed in the paper are quantum lattice systems introduced in the recent paper [1]. They are a mixture of systems defined on $\mathbb{Z}^d$ and $\mathbb{R}^d$.

In all the three cases our basic state space is a weighted Hilbert space. Choosing the weight properly we obtain as a byproduct additional information on the support of the invariant measure μ.

Our main results are existence theorems and theorems on exponential estimates formulated and proved for the three types of systems discussed above. They will be consequences of two general theorems on stochastic equations

$$\begin{aligned} dX &= (AX + F(X))\,dt + B\,dW\,, \\ X(0) &= x\,. \end{aligned} \tag{1.6}$$

presented in Sect. 2. Theorem 2.1 gives sufficient conditions for existence and uniqueness of solutions to (1.6), with A and F having appropriate dissipativity properties in a Hilbert space H, W being a cylindrical Wiener process on a Hilbert space U and B a linear operator from U into H. Estimates of (1.1) type, for general equations (1.6), are the content of Theorem 2.3.

The idea to model spin systems by stochastic equations is rather old and goes back to the papers [9, 11, 12, 16, 17, 21, 22]. However, our general approach allows to obtain new results on existence and on asymptotic behaviour of solutions. We treat in a unified way spin systems on $\mathbb{Z}^d$ and $\mathbb{R}^d$ as well as quantum spin systems. This paper is an enlarged and rewritten version of the paper [7].

2 Stochastic dissipative systems

Let H and U be separable Hilbert spaces, with norms $\|\cdot\|$, $\|\cdot\|_U$ and scalar products $\langle\,\cdot,\cdot\,\rangle$ and $\langle\,\cdot,\cdot\,\rangle_U$. The spaces of all linear bounded operators from U into H and from U into U will be denoted by $L(U,H)$ and $L(U)$.

Moreover let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a filtration $\mathscr{F}_t$ and $W(t)$, $t \geqq 0$, an $\mathscr{F}_t$ adapted Wiener process defined on Ω with values in U and with the covariance operator $Q \in L(U)$. Thus for arbitrary a, $b \in U$ and t, $s \geqq 0$:

$$\mathbb{E}(\langle W(t), a\rangle_U \langle W(t), b\rangle_U) = t \vee s \langle Qa, b\rangle_U\,.$$

We will be concerned here with Eq. (1.6) where $B \in L(U,H)$ and A and F are respectively linear and nonlinear mappings from $D(A) \subset H$, $D(F) \subset H$ into H satisfying appropriate dissipativity conditions which will be introduced below.

Let $(E, \|\cdot\|_E)$ be a Banach space and E^* its dual. For arbitrary $x \in E$ the subdifferential $\partial\|x\|_E$ of the norm $\|\cdot\|_E$ at x is given by the formula

$$\partial\|x\|_E = \{x^* \in E^*\colon\ \|x + y\|_E - \|x\|_E \geqq x^*(y)\ \forall y \in E\}\,.$$

A mapping G from $D(G) \subset E$ into E is said to be dissipative in E if for arbitrary $x,\ y \in D(G)$ there exists $z^* \in \partial\|x - y\|_E$ such that

$$z^*(G(x) - G(y)) \leqq 0\,.$$

If in addition, for some $\alpha > 0$ the mapping $I - \alpha G$ is surjective then G is called m-dissipative.

If $K \subset E$ is a Banach space embedded into E then the part G_K of G in K is defined as follows:

$$D(G_K) = \{x \in D(G) \cap K\colon\ G(x) \in K\}\,,$$
$$G_K(x) = G(x) \quad \text{for} \quad x \in D(G_K)\,.$$

It is convenient to introduce the following hypotheses on the operators A, F, B, the process $W(t)$, $t \geqq 0$ and on a Banach space $(K, \|\cdot\|_K)$ continuously and densely embedded into H.

Hypothesis 1
(i) *There exists $\eta \in \mathbb{R}$ such that operators $A + \eta$ and $F + \eta$ are m-dissipative on H.*
(ii) *The parts in K of $A + \eta$ and $F + \eta$ are m-dissipative on K.*
(iii) *$D(F) \supset K$ and F maps bounded sets in K into bounded sets in H.*
(iv) *K is a reflexive space.*
(v) *$B \in L(U, H)$.*

Let $W_A(t)$, $t \geqq 0$ be the solution to the linear equation

$$dZ(t) = AZ(t)\,dt + B\,dW(t)\,,$$
$$Z(0) = 0\,,$$

given by

$$Z(t) = W_A(t) = \int_0^t S(t-s)B\,dW(s), \quad t \geqq 0$$

where $S(t)$, $t \geqq 0$, is the semigroup generated by A on H.

Hypothesis 2 *The process $W_A(t)$, $t \geqq 0$, is continuous in H, takes values in the domain $D(F_K)$ of the part of F in K and for any $T > 0$ we have*

$$\sup_{t \in [0,T]} (\|W_A(t)\|_K + \|F_K(W_A(t))\|_K) < +\infty, \quad \mathbb{P}\ \text{a.s.}$$

An H-continuous, $\mathscr{F}_t$-adapted process $X(t)$, $t \geqq 0$, is said to be a *strong solution* to (1.6) if it satisfies $\mathbb{P}$-a.s. the equation

$$X(t) = x + \int_0^t (AX(s) + F(X(s)))\,ds + BW(t), \quad t \geqq 0\,,$$

and it is a mild solution if it satisfies the following integral equation:

$$X(t) = S(t)x + \int_0^t S(t-s)F(X(s))\,ds + W_A(t), \quad t \geqq 0\,.$$

If for a H-valued process X, there exists a sequence X_n, of mild solutions to (1.6) such that $\mathbb{P}$-a.s., $X_n \to X$ uniformly on (any) interval $[0,T]$, then X is said to be a *generalized solution* to (1.6). Note that each strong solution is mild and each mild solution is a generalized solution.

Theorem 2.1 *Assume that Hypotheses* 1 *and* 2 *are fulfilled. Then for arbitrary* $x \in K$ *there exists a unique mild solution of* (1.6) *and for arbitrary* $x \in H$ *there exists a unique generalized solution* $X(t,x)$, $t \geqq 0$ *of* (1.6). *If operator* A *and its part in* K *are bounded then solutions for* $x \in K$ *are strong.*

Remark. 2.2 Theorem 2.1 is a slight modification of Theorem 4.1 in [6] and therefore its proof is omitted. It follows from the proof that processes $X(t,x)$, $t \geqq 0$, are Markov with a Feller transition semigroup P_t, $t \geqq 0$, given by

$$P_t\varphi(x) = \mathbb{E}(\varphi(X(t,x))), \quad t \geqq 0,\; x \in H,\; \varphi \in C_b(H)\,,$$

where $C_b(H)$ denotes the space of all uniformly continuous and bounded functions on H.

The next theorem is our main result concerned with invariant measures for (1.6) and with asymptotic properties of the semigroup P_t, $t \geqq 0$.

Theorem 2.3 *If in addition to Hypotheses* 1 *and* 2
(i) *there exist* $\eta_1, \eta_2 \in \mathbb{R}$ *such that* $\omega = \eta_1 + \eta_2 > 0$ *and operators* $A + \eta_1$, $F + \eta_2$ *are dissipative in* H,
(ii) *one has*

$$\sup_{t \geqq 0} \mathbb{E}(\|W_A(t)\| + \|F(W_A(t))\|) < +\infty\,. \tag{2.1}$$

Then there exists a unique invariant measure μ *for the semigroup* P_t, $t \geqq 0$. *Moreover, for all bounded and Lipschitz continuous functions* φ *on* H *one has*

$$\left| P_t\varphi(x) - \int_H \varphi(y)\mu(dy) \right| \leqq (c + 2\|x\|)e^{-\omega t}\|\varphi\|_{\mathrm{Lip}}\,, \tag{2.2}$$

where

$$c = \sup_{t \geqq 0} \mathbb{E}\left(\|W_A(t)\| + \frac{1}{\omega}\|F(W_A(t))\| \right)\,.$$

Proof. Although some ingredients of the proof are contained in [4, 6], the estimate (2.2) was not derived there. For the completeness of the presentation we sketch the proof.

To prove existence of an invariant measure for P_t, $t \geqq 0$, define a Wiener process $\overline{W}(t)$, $t \in \mathbb{R}$, on the whole real line by setting

$$\overline{W}(t) = \begin{cases} W(t) & \text{if } t \geqq 0\,, \\ V(-t) & \text{if } t \leqq 0\,, \end{cases}$$

$$\overline{\mathscr{F}}_t = \sigma(\overline{W}(s),\ s \leqq t), \quad t \in \mathbb{R}\,,$$

where $V(t)$, $t \geqq 0$, is an independent copy of $W(t)$, $t \geqq 0$. Denote by $X(t,\lambda,x)$, $t \geqq \lambda$, $x \in H$, the generalized solution of

$$\begin{aligned} &dX(t) = (AX(t) + F(X(t)))\,dt + B\,d\overline{W}(t), \quad t \geqq \lambda\,, \\ &X(\lambda) = x\,. \end{aligned}$$

Assume for a moment that $x \in K$. Then, by Theorem 2.1

$$X(t,\lambda,x) = S(t-\lambda)x + \int_\lambda^t S(t-r)F(X(r,\lambda,x))\,ds + W_{A,\lambda}(t)\,,$$

where

$$W_{A,\lambda}(t) = \int_\lambda^t S(t-r)\,d\overline{W}(r), \quad t > \lambda\,.$$

We show now that

$$\mathbb{E}\|X(t,\lambda,x)\| \leqq \|x\| + c, \quad t \geqq \lambda, \ x \in H\,, \tag{2.3}$$

where c is the constant from the theorem. Remark first that

$$Z_\lambda(t) = X(t,\lambda,x) - W_{A,\lambda}(t), \quad t \geqq \lambda\,,$$

is the mild solution of the problem

$$\begin{aligned} &\frac{d}{dt}Z(t) = AZ(t) + F(Z(t) + W_{A,\lambda}(t))\,, \\ &Z(\lambda) = x\,. \end{aligned}$$

Denote:

$$x^*_{\lambda,t} = \begin{cases} \frac{z_\lambda(t)}{\|z_\lambda(t)\|} & \text{if } z_\lambda(t) \neq 0\,, \\ 0 & \text{if } z_\lambda(t) = 0\,. \end{cases}$$

Then, by the chain rule (see [5, Proposition D.4]), and by (i),

$$\begin{aligned} &\frac{d^-}{dt}\|Z_\lambda(t)\| \\ &\leqq \langle AZ_\lambda(t) + F(Z_\lambda(t) + W_{A,\lambda}(t)), x^*_\lambda(t)\rangle \\ &\leqq \langle (A+\eta_1)Z_\lambda(t), x^*_\lambda(t)\rangle - \eta_1\langle Z_\lambda(t), x^*_\lambda(t)\rangle \\ &\quad + \langle (F+\eta_2)(Z_\lambda(t) + W_{A,\lambda}(t)) - (F+\eta_2)(W_{A,\lambda}(t)), x^*_\lambda(t)\rangle \\ &\quad - \eta_2\langle Z_\lambda(t), x^*_\lambda(t)\rangle + \langle F(W_{A,\lambda}(t)), x^*_\lambda(t)\rangle \\ &\leqq -\omega\langle Z_\lambda(t), x^*_\lambda(t)\rangle + \langle F(W_{A,\lambda}(t)), x^*_\lambda(t)\rangle \\ &\leqq -\omega\|Z_\lambda(t)\| + \|F(W_{A,\lambda}(t))\|, \quad t \geqq \lambda\,. \end{aligned}$$

Consequently

$$\|Z_\lambda(t)\| \leqq e^{-\omega(t-\lambda)}\|x\| + \int_\lambda^t e^{-\omega(t-s)}\|F(W_{A,\lambda}(s))\|\, ds, \quad t \geqq \lambda ,$$

and

$$\mathbb{E}\|Z_\lambda(t)\| \leqq \|x\| + \frac{1}{\omega}\sup_{t\geqq\lambda}\ \mathbb{E}\|F(W_{A,\lambda}(t))\|, \quad t \geqq \lambda .$$

This implies (2.3) for $x \in K$ and, by a limit argument, for all $x \in H$.

In a similar way one shows that for arbitrary $x,\ y \in H$

$$\|X(t,\lambda,x) - X(t,\lambda,y)\| \leqq e^{-\omega(t-\lambda)}\|x-y\|, \quad t > \lambda .$$

Consequently for $\gamma \leqq \lambda \leqq t$

$$\begin{aligned}\mathbb{E}\|X(t,\lambda,x) - X(t,\gamma,x)\| &= \mathbb{E}\|X(t,\lambda,x) - X(t,\lambda,X(\lambda,\gamma,x))\| \\ &\leqq e^{-\omega(t-\lambda)}\mathbb{E}\|X(\lambda,\gamma,x) - x\|\end{aligned}$$

and, by (2.3)

$$\mathbb{E}\|X(t,\lambda,x) - X(t,\gamma,x)\| \leqq e^{-\omega(t-\lambda)}(2\|x\| + c) . \tag{2.4}$$

Therefore there exists a random variable ζ, the same for all $x \in H$, such that

$$\lim_{\lambda\to-\infty} \mathbb{E}\|X(0,\lambda,x) - \zeta\| = 0 . \tag{2.5}$$

We claim that the law $\mu = \mathscr{L}(\zeta)$ is the unique invariant measure for P_t, $t \geqq 0$. To see this it is enough to remark that, by (2.5), for arbitrary $x \in H$,

$$P_t(x,\cdot\,) = \mathscr{L}(X(t,x)) = \mathscr{L}(X(0,-t,x)) \to \mu ,$$
$$\text{weakly as } t \to +\infty .$$

Finally, let φ be a bounded Lipschitz function on H, then, by (2.5), for $s \geqq t \geqq 0$

$$\begin{aligned}&|P_t\varphi(x) - P_s\varphi(x)| \\ &= |\mathbb{E}(\varphi(X(t,0,x)) - \varphi(X(s,0,x)))| \\ &= |\mathbb{E}(\varphi(X(0,-t,x)) - \varphi(X(0,-s,x)))| \\ &\leqq \|\varphi\|_{\mathrm{Lip}}\mathbb{E}\|X(0,-t,x)) - X(0,-s,x)\| \\ &\leqq \|\varphi\|_{\mathrm{Lip}}e^{-\omega t}(2\|x\| + c) .\end{aligned}$$

Letting $s \to \infty$ we obtain the desired inequality. □

3 Classical systems on discrete lattices

We apply here results from the preceding section to Eq. (1.3).

We assume that $W(t) = \{W_\gamma(t)\}_{\gamma\in\mathbb{Z}^d}$, $t \geqq 0$, is a Wiener process on $(\Omega, \mathscr{F}, \mathbb{P})$ with values in $U = \ell^2$ and with the covariance operator $Q \in L(U)$.

In particular if Q is the identity operator then processes W_γ, $\gamma \in \mathbb{Z}^d$, are independent standard real valued Wiener processes. However Q can be any non-negative definite operator on U and for any $t > 0$ the family $\{W_\gamma(t)\}_{\gamma\in\mathbb{Z}^d}$ can form a general Gaussian random field including stationary ones if Q is shift invariant.

The operators A and F will be given by the formulae

$$\begin{aligned} A(x_\gamma) &= \left(\sum_{j\in\mathbb{Z}^d} a_{j\gamma} x_j \right), \quad x = (x_\gamma) \in H , \\ F(x_\gamma) &= (f(x_\gamma)), \quad x = (x_\gamma) \in H . \end{aligned} \tag{3.1}$$

and $H = \ell^2_\rho(\mathbb{Z}^d)$ will be a weighted Hilbert space of sequences (x_γ) with positive summable weight $\rho : \mathbb{Z}^d \to \mathbb{R}^+$. B will denote the embedding operator from U into H. $C_b(H)$ and $B_b(H)$ are respectively the space of all bounded continuous functions in H and the space of all bounded Borel functions in H.

The following proposition, which goes back to Schur [13], gives sufficient conditions under which matrix $(a_{\gamma j})$ defines a bounded operator on $\ell^p_\rho(\mathbb{Z}^d)$ for arbitrary $p \in [1, +\infty]$.

Proposition 3.1 *Assume that*
(i) $\sup_{\gamma\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} |a_{\gamma j}| = \alpha < +\infty.$

(ii) *There exists* $\beta > 0$ *such that*

$$\sum_{\gamma\in\mathbb{Z}^d} |a_{\gamma j}|\rho(\gamma) \leqq \beta\rho(j), \quad j \in \mathbb{Z}^d .$$

Then the formula

$$A_p x = \left(\sum_{j\in\mathbb{Z}^d} a_{\gamma j} x_j \right) , \quad x \in \ell^p_\rho(\mathbb{Z}^d) , \tag{3.2}$$

defines a linear bounded operator on $\ell^p_\rho(\mathbb{Z}^d)$, *for all* $p \in [1, +\infty]$, *with the norm not greater than*

$$\alpha^{1/q}\beta^{1/p}, \quad \frac{1}{p} + \frac{1}{q} = 1 .$$

Proof. Assume, for instance, that $p \in]1, +\infty[$ and that $x = (x_j)$ has only a finite number of coordinates different from 0. By Hölder inequality and (i), for arbitrary $\gamma \in \mathbb{Z}^d$

$$\begin{aligned} \left| \sum_{j\in\mathbb{Z}^d} a_{\gamma j} x_j \right|^p &\leqq \left(\sum_{j\in\mathbb{Z}^d} |a_{\gamma j}|\ |x_j| \right)^p \leqq \left(\sum_{j\in\mathbb{Z}^d} (|a_{\gamma j}|^{1/p}|x_j|)|a_{\gamma j}|^{1/q} \right)^p \\ &\leqq \left(\sum_{j\in\mathbb{Z}^d} |a_{\gamma j}|\ |x_j|^p \right) \left(\sum_{j\in\mathbb{Z}^d} |a_{\gamma j}| \right)^{p/q} \leqq \alpha^{p/q} \sum_{j\in\mathbb{Z}^d} |a_{\gamma j}|\ |x_j|^p . \end{aligned}$$

Consequently, by (ii)

$$\sum_{\gamma\in\mathbb{Z}^d}\rho(\gamma)\left|\sum_{j\in\mathbb{Z}^d}a_{\gamma j}x_j\right|^p \leqq \alpha^{p/q}\sum_{\gamma,j\in\mathbb{Z}^d}|a_{\gamma j}|\rho(\gamma)|x_j|^p$$
$$\leqq \alpha^{p/q}\beta\sum_{j\in\mathbb{Z}^d}\rho(j)|x_j|^p\,,$$

and the result follows. □

We will impose several conditions on the matrix $\mathscr{A}$ and on the positive weight ρ, compare [1, 26]. We will assume that

$$\begin{aligned} &a_{\gamma,j}=0 \quad \text{if } |\gamma-j|>R\,,\\ &|a_{\gamma,j}|\leqq M \quad \text{for all } \gamma,j\in\mathbb{Z}^d \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\left|\frac{\rho(\gamma)}{\rho(j)}\right|\leqq M \quad \text{if } |\gamma-j|\leqq R\,,\\ &\sum_{\gamma\in\mathbb{Z}^d}\rho(\gamma)<+\infty\,, \end{aligned} \tag{3.4}$$

where $R>0$ and $M>0$ are positive constants.

It follows directly from Proposition 3.1 that

Proposition 3.2 *If conditions* (3.3) *and* (3.4) *hold then the operators* A_p *given by* (3.1) *are bounded.*

Condition (3.4) is satisfied for two specific families of weights,

$$\rho^\kappa(\gamma)=e^{-\kappa|\gamma|},\quad \gamma\in\mathbb{Z}^d,\ \kappa>0 \tag{3.5}$$

and

$$\rho_\kappa(\gamma)=\rho_{\kappa r}(\gamma)=\frac{1}{1+\kappa|\gamma|^r},\quad \gamma\in\mathbb{Z}^d,\ \kappa>0,\ r>d\,. \tag{3.6}$$

For the second family the parameter $r>d$ will be fixed once and for all, therefore we will write shortly ρ_κ.

Remark. 3.3 It is useful to notice that for arbitrary $\kappa>0$ and $r>d$:

$$\ell^2_{\rho^\kappa}(\mathbb{Z}^d)\supset\mathscr{H}_\infty\supset\ell^2_{\rho_\kappa}(\mathbb{Z}^d)\,.$$

In addition, for any $r>d$, sets $\ell^2_{\rho_\kappa}(\mathbb{Z}^d)$ are identical for all $\kappa>0$.

As far as the function $f:\mathbb{R}\to\mathbb{R}$ is concerned we will require that

$$f=f_0+f_1 \tag{3.7}$$

where:

$$f_1 \text{ is Lipschitz continuous}\,, \tag{3.8}$$

$$\begin{aligned} &f_0(\xi)+\eta\xi,\ \xi\in\mathbb{R} \text{ is continuous and decreasing for some}\\ &\quad\eta\in\mathbb{R}, \text{ and for some } s\geqq 1 \text{ and } c_0>0\,,\\ &|f_0(\xi)|\leqq c_0(1+|\xi|^s),\quad \xi\in\mathbb{R}\,. \end{aligned} \tag{3.9}$$

For instance if f is Lipschitz continuous and

$$f_0(\xi) = -\xi^{2n+1} + \sum_{k=0}^{2n} b_k \xi^{2n-k}, \quad \xi \in \mathbb{R}, \; n \in \mathbb{N} \cup \{0\},$$

then (3.8) and (3.9) are satisfied with $s = 2n+1$.

Our first theorem is the following existence result.

Theorem 3.4 *Assume that conditions* (3.3), (3.4), *and* (3.7)–(3.9) *are satisfied. Let* $H = \ell^2_\rho(\mathbb{Z}^d)$ *and let the operators* A *and* F *be given by* (3.1). *Then*

(i) *For arbitrary* $x \in \ell^{2s}_\rho(\mathbb{Z}^d)$, *there exists a unique strong solution* $X(t,x)$, $t \geqq 0$, *of* (1.6) *and* (1.3).

(ii) *For arbitrary* $x \in H = \ell^2_\rho(\mathbb{Z}^d)$, *there exists a unique generalized solution* $X(t,x)$, $t \geqq 0$, *of* (1.6) *and* (1.3), *and the transition semigroup*

$$P_t\varphi(x) = \mathbb{E}(\varphi(X(t,x))), \quad t \geqq 0, \; x \in H, \; \varphi \in C_b(H),$$

is Feller.

Proof. We will apply Theorem 2.1 with

$$H = \ell^2_\rho(\mathbb{Z}^d) \quad \text{and} \quad K = l^{2s}_\rho(\mathbb{Z}^d).$$

Note that, by Proposition 3.1, the linear operators A and A_{2s} are bounded on H and K respectively and therefore, without any loss of generality, we can assume that $A = 0$. The operator $F + \eta_1$, with the domain

$$D(F) = \left\{ x \in \ell^2_\rho : \sum_{\gamma \in \mathbb{Z}^d} \rho(\gamma) |f(x_\gamma)|^2 < +\infty \right\}$$

is m-dissipative in H and its part in K is m-dissipative in K provided η_1 is small enough. It is also clear that F maps bounded sets in K into bounded sets in H. So Hypothesis 1 is satisfied in our situation.

We show now that Hypothesis 2 holds as well. Let $Q^{1/2} = (q_{\gamma j})$ be the nonnegative square root of Q. Then

$$W_\gamma(t) = \sum_{j \in \mathbb{Z}^d} q_{\gamma j} \widehat{W}_\gamma(t), \quad \gamma \in \mathbb{Z}^d,$$

where $\widehat{W}_\gamma$, $\gamma \in \mathbb{Z}^d$, are independent real valued Wiener processes. By Gaussianity of $\widehat{W}$, for any $p \geqq 1$,

$$\begin{aligned} \mathbb{E}|W_\gamma(t)|^p &= c_p (\mathbb{E}|W_\gamma(t)|^2)^{p/2} \\ &= c_p t^{p/2} \left(\sum_{j \in \mathbb{Z}^d} q^2_{\gamma j} \right)^{p/2}, \quad t \geqq 0. \end{aligned}$$

Consequently

$$\mathbb{E}\|W(t)\|^p_{\ell^p_\rho(\mathbb{Z}^d)} = C_p t^{p/2} \sum_{\gamma\in\mathbb{Z}^d} \rho(\gamma) \left(\sum_{j\in\mathbb{Z}^d} q^2_{\gamma j} \right)^{p/2}, \quad t \geqq 0$$

Since $Q^{1/2}$ is a bounded operator on $\ell^2(\mathbb{Z}^d)$, therefore

$$\sup_{\gamma\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} q^2_{\gamma j} < +\infty ,$$

and $W(t)$, $t \geqq 0$, is a $\ell^p_\rho(\mathbb{Z}^d)$-valued Wiener process. By Kolmogorov's theorem W has a continuous version in any $\ell^p_\rho(\mathbb{Z}_d)$, $p \geqq 1$, and Hypothesis 2 is satisfied. □

Remark. 3.5 Similar results, but with different concepts of solutions are contained e.g. in [9,22]. Existence of strong solutions seems to be new. Conditions we impose on f are slightly weaker than those in [26] as we do not require that f_0 and f_1 are differentiable.

The following theorem gives a precise information on the rate of convergence of equilibrium.

Theorem 3.6 *Assume, in addition to conditions of Theorem* 3.4, *that for an* $\eta > 0$, *operator* $A + \eta$, *restricted to* $\ell^2(\mathbb{Z}^d)$ *is dissipative, that* f_0 *is decreasing and* $\eta - \|f_1\|_{\text{Lip}} > \omega > 0$. *Then there exists* $\kappa_0 > 0$ *such that in the spaces* $\ell^2_\rho(\mathbb{Z}^d) = H$, *with* $\rho = \rho^\kappa$ *given by* (3.5), *or with* $\rho = \rho_\kappa$ *given by* (3.6), *and* $\kappa \in]0, \kappa_0[$, *Eq.* (1.6) *has unique generalized solutions. The semigroup* P_t, $t \geqq 0$ *has a unique invariant measure* μ *on* H *and there exists* $c > 0$ *such that, for any bounded and Lipschitz function* φ *on* H, *all* $t > 0$ *and all* $x \in H$:

$$\left| P_t\varphi(x) - \int_H \varphi(x)\mu(dx) \right| \leqq (c + 2|x|)e^{-\omega t}\|\varphi\|_{\text{Lip}} .$$

To prove the theorem we need the following two elementary lemmas.

Lemma 3.7 *Assume that conditions* (3.3) *and* (3.4) *are satisfied and in addition*

$$\left| \sqrt{\frac{\rho(\gamma)}{\rho(j)}} - 1 \right| \leqq \delta \quad \text{for } |\gamma - j| \leqq R .$$

If

$$\langle Ax, x\rangle_{\ell^2(\mathbb{Z}^d)} \leqq \alpha\|x\|^2_{\ell^2(\mathbb{Z}^d)}, \quad x \in \ell^2(\mathbb{Z}^d)$$

then

$$\langle Ax, x\rangle_{\ell^2_\rho(\mathbb{Z}^d)} \leqq \left(\alpha + \frac{\delta}{2M}(2R+1)^d \right) \|x\|^2_{\ell^2_\rho(\mathbb{Z}^d)}, \quad x \in \ell^2_\rho(\mathbb{Z}^d) .$$

Proof. Take $x \in \ell^2_\rho(\mathbb{Z}^d)$ then

$$\langle Ax, x\rangle_{\ell^2_\rho(\mathbb{Z}^d)} = \sum_\gamma \rho(\gamma)\left(\sum_j a_{\gamma,j}x_j\right)x_\gamma$$

$$= \sum_{|\gamma-j|\leqq R}\sqrt{\frac{\rho(\gamma)}{\rho(j)}}a_{\gamma,j}\sqrt{\rho(j)}x_j\sqrt{\rho(\gamma)}x_\gamma$$

$$\leqq \sum_{|\gamma-j|\leqq R}\left(\sqrt{\frac{\rho(\gamma)}{\rho(j)}}a_{\gamma,j} - a_{\gamma,j}\right)\sqrt{\rho(j)}x_j\sqrt{\rho(\gamma)}x_\gamma$$

$$+ \sum_{|\gamma-j|\leqq R} a_{\gamma,j}\sqrt{\rho(j)}x_j\sqrt{\rho(\gamma)}x_\gamma$$

$$\leqq \frac{\delta}{2M}\left[\sum_{|\gamma-j|\leqq R}(\rho(j)x_j^2 + \rho(\gamma)x_\gamma^2\right] + \alpha\|x\|^2_{\ell^2_\rho(\mathbb{Z}^d)}$$

$$\geqq \frac{\delta}{2M}(2R+1)^d\|x\|^2_{\ell^2_\rho(\mathbb{Z}^d)} + \alpha\|x\|^2_{\ell^2_\rho(\mathbb{Z}^d)}\,. \qquad \square$$

Lemma 3.8 *For weights ρ^κ and ρ_κ given by* (3.5) *and by* (3.6) *we have*

$$\lim_{\kappa\to 0}\left[\sup_{|\gamma-j|\leqq R}\left|\sqrt{\frac{\rho^\kappa(\gamma)}{\rho^\kappa(j)}} - 1\right|\right] = 0, \quad \lim_{\kappa\to 0}\left[\sup_{|\gamma-j|\leqq R}\left|\sqrt{\frac{\rho_\kappa(\gamma)}{\rho_\kappa(j)}} - 1\right|\right] = 0\,.$$

Proof follows by direct calculations.

Proof of Theorem 3.6. We will consider only the weights ρ^κ as the case of $\rho = \rho_\kappa$ can be treated in the same way.

It is clear that

$$\langle F(x) - F(y), x - y\rangle \leqq \|f_1\|_{\mathrm{Lip}}\|x-y\|^2, \quad x, y \in D(F)\,,$$

and therefore $F - \|f_1\|_{\mathrm{Lip}}$ is m-dissipative. By Lemma 3.7 and Lemma 3.8 for arbitrary $\varepsilon > 0$ there exists $\kappa_\varepsilon > 0$ such that for $\kappa \in]0, \kappa_\varepsilon[$

$$\langle A(x-y) + F(x) + (\eta - \varepsilon(x-y), x-y\rangle_{\ell^2_{\rho^\kappa}(\mathbb{Z}^d)} \leqq 0, \quad x, y \in \ell^2_{\rho^\kappa}(\mathbb{Z}^d)\,.$$

Consequently if $\kappa \in]0, \kappa_\varepsilon[$ the operator $A + \eta - \varepsilon$ is m-dissipative. Since $-\|f_1\|_{\mathrm{Lip}} + \eta - \varepsilon > \omega$ for sufficiently small $\varepsilon > 0$ condition (i) of Theorem 2.3 is fulfilled.

We will show now that the process W_A has bounded p-moments in $\ell^p_\rho(\mathbb{Z}^d)$ spaces $p \geqq 1$:

$$\sup_{t>0}\mathbb{E}|W_A(t)|^p_{\ell^p(\mathbb{Z}^d)} < +\infty\,. \tag{3.10}$$

Denote $W_A(t) = (Z_\gamma(t))^d_{\gamma\in\mathbb{Z}}$ and $S(t) = (s_\gamma(t))^d_{j\gamma\in\mathbb{Z}}$. Then

$$Z_\gamma(t) = \sum_{j\in\mathbb{Z}^d} \int_0^t s_{\gamma j}(u)\, dW_j(u),$$

and

$$\mathbb{E}\|W_A(t)\|^p_{\ell^p_\rho(\mathbb{Z}^d)} = \sum_{\gamma\in\mathbb{Z}^d} \mathbb{E}(|Z_\gamma(t)|^p).$$

By Gaussianity and assuming, to simplify notation, that $Q = I$,

$$\mathbb{E}|Z_\gamma(t)|^p = c_p(\mathbb{E}|Z_\gamma(t)|^2)^{p/2} \leqq c_p \left(\int_0^t \sum_{j\in\mathbb{Z}^d} s^2_{\gamma j}(u)\, du\right)^{p/2}.$$

However

$$\sum_{j\in\mathbb{Z}^d} s^2_{\gamma j}(u) \leqq \|S(u)\|^2_{L(\ell^2(\mathbb{Z}^d))}$$

and therefore

$$\mathbb{E}\|W_A(t)\|^p_{\ell^p_\rho(\mathbb{Z}^d)} \leqq c_p \sum_{\gamma\in\mathbb{Z}^d} \rho(\gamma) \left(\int_0^t \|S(u)\|^2_{L(\ell^2(\mathbb{Z}^d))}\right)^{p/2}.$$

Since

$$\|S(u)\|_{L(\ell^2(\mathbb{Z}^d))} \leqq e^{-\eta u}, \quad u \geqq 0,$$

the estimate (3.10) follows. From (3.10)

$$\sup_{t>0}(\mathbb{E}\|W_A(t)\| + \|F(W_A(t))\|) < +\infty.$$

This completes the proof. □

Remark. 3.9 A theorem similar to Theorem 3.6 was proved earlier by Zegarlinski [26, Theorem 4.2]. His method, based on logarithmic Sobolev inequalities does not require that $\|f_1\|_{\mathrm{Lip}}$ is small. On the other hand we can cover less regular local interactions f like $f(\xi) = -\mathrm{sign}\,\xi|\xi|^{1/2}$, $\xi \in \mathbb{R}$, and our basic estimate holds for a more general class of functions φ. Since for all $\kappa > 0$ the spaces $\ell^2_{\rho_\kappa}(\mathbb{Z}^d)$ are identical as sets, see Remark 3.3, the invariant measure μ is supported by sequences (x_γ) such that

$$\sum_{\gamma\in\mathbb{Z}^d} \frac{|x_\gamma|^2}{1+|\gamma|^r} < +\infty$$

for arbitrary $r > d$. This set is smaller than $\mathscr{H}_\infty$. Thus working with weighted Hilbert spaces is not only technically convenient but gives additional information about the spin system. More general nonlinearities could be treated with the help of appropriate Orlicz spaces (replacing $\ell^p_\rho(\mathbb{Z}^d)$).

We finally remark that the noise process W can be absent in our approach, but not in the one based on logarithmic Sobolev inequalities.

4 Spin systems on Euclidean spaces

We pass now to systems on $\mathbb{R}^d$. We will restrict our considerations only to equations of the form:

$$\begin{aligned} dX(t,\xi) &= [(\Delta - \alpha)X(t,\xi) + f(X(t,\xi))]\,dt + dW(t,\xi)\,, \\ X(0,\xi) &= x(\xi), \quad \xi \in \mathbb{R}^d, \ t > 0\,, \end{aligned} \tag{4.1}$$

where Δ is the Laplace operator, α a nonnegative constant and $W(t,\xi)$, $\xi \in \mathbb{R}^d$, $t > 0$, an infinite dimensional Wiener process with covariance operator Q, defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We will assume that either

$$Q \text{ is the identity operator on } U = L^2(\mathbb{R}^d)\,, \tag{4.2}$$

or

$$\begin{aligned} &Q \text{ is a convolution operator on } U = L^2(\mathbb{R}^d): \\ &Qu(\xi) = q * u(\xi), \quad \xi \in \mathbb{R}^d, \ u \in L^2(\mathbb{R}^d)\,, \end{aligned} \tag{4.3}$$

where

$$q(\xi) = \int_{\mathbb{R}^d} e^{i\langle \xi,\eta\rangle_{\mathbb{R}^d}} g(\eta)\,d\eta, \quad \xi \in \mathbb{R}^d\,,$$

with $g \geqq 0$ and $q, g \in L^1(\mathbb{R}^d)$.

Thus in the former case the stochastic term in (4.1) represents the space-time white noise. in the latter case, see [8], for arbitrary $t > 0$ the random field $W(t,\xi)$, $\xi \in \mathbb{R}^d$, $t > 0$, is stationary and Gaussian with the correlation function $\sqrt{t}q$:

$$\mathbb{E}(W(t,\xi)W(t,\eta)) = \sqrt{t}q(\xi - \eta), \quad \xi, \eta \in \mathbb{R}^d\,.$$

Typical examples are provided by functions

$$q(\xi) = e^{-|\xi|^\delta}, \quad \xi \in \mathbb{R}^d, \ \delta \in]0,2]\,,$$

see [10, Vol. II]. Note that two extreme cases are included in our considerations: $Q = I$ and $Q = 0$ (noise absent), compare [21; 26, Sect. 5].

Similarly as in the discrete case Eq. (4.1) will be studied in weighted Hilbert spaces $H = L^2_\rho(\mathbb{R}^d)$ with two types of weights:

$$\rho^\kappa(\xi) = e^{-\kappa|\xi|}, \quad \xi \in \mathbb{R}^d, \ t > 0\,, \tag{4.4}$$

and

$$\rho_\kappa(\xi) = \rho_{\kappa r}(\xi) = \frac{1}{1 + \kappa|\xi|^r}, \quad \xi \in \mathbb{R}^d\,, \tag{4.5}$$

where $\kappa > 0$ and $r > d$ (to assure integrability of ρ_κ.) Parameter r is fixed for the whole section.

Let

$$\begin{aligned} p_t(\xi) &= (4\pi t)^{-d/2} e^{-|\xi|^2/4t}, \quad \xi \in \mathbb{R}^d, \ t > 0\,, \\ p_0(\xi) &= \delta_0(\xi)\,. \end{aligned}$$

be the heat kernel and set

$$S(t)x = e^{-\alpha t} p_t * x, \quad t \geqq 0 \,. \tag{4.6}$$

We start from propositions on the semigroup $S(t)$, $t \geqq 0$, in the space $L^p_\rho(\mathbb{R}^d)$. The first result is a continuous analogue of Proposition 3.1

Proposition 4.1 *Assume that ρ is a positive, continuous and integrable function on $\mathbb{R}^d$ such that for some $\omega \in \mathbb{R}$ and all $t \geqq 0$*

$$p_t * \rho \leqq e^{\omega t} \rho \,. \tag{4.7}$$

Then, for arbitrary $p \in [1, +\infty]$ and $\alpha \in \mathbb{R}$ the formula (4.6) *defines a C_0-semigroup on $L^p_\rho(\mathbb{R}^d)$ such that*

$$\|S(t)\|_{L(L^p_\rho(\mathbb{R}^d))} \leqq e^{(-\alpha+\omega/p)t}, \quad t \geqq 0 \,.$$

Proof. We consider only the case $p \in]1 + \infty[$. Setting $q = p/(p-1)$, and

$$\int_{\mathbb{R}^d} p_t(\xi - \zeta) x(\zeta)\, d\zeta = \int_{\mathbb{R}^d} p_t^{1/2}(\xi - \zeta) x(\zeta) p_t^{1/2}(\xi - \zeta)\, d\zeta$$

we have

$$\left| \int_{\mathbb{R}^d} p_t(\xi - \zeta) x(\zeta)\, d\zeta \right|^p$$

$$\leqq \left(\int_{\mathbb{R}^d} p_t(\xi - \zeta) |x(\zeta)|^p d\zeta \right) \left(\int_{\mathbb{R}^d} p_t(\xi - \zeta)\, d\zeta \right)^{p/q} .$$

Consequently

$$\begin{aligned} \|S(t)x\|^p_{L^p_\rho(\mathbb{R}^d)} &\leqq e^{-\alpha p t} \int_{\mathbb{R}^d \times \mathbb{R}^d} p_t(\xi - \zeta) |x(\zeta)|^p \rho(\xi)\, d\xi\, d\zeta \\ &\leqq e^{(-\alpha p + \omega)t} \int_{\mathbb{R}^d} |x(\zeta)|^p \rho(\zeta)\, d\zeta \\ &\leqq e^{(-\alpha p + \omega)t} \|x\|^p_{L^p_\rho(\mathbb{R}^d)} \,. \qquad \square \end{aligned}$$

Nonnegative functions ρ satisfying (4.7) with $\omega \geqq 0$ are called *ω-excessive* in the classical potential theory, see [3]. They are solutions of the inequality

$$\Delta \rho \leqq \omega \rho \,, \tag{4.8}$$

understood in the sense of distributions, see [24].

Proposition 4.2 *Assume that*

$$\rho(\xi) = \psi(|\xi|), \quad \xi \in \mathbb{R}^d,$$

where ψ is a C^1 function on $[0, +\infty[$ with ψ'' continuous and integrable on bounded, closed, subintervals of $]0, +\infty[$. Then (4.7) *holds if and only if*

(i) $d = 1$ *and*

$$\psi'(0) \leqq 0, \qquad \psi''(s) \leqq \omega\psi(s), \ s > 0\,,$$

(ii) $d > 1$ *and*

$$\psi''(s) + \frac{d-1}{s}\psi'(s) \leqq \omega\psi(s), \quad s > 0\,.$$

Proof. If $d = 1$ then

$$\Delta\rho(\xi) = \psi''(|\xi|) + 2\psi'(0)\delta_{\{0\}}, \quad \xi \in \mathbb{R}\,,$$

and if $d > 1$ then

$$\Delta\rho(\xi) = \psi''(|\xi|) + \frac{d-1}{|\xi|}\psi'(|\xi|), \quad \xi \in \mathbb{R}\,,$$

see [24], so the result follows. □

Generator of the semigroup $S(t)$, $t \geqq 0$, in $L^p_\rho(\mathbb{R}^d)$ will be denoted by $A_{p\rho}$. Note that for $x \in C^2_b(\mathbb{R}^d)$, $A_{p\rho} = \Delta x - \alpha x$.

As a corollary from Propositions 4.1 and 4.2, by direct calculations, we obtain the following crucial property.

Proposition 4.3 *Operators $A_{p\rho} + \eta$ are m-dissipative in $L^p_\rho(\mathbb{R}^d)$ if*
(i) $\rho = \rho^\kappa$ *and* $\eta \leqq \alpha - \kappa^2/p$, *or*
(ii) $\rho = \rho_\kappa$ *and* $\eta \leqq \alpha - \kappa^{1/r}r^2/p$.

We are ready now to state and prove our main results. We start from an existence theorem.

Theorem 4.4 *Assume that f satisfies* (3.7)–(3.9) *and either*
(i) $d = 1$ *and* (4.2) *or* (4.3) *holds or*
(ii) $d > 1$ *and* (4.3) *holds.*
Then Eq. (4.1) *has a unique generalized solution in $L^2_\rho(\mathbb{R}^d)$ where ρ is given either by* (4.4) *or by* (4.5). *If $x \in L^{2s}_\rho(\mathbb{R}^d)$ then the generalized solution is mild.*

Proof. We apply Theorem 2.1 with $H = L^2_\rho(\mathbb{R}^d)$, $K = L^{2s}_\rho(\mathbb{R}^d), \rho = \rho^\kappa$ or $\rho = \rho_\kappa$. It follows from Proposition 4.3 that operators $A_{2,\rho} + \eta$ and $A_{2s,\rho} + \eta$ are m-dissipative in H and K respectively for sufficiently small η. To see that the operator $F + \eta$ where

$$F(x)(\xi) = f(x(\xi)), \quad \xi \in \mathbb{R}^d\,,$$
$$D(F) = \{x \in L^2_\rho(\mathbb{R}^d) \colon f(x(\,\cdot\,)) \in L^2_\rho(\mathbb{R}^d)\}$$

satisfies Hypothesis 1 note that $D(F) \supset K$ and the domain $D(F_K)$ of the part F_K of F in K contains $L^{2p}_\rho(\mathbb{R}^d)$. It is also easy to see that $F + \eta$ and $F_K + \eta$ are m-dissipative for small η and that F maps bounded sets in K into bounded sets in H. Then Hypothesis 1 holds. The following proposition shows that Hypothesis 2 is satisfied as well and consequently finishes the proof of Theorem 4.4.

Proposition 4.5 *Processes $W_{A_{pp}}(t)$, $t \geqq 0$, have continuous trajectories in any $L^p_\rho(\mathbb{R}^d)$, $p \geqq 1$, with $\rho = \rho^\kappa$ or $\rho = \rho_\kappa$ provided that*
(i) $d = 1$ *and* (4.4) *or* (4.5) *holds*
or
(ii) $d > 1$ *and* (4.4) *holds.*

Proof. We will consider only the more difficult case (ii). Without any loss of generality, we can assume that $p \geqq 2$ and that

$$W(t,\xi) = \sum_{j=1}^{\infty} Q^{1/2} h_j(\xi)\beta_j(t), \quad t \geqq 0, \ \xi \in \mathbb{R}^d ,$$

where $\{h_j\}$ is an orthonormal and complete basis in $U = L^2(\mathbb{R}^d)$ and $\{\beta_j\}$ is a sequence of independent real valued Wiener processes. We will use the factorization formula, see [5, p.128], and investigate first the process

$$Y_\delta(t) = \int_0^t S(t-s)(t-s)^{-\delta}\, dW(s), \quad t \geqq 0 ,$$

where $\delta \in]1/p, 1/2[$. By Gaussianity of $Y_\delta(t,\xi)$, $t \geqq 0$, $\xi \in \mathbb{R}^d$, for a proper constant $c_1 > 0$,

$$\begin{aligned}
&\mathbb{E}|Y_\delta(t)|^p_{L^p_\rho(\mathbb{R}^d)} \\
&= \mathbb{E} \int_{\mathbb{R}^d} \rho(\xi)|Y_\delta(t,\xi)|^p\, d\xi \\
&= \int_{\mathbb{R}^d} \rho(\xi)\mathbb{E}\left|\int_0^t (t-s)^{-\delta}\sum_{j=1}^{\infty}(S(t-s)Q^{1/2}h_j)(\xi)\, d\beta_j(s)\right|^p d\xi \\
&\leqq c_1 \int_{\mathbb{R}^d} \rho(\xi)\left(\mathbb{E}\left|\int_0^t (t-s)^{-\delta}\sum_{j=1}^{\infty}(S(t-s)Q^{1/2}h_j)(\xi)\, d\beta_j(s)\right|^2\right)^{p/2} d\xi \\
&\leqq c_1 \int_{\mathbb{R}^d} \rho(\xi)\left(\int_0^t \sigma^{-2\delta}\sum_{j=1}^{\infty}|S(s)Q^{1/2}h_j(\xi)|^2\, ds\right)^{p/2} d\xi .
\end{aligned}$$

But

$$S(s)Q^{1/2}h_j(\xi) = e^{-\alpha s} p_s * q_1 * h_j(\xi), \quad \xi \in \mathbb{R}^d ,$$

where

$$Q^{1/2}u = q_1 * u, \quad u \in U .$$

Consequently

$$\left|S(s)Q^{1/2}h_j(\xi)\right|^2 = e^{-2\alpha s}\langle p_s * q_1(\xi - \,\cdot\,), h_j \,\cdot\, \rangle_U$$

and by Parseval's and Plancherel's identities

$$\sum_{j\in\mathbb{R}^d} \left|S(s)Q^{1/2}h_j(\xi)\right|^2 = e^{-2\alpha s}\|p_s * q_1\|_U$$

$$= e^{-2\alpha s}\int_{\mathbb{R}^d} e^{-2s|\eta|^2} g(\eta)d(\eta)\,.$$

Therefore, for a constant c_2,

$$\mathbb{E}|Y_\delta(t)|^p_{L^p_\rho(\mathbb{R}^d)} \leqq c_2\left[\int_{\mathbb{R}^d} g(\eta)\left(e^{-2s(m+|\eta|^2)}s^{-2\alpha}ds\right)d\eta\right]^{p/2}.$$

Since $m > 0$, for another constant c_3,

$$\mathbb{E}|Y_\delta(t)|^p_{L^p_\rho(\mathbb{R}^d)} \leqq \left[c_3\int_{\mathbb{R}^d}\frac{g(\eta)}{(m+|\eta|^2)^{2\delta+1}}d\eta\right]^{p/2} < +\infty$$

for arbitrary $t \geqq 0$.

Taking into account, see [5, p.128], that

$$W_{A_{p\rho}}(t) = \frac{\sin\ \pi\delta}{\pi}G_\delta Y_\delta(t), \quad t \geqq 0\,,$$

where G_δ is the operator given by

$$G_\delta\varphi(t) = \int_0^t S(t-s)(t-s)^{\delta-1}\varphi(s)\,ds, \quad t\in[0,T],\ \ \varphi\in L^p_\rho(0,T;\ L^p(\mathbb{R}^d))\,,$$

we see that

$$\left\|W_{A_{p\rho}}(t)\right\|_{L^p_\rho(\mathbb{R}^d)} \leqq \frac{\sin\ \pi\delta}{\pi}\left(\int_0^t s^{q(\delta-1)}\|S(s)\|^q_{L(L^p_\rho(\mathbb{R}^d))}ds\right)^{1/q}$$

$$\times\left(\int_0^t \|Y_\delta(s)\|^p_{L^p_\rho(\mathbb{R}^d)}ds\right)^{1/q},$$

where $q = p/(p-1)$.

The function $\|S(s)\|^q_{L(L^p(\mathbb{R}^d))}$, $s \geqq 0$, is locally bounded and $q(\delta-1) > -1$. Therefore for arbitrary $T > 0$ there exists a constant c_4 such that

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left\|W_{A_{p\rho}}(t)\right\|^p_{L^p_\rho(\mathbb{R}^d)}\right) \leqq c_4\int_0^T\mathbb{E}\|Y_\delta(s)\|_{L^p_\rho(\mathbb{R}^d)}ds < +\infty\,,$$

and the required continuity follows. □

Remark. 4.6 In a similar way an identical result can be proved also if $m \leqq 0$ and $d = 1$ or if $d > 1$ and

$$\int_{|\eta|\leqq 1}\frac{g(\eta)}{|\eta|^2}d\eta < +\infty\,.$$

Remark. 4.7 Similar theorems were proved earlier in [19, 21] in the case $d = 1$. Our conditions on f are more special than the ones in [19, 21]. However our dissipativity approach allows to cover the case $d \geqq 1$ and to construct transition semigroups on all weighted spaces $L_{\rho^\kappa}(\mathbb{R}^d)$, $L^2_{\rho_\kappa}(\mathbb{R}^d)$.

We will finally prove the theorem on exponential convergence.

Theorem 4.8 *In addition to conditions of Theorem* 4.4 *assume that the function f_0 is decreasing and*

$$\alpha - \|f_1\|_{\mathrm{Lip}} > \omega > 0 .$$

Then there exist $\kappa_0 > 0$ such that the semigroup P_t, $t \geqq 0$, corresponding to the solution of (4.1) *has a unique invariant measure both in $H = L^2_{\rho^\kappa}(\mathbb{R}^d)$ and $H = L^2_{\rho_\kappa}(\mathbb{R}^d)$, for any $\kappa \in]0, \kappa_0[$. Moreover there exists $c > 0$ such that for any bounded Lipschitz function φ on H, all $t > 0$ and all $x \in H$*

$$\left| P_t\varphi(x) - \int_H \varphi(x)\mu(dx) \right| \leqq (c + 2)\|x\|)e^{-\omega t}\|\varphi\|_{\mathrm{Lip}} .$$

Proof. We apply Theorem 2.3. Arguing as in the proof of Theorem 3.4 one easily shows that condition (i) of Theorem 2.3 holds. It remains to check that (ii) holds as well. As for the discrete lattice we will prove that for arbitrary $p \geqq 1$

$$\sup_{t \geqq 0} \mathbb{E}\left(\|W_{A_{pp}}(t)\|_{L^p(\mathbb{R}^d)} \right) < +\infty .$$

Note, compare the proof of Theorem 4.4, that for all $t \geqq 0$

$$\begin{aligned}
\mathbb{E}\,\|W_{A_{pp}}(t)\|^p_{L^p_\rho(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \rho(\xi)\left(\left| \int_0^t \sum_{j=1}^\infty S(t-s)Q^{1/2}h_j(\xi)\,d\beta_j(s) \right|^p \right) d\xi \\
&\leqq c_1 \left(\int_{\mathbb{R}^d} \rho(\xi) \right) \left(\int_0^t \sum_{j=1}^\infty |S(t-s)Q^{1/2}h_j(\xi)|^2 ds \right)^{p/2} d\xi \\
&\leqq c_1 \left(\int_{\mathbb{R}^d} \rho(\xi) \frac{g(\eta)}{2(\alpha + |\eta|^2)} d\eta \right)^{p/2} d\xi . \qquad \square
\end{aligned}$$

Remark. 4.9 Existence of a stationary solution of (4.1), but not of invariant measure, was obtained by Marcus [21] if $d = 1$. Our results about the exponential convergence of the transition semigroup are new.

5 Quantum lattice systems

Quantum lattice systems are a mixture of systems described in Sect. 3 and Sect. 4. They were introduced in [1] and are described by systems of equations of the form

$$dX_\gamma(t) = \left(\mathscr{A}X_\gamma(t) + \sum_{i\in\mathbb{Z}^d} a_{\gamma j}X_j(t) + \mathscr{F}(X_\gamma(t)) \right) dt + dW_\gamma(t)$$

$$X_\gamma(0) = x_\gamma \in \mathscr{H}, \quad \gamma \in \mathbb{Z}^d, \ t \geqq 0 . \tag{5.1}$$

In Eq. (5.1), $\mathscr{A}$ and $\mathscr{F}$ are respectively linear and nonlinear, in general unbounded, operators on a Hilbert space $\mathscr{H}, (a_{\gamma j})_{\gamma, j\in\mathbb{Z}^d}$ is a given matrix with real elements and $(W_\gamma)_{\gamma\in\mathbb{Z}^d}$ is a family of independent, cylindrical Wiener processes on $\mathscr{H}$. Following [1] we will require that the space $\mathscr{H}$ and the operators $\mathscr{A}$ and $\mathscr{F}$ are of special character although it will be clear that our general schema works in the more abstract setting. We will thus assume that

$$\mathscr{H} = L^2(0,1)\,, \tag{5.2}$$

$$\begin{aligned} \mathscr{A} &= \frac{d^2}{d\xi^2} - \alpha\,, \\ D(\mathscr{A}) &= \{x \in H^2(0,1)\colon x(0) = x(1), \quad x'(0) = x'(1)\}\,, \end{aligned} \tag{5.3}$$

$$\begin{aligned} \mathscr{F}(x)(\xi) &= f(x(\xi)), \quad \xi \in [0,1]\,, \\ D(\mathscr{F}) &= \{x \in L^2(0,1)\colon f(x) \in L^2(0,1)\}\,, \end{aligned} \tag{5.4}$$

and on f and on the matrix $(a_{\gamma j})_{\gamma, j\in\mathbb{Z}^d}$ we will impose conditions (3.7)–(3.9) and (3.3).

To see that (5.1) is of the general form studied in Sect. 2 define

$$H = \ell^2_\rho(L^2(0,1)) = \left\{ (x_\gamma) \in \mathscr{H}^{(\mathbb{Z}^d)}\colon \sum_{\gamma\in\mathbb{Z}^d} \rho(\gamma)\|x_\gamma\|^2_{\mathscr{H}} < +\infty \right\}\,,$$

$$K = \ell^{2s}_\rho(L^{2s}(0,1)) = \left\{ (x_\gamma) \in (L^{2s}(0,1))^{(\mathbb{Z}^d)}\colon \sum_{\gamma\in\mathbb{Z}^d} \rho(\gamma)\|x_\gamma\|^{2s}_{L^{2s}(0,1)} < +\infty \right\}\,,$$

where $\rho = \rho^\kappa$ or $\rho = \rho_\kappa$, $\kappa > 0$, see Sect. 3.

Let $A = A_0 + A_1$ where A_1 is a bounded linear operator on H given by

$$A_1(x_\gamma) = \left(\sum_{j\in\mathbb{Z}^d} a_{\gamma j}x_j \right), \quad x \in D(A_1) = H\,,$$

and

$$A_0(x_\gamma) = (\mathscr{A}x_\gamma), \quad x = (x_\gamma) \in D(A_0)\,,$$

$$D(A_0) = \left\{ (x_\gamma) \in H\colon \sum_{\gamma\in\mathbb{Z}^d} \rho(\gamma)\|\mathscr{A}x_\gamma\|^2_{\mathscr{H}} < +\infty \right\}\,.$$

Boundedness of A_1 follows from an obvious generalization of Proposition 3.1. Let, in addition,

$$F(x_\gamma) = (\mathscr{F} x_\gamma), \quad x = (x_\gamma) \in D(F),$$

$$D(F) = \left\{ (x_\gamma) \in H : \sum_{\gamma \in \mathbb{Z}^d} \rho(\gamma) \|\mathscr{F} x_\gamma\|^2_{\mathscr{H}} < +\infty \right\}.$$

It is easy to check that operator $A_0 + \eta$ on H and its restriction $A_{0p} + \eta$ to K are m-dissipative for sufficiently small η.

Let $S(t)$, $S_0(t)$, $t \geqq 0$, be C_0-semigroups on H generated by A and A_0 and denote

$$W_A(t) = \int_0^t S(t-s)\, dW(s), \quad t \geqq 0,$$

$$W_{A_0}(t) = \int_0^t S_0(t-s)\, dW(s), \quad t \geqq 0,$$

where $W(t)$, $t \geqq 0$, is the Wiener process $(W_\gamma(\,\cdot\,))_{\gamma \in \mathbb{Z}^d}$ embedded into H.

Proposition 5.1 *Processes $W_{A_0}(t)$, $t \geqq 0$, $W_A(t)$, $t \geqq 0$, have continuous versions with values in $\ell^p_\rho(L^p(0,1))$, $p \geqq 1$.*

Proof. Existence of a continuous version of $W_{A_0}(t)$, $t \geqq 0$, in $\ell^p_\rho(L^p(0,1))$ can be obtained by factorization, as in the proof of Proposition 4.5, and using the diagonal character of the semigroup $S_0(\,\cdot\,)$.

Note that for $Z(t) = W_A(t)$, $t \geqq 0$,

$$Z(t) = W_{A_0}(t) + \int_0^t S_0(t-s) A_1 Z(s)\, ds, \quad t \geqq 0. \tag{5.5}$$

Since the semigroup $S_0(\,\cdot\,)$ has a C_0-continuous restriction to $\ell^p_\rho(L^p(0,1))$, Eq. (5.1) has a unique solution in $C([0,T]; \ell^p_\rho(L^p(0,1)))$ for arbitrary $T > 0$ by an elementary fixed point argument. This proves the result. □

As a corollary from our discussion and Proposition 5.1

Theorem 5.2 *Assume that conditions* (3.7)–(3.9) *and conditions* (5.2)–(5.4) *hold. Then for arbitrary $x \in H$ Eq.* (5.1) *has a unique generalized solution $X(\,\cdot\,,x)$. If $x \in K$ the solution is mild.*

As in Sects. 3 and 4 one can derive a result on exponential decay for the transition semigroup P_t, $t \geqq 0$ corresponding to the solution $X(\,\cdot\,,x)$ of (5.1).

Theorem 5.3 *In addition to conditions of Theorem* 5.2, *assume that f_0 is decreasing and that $\alpha - \|f_1\|_{\mathrm{Lip}} > \omega > 0$. Then there exists $\kappa_0 > 0$ such that the semigroup P_t, $t \geqq 0$, corresponding to the solution of* (5.1) *has a unique invariant measure μ both on $H = L^2_{\rho^\kappa}$ and on $H = L^2_{\rho_{\kappa,r}}$, for all $\kappa \in]0, \kappa_0[$ and $r > d$. Moreover there exists $c > 0$ such that for any bounded and*

Lipschitz function φ on H, all $t > 0$ and all $x \in H$:

$$\left|P_t\varphi(x) - \int_H \varphi(x)\mu(dx)\right| \leqq (c + 2\|x\|)e^{-\omega t}\|\varphi\|_{\mathrm{Lip}} .$$

Proof. As in the proof of Theorem 4.8 it is enough to check assumptions of Theorem 2.3. We will show that

$$\sup_{t \geqq 0} \mathbb{E}(\|W_A(t)\|_H) < +\infty . \tag{5.6}$$

$$\sup_{t \geqq 0} \mathbb{E}(\|F(W_A(t))\|_H) < +\infty , \tag{5.7}$$

as the remaining conditions hold in an obvious way.

To prove (5.6) note that for $t \geqq 1$,

$$\begin{aligned}
&\mathbb{E}\|W_A(t)\|_H \\
&= \mathbb{E}\left\|\int_0^t S(t-s)\,dW(s)\right\|_H = \mathbb{E}\left\|\int_0^t S(s)\,dW(s)\right\|_H \\
&\leqq \sum_{k=0}^{[t]-1} \mathbb{E}\left\|\int_k^{k+1} S(s)\,dW(s)\right\|_H + \mathbb{E}\left\|\int_{[t]}^t S(s)\,dW(s)\right\|_H \\
&\leqq \sum_{k=0}^{[t]-1} \|S(k)\|_H \mathbb{E}\left\|\int_0^1 S(s)\,dW(s)\right\|_H + \sup_{u \leqq 1} \mathbb{E}\left\|\int_0^u S(s)\,dW(s)\right\|_H \\
&\leqq \sum_{k=0}^{[t]-1} e^{-\alpha k} \mathbb{E}\left\|\int_0^1 S(s)\,dW(s)\right\|_H + \sup_{u \leqq 1} \mathbb{E}\left\|\int_0^u S(s)\,dW(s)\right\|_H \\
&\leqq \frac{2e^\alpha}{e^\alpha - 1} \sup_{u \leqq 1} \mathbb{E}\left\|\int_0^u S(s)\,dW(s)\right\|_H , \quad t \geqq 1 .
\end{aligned}$$

To show (5.7) remark first that

$$\mathbb{E}\|F(W_A(t))\|_H \leqq (\mathbb{E}\|F(W_A(t))\|_H^2)^{1/2} \leqq (\mathbb{E}\|W_A(t)\|_K^s)^{1/2} .$$

Since the distribution of $W_A(t)$ on K is Gaussian, therefore there exists a constant $\widehat{c}$ such that

$$\mathbb{E}\|W_A(t)\|_K^s \leqq \widehat{c}(\mathbb{E}\|W_A(t)\|_K)^s .$$

Moreover there exists a contant $\widehat{\alpha} > 0$ such that for the restriction $\widehat{S}(t)$, $t \geqq 0$, of the semigroup $S(t)$, $t \geqq 0$ to K, one has

$$\|\widehat{S}(t)\|_K \leqq e^{-\widehat{\alpha} t}, \quad t \geqq 0 .$$

It remains now to repeat the proof of (5.6). □

Remark. 5.4 Theorem 5.2, under slightly stronger conditions was proved earlier in [1] using specific properties of the one-dimensional heat equation. It is however clear that the dissipativity method allows to treat spaces $\mathscr{H}$ more general than $L^2(0,1)$. Existence of invariant measure μ for (5.1) was also obtained in [1] with more stringent conditions imposed on f, ρ and $(a_{\gamma j})$. Exponential estimates in Theorem 5.3 are new.

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