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Probability Theory and Related Fields

Stationary self-similar extremal processes

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Summary. Let $(\xi_k)_{k=-\infty}^{\infty}$ be a stationary sequence of random variables, and, for $A \subset \mathbb{R}$, let $M_n(A) := \bigvee_{k/n \in A} \gamma_n(\xi_k)$ where γ_n is an affine transformation of $\mathbb{R}$ (has the form $a_n \cdot + b_n$, $a_n > 0$, $b_n \in \mathbb{R}$). Then M_n is a random sup measure, that is, $M_n(\bigcup_\alpha G_\alpha) = \bigvee_\alpha M_n(G_\alpha)$ for arbitrary collections of open sets G_α. We show that the possible limiting random sup measures for such sequences (M_n) are those which are stationary ($M(\cdot + b) =_d M$ for $b \in \mathbb{R}$) and self-similar ($M(a\cdot) =_d \delta^{\log a}(M)$ for $a > 0$, where δ is an affine transformation of $\mathbb{R}$). By applying simple transformations, we need only study stationary M such that $M(a\cdot) =_d aM$ for $a > 0$. We show that these processes retain some but not all of the properties of the classical case. In particular, we display a nontrivial example such that $t \mapsto M(0, t]$ is continuous wp1. The classical planar point process representation of extremal processes is a special case of the present approach, but is not adequate for describing all possible limits.

1 Introduction

Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of random variables, and let

$$M_n := \bigvee_{k=1}^{n} \xi_k$$

denote the partial maxima. Research on the asymptotics of M_n, or rather $a_n M_n + b_n$ for suitably chosen constants $a_n > 0$, b_n, has a long tradition in probability theory. The resulting theory has developed along the same lines as its counterpart for the partial sums $\sum_{k=1}^{n} \xi_k$. As in the latter case, most is known with the ξ_k

independent and identically distributed. All possible limit laws for $a_n M_n + b_n$ were already known by Fisher and Tippett (1928). The domains of attraction (= the collection of distributions of ξ_k producing a specified limit law for $a_n M_n + b_n$) were characterized by Gnedenko (1943) and de Haan (1970). Functional limit theorems for

$$a_n M_{\lfloor nt \rfloor} + b_n$$

as random functions of t were obtained first by Dwass (1964) and Lamperti (1964). The limit processes were called 'the' extremal processes and became a separate topic of research. Pickands (1971) was the first who regarded them as functionals of planar Poisson processes. This approach was then adopted by many authors and has been codified in Resnick's survey (1986) and monograph (1987). As a result, convergence considerations shifted from random functions to planar point processes.

As could be expected, a second line of research has developed in the last decades on asymptotics for M_n with the assumption of independent identically distributed ξ_k relaxed to specific forms of dependence, mostly including stationarity. Research has been very productive. Without hope of being complete we mention work by Berman (1964), Loynes (1965), Newell (1964), Adler (1978), O'Brien (1987), the monograph by Leadbetter, Lindgren and Rootzén (1983) and the survey by Leadbetter and Rootzén (1988).

However, if one looks at the results with special interest in finding new limiting extremal processes, one gets a bit disappointed. With few exceptions all limit processes are the same as in the iid case and the dependence influences only the scaling, or marginal limit results are obtained for M_n, whose functional analogues for $M_{\lfloor nt \rfloor}$ have random constant functions as limit (so there is too strong dependence in the limit), or the limits are random mixtures of all these processes. Without stationarity it is not so hard to find other limiting extremal processes, cf. Weissman (1975) for the case of independent ξ_k, and Hüsler (1986).

There is more variety in the literature if nth largest values ($n \geqq 2$) are being considered. In many cases of dependence these tend to cluster below the largest values (cf. Mori (1977) for an early case, and Hsing (1987, 1988)).

In the present paper we restrict our attention to largest values. We characterize, in a certain interpretation, *all* limiting processes of $a_n M_{\lfloor n \cdot \rfloor} + b_n$ for stationary (ξ_k). To this end we must be able to identify a stochastic process as being extremal by intrinsic characterization. The need for this has triggered an extensive study of random semicontinuous functions and random closed sets by Vervaat (1988), and the needed results are quoted from that paper in Sects. 2–4. For a similar approach to extremal processes with different applications, see Norberg (1987).

The consequences of stationarity of (ξ_k) for the limiting extremal processes are treated in Sect. 5. An important ingredient is Lamperti's (1962) fundamental theorem on self-similarity which applies to all stochastic processes, not just the extremal ones (Sect. 6).

The main result of this paper (Sect. 8) is that all nondegenerate limiting processes of $a_n M_{\lfloor n \cdot \rfloor} + b_n$ for stationary (ξ_k) correspond to stationary self-similar random upper semicontinuous functions (the combination of 'stationary' and 'self-similar' is referred to as 'self-affine'). We discuss examples and explore general properties (Sects. 9–11). Many of the examples are produced by the technique of subordination (Sect. 10). One of them has continuous sample paths wp1 (Sect. 11).

A broad class of such extremal processes arise from planar point processes (Poisson in the classical iid case). However, not all processes arise this way. Consequently, the point process approach, which was so successful and elucidating in the last decade, is not general enough for handling all limiting processes arising from stationary sequences.

The completion of this paper has taken a long time, with many of the results obtained already in 1980. This is partly due to the need to develop the topological background in Vervaat (1988). A preview of the present results without proofs appeared in Vervaat (1986).

2 Sup measures

Formerly, extremal processes were stochastic processes $(M(t))_{t \geqq 0}$, where $M(t)$ was to be interpreted as the supremum of a random phenomenon observed in $(0, t]$. In the last decade it has become more common to regard extremal processes as random functions on families $\mathscr{A}$ of subsets of the time domain: $(M(A))_{A \in \mathscr{A}}$. Often $\mathscr{A}$ was a collection of intervals in $[0, \infty)$, and $M(A)$ was to be interpreted as the supremum of a random phenomenon observed during A. A smooth theory can be developed if we take for $\mathscr{A}$ the collection $\mathscr{G}$ of open subsets of $\mathbb{R}$ and require

$$M\Big(\bigcup_\alpha G_\alpha\Big) = \bigvee_\alpha M(G_\alpha) \tag{2.1}$$

for arbitrary collections of open sets (G_α) in $\mathbb{R}$. We want to define an extremal process as a random variable taking values in a space of functions on $\mathscr{G}$ satisfying (2.1). So it is useful to study such function spaces.

Definition 2.1. Let $\mathbb{I}$ be a compact interval in $\overline{\mathbb{R}} := [-\infty, \infty]$. An ($\mathbb{I}$-valued) *sup measure* is a mapping $m: \mathscr{G} \to \mathbb{I}$ such that $m(\emptyset) = \min \mathbb{I}$ and

$$m\Big(\bigcup_\alpha G_\alpha\Big) = \bigvee_\alpha m(G_\alpha) \tag{2.2}$$

for all collections (G_α) in $\mathscr{G}$.

A way of producing a sup measure is to start with a function $f: \mathbb{R} \to \mathbb{I}$ and set $f^\vee(G) := \bigvee_{t \in G} f(t)$ for $G \in \mathscr{G}$ (here and in the sequel, $\bigvee_{t \in \emptyset} := \min \mathbb{I}$). Then $f^\vee$ is a sup measure. We call $f^\vee$ the *sup integral* of f, and occasionally write $f^\vee = i^\vee f$. Obviously, different f may produce the same sup measure, e.g., $1_{\mathbb{R}}^\vee = 1_{\mathbb{Q}}^\vee$. The question comes up whether all sup measures are sup integrals, and to what extent sup integrands are unique. In the remainder of this section we present material from Vervaat (1988, Sects. 1, 2), where the results are proved in the larger generality of $\mathscr{G}$ being the open sets of an arbitrary topological space (not necessarily Hausdorff).

Definition 2.2. If $m: \mathscr{G} \to \mathbb{I}$ is increasing ($m(G_1) \leqq m(G_2)$ if $G_1 \subset G_2$), then the *sup derivative* $d^\vee m$ is the function $\mathbb{R} \to \mathbb{I}$ defined by

$$d^\vee m(t) := \bigwedge_{G \ni t} m(G) \quad \text{for } t \in \mathbb{R},$$

or equivalently,

$$d^\vee m := \bigwedge_G \{m(G)1_G \vee (\max \mathbb{I})1_{\mathbb{R}\setminus G}\}.$$

There is a close relationship between sup measures and upper semicontinuous (usc) functions. We first recall the definition and some properties of upper semicontinuity.

Definition 2.3. A function $f: \mathbb{R} \to \mathbb{I}$ is *upper semicontinuous (usc)* if $\{t: f(t) < x\}$ is open for $x \in \mathbb{I}$.

Properties 2.4. (a) If f is two-valued, then f is usc iff f assumes its larger value on a closed set. In particular, 1_A is usc iff A is closed.

(b) Pointwise infima of arbitrary collections of usc functions are usc.

(c) For $f: \mathbb{R} \to \mathbb{I}$ the function $d^\vee i^\vee f$ is the smallest usc function larger than or equal to f. In particular, f is usc iff $f = d^\vee i^\vee f$.

By (a) and (b) we see that $d^\vee m$ in Definition 2.2 is usc. Compare Property 2.4(c) with (b) in the next theorem.

Theorem 2.5. *Let m be as in Definition 2.2. Then*

(a) *$d^\vee m$ is usc;*

(b) *$i^\vee d^\vee m$ is the largest sup measure smaller than or equal to m. In particular, m is a sup measure iff $m = i^\vee d^\vee m$.*

Let SM denote the collection of all sup measures $\mathscr{G} \to \mathbb{I}$ and US the collection of all usc functions $\mathbb{R} \to \mathbb{I}$. Then Property 2.4(c), Theorem 2.5 and the fact that $f^\vee$ is a sup measure imply that $d^\vee: \mathrm{SM} \to \mathrm{US}$ is a bijection with inverse $i^\vee$.

For sup measures $m: \mathscr{G} \to \mathbb{I}$ there is a canonical extension to all subsets of $\mathbb{R}$ because

$$\bigvee_{t \in A} d^\vee m(t) = \bigwedge_{G \supset A} m(G) \quad \text{for } A \subset \mathbb{R},\ A \neq \emptyset. \tag{2.3}$$

The common value of both sides is denoted by $m(A)$. For singletons $A = \{t\}$, formula (2.3) reduces to Definition 2.2.

We now establish topologies on SM and US. The following results and those in the next section up to Lemma 3.3 are quoted from Vervaat (1988) (cf. also Norberg (1986) and Salinetti and Wets (1986)).

Definition 2.6. A sequence of sup measures (m_n) converges *sup vaguely* to a sup measure m if

$$\limsup_n m_n(K) \leq m(K) \quad \text{for compact } K \subset \mathbb{R}, \tag{2.4a}$$

$$\liminf_n m_n(G) \geq m(G) \quad \text{for open } G \subset \mathbb{R}. \tag{2.4b}$$

A sequence of usc functions (f_n) converges sup vaguely to an usc function f if $f_n^\vee \to f^\vee$ in SM.

These convergence concepts determine the *sup vague topologies* on SM and US, and make $d^\vee$ and $i^\vee$ homeomorphisms. We continue our discussion only for SM; the translation to US is straightforward.

In the first instance, (2.4) should be required for nets, or rather one should define the sup vague topology as the coarsest topology making the evaluations $m \mapsto m(K)$ usc and $m \mapsto m(G)$ lower semicontinuous, but the resulting topology turns out to be *compact* and *metrizable*, so is determined by sequential convergence. We can relax the requirements in (2.4) to compact intervals K and bounded open intervals G, or more generally, to K from a base of compact sets and to G from a base of open sets. Moreover, sup vague convergence can be characterized by convergence on continuity sets. We say that $A \subset \mathbb{R}$ is a *continuity set* of m if $m(\operatorname{int} A) = m(\operatorname{clos} A)$. Now (2.4) holds iff

$$m_n(A) \to m(A) \quad \text{for all bounded continuity sets } A \text{ of } m, \tag{2.6}$$

iff (2.6) holds with A restricted further to be a bounded continuity interval.

3 Extremal processes as random sup measures

We now make SM a measurable space. The Borel field on SM (Bor SM) generated by the sup vague topology turns out to be the smallest that makes the evaluations $m \mapsto m(A)$ measurable for all open A, or all compact A, or all compact intervals A, or all bounded open intervals A, or all intervals as in previous cases with rational endpoints. In particular, a mapping $M: \Omega \to \mathrm{SM}$ with $(\Omega, \mathscr{F}, \mathbb{P})$ a probability space is a random sup measure, i.e., a random variable in SM, iff $M(A)$ is a random variable in $\mathbb{I}$ for all A in one of the previous collections.

Definition 3.1. An *extremal process* is a random sup measure.

The probability distribution over Bor SM of an extremal process M is determined by the finite-dimensional distributions of $(M(A))_{A \in \mathscr{A}}$, where $\mathscr{A}$ may be any of the collections indicated above.

We are now ready for convergence in distribution. Because SM is compact, characterizations are relatively easy. We denote the collection of nonempty bounded open intervals by $\mathscr{I}$. For an extremal process M, let $\mathscr{I}(M)$ be the collection of *continuity intervals* of M, defined by

$$\mathscr{I}(M) := \{I \in \mathscr{I} : M(I) = M(\operatorname{clos} I) \text{ wp1}\}. \tag{3.1}$$

The class $\mathscr{I}(M)$ is rather large, see Corollary 4.3. The next result is the last one quoted from Vervaat (1988).

Theorem 3.2. *Let M_n $(n \geq 1)$ and M be extremal processes. Then $M_n \to_d M$ (convergence in distribution in* SM*) iff the finite-dimensional distributions of $(M_n(I))_{I \in \mathscr{I}(M)}$ converge weakly to those of $(M(I))_{I \in \mathscr{I}(M)}$.*

Occasionally we will consider $M_n(I_n)$, where I_n varies also with n. So the following is useful.

Lemma 3.3. *If* $m_n \to m$ *in* SM *and* I_n, I *are bounded intervals in* $\mathbb{R}$ *such that* $\inf I_n \to \inf I$ *and* $\sup I_n \to \sup I$, *then*

$$\limsup_n m_n(I_n) \leqq m(\operatorname{clos} I),$$

$$\liminf_n m_n(I_n) \geqq m(\operatorname{int} I).$$

Proof. Let (K_k) be a sequence of compact intervals such that $K_{k+1} \subset \operatorname{int} K_k$ for $k=1, 2, \ldots$ and $\bigcap_k K_k = \operatorname{clos} I$. Let (G_k) be an increasing sequence of open intervals such that $\operatorname{clos} G_k \subset G_{k+1}$ and $\bigcup_k G_k = \operatorname{int} I$. Then $m(K_k) \to m(\operatorname{clos} I)$ by (2.3), and $m(G_k) \to m(\operatorname{int} I)$ because m is a sup measure. For fixed k we have $G_k \subset I_n \subset K_k$ for all large n, so

$$\limsup_n m_n(I_n) \leqq \limsup_n m_n(K_k) \leqq m(K_k),$$

$$\liminf_n m_n(I_n) \geqq \liminf_n m_n(G_k) \geqq m(G_k).$$

Letting $k \to \infty$ we obtain the lemma. $\square$

4 Random upper semicontinuous functions

Whenever convenient we will study extremal processes M via their sup derivatives

$$X(t) := d^\vee M(t) = \bigwedge_{G \ni t} M(G) = M(\{t\}),$$

where G must vary in the first instance through the open sets. It is obvious that X does not change if G varies through the open intervals, or through the compact intervals with t as interior point. The process X is a random usc function, i.e., a mapping X from the underlying probability space into the usc functions such that $X^\vee(A)$ is a random variable in $\mathbb{I}$ for all A in one of the collections at the beginning of Sect. 3.

Some caution is needed when thinking about the classical procedures of selecting smooth versions of stochastic processes with the same finite-dimensional distributions. They apply to M rather than X. To see this, let ξ have a uniform distribution over $[0, 1]$, and set $X := 1_{\{\xi\}}$. Then X has the same finite-dimensional distributions as $Y \equiv 0$, but $X^\vee$ and $Y^\vee$ do not have the same distribution over Bor SM since $X^\vee[0, 1] = 1$ wp1 and $Y^\vee[0, 1] = 0$ wp1.

Random usc functions X are measurable processes, i.e., the mapping $(t, \omega) \mapsto X(t, \omega)$ is jointly measurable. To see this, note that $X(t) = \lim_n X^\vee(t - n^{-1}, t + n^{-1})$ and that the mapping $((s, t), \omega) \mapsto X^\vee((s, t), \omega)$ is jointly measurable, because $X^\vee$ depends monotonically on s and t $(s < t)$.

With (random) functions $f: \mathbb{R} \to \mathbb{I}$ we will also consider their graphs and hypographs. It is convenient to regard them as subsets of $\mathbb{R} \times \mathbb{J}$ rather than $\mathbb{R} \times \mathbb{I}$, where $\mathbb{J} := \mathbb{I} \setminus \min \mathbb{I}$. So we define the *graph* of f to be

$$\Gamma(f) := \{(t, x) \in \mathbb{R} \times \mathbb{J} : f(t) = x\}.$$

Note that $\Gamma(f) = \emptyset$ in case $f \equiv \min \mathbb{I}$. For points $(t, x) \in \mathbb{R} \times \mathbb{J}$ we define the *umbra*

$$\downarrow(t, x) := \{t\} \times (\min \mathbb{I}, x],$$

and we extend this notion to subsets F of $\mathbb{R} \times \mathbb{J}$ by

$$\downarrow F := \bigcup_{z \in F} \downarrow z.$$

For functions $f: \mathbb{R} \to \mathbb{I}$ we call $\downarrow \Gamma(f)$ the *hypograph* of f. It is well-known that functions $f: \mathbb{R} \to \mathbb{I}$ are usc iff their hypographs $\downarrow \Gamma(f)$ are closed in $\mathbb{R} \times \mathbb{J}$.

An important particular case occurs if F is a locally finite subset of $\mathbb{R} \times \mathbb{J}$, i.e., $F \cap (K \times [x, \max \mathbb{I}])$ is finite for compact intervals K in $\mathbb{R}$ and $x \in \mathbb{J}$. Then both F and $\downarrow F$ are closed in $\mathbb{R} \times \mathbb{J}$, and $\downarrow F$ is the hypograph of an usc function f with countable graph. We have $F = \Gamma(f)$ in case F has at most one point on verticals. If F is a random closed subset of $\mathbb{R} \times \mathbb{J}$ which is locally finite wp1, then it can be regarded as the support of a simple point process Π, a random integer-valued Radon measure on $\mathrm{Bor}(\mathbb{R} \times \mathbb{J})$ such that $\bigvee_z \Pi\{z\} \leq 1$ wp1. In this case we can define a random usc function X by $\downarrow \Gamma(X) = \downarrow$(support of Π). In particular, we reobtain the well-known representation of classical extremal processes as functionals of planar Poisson processes by considering $X^\vee$ starting from Poisson Π.

Hypographs are a more natural characteristic of usc functions than graphs. From a theoretical point of view this is made clear in Salinetti and Wets (1986), Norberg (1986) and Vervaat (1988). From a technical point of view they turn out indispensable with subordination (Sect. 10). Finally, random hypographs take values in the well-defined measurable space of closed subsets of $\mathbb{R} \times \mathbb{J}$ (cf. Matheron (1975) and the three previous references). For random graphs there is no obvious measurable codomain, unless via their closures. The operation of recovering the graph of an usc function from its closure is the same as recovering it from its full umbra, the hypograph.

In the sequel we need more detailed knowledge about how and in how many places (random) usc functions can have peak values that are substantially larger than any other values in (one-sided) neighborhoods.

Definition 4.1. A function $f: \mathbb{R} \to \mathbb{I}$ is said to be *left sup continuous* at $t \in \mathbb{R}$ if $f(t) = \limsup_{s \uparrow t} f(s)$, *right sup continuous* at t if $f(t) = \limsup_{s \downarrow t} f(s)$, and *sup continuous* at t if f is both left and right sup continuous at t. We say that f is sup continuous (left, right sup continuous) if f is sup continuous (left, right sup continuous) at each $t \in \mathbb{R}$.

A sup continuous function is usc, but an usc function need not be sup continuous.

Lemma 4.2. (a) *If* $f: \mathbb{R} \to \mathbb{I}$ *is usc, then the set on which* f *is not sup continuous (left, right sup continuous) is countable.*

(b) *If* X *is a random usc function* $\mathbb{R} \to \mathbb{I}$, *then the set of* t *for which there is positive probability that* X *is not sup continuous (left, right sup continuous) is countable.*

Proof. If is sufficient to prove the cases of left sup continuity.

(a) If f is not left sup continuous at t, then we can select an open rectangle $R_t := (t-\delta_t, t) \times (f(t)-\varepsilon_t, f(t))$ such that $R_t \cap \downarrow \Gamma(f) = \emptyset$. Then all selected rectangles R_t are disjoint. Since $\mathbb{R} \times \mathbb{I}$ is separable, there can be only countably many such rectangles.

(b) Adapt the ε-δ arguments in the second paragraph of p. 124 of Billingsley (1968). □

It is obvious that I is a continuity interval of an extremal process M (cf. (3.1)) if $I \in \mathscr{I}$ and $d^\vee M$ is sup continuous wp1 at $\inf I$ and $\sup I$. So Lemma 4.2(b) implies

Corollary 4.3. *Let* M *be an extremal process. Then there is a minimal countable subset* D *of* $\mathbb{R}$ *such that*

$$\mathscr{I}(M) \supset \{I \in \mathscr{I}: \inf I, \sup I \notin D\}.$$

5 Stationary extremal processes

We now formulate the main subject of this paper in terms of notions introduced in the previous sections. Let $(\xi_k)_{k\in\mathbb{Z}}$ be a stationary sequence of random variables in $\mathbb{R}$. For $A \subset \mathbb{R}$, set

$$(5.1\text{a}) \qquad M_1(A) := \bigvee_{k \in A} \xi_k,$$

$$(5.1\text{b}) \qquad M_n(A) := a_n M_1(nA) + b_n = \bigvee_{k: k/n \in A} (a_n \xi_k + b_n) \qquad \text{for } n = 2, 3, \ldots,$$

where (a_n) is a sequence in $(0, \infty)$ and (b_n) in $\mathbb{R}$. We want to characterize those extremal processes M that are limits in distribution of M_n as $n \to \infty$ for some stationary sequence (ξ_k) and some choice of normalizing sequences (a_n) and (b_n).

Although an essential ingredient (Lamperti's theorem in the next section) is still missing, we can draw some conclusions about M at this stage. For the remainder of this section we take $\mathbb{I} = \bar{\mathbb{R}}$. Recall that $\mathscr{I}$ is the collection of non-empty bounded open intervals.

Theorem 5.1. *If* $M_n \to_d M$ *in* SM *with* M_n *as in* (5.1), *then*

(a) M *is stationary:* $M =_d M(\cdot + b)$ *for* $b \in \mathbb{R}$;

(b) $\mathscr{I}(M) = \mathscr{I}$, *that is,* $M(I) = M(\operatorname{clos} I)$ wp1 *for* $I \in \mathscr{I}$;

(c) *the finite-dimensional distributions of* $(M_n(I))_{I\in\mathscr{I}}$ *converge weakly to those of* $(M(I))_{I\in\mathscr{I}}$ *as* $n \to \infty$.

For the proof of Theorem 5.1 we need the following lemma of independent interest.

Lemma 5.2. *The mappings* $\mathrm{SM} \times \mathbb{R} \ni (m, b) \mapsto m(\cdot + b) \in \mathrm{SM}$ *and* $\mathrm{SM} \times (0, \infty) \ni (m, a) \mapsto m(a\cdot) \in \mathrm{SM}$ *are sup vaguely continuous.*

Proof. Let $m_n \to m$ in SM and $b_n \to b$ in $\mathbb{R}$. For compact intervals K, Lemma 3.3 with $I_n = K + b_n$, $I = K + b$ gives $\limsup_n m_n(K + b_n) \leqq m(K + b)$. For bounded open intervals G, Lemma 3.3 with $I_n = G + b_n$, $I = G + b$ gives $\liminf_n m_n(G + b_n) \geqq m(G + b)$. So $m_n(\cdot + b_n) \to m(\cdot + b)$ by (2.4) restricted to compact and to bounded open intervals. The proof for the second mapping is similar. □

Proof of Theorem 5.1. (a) Obviously, $M_n(\cdot + b) =_d M_n$ in SM for $b \in n^{-1}\mathbb{Z}$. For each $k \in \mathbb{Z}_+$ and $b \in k^{-1}\mathbb{Z}$ we have

$$M \leftarrow_d M_{nk} =_d M_{nk}(\cdot + b) \to_d M(\cdot + b) \qquad \text{as } n \to \infty$$

(the latter convergence by Lemma 5.2). Hence

$$M =_d M(\cdot + b) \tag{5.2}$$

for such b. This result holds for $k = 1, 2, \ldots$, so (5.2) holds for $b \in \mathbb{Q}$. Again by Lemma 5.2, the mapping $b \mapsto \operatorname{law} M(\cdot + b)$ is weakly continuous, so (5.2) holds for all $b \in \mathbb{R}$.

(b) Let D be as in Corollary 4.3. Then D is invariant under translations since M is stationary, and countable by Lemma 4.2(b), hence empty. So $\mathcal{I} = \mathcal{I}(M)$ by Corollary 4.3.

(c) Follows from (b) and Theorem 3.2. □

6 Self-similarity and self-affineness

Consider M_n as in (5.1) and let $\mathbb{I} = \mathbb{R}$. In the previous section we have explored the consequences for the limiting process M of the underlying sequence (ξ_k) being stationary. In the present section we consider the consequences of M being a limit in distribution of processes of the form $a_n M_1(n\cdot) + b_n$.

At this point it is convenient to introduce a condensed notation for the affine transformations $a_n \cdot + b_n$ with $a_n > 0$. We denote by Aff the set of all transformations $\gamma: x \mapsto ax + b$ of $\mathbb{R}$ with $a \in (0, \infty)$ and $b \in \mathbb{R}$. We also write $\gamma = a \cdot + b$. The set Aff is a noncommutative group with composition as binary map:

$$\gamma_1 \gamma_2 := \gamma_1 \circ \gamma_2 = a_1 a_2 \cdot + a_1 b_2 + b_1,$$

unit element $1 \cdot + 0$ and inverse $\gamma^{-1} = a^{-1}(\cdot - b)$. We make Aff a topological group by declaring $a \cdot + b \mapsto (\log a, b) \in \mathbb{R}^2$ a homeomorphism, so $a_n \cdot + b_n \to a \cdot + b$ in Aff iff $a_n \to a$ in $(0, \infty)$ and $b_n \to b$ in $\mathbb{R}$. We define real powers γ^t of $\gamma = a \cdot + b$ by

$$\gamma^t = \begin{cases} a^t(\cdot - c) + c & \text{if } a \neq 1, \quad \text{where } c := b/(1-a), \\ \cdot + tb & \text{if } a = 1 \end{cases}$$

(note that this is a continuous extension to $t \in \mathbb{R}$ of what one has to define first for $t \in \mathbb{Z}$ and then for $t \in \mathbb{Q}$).

We now reformulate the starting point around (5.1). For a stationary sequence $(\xi_k)_{k\in\mathbb{Z}}$ of random variables in $\mathbb{R}$ and a sequence $(\gamma_n)_{n=2}^{\infty}$ in Aff we set:

(6.1 a) $$M_1(A):=\bigvee_{k\in A}\xi_k,\qquad A\subset\mathbb{R},$$

(6.1 b) $$M_n(A):=\gamma_n(M_1(nA))=\bigvee_{k:k/n\in A}\gamma_n(\xi_k)\qquad\text{for } n=2,3,\ldots$$

Recall that sup measures and extremal processes can take values in $\bar{\mathbb{R}}$.

Theorem 6.1. (a) *Let M_n be as in* (6.1). *If $M_n\to_d M$ in* SM, *then M is stationary.*

(b) *If in addition $M(0,1]$ is nondegenerate and finite-valued* wp1, *then there is a $\delta\in$*Aff *such that*

(b1) $M(a\cdot)=_d\delta^{\log a}(M(\cdot))$ *in* SM *for $a>0$, and*

(b2) $\gamma_n\gamma_{\lfloor nt\rfloor}^{-1}\to\delta^{\log t}$ *in* Aff *as $n\to\infty$ for $t>0$.*

(c) *Conversely, if an extremal process M is stationary and satisfies* (b1) *for some $\delta\in$*Aff, *then there exists a stationary sequence (ξ_k) of random variables in $\mathbb{R}$ and a sequence (γ_n) in* Aff *such that M_n as defined in* (6.1) *converges in distribution to M. If $M(0,1]$ is in addition finite* wp1, *then the ξ_k are finite* wp1.

Proof. (a) This is Theorem 5.1 (a).

(b) By Skorohod's representation theorem there are random variables M'_n and M' in SM such that $M'_n=_d M_n$, $M'=_d M$ and $M'_n\to M'$ wp1. By Lemma 3.3 and Theorem 5.1 (b) it follows that $M'_n(0,\lfloor nt\rfloor/n]\to M'(0,t]$ wp1. Consequently, $M_n(0,\lfloor nt\rfloor/n]\to_d M(0,t]$ in $\bar{\mathbb{R}}$. Substituting

$$M_n(0,\lfloor nt\rfloor/n]=\gamma_n(M_1(0,\lfloor nt\rfloor])$$

and noting that $M(0,1]$ is finite wp1 and nondegenerate, we are ready to apply the sequential version of Lamperti's theorem: Theorem 8.5.3 in Bingham, Goldie and Teugels (1987). By this theorem there is a $\delta\in$Aff such that (b2) holds. For $a>0$ we have $M_n(a\cdot)\to_d M(a\cdot)$ in SM as $n\to\infty$. On the other hand,

$$\begin{aligned}M_n(a\cdot)&=\gamma_n(M_1(na\cdot))\\&=\gamma_n\gamma_{\lfloor na\rfloor}^{-1}\gamma_{\lfloor na\rfloor}\left(M_1\left(\lfloor na\rfloor\frac{na}{\lfloor na\rfloor}\cdot\right)\right)\to_d\delta^{\log a}(M(\cdot))\end{aligned}$$

by (b2), Lemma 5.2 and the continuity of the mapping $\mathrm{Aff}\times\bar{\mathbb{R}}\ni(\gamma,x)\mapsto\gamma(x)\in\bar{\mathbb{R}}$. Comparing the limits, we find (b1).

(c) Set $\xi_k:=M(k-1,k]$ for $k\in\mathbb{Z}$ and $\gamma_n:=\delta^{-\log n}$. Then (ξ_k) is a stationary sequence in $\bar{\mathbb{R}}$ (in $\mathbb{R}$ if $M(0,1]$ is finite wp1), and $M_n(I)=M(I_n)$, where $I\in\mathscr{I}$ and $I_n:=\bigcup_{k:k/n\in I}(k-1,k]/n$. So $M_n(I)=M(I_n)\to M(I)$ wp1 by Lemma 3.3 and Theorem 5.1 (b). □

Extremal processes M satisfying (b1) are called δ *self-similar*. Extremal processes that are both δ self-similar and stationary, are called δ *self-affine*, because the group of transformations of SM

(6.2) $$m \mapsto \delta^{-\log a}(m(a\cdot + b)), \qquad a \in (0, \infty),\ b \in \mathbb{R},$$

that leave the distribution of M invariant is isomorphic to Aff.

If M is δ self-affine and $M(0, 1]$ is finite wp1 and nondegenerate, then so is $M(b, b+a]$ for all $a \in (0, \infty)$ and $b \in \mathbb{R}$. So $M(I)$ is finite wp1 and nondegenerate for all bounded intervals. From Theorem 6.1 we conclude that all nondegenerate limits in distribution of M_n as in (6.1) with $M(0, 1]$ finite are given by the nondegenerate self-affine extremal processes with finite values on $\mathscr{I}$.

If δ is a pure multiplication, say $\delta(x) = e^H x$ $(H \in \mathbb{R})$, then (b1) reads:

$$M(a\cdot) =_d a^H M \qquad \text{for } a > 0.$$

In this particular case we say that M is self-similar with exponent H or H self-similar. Likewise, H self-similar stationary extremal processes are called H self-affine. This special case of self-similarity is the self-similarity encountered in the literature.

7 Fixed points and invariant set of affine transformations

Although Theorem 6.1 gives lots of information about the limiting extremal processes arising from (6.1), the results only become complete after an investigation into how δ self-affine extremal processes behave at fixed points and in invariant sets of δ. For this section it is just as easy to consider extremal processes with values in $\bar{\mathbb{R}}$. The application will be to the finite-valued case, though.

For $\delta \in \text{Aff}$ we define the invariant σ-field $\mathscr{V}_\delta$ by $\mathscr{V}_\delta := \{V \in \text{Bor}\,\bar{\mathbb{R}} : \delta V = V\}$. Then $\mathscr{V}_\delta$ is a σ-field with atoms $\{-\infty\}$, $(-\infty, c)$, $\{c\}$, (c, ∞), $\{\infty\}$ in case δ is not a translation, c being its fixed point, and with atoms $\{-\infty\}$, $\mathbb{R}$, $\{\infty\}$ in case δ is a translation, except when δ is the identity map, in which case $\mathscr{V}_\delta = \text{Bor}\,\bar{\mathbb{R}}$.

In the remainder of this section, $\doteq$ denotes equality of events up to null events.

Theorem 7.1. *If M is a δ self-affine extremal process and $I = (0, 1]$, then*

$$[M(I) \in V] \doteq \bigcap_{J \in \mathscr{I}} [M(J) \in V] \doteq \bigcup_{J \in \mathscr{I}} [M(J) \in V] \qquad \textit{for } V \in \mathscr{V}_\delta.$$

Remark 7.2. It follows that the events $[M(I) \in V]$ are invariant up to null events under all transformations in (6.2) applied to M. Hence M remains δ self-affine under the conditional distribution given $[M(I) \in V]$, in case $\mathbb{P}[M(I) \in V] > 0$.

Proof of Theorem 7.1. Consider first $V \in \mathscr{V}_\delta$ of the form $[-\infty, v)$ or $[-\infty, v]$. Let $J_1, J_2 \in \mathscr{I}$. Then $J_2 = a(J_1 - b)$ for some reals a and b, $a > 0$, so

$$\mathbb{P}[M(J_2) \in V] = \mathbb{P}[M(J_1) \in \delta^{\log a} V] = \mathbb{P}[M(J_1) \in V].$$

If $J_1 \subset J_2$, then $[M(J_2) \in V] \subset [M(J_1) \in V]$, so $[M(J_2) \in V] \doteq [M(J_1) \in V]$. We get the same result for all $J_1, J_2 \in \mathscr{I}$ by comparing J_1, J_2 with the convex hull of $J_1 \cup J_2$. Consequently, with $\mathscr{I}_0$ denoting the intervals from $\mathscr{I}$ with rational end-points,

$$\bigcap_{J \in \mathscr{I}} [M(J) \in V] = \bigcap_{J \in \mathscr{I}_0} [M(J) \in V] \doteq \bigcup_{J \in \mathscr{I}_0} [M(J) \in V] = \bigcup_{J \in \mathscr{I}} [M(J) \in V].$$

A comparison $J_1 \subset I \subset J_2$ proves the theorem for V of the above form. The result for general $V \in \mathscr{V}_\delta$ follows by set subtraction, or if δ is the identity map by verifying that M is constant wp1 on $\mathscr{I}$. □

The next lemma singles out the atoms of the σ-field $\mathscr{V}_\delta$ that are avoided wp1 by δ self-affine extremal processes.

Lemma 7.3. *Let M be a δ self-affine extremal process.*

(a) *If δ is not a translation, say $\delta = e^H(\cdot - c) + c$ with $H \neq 0$, then M has* wp1 *no values in $(-\infty, c)$ in case $H > 0$, and no values in (c, ∞) in case $H < 0$.*

(b) *If δ is a translation: $\delta = \cdot + b$ with $b < 0$, then M has* wp1 *no values in $\mathbb{R}$.*

Proof. (a) First consider the special case $c = 0$, $H < 0$. Assume that $\mathbb{P}[M(I) \in (0, \infty)] > 0$. By Remark 7.2 we may assume after conditioning that $\mathbb{P}[M(I) \in (0, \infty)] = 1$. We now have for $0 < a < 1$

$$M(0, 1] =_d a^{-H} M(0, a] \leqq a^{-H} M(0, 1] \to 0 \qquad \text{as } a \downarrow 0,$$

so $M(0, 1] = 0$ wp1, a contradiction. The remaining cases are transformed into the special case by considering $M - c$ in case $H < 0$ and $-1/(M - c)$ in case $H > 0$.

(b) If M is δ self-affine with $\delta = \cdot + b$, then e^M is δ' self-affine, with δ' as in (a) for $H = b$ and $c = 0$. If $b < 0$, then e^M has wp1 no values in $(0, \infty)$ by (a), so M has wp1 no values in $\mathbb{R}$. □

8 Reduction to the standard case

If M is an extremal process with values in $\mathbb{I}$ and $\varphi: \mathbb{I} \to \bar{\mathbb{R}}$ is nondecreasing and left-continuous, then $\varphi \circ M$ is an extremal process. If in addition M is stationary, then so is $\varphi \circ M$. Special choices of φ can be used to transform δ self-affine extremal processes into 1 self-affine extremal processes. This allows us to make 1 self-affine extremal processes M such that $M(0, 1]$ has values in $(0, \infty)$ wp1 the central object of study in the rest of the paper. They satisfy

(8.1 a) $$M(I) \in (0, \infty) \quad \text{for } I \in \mathscr{I} \text{ wp1},$$

(8.1 b) $$M(a \cdot + b) =_d aM \quad \text{for } a \in (0, \infty),\ b \in \mathbb{R}$$

We reserve the term *self-affine extremal process* (without further prefix) for 1 self-affine extremal processes, i.e., extremal processes that satisfy (8.1 b). If we insist on (8.1 a) holding in addition, then we talk about *proper* self-affine extremal processes. We write $M \equiv c$ if $M(I)=c$ for all $I \in \mathcal{I}$. Then (8.1a) excludes positive probabilities for $[M \equiv 0]$ and $[M \equiv \infty]$ in presence of (8.1b).

The following theorem combines everything of the two previous sections for the finite-valued case. The statement "wp1 either A or B" means that the events AB and $A^c B^c$ are null events.

Theorem 8.1. *Let M_n be as in* (6.1) *based on a stationary sequence* (ξ_k). *If $M_n \to_d M$ in* SM *and $M(0, 1]$ is nondegenerate and finite-valued* wp1, *then M is δ self-similar for a unique $\delta \in$* Aff *and one and only one of the following statements holds.*

(a) $\delta = 1 \cdot + 0$ *and* $M \equiv M(0, 1]$ wp1;

(b) $\delta = e^H(\cdot - c) + c$ *with* $H > 0$, *and* wp1 *either* $M \equiv c$ *or* $M = c + N^H$ *for a proper self-affine extremal process* N;

(c) $\delta = e^H(\cdot - c) + c$ *with* $H < 0$, *and* wp1 *either* $M \equiv c$ *or* $M = c - N^H$ *for a proper self-affine extremal process* N;

(d) $\delta = \cdot + b$ *with* $b > 0$, *and* wp1 $M = b \log N$ *for a proper self-affine extremal process* N.

Conversely, in each case every M of the indicated form is δ self-affine.

In (6.1) we started with a stationary sequence (ξ_k) to construct M_1. Instead one can start with a stationary extremal process M_1 (stationary for all real shifts, not just the shifts over integers), and consider M_n in (6.1b) based on such an M_1.

Theorem 8.2. *Let M_1 be a stationary extremal process, and let M_n be defined by the first identity in* (6.1b). *Then Theorem 8.1 holds with M_n in its present meaning.*

Proof. We extend the integer-part function $\lfloor \cdot \rfloor$ to sets A of reals by $\lfloor A \rfloor := \{\lfloor t \rfloor : t \in A\}$. Considering $\xi_k := M_1(k-1, k]$ we see that Theorem 8.1 applies to M'_n based on $M'_1 := M_1(\lfloor \cdot \rfloor)$. Note that $M'_n = M_n(\lfloor n \cdot \rfloor / n)$. From Lemma 3.3 it follows that M'_n converges wp1 in SM iff M_n does. By Skorokhod's representation theorem (cf. its application in the proof of Theorem 6.1(b)) the same equivalence follows for convergence in distribution. □

As indicated in Sect. 4, we will study M also by $X := d^\vee M$ and its graph $\Gamma(X)$ and hypograph $\downarrow\Gamma(X)$. It is immediate that M is self-affine iff X is, i.e., (8.1b) holds with X instead of M. Furthermore, X is self-affine iff $\Gamma(X)$ and $\downarrow\Gamma(X)$ are self-affine, by which we mean that $\Gamma(X)$ and $\downarrow\Gamma(X)$ as random subsets of $\mathbb{R} \times \mathbb{J}$ ($\mathbb{J} = (0, \infty]$) are invariant in distribution for the transformations

$$(8.2) \qquad (t, x) \mapsto (at + b, ax) \quad \text{for } a \in (0, \infty),\ b \in \mathbb{R}$$

of $\mathbb{R} \times \mathbb{J}$. The exclusion of ∞ as a value in (8.1a) corresponds to the requirement that X be finite-valued (as an usc function, X attains its supremum in compact intervals). The exclusion of 0 as a value in (8.1a) is characterized for X in Theorem 9.1.

If, more particularly, $\downarrow\Gamma(X)$ is the umbra of the support of a point process Π in $\mathbb{R} \times \mathbb{J}$, then X is self-affine if Π is self-affine, by which we mean that Π is invariant in distribution under the transformations in (8.2). Self-affine point processes are called *Poincaré* in O'Brien and Vervaat (1985), to which we refer for a large collection of examples of such processes (the historical order of our research was first sup self-affine processes, then additively self-affine processes, and many examples in O'Brien and Vervaat (1985) were constructed initially to serve in the present context). We want to draw attention to the g-adic lattice process, of which each atom determines the location of all other atoms at the same or a lower level. One production rule for new self-affine point processes, subordination (cf. also Vervaat 1985), will be discussed again in Sect. 10.

All self-affine processes Π have a mean measure $\mathbb{E}\Pi$ which is invariant under the transformations in (8.2). Consequently (cf. O'Brien and Vervaat 1985), there are constants c_1 and c_2 in $[0, \infty]$ such that

(8.3a) $$\mathbb{E}\Pi(dt, dx) = c_1\, dt \frac{dx}{x^2} \quad \text{for } 0 < x < \infty,$$

(8.3b) $$\mathbb{E}\Pi(dt, \{\infty\}) = c_2\, dx.$$

We say that Π has *finite intensity* if $c_1 < \infty$ and $c_2 = 0$.

The most important example of self-affine point processes with finite mean measure is the self-affine Poisson process. The resulting extremal process M has *independent peaks*, i.e., the random variables $M(I)$ are independent for disjoint I. It is a classical result of extreme value theory that all limiting processes arising from a stationary sequence (ξ_k) of *independent* random variables can be brought into this form by the transformations in Theorem 8.1.

So far all examples in the literature of limiting extremal processes arising from stationary sequences (ξ_k) with dependent terms turned out to be generated by point processes, so that $\Gamma(X)$ is countable wp1. So the question whether all self-affine extremal processes are generated by point processes becomes important. We obtain a negative answer in Example 10.10, where a proper self-affine extremal process with uncountable $\Gamma(X)$ is constructed.

9 General properties of proper self-affine extremal processes

As a basis for comparison, we consider the case that M is generated by a self-affine Poisson process Π, and is therefore a classical extremal process (cf. Resnick 1986; 1987). Then

$$X(t) = \sup\{x : \Pi\{(t, x)\} \geqq 1\}, \qquad \text{where } \sup \emptyset := 0,$$

$$[M(I) \leqq x] = [\Pi(I \times (x, \infty)) = 0].$$

Also, wp1, $X(t) = 0$ for all t except for a countable dense set. As a function of t, $M(0, t]$ is nondecreasing, $M(0, t] \to 0$ as $t \downarrow 0$ and $M(0, t] \to \infty$ as $t \to \infty$. Furthermore, there is a random doubly-infinite increasing sequence $(\tau_k)_{k=-\infty}^{\infty}$ with $\tau_k \to 0$ as $k \to -\infty$ and $\tau_k \to \infty$ as $k \to \infty$ such that $M(0, t]$ jumps at each τ_k but $M(0, t]$ is constant on each interval $[\tau_{k-1}, \tau_k)$. Finally, $M(0, t]$ has a density of the form

$$f_t(x) = \frac{ct}{x^2} e^{-ct/x}, \qquad x > 0$$

for some $c > 0$.

Not all these properties hold in the general case, as some later examples will show. Some weaker properties do hold in general, however, as we now demonstrate. Recall that in the remainder of this paper proper self-affine extremal processes M satisfy (8.1), and self-affine random subsets of and self-affine point processes on $\mathbb{R} \times \mathbb{J}$ satisfy (8.2). We write

$$X := d^{\vee} M$$

Theorem 9.1. *If M is a proper self-affine extremal process, then $\{t: X(t)>0\}$ is* wp1 *a dense F_σ Lebesgue null set in $\mathbb{R}$.*

Proof. The event that $\{t: X(t)>0\}$ is dense is the same as that of $M(I)$ being positive for all $I \in \mathscr{I}$ with rational endpoints. Let I be a bounded open interval around 0. Then $M(aI) \downarrow d^{\vee} M(0) = X(0)$ as $a \downarrow 0$, and we find for $0 < x < \infty$

$$\begin{aligned} \mathbb{P}[X(0)>x] &= \lim_{a \downarrow 0} \mathbb{P}[M(aI)>x] \\ &= \lim_{a \downarrow 0} \mathbb{P}[M(I)>x/a] = \mathbb{P}[M(I)=\infty] = 0. \end{aligned}$$

So

$$\mathbb{P}[X(0)>0]=0. \tag{9.1}$$

Since X is measurable (Sect. 3) and stationary, we have by Fubini

$$\begin{aligned} 0 = \mathbb{P}[X(0)>0] &= \mathbb{P}[X(t)>0] \\ &= \int_{\mathbb{R}} \mathbb{P}[X(t)>0]\, dt = \mathbb{E}\,\mathrm{Leb}\{t \in \mathbb{R}: X(t)>0\}, \end{aligned}$$

so $\mathrm{Leb}\{t \in \mathbb{R}: X(t)>0\}=0$ wp1. Finally, $\{t: X(t)>0\} = \bigcup_{n=1}^{\infty} \{t: X(t) \geqq n^{-1}\}$ is an F_σ set, because X is usc. □

Corollary 9.2. *If M is a proper self-affine extremal process, then*

(a) $\lim_{\varepsilon \downarrow 0} M(-\varepsilon, \varepsilon) = \lim_{\varepsilon \downarrow 0} M(0, \varepsilon) = 0$ wp1, and

(b) $\mathrm{Leb}\{t>0: M(0, t-\varepsilon] \neq M(0, t+\varepsilon]$ for all $\varepsilon>0\}=0$ wp1.

Proof. (a) This follows from (9.1) and the fact that X is usc.

(b) Fix a sample point for which $\{t: X(t)>0\}$ is a Lebesgue null set. Recall that $M(0, t]>0$ wp1. Let $t>0$ be such that $0=X(t)<M(0, t]$. Since X is usc, $X(s)<M(0, t]$ for all s in some open neighborhood of t. Then $M(0, s]$ is constant for such s. □

Theorem 9.3. *If M is a proper self-affine extremal process, then the following hold true.*

(a) *For each* $t>0$, $M(0,t]$ *has a density* f_t *such that* $x^2 f_t(x)$ *is nondecreasing in* x. *Equivalently*, $1/M(0,t]$ *has a density* g_t *such that* $g_t(x)$ *is nonincreasing in* x. *Furthermore*,

$$(9.2)\qquad f_t(x)=\frac{1}{t}f_1\left(\frac{x}{t}\right).$$

(b) *Let* F *be the hypograph of* X *and let* $F_1:=\{t\in\mathbb{R}:(t,1)\in F\}$ *be its level set at height* 1. *Then*

$$(9.3)\qquad \lim_{x\to\infty} x^2 f_1(x)=\lim_{x\downarrow 0} g_1(x)=\mathbb{E}\#(F_1\cap(0,1)].$$

Proof. (a) We first show that $1/M(0,1]$ has a nonincreasing density on $(0,\infty)$. We have

$$\begin{aligned}\mathbb{P}[1/M(0,1]<x]&=\mathbb{P}[xM(0,1]>1]=\mathbb{P}[M(0,x]>1]\\&=\mathbb{P}[(0,x]\cap F_1\neq\emptyset].\end{aligned}$$

The random closed subset F_1 of $\mathbb{R}$ is stationary, i.e., $F_1+b=_d F_1$ for $b\in\mathbb{R}$, so we have for $0\leqq y<y+h<x$:

$$\begin{aligned}\mathbb{P}[1/M(0,1]\in[x,x+h)]&=\mathbb{P}[(0,x]\cap F_1=\emptyset,\quad (x,x+h]\cap F_1\neq\emptyset]\\&=\mathbb{P}[(y-x,y]\cap F_1=\emptyset,\quad (y,y+h]\cap F_1\neq\emptyset]\\&\leqq\mathbb{P}[(0,y]\cap F_1=\emptyset,\quad (y,y+h]\cap F_1\neq\emptyset]=\mathbb{P}[1/M(0,1]\in[y,y+h)].\end{aligned}$$

The second statement of (a) now follows by standard real analysis, the first by an obvious transformation, and self-similarity yields (9.2).

(b) The first identity in (9.3) is a standard transformation. For the second, set $I:=(0,1]$. Then, by stationarity of F_1 and Fubini,

$$\begin{aligned}(9.4)\qquad \lim_{x\downarrow 0} g_1(x)&=\lim_{x\downarrow 0} x^{-1}\,\mathbb{P}[1/M(I)<x]\\&=\lim_{x\downarrow 0} x^{-1}\,\mathbb{P}[xI\cap F_1\neq\emptyset]=\lim_{x\downarrow 0} x^{-1}\,\mathbb{P}[(t+xI)\cap F_1\neq\emptyset]\\&=\lim_{x\downarrow 0} x^{-1}\int_I \mathbb{P}[(t+xI)\cap F_1\neq\emptyset]\,dt\\&=\lim_{x\downarrow 0} x^{-1}\,\mathbb{E}\mathrm{Leb}((F_1-xI)\cap I).\end{aligned}$$

We have

$$(9.5)\qquad x^{-1}\,\mathrm{Leb}((F_1-xI)\cap I)\uparrow\#(F_1\cap I)\quad \text{as } x\downarrow 0,$$

even if $\#(F_1\cap I)=\infty$. To see this, note that for almost all sample points $\mathrm{Leb}\,F_1=0$ and $\min(F_1\cap I)>0$ by (9.1). The contribution to the left-hand side of (9.5) from each open connected component of the complement of F_1 in $(0,\max(F_1\cap I)]$ is increasing in x. Now (9.3) follows by (9.4), (9.5) and the monotone convergence theorem. □

Remarks 9.4. There is no extension of Theorem 9.3 to joint distributions of $M(I)$ for two intervals I. The example of the g-adic lattice process (O'Brien and Vervaat 1985, Ex. 3.2) makes it obvious that they need not be absolutely continuous. The same example shows that f_t need not be positive on all of $(0, \infty)$, but may vanish near 0. So in the case of H self-affine extremal processes with $H<0$, the distribution of $M(0, t]$ may have compact support.

Corresponding results for additively self-affine processes (= self-similar processes with stationary increments) are only a conjecture. O'Brien and Vervaat (1983) and Maejima (1986) obtain only very partial results in this direction. These have not been improved so far.

10 Subordination

In this section we restrict our attention to proper self-affine random usc functions X, i.e., self-affine random usc functions that are in addition finite and not identically zero wp1. Let U be the hypograph of another finite, nonnegative, not identically zero usc function, this time nonrandom. Then

$$F := \operatorname{clos} \bigcup_{(t,x)\in\Gamma(X)} ((t,0)+xU) \tag{10.1}$$

is a closed subset of $\mathbb{R}\times(0,\infty]$, which is equal to its umbra, and therefore it is the hypograph $\downarrow\Gamma(Y)$ of a random usc function Y. We shall see shortly that Y is also self-affine, but not necessarily proper, a major topic in this section. We say Y (and the extremal process $Y^\vee$) is *subordinated* to X ($X^\vee$) by U. It is instructive to consider the particular case that $\Gamma(X)$ is the support of a point process Π, for instance Poisson, and that U is the umbra of some finite cloud C of points around $(0, 1)$, so that $U=\downarrow C$. Then F is the umbra of the union of clouds around the atoms of the point process, where the clouds are not only shifted to these atoms but also expanded proportionally to their heights.

For a further analysis it is convenient to regard $\mathbb{R}\times(0,\infty)$ as a noncommutative group isomorphic to Aff via $(t, x)\mapsto x\cdot+t$, so

$$(t_1, x_1)(t_2, x_2)=(t_1+x_1t_2, x_1x_2)$$

with unit element $(0, 1)$ and inverse $(t, x)^{-1}=(-tx^{-1}, x^{-1})$. We extend the operations of multiplication and inversion to subsets of $\mathbb{R}\times(0,\infty)$:

$$FG:=\{z_1z_2: z_1\in F, z_2\in G\}; \qquad F^{-1}:=\{z^{-1}: z\in F\}.$$

Note that $F\mapsto F^{-1}$ is not an inverse for this multiplication of sets. Nevertheless we have $HF\cap G=\emptyset$ iff $H\cap GF^{-1}=\emptyset$ (multiplication of sets has priority above set operations in this section). In this setting, random subsets F of $\mathbb{R}\times(0,\infty)$ are self-affine iff $F=_d zF$ for $z\in\mathbb{R}\times(0,\infty)$ (cf. lines around (8.2)), so a random usc function X is self-affine iff $\Gamma(X)=_d z\Gamma(X)$ or $\downarrow\Gamma(X)=_d z\cdot\downarrow\Gamma(X)$ for $z\in\mathbb{R}\times(0,\infty)$. Formula (10.1) now reads

$$F=\operatorname{clos}(\Gamma(X)U). \tag{10.2}$$

It is now obvious that F is self-affine if $\Gamma(X)$ is.

Subordination is a convenient way of producing new examples, but there is some risk that the result is unintentionally trivial. Specifically, if $\Gamma(X)U$ is dense in $\mathbb{R}\times(0,\infty]$, then $F=\mathbb{R}\times(0,\infty]$, so for Y with $\downarrow\Gamma(Y)=F$ we have $Y\equiv\infty$. We therefore look for conditions that guarantee Y to be proper wp1.

The following lemma gives the central criterion, which will be developed further in the most important special case. It involves the open rectangles

$$G_y:=(0,1)\times(y,\infty),\qquad y>0.$$

Note that for open G

$$GU^{-1}\cap\Gamma(X)=\emptyset\quad\text{iff}\quad G\cap\Gamma(Y)=\emptyset.$$

Lemma 10.1. *Let X be a self-affine random usc function which is proper* wp1, *and let U be the hypograph of a finite usc function. Then the random usc function Y subordinated to X by U is proper* wp1 *iff $U\neq\emptyset$ and*

$$\lim_{y\to\infty}\mathbb{P}[\Gamma(X)\cap G_y U^{-1}=\emptyset]=1. \tag{10.3}$$

Proof. Let $F:=\downarrow\Gamma(Y)$, so that (10.2) holds. By Theorem 7.1, Y is finite wp1 iff $\mathbb{P}[Y^\vee(0,1)<\infty]=1$. We have

$$[Y^\vee(0,1)<\infty]=\bigcup_{y>0}[F\cap G_y=\emptyset],$$

where the union applies to events that increase with y, so

$$\mathbb{P}[Y^\vee(0,1)<\infty]=\lim_{y\to\infty}\mathbb{P}[F\cap G_y=\emptyset].$$

In general we have for open sets G in $\mathbb{R}\times(0,\infty)$

$$\mathbb{P}[F\cap G=\emptyset]=\mathbb{P}[\Gamma(X)U\cap G=\emptyset]=\mathbb{P}[\Gamma(X)\cap GU^{-1}=\emptyset].$$

Combining the last two formulae we obtain

$$\mathbb{P}[Y^\vee(0,1)<\infty]=\lim_{y\to\infty}\mathbb{P}[\Gamma(X)\cap G_y U^{-1}=\emptyset],$$

and (10.3) follows. Finally, Y is wp1 not identically zero iff $U\neq\emptyset$. □

We now turn to the most common case. We write

$$\pi(dt,dx):=dt\,x^{-2}\,dx,$$

so that self-affine point processes with finite positive intensity have intensity $c\pi$ for some $c\in(0,\infty)$ (cf. (8.3)).

Assumption 10.2. $\Gamma(X)$ is the support of a self-affine point process Π (so $\Pi(A)=\#(\Gamma(X)\cap A)$) with finite positive intensity $c\pi$.

Assumption 10.3. In addition to Assumption 10.2, we have $\Pi(A)=\infty$ wp1 if $\pi(A)=\infty$ for Borel sets A in $\mathbb{R}\times(0,\infty)$.

We are able to recognize π as the left Haar measure of $\mathbb{R}\times(0,\infty)$ regarded as a group isomorphic to Aff, so $\pi(zA)=\pi(A)$ for $z\in\mathbb{R}\times(0,\infty)$ and $A\subset\mathbb{R}\times(0,\infty)$. Assumption 10.3 is satisfied in case Π is Poisson, but not for all self-affine point processes, as we shall see shortly (examples for which Assumption 10.3 fails are the g-adic lattice process and Ex. 3.5(f) in O'Brien and Vervaat 1985). Criterion (10.3) is simplified considerably in the following theorem.

Theorem 10.4. (a) *If*

$$\pi(G_y U^{-1}) < \infty \tag{10.4}$$

for some $y > 0$, *then* (10.4) *holds for all* $y > 0$.

(b) *Under Assumption 10.2 the equivalent statements in Lemma 10.1 are implied by* (10.4).

(c) *Under Assumption 10.3 the equivalent statements in Lemma 10.1 are equivalent to* (10.4).

Remark 10.5. If (10.4) holds, then there are wp1 finitely many atoms of Π that give rise to shifts of U that intersect G_y. So the operation of taking closure can be omitted in (10.1) and (10.2).

Proof of Theorem 10.4. (a) It is obvious that (10.4) for some $y > 0$ implies (10.4) for all larger y. Therefore it suffices to prove that (10.4) for some $y > 0$ implies (10.4) with y replaced by $y/2$. This follows from

$$\begin{aligned}\infty > \pi(G_y U^{-1}) &= \pi((0, \tfrac{1}{2}) G_y U^{-1}) = \pi\{((0, \tfrac{1}{2}) \times (\tfrac{1}{2} y, \infty)) U^{-1}\}\\ &= \tfrac{1}{2}(\pi\{((0, \tfrac{1}{2}) \times (\tfrac{1}{2} y, \infty)) U^{-1}\} + \pi\{((\tfrac{1}{2}, 1) \times (\tfrac{1}{2} y, \infty)) U^{-1}\})\\ &\geqq \tfrac{1}{2} \pi(G_{y/2} U^{-1}).\end{aligned}$$

(b) In Lemma 10.7 we will prove that $G_y U^{-1} \downarrow \emptyset$ as $y \to \infty$. If (10.4) holds, this implies that $\pi(G_y U^{-1}) \to 0$, i.e., that $\Pi(G_y U^{-1}) \to 0$ as $y \to \infty$ in expectation, and hence also in probability, which is (10.3).

(c) In presence of (b) it remains to prove that (10.3) implies (10.4). By (10.3), $\mathbb{P}[\Pi(G_y U^{-1}) < \infty] > 0$ for some y. This implies (10.4) for the same y by Assumption 10.3. □

Before discussing examples and filling the gap left in the proof of Theorem 10.4, let us calculate a set that will be needed a few times. Let U be a 'rectangle' in $\mathbb{R} \times (0, \infty)$, $U := C \times D$. Then, with $I := (0, 1)$,

$$U^{-1} = \bigcup_{y \in D} ((-y^{-1} C) \times \{y^{-1}\}) = \bigcup_{z \in D^{-1}} ((-z C) \times \{z\}); \tag{10.5}$$

$$\begin{aligned}G_y U^{-1} &= \bigcup_{u \in (y, \infty)} \bigcup_{z \in D^{-1}} ((I - uzC) \times \{uz\})\\ &= \bigcup_{z > y/\sup D} ((I - zC) \times \{z\}).\end{aligned}$$

Example 10.6 (to show that (10.4) is not necessary under Assumption 10.2 only). Let Π be the g-adic lattice process of Example 3.2 in O'Brien and Vervaat (1985) scaled such that the support at level x conditioned on having atoms at level x consists of a uniformly distributed translation of $x\mathbb{Z}$. Let $U := \mathbb{Z} \times (0, 1]$.

Then subordination by U leaves the g-adic lattice process unchanged. So the equivalent statements of Lemma 10.1 hold. For $n \in \mathbb{Z}$, set

$$L_n(y) := \{(t, x): nx < t < nx+1, x > y\}.$$

Then $G_y U^{-1} = \bigcup_n L_n(y)$ by (10.5), and $\pi(\bigcup_n L_n(y)) = \infty$ ($\pi(L_n(y)) = 1/y$, though).

We now fill the gap in the proof of Theorem 10.4.

Lemma 10.7. *If U is the hypograph of a finite usc function, then $G_y U^{-1} \downarrow \emptyset$ as $y \to \infty$.*

Proof. Let $J := (-\infty, -1] \cup [1, \infty)$ and $I := (0, 1)$. Let $(t, x) \in \mathbb{R} \times (0, \infty)$. By (10.5) we have

$$G_y(sJ \times (0, \infty))^{-1} = \bigcup_{z \in (0, \infty)} ((I + zsJ) \times \{z\}),$$

which does not contain (t, x) if s is large enough that $(xs-1)^+ > |t|$. For such an s, choose v such that

$$U \subset (sJ \times (0, \infty)) \cup (\mathbb{R} \times (0, v]). \tag{10.6}$$

Since $G_y(\mathbb{R} \times (0, v])^{-1} = \mathbb{R} \times (yv^{-1}, \infty)$ does not contain (t, x) for $y > tv$, the result follows from (10.6). □

We now investigate criterion (10.4) for rectangular U.

Theorem 10.8. *Let Assumption 10.2 hold, and let $U = C \times (0, x]$, where C is a closed subset of $\mathbb{R}$. Then* (10.4) *holds iff*

$$\int_0^1 \mathrm{Leb}(C - yI) \frac{dy}{y} < \infty, \tag{10.7}$$

where $I := (0, 1)$, iff

$$C \text{ is bounded}, \ \mathrm{Leb}\, C = 0, \quad \text{and} \quad \sum_{n=1}^{\infty} l_n |\log l_n| < \infty, \tag{10.8}$$

where (l_n) is an enumeration of the lengths of the disjoint open intervals whose union is $[\min C, \max C] \setminus C$.

Proof. By (10.5) and $I - xC = -x(C - x^{-1}I)$ we have

$$\pi(G_x U^{-1}) = \int_1^\infty x \,\mathrm{Leb}(C - x^{-1}I) \frac{dx}{x^2} = \int_0^1 \mathrm{Leb}(C - yI) \frac{dy}{y}.$$

We find that (10.4) is equivalent to (10.7). The same criterion occurs in Vervaat (1985, Lemma 4.3). By direct calculation or the lemma on p. 326 of Carleson (1952) it follows that (10.7) is equivalent to (10.8). □

The following corollary is based on the observation that for fixed X and varying U, Y depends in an increasing way on U.

Corollary 10.9. (a) *Under Assumption 10.2, let $\emptyset \neq U \subset C \times (0, x]$, C closed in $\mathbb{R}$, and let C satisfy* (10.7) *or* (10.8). *Then* (10.4) *holds and Y subordinated to X by U is proper* wp1.

(b) *Under Assumption 10.3, let $C \times (0, x] \subset U$, C closed in $\mathbb{R}$, and let C violate* (10.7) *or* (10.8). *Then,* wp1, *Y subordinated to X by U is not finite.*

Example 10.10. The ternary Cantor set C satisfies (10.8), so subordination by $C\times(0, 1]$ to an X as in Assumption 10.2 produces examples of proper random self-affine usc functions Y with uncountable graphs (cf. Sect. 4).

Remarks 10.11. (a) Obviously, (10.8) is satisfied in case $\# C<\infty$. For infinite C it does not matter whether C is countable or not. In both cases (10.8) may or may not hold. It is even possible to violate (10.8) for C consisting of a convergent sequence with its limit (aim at $l_n=n^{-1}(\log n)^{-2}$).

(b) We have assumed U to be the hypograph of a finite usc function. We are unable to prove or disprove our conjecture that U must be the hypograph of a *bounded* usc function for Y to be finite wp1.

(c) In case $\Gamma(X)$ is countable, it is possible to generalize subordination by one fixed nonrandom U to subordination by random U_n's, for instance independent and identically distributed. See Vervaat (1985) for details in an analogous situation.

11 Sup continuous self-affine extremal processes

In this section we show the existence of proper self-affine extremal processes M such that the function $(t, u)\mapsto M(t, u)$ on $\{(t, u)\in\mathbb{R}^2: t<u\}$ is continuous wp1. Recall the definition of sup continuity (Def. 4.1). The following hold.

Properties 11.1. (a) For $f\colon\mathbb{R}\to\mathbb{I}$ we have that f is sup continuous iff f is usc and $(t, u)\mapsto f^\vee(t, u)$ is continuous on $\{(t, u)\in\mathbb{R}^2: t<u\}$.

(b) The supremum or infimum of finitely many sup continuous functions is again sup continuous.

(c) If $f\colon\mathbb{R}\to\mathbb{I}$ is sup continuous, then the set $L:=\{f(t): a<t<b\}$ is connected for any $a<b$.

Proof. (a) and (b) are obvious. We prove (c). Let x and $y\in\{f(t): a<t<b\}$ and let z satisfy $x<z<y$. Suppose $x=f(u)$ and $y=f(v)$ where $u<v$, the other case being similar. Since f is sup continuous, $f(\inf\{t>u: f(t)>z\})=z$. □

Let $f\colon\mathbb{R}\to\mathbb{I}:=[0, \infty]$ be sup continuous, and suppose that $U:=\downarrow\Gamma(f)$ satisfies the condition of Corollary 10.9(a). Let Π be a self-affine point process satisfying Assumption 10.2. Then the random usc function subordinated to Π by U is a proper self-affine random usc function that is sup continuous wp1, by Properties 11.1 and Remark 10.5. In particular, the random functions $t\mapsto M(0, t]$ and $t\mapsto M(-t, 0]$ are continuous wp1.

So in order to show the existence of extremal processes as claimed in the beginning of this section, if suffices to exhibit a nonzero sup continuous function f such that $\downarrow\Gamma(f)\subset C\times(0, 1]$, where C is a closed set such that (10.8) holds. Since we want f to be sup continuous, we see from Property 11.1(c) that the support of f must be uncountable. Thus we are led to consider sets C of Cantor type.

Example 11.2. The support of f is contained in the Cantor-like set C of all x in [0, 1] of the form $\sum_{n=1}^{\infty} c_n 5^{-n}$ with $c_n=0$, 2 or 4. For each such x let K_x

denote the set $K_x := \{n \in \mathbb{N}: n \geqq 2, c_n \neq 2\}$. Let $f_n(x) := \prod_{k \in K_x: k \leqq n} \left(1 - \frac{1}{k}\right)$ for x in C. Then $f_n(x)$ decreases to $f(x) := \prod_{k \in K_x} \left(1 - \frac{1}{k}\right)$. The function f_n is locally constant, so continuous on C. Hence f_n and $f = \inf f_n$ are usc on $\mathbb{R}$ if we define $f_n(x) = 0$ off C.

In order to prove sup continuity, it suffices to show that for any x and any $\varepsilon > 0$ there exist $x_1 < x < x_2$ with $x_2 - x_1 < \varepsilon$ and $f(x_j) > f(x) - \varepsilon$ for $j = 1, 2$. We need to consider only the case that f is strictly positive at x. This implies that $\sum_{k \in K_x} 1/k$ converges, and hence that the expansion of x contains infinitely many 2's. Suppose $c_m = 2$, and define x_1 and x_2 by replacing this digit by 0 and 4 respectively. Then $x_2 - x_1 = 4 \cdot 5^{-m}$ and f decreases by a factor $\left(1 - \frac{1}{m}\right)$. Since f is bounded by 1, the condition above is satisfied if we choose m sufficiently large.

Remarks 11.3. (a) If the subordinated f is only right sup continuous, then $t \mapsto M(-t, 0]$ is continuous wp1, but not necessarily so $t \mapsto M(0, t]$.

(b) The subordinated f suggested at the end of Sect. 9 in Vervaat (1986) is too simple. It produces a sup continuous improper self-affine random usc function: the function $X \equiv \infty$ wp1.

(c) Let Π be a self-affine Poisson process and let $f(t) := 1 + t$ for $t = 0$ and $t = \pm n^{-1}$ $(n \geqq 2)$, $:= 0$ else. We can prove for the subordinated extremal process M that $t \mapsto M(0, t]$ has a positive derivative at some t wp1. Note that $\frac{d}{dt} M(0, t] = 0$ wp1 for each fixed t, by Corollary 9.2(b).

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