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Probability Theory and Related Fields

The probabilistic solution of the third boundary value problem for second order elliptic equations

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Summary. Using standard reflected Brownian motion (SRBM) and martingales we define (in the spirit of Stroock and Varadhan – see [S-V]) the probabilistic solution of the boundary value problem

$$\tfrac{1}{2}\Delta u+qu=0, \quad \text{in } D;$$

$$\frac{\partial u}{\partial n}+cu=-f, \quad \text{on } \partial D,$$

where D is a bounded domain with C^3 boundary and n is the inward unit normal vector on ∂D. The assumptions for q, c and f are quite general.

The corresponding Dirichlet problem was studied by Chung, Rao, Zhao and others (see [C-R1] and [Z-M]) and the corresponding Neumann by Pei Hsu in [H2]. Here we show that the probabilistic solution of our problem exists, is unique (unless we hit an eigenvalue), continuous on $\bar{D}$ and equivalent to the weak analytic solution. The method we use is to reduce the problem to an integral equation in D that involves the associated semigroup and, hence, to the study of the properties of this semigroup. In this way we do not have to assume that the spectrum is negative (almost every previous work on these probabilistic solutions makes this assumption). We construct the kernel of this semigroup and we prove certain estimates for it which help us to establish many other results, including the gauge theorem. We also show that, if the boundary function c is continuous, our semigroup is a uniform limit of Neumann semigroups and, furthermore, that the Dirichlet semigroup is a uniform limit of semigroups of our type. Therefore the Dirichlet spectrum is a "monotone" limit of spectra of mixed problems (see Sect. 5B), a fact which is mentioned without proof in Vol 1, Ch. IV, Sect. 2 of the *Methods of Mathematical Physics* by Courant and Hilbert. This establishes the interrelation of the three boundary value problems. Finally, we add a drift term to our differential equation, which becomes

$$\tfrac{1}{2}\Delta u+b\cdot\nabla u+qu=0$$

and we solve the third boundary value problem for this equation probabilistically, with the help of Girsanov's transformation.

1 Preliminaries

Let $D \subset R^d$ be a bounded domain (a domain is an open, connected and nonempty set) with C^3 boundary ∂D. A (continuous) path in $\bar{D}$ is a (continuous) function defined on $[0, \infty)$ and taking values in $\bar{D}$. Now let $B = B(t)$ be a continuous path in R^d with $B(0)$ in $\bar{D}$. We say that a pair (X, L) is a solution to the *problem of reflection (or Skorohod equation)*, if the following conditions are satisfied:

(i) $X = X(t)$ is a continuous path in $\bar{D}$;

(ii) $L = L(t)$ is a continuous nondecreasing function (with $(0)=0$) which increases only when $X(t)$ is on ∂D, namely

$$L(t) = \int_0^t 1_{\partial D}[X(s)]\, dL_s; \tag{1.1}$$

(iii) The following relation (Skorohod equation) holds

$$X(t) = B(t) + \int_0^t n[X(s)]\, dL_s, \tag{1.2}$$

where $n(x)$ is the *inward* unit normal vector of ∂D at x. It is known (see [S.Y], [L-S] or [H1]) that, under the above assumptions for D and B, the problem of reflection has a unique solution (X, L). Furthermore X has the same modulus of continuity as B, where the modulus of continuity of B is defined to be

$$\Delta_s(\sigma; B) = \sup\{|B(a) - B(b)| : a, b \in [0, s+\sigma], |b-a| \leqq \sigma\}$$

(for $x \in R^d$, we denote by $|x|$ its Euclidean norm).

If $B = (B_t, \mathscr{F}_t, P^x)$ is a (standard) Brownian motion (BM) in R^d starting at $x \in \bar{D}$ and (X, L) is the solution to the problem of reflection for B and D, then $X = \{X_t, t \geqq 0\}$ is a diffusion (i.e. a strong Markov process with continuous paths) living in $\bar{D}$ called the standard reflected Brownian motion (SRBM) in D starting at x and $L = \{L_t, t \geqq 0\}$ is an increasing process called the boundary local time of X. The transition densities $p(t, x, y)$ of X satisfy the following initial-boundary value (parabolic) problem (see [H1] or Theor. 6.1 later in this article)

$$\begin{aligned} &\frac{\partial}{\partial t} p(t,x,y) = \tfrac{1}{2}\Delta_x p(t,x,y), && (t,x,y) \in (0,\infty) \times D \times D; \\ &\lim_{t \downarrow 0} p(t,x,y) = \delta_y(x), && (x,y) \in \bar{D} \times \bar{D}; \\ &\frac{\partial}{\partial n_x} p(t,x,y) = 0, && (t,x,y) \in (0,\infty) \times \partial D \times \bar{D}. \end{aligned} \tag{1.3}$$

Here Δ_x is Δ acting on the x variables, $\partial/\partial n_x = n(x) \cdot \nabla_x$ where $\nabla_x = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$ and δ_y is the Dirac δ-function at y, so that the second equation above means that, for any $f \in C(\bar{D})$, $y \in \bar{D}$, we have

$$\lim_{t \downarrow 0} \int_{\bar{D}} f(x)\, p(t,x,y)\, dx = f(y).$$

It is shown in [I] that (under our assumptions for D) there exist a unique solution $p(t, x, y)$ of (1.3) defined on $(0, \infty) \times \bar{D} \times \bar{D}$ and having the following properties.

P1. $p(t, x, y)$ is strictly positive and symmetric in x and y, for all $t>0$ and x, y in $\bar{D}$.

P2. As a function of x, $p(t, x, y)$ belongs to $C^2(D) \cap C^1(\bar{D})$, for fixed $(t, y) \in (0, \infty) \times \bar{D}$. In fact, we have joint continuity (and differentiability) in (t, x, y), if $t>0$, by standard theorems regarding continuous dependence on the parameters (see [S.L]).

The continuity of $p(t, x, y)$ ond $\bar{D}$ implies that X has the strong Feller property, namely for any bounded Borel measurable function f defined on $\bar{D}$ and any fixed $t>0$, the function $x \mapsto E^x\{f(X_t)\}$ is continuous on $\bar{D}$.

P3. Given any positive constant t_0, there are constants $K=K(t_0)>0$ and $c=c(t_0)>0$ such that, if $0<t \leqq t_0$, then for all x in $\bar{D}$ we have

(1.4) $$p(t, x, y) \leqq K t^{-d/2} e^{-c|x-y|^2/t}.$$

For a proof see [H3].

If we denote by $\{P_t, t \geqq 0\}$ the transition semigroup of X, then the above statements imply that P_t has the Feller property (i.e. P_t is a continuous operator on $C(\bar{D})$ and converges strongly to the identity as $t \downarrow 0$). Its infinitesimal generator $\mathscr{A}$ is $\Delta/2$ with domain the closure of

$$D_0(\mathscr{A}) = \{f \in C^2(\bar{D}) \colon \partial f/\partial n = 0, \text{ on } \partial D\}$$

with respect to some natural Sobolev type norm. For more details see [H3] or [I-W].

P4. There are positive constants C and β such that, if $t \geqq t_1 > 0$, then

(1.5) $$\sup_{x, y \in \bar{D}} \left| p(t, x, y) - \frac{1}{m(D)} \right| \leqq C e^{-\beta t},$$

where $m(D)$ is the R^d-Lebesgue measure of D (this estimate means that $p(t, x, y)$ approaches the uniform distribution on D, exponentially fast, as $t \to \infty$). Also, the function

(1.6) $$p(t) = \sup_{x, y \in \bar{D}} p(t, x, y), \qquad t>0$$

in finite and nonincreasing (see [H1]).

If D is as above, then it can be shown by elementary differential geometry that there is a constant $\delta_0>0$ such that, for $0 \leqq \delta \leqq \delta_0$ the map

(1.7a) $$x \mapsto x + \delta n(x), \qquad x \in \partial D$$

is a one-to-one continuous map from ∂D onto the hypersurface

(1.7b) $$D(\delta) = \{x + \delta n(x) \colon x \in \partial D\}$$

with a continuous inverse (by compactness). We denote by D_δ, the domain bounded by ∂D and $D(\delta)$, namely

$$D_\delta = \{x \in D : d(x, \partial D) < \delta\}, \tag{1.8}$$

where $d(\cdot, \cdot)$ is the Euclidean distance in R^d. Then, *P3* and *P4* imply

P5. There are positive constants A, C and δ_0 such that, if $0 < \delta \leqq \delta_0$, then for all (t, x) in $(0, \infty) \times \bar{D}$ we have

$$\frac{1}{\delta} \int_{D_\delta} p(t, x, y)\, dy \leqq \frac{K}{\sqrt{t}} + C \tag{1.9}$$

and also

$$\int_{\partial D} p(t, x, y)\, \sigma(dy) \leqq \frac{K}{\sqrt{t}} + C, \tag{1.9'}$$

where $\sigma(dy)$ is the $(d-1)$-dimensional volume element on ∂D. In the case $D = (a, b)$ $(d=1)$ we agree that

$$\int_{\partial D} f(y)\, \sigma(dy) = f(a) + f(b).$$

We now give some properties of the process L. First we set

$$L^\delta(t) = \frac{1}{2\delta} \int_0^t I_{D_\delta}(X_s)\, ds. \tag{1.10}$$

Obviously, L^δ is a continuous additive functional (CAF).

P6. Let D, B, X and L be as previously defined. Then

(i) For any fixed $t \geqq 0$,

$$\lim_{\delta \downarrow 0} \sup_{x \in \bar{D}} E^x\{|L(t) - L^\delta(t)|^2\} = 0,$$

i.e. L^δ converges to L in $L^2(P^x)$, uniformly in x.

(ii) There is a set Ω_0 (independent of x) with $P^x(\Omega_0) = 1$ such that, if $\omega \in \Omega_0$, then

$$L(t, \omega) = \lim_{\delta \downarrow 0} L^\delta(t, \omega).$$

In both *(i)* and *(ii)* the convergence is uniform in t on compact subsets of $[0, \infty)$.

Remarks. (a) The fact that the L^2 convergence is uniform in x, is not mentioned explicitely in our reference [H1], but it clearly follows from the proof presented there (see also the Appendix).

(b) Part *(ii)* together with the fact that the convergence is uniform in t imply that L is a CAF of X (since L^δ is).

(c) In fact, the limit in *(i)* exists in a much stronger sense (see Appendix).

(d) Finally, let's discuss the factor 1/2 in (1.6). To understand its meaning consider the one dimensional Brownian motion B and reflect it at 0, so that we get the reflected process $X=|B|$. Then $D=(0,\infty)$, $D_\delta=(0,\delta)$ and so

$$\frac{I_{D_\delta}(X_s)}{2\delta}=\frac{I_{D_\delta}(|B_s|)}{2\delta}=\frac{I_{(-\delta,\delta)}(B_s)}{2\delta}.$$

Notice that 2δ is the length of the interval $(-\delta,\delta)$. Another argument, that we can say (for the general case $D\subset R^d$), is that the time that the reflected path spends in D_δ (for small δ) is approximately *twice* the time that the (locally) nonreflected path spends in it.

The rest of this section contains propositions that we need in the sequel but we were not able to find in the existing literature.

Let c be in $\mathscr{B}(\partial D)$, the Borel functions on ∂D. We assume that $|c|<\infty$ and we set

$$A_c(t)=\int_0^t c(X_s)\,dL_s, \tag{1.11}$$

where the integral is in the Lebesgue-Stieltjes sense and, in general, it may not be defined ($A_c=A_{c^+}-A_{c^-}$ so we may have $\infty-\infty$).

If c is in $C(\partial D)$, then it can be extended to a function $\bar{c}$ in $C(\bar{D})$, such that $\bar{c}=c$ on ∂D. So, without loss of generality, we may assume $c\in C(\bar{D})$. We define (in order to avoid long expressions)

$$A_c^\delta(t)=\int_0^t c(X_s)\,dL^\delta(s)=\frac{1}{2\delta}\int_0^t c(X_s)\,1_{D_\delta}(X_s)\,ds. \tag{1.12}$$

Proposition 1.1. *Let $c\in\mathscr{B}(\partial D)$ be nonnegative (in order to avoid the possibility $\infty-\infty$). Then*

$$E^x\{A_c(t)\}=\frac{1}{2}\int_0^t\int_{\partial D} p(s,x,y)\,c(y)\,\sigma(dy)\,ds. \tag{1.13}$$

Proof. Assume first that c is continuous on ∂D. By $P6$ we have that $L^\delta(t)\to L(t)$ for P^x-a.e. ω, as $\delta\downarrow 0$. Now, $c(X_t)$ is continuous for a.e. ω and so, by "vague convergence" (a standard 3ε argument: for $s\in[0,t]$, we approximate $c(X_s)$ by step functions, etc.),

(1.14) $\quad A_c^\delta(t)\to A_c(t)$, $\quad$ for a.e. ω (uniformly on bounded intervals of t).

Since $L^\delta(t)\to L(t)$ in L^2 (by $P6$) and so in L^1, by (1.12) and an extended dominated convergence theorem (see [R], Ch. 4, Theorem 16), we get

$$E^x\{A_c(t)\}=\lim_{\delta\downarrow 0}E^x\{A_c^\delta(t)\}.$$

But,

$$E^x\{A_c^\delta(t)\}=E^x\left\{\frac{1}{2\delta}\int_0^t c(X_s)\,I_{D_\delta}(X_s)\,ds\right\}=\frac{1}{2}\int_0^t\frac{1}{\delta}\int_{D_\delta} p(s,x,y)\,c(y)\,dy\,ds.$$

Letting $\delta\downarrow 0$ we obtain (1.13).

The general case follows by standard dominated and monotone convergence arguments (first we take c to be bounded, etc.). □

Remark. The above proposition together with $P2$ imply that if $A\subset\partial D$ such that $\sigma(A)=0$, then

$$E^x\left\{\int_0^t 1_A(X_s)\,dL_s\right\}=0.$$

(In the Appendix we compute the higher moments $E^x\{A_c(t)^n\}$.)

The next proposition can be considered as an attempt to extend André's Reflection Principle (which is a property of the one dimensional BM – see [C1], Sect. 4.2, Exer. 12) to higher dimensions.

Proposition 1.2. *Let $t>0$ be fixed (and D as usual). We set*

$$M(t)=\sup_{0\leqq s\leqq t} d(B_s,\bar{D}), \tag{1.15}$$

where B is the BM in R^d. Then, given any $\alpha>0$, there is an $\varepsilon>0$ such that

$$\inf_{x\in\partial D} P^z\{M(t)>\varepsilon\}>1-\alpha. \tag{1.16}$$

Proof. Let

$$E_n=\left\{x\in R^d: d(x,\bar{D})<\frac{1}{n}\right\}.$$

By elementary differential geometry we know that, since ∂D is C^3, there is some n_0 such that, for $n\geqq n_0$, we have that ∂E_n is smooth (and so E_n is regular). From now on we assume $n\geqq n_0$. The exit times of the E_n's are defined by

$$\tau_n=\inf\{t>0: B_t\in E_n^c\}.$$

Now let

$$f_n(x)=P^x\{\tau_n<t\}.$$

Each f_n is continuous on R^d (see [C2]) and equals to 1 on E_n^c. Also, $f_{n+1}(x)\geqq f_n(x)$.

Furthermore, the continuity of the paths and the regularity of D (and of D^c) imply that for all x in $\bar{D}$ we have

$$\tau_n\downarrow\tau_{\bar{D}}=\tau_D,\qquad P^x\text{-a.s.},$$

hence

$$f_n(x)\uparrow P^x\{\tau_D<t\},\quad \text{as } n\to\infty.$$

Now $P^x\{\tau_D<t\}$ is continuous on R^d (as a function of x, of course) and so the convergence in the above formula is uniform on $\bar{D}$ (in fact on R^d), by Dini's

Theorem. In particular, we have uniform convergence on ∂D. But $z \in \partial D$ implies that $P^z\{\tau_D < t\} = 1$, because D is regular. Therefore, by taking n sufficiently large, we can make

$$\inf_{x \in \partial D} P^z\{\tau_n < t\} > 1 - \alpha.$$

From this, our proposition follows immediately (just take $\varepsilon < 1/n$}. □

Next, using Proposition 1.2 we prove a useful estimate regarding the boundary local time L. This estimate is needed for the analysis done in Sect. 5.

Theorem 1.3. *Let $t > 0$ be fixed. Then, given any $\alpha > 0$, there is an $\varepsilon > 0$ such that*

$$\inf_{z \in \partial D} P^z\{L_t > \varepsilon\} > 1 - \alpha. \tag{1.17}$$

Proof. From (1.2) we get that

$$L_s = \int_0^s dL_u \geqq \left| \int_0^s n(X_u)\, dL_u \right| = |X_s - B_s| \geqq d(B_s, \bar{D}).$$

The rest follows from the monotonicity of L_t and the previous proposition. □

Remark. An immediate corollary of Theorem 1.3 is that $\inf_{z \in \partial D} P^z\{L_t > 0\} = 1$. Therefore ($P^x$-a.s.)

$$\tau_D \overset{\text{def}}{=} \inf\{t > 0 : B_t \in D^c\} = \inf\{t > 0 : X_t \in \partial D\} = \inf\{t > 0 : L_t > 0\}. \tag{1.18}$$

2 The classes $K_d(D)$ and $\Sigma_d(\partial D)$

Here we describe two classes of functions which are suitable for our boundary value problem. The first is a well known class but the second is introduced here for the first time.

In what follows, all the functions are assumed to be Borel measurable in their domains of definition. For $A \subset R^d$, we denote by $b\mathscr{B}(A)$ the class of the (real- or complex-valued, depending on the context) bounded Borel functions on A.

Definition I. Let q be a real-valued (or complex-valued) function, defined on R^d. We say that q is in the class K_d (the *Kato-Stummel* class) if

$$\lim_{\alpha \downarrow 0} \sup_{x \in R^d} \int_{|x-y| \leqq \alpha} G_d(x, y) |q(y)|\, dy = 0, \tag{2.1}$$

where

$$G_d(x, y) = \begin{cases} |x-y|, & \text{if } d = 1; \\ -\ln|x-y|, & \text{if } d = 2; \\ |x-y|^{2-d}, & \text{if } d \geqq 3. \end{cases}$$

Modulo a constant factor, G_d is the potential kernel for R^d (G_3 is the standard Newtonian potential and G_2 is the so called logarithmic potential).

If $qI_D \in K_d$ we say that $q \in K_d(D)$ (thus, $K_d(R^d) = K_d$). In this case q need not be defined outside the set D.

Remarks. (a) $K_d(D)$ is a vector space and if $q \in K_d(D)$, then $|q|$, q^+ and q^- are in $K_d(D)$ (recall that: $q^+ = q \vee 0$, $q^- = q \wedge 0$).

(b) $b\mathscr{B}(D) \subset K_d(D)$ for any $D \subset R^d$. If D is bounded, $d > 1$ and $p > d/2$, then $L^p(D) \subset K_d(D) \subset L^1(D)$. For the one-dimensional case we have $K_1(a, b) = L^1(a, b)$, if a and b are finite (easy).

(c) There are several equivalent definitions for the class K_d. Some are given below (also, see [A-S], [C-Z], [H1], [S.B]).

Proposition 2.1. *Let D be a bounded domain. We have that $q \in K_d(D)$ if and only if*

$$x \mapsto \int_D G_d(x, y)|q(y)|\,dy \quad \text{is continuous (on } R^d). \tag{2.2}$$

For a proof see [C-Z].

Remarks. (a) Changing q in (2.2), first to q^+, then to q^-, and subtracting, we get that, given $q \in K_d(D)$, the function that we obtain by replacing $|q|$ by q in (2.2), is still continuous. This remark applies also to Proposition 2.4 below.

(b) If $q \in b\mathscr{B}(D)$, then the function in (2.2) is in $C^{1,\alpha}(R^d)$ for all $\alpha < 1$. On the other hand, if $q \in C^\alpha(D)$ for some $\alpha > 0$, then the function in (2.2) is in $C^2(R^d)$. The verifications are easy (see [G-T]).

The following theorems relate K_d with the Brownian motion (BM) in R^d and with the SRBM in a domain D.

Theorem 2.2. *Let B be a BM in R^d. Then $q \in K_d$ if and only if*

$$\lim_{t \downarrow 0} \sup_{x \in R^d} E^x \left\{ \int_0^t |q(B_s)|\,ds \right\} = 0. \tag{2.3}$$

For a proof see [C-Z]. Here we see why we need q to be Borel measurable, namely to guarantee that $q(B_t)$ is measurable.

Theorem 2.3. *Let $D \subset R^d$ be a bounded domain with C^3 boundary and X the SRBM in D. Then $q \in K_d(D)$ if and only if*

$$\lim_{t \downarrow 0} \sup_{x \in \bar{D}} E^x \left\{ \int_0^t |q(X_s)|\,ds \right\} = 0. \tag{2.4}$$

For a proof see [C-H]. Notice that the "only if" part follows from *P3* of Sect. 1.

Remark. Equation (2.4) together with the additivity imply that

$$\sup_{x \in \bar{D}} E^x \left\{ \int_0^t |q(X_s)|\,ds \right\} < \infty, \tag{2.5}$$

for any $t \geqq 0$.

The following proposition seems missing from the existing literature. (We don't really claim something new here. The proposition must be an easy consequence of the properties of the semigroup for the Neumann problem, given in [H 3].)

Proposition 2.4. *Same assumptions as in the previous theorem. Then,* $q \in K_d(D)$ *if and only if, for some* $t>0$, *the function*

$$F(x)=F_q(t;x)=E^x\left\{\int_0^t |q(X_s)|\, ds\right\} \tag{2.6}$$

is continuous on $\bar{D}$.

Proof. First assume $q \in K_d(D)$. Then, for any fixed t, $F_q(t;x)$ is bounded on $\bar{D}$, because of the previous remark. Now, for $0 \leqq r \leqq t$ we have

$$\begin{aligned} F_q(t;x) &= F_q(r;x)+E^x\left\{\int_r^t |q(X_s)|\, ds\right\} \\ &= F_q(r;x)+E^x\left\{\left[\int_0^{t-r} |q(X_s)|\, ds\right]\circ\theta_r\right\} \\ &= F_q(r;x)+E^x\{F_q(t-r;X_r)\} \\ &= F_q(r;x)+[P_r F_q(t-r;\cdot)](x), \end{aligned} \tag{2.7}$$

where P_t is the transition semigroup of X. Since X has the strong Feller property (see Sect. 1) and $F_q(t;\cdot)$ is bounded on $\bar{D}$ (by the previous remark), we get that the second term in the right-hand side of (2.7) is continuous on $\bar{D}$. So, if we let $r \downarrow 0$, since $F_q(r;x)$ approaches 0 uniformly on $\bar{D}$ by (2.4), we get that $F_q(t;x)$ is continuous on $\bar{D}$, for any fixed $t>0$.

Conversely, assume that, for some fixed $t>0$, $F_q(t;x)$ is continuous on $\bar{D}$. Observe that, if $0 \leqq r \leqq t$, then $F_q(t-r;x)$ is bounded by $F_q(t;x)$ and so (2.7) implies that $F_q(r;x)$ is continuous on $\bar{D}$. Next, define

$$q_n = |q| \wedge n, \qquad \tilde{q}_n = |q| - q_n.$$

We will simplify our notation, a little, by writing $F(t;x)$, $F_n(t;x)$ and $\tilde{F}_n(t;x)$, instead of $F_q(t;x)$, $F_{q_n}(t;x)$ and $F_{\tilde{q}_n}(t;x)$ respectively.

Now, q_n is bounded and so $q_n \in K_d(D)$. Thus, $F_n(t;x)$ is continuous on $\bar{D}$, for any fixed t, by the first part of this proof. Moreover,

$$F(r;x)=F_n(r;x)+\tilde{F}_n(r;x) \tag{2.8}$$

and so $\tilde{F}_n(r;x)$ is continuous on $\bar{D}$, for any fixed $r \in [0,t]$.

Finally, since $q_n \in K_d(D)$, Theorem 2.3 and (2.8) give

$$\limsup_{r \downarrow 0}\, \sup_{x \in \bar{D}} F(r;x) \leqq \sup_{x \in \bar{D}} \tilde{F}_n(r;x). \tag{2.9}$$

But, $\tilde{q}_n \downarrow 0$ a.s. on $\bar{D}$, so

$$\tilde{F}_n(r;x) = E^x\left\{\int_0^r \tilde{q}_n(X_s)\,ds\right\}$$

decreases to 0, as $n \to \infty$, for each $x \in \bar{D}$, by dominated convergence (notice that the occupation time of a set of R^d-measure zero is zero a.s.). In fact, it decreases uniformly in x by Dini's theorem. So, the right-hand side of (2.9) can be made as small (positive) as we wish, by taking n sufficiently large, and so, we are done (by Theorem 2.3). □

Definition II. Let X be the SRBM in $D \subset R^d$, where $d \geqq 2$, and c be a Borel function on ∂D. We say that c is in the class $\Sigma_d(\partial D)$, if

$$\limsup_{t \downarrow 0}\ \sup_{x \in \bar{D}} E^x\left\{\int_0^t |c(X_s)|\,dL_s\right\} = 0, \tag{2.10}$$

where L is (as usual) the boundary local time of X and the integral inside the expectation is in the Lebesgue-Stieltjes sense. (Notice that $c(X_s)$ is Borel measurable a.s. as a function of s by the continuity of the paths and the fact that every Borel subset of ∂D is also Borel in R^d.)

Remarks. (a) $b\mathscr{B}(\partial D) \subset \Sigma_d(\partial D) \subset L^1(\partial D)$ by Proposition 1.1 and $P2$, $P4$, $P5$ of Sect. 1.

(b) If $c \in \Sigma_d(\partial D)$, then

$$\sup_{x \in \bar{D}} E^x\left\{\int_0^t |c(X_s)|\,dL_s\right\} < \infty, \tag{2.11}$$

for any fixed $t \geqq 0$.

Proposition 2.5. *Let $d \geqq 2$. Then, $c \in \Sigma_d(\partial D)$ if and only if the function*

$$\hat{F}(x) = \hat{F}_c(t;x) = E^x\left\{\int_0^t |c(X_s)|\,dL_s\right\}, \tag{2.12}$$

is continuous on $\bar{D}$ for some fixed $t > 0$.

Proof. We can just imitate the proof of Proposition 2.4. □

Remark. As in Proposition 2.4, if there is a $t > 0$ such that $\hat{F}_c(t;\cdot)$ is continuous on $\bar{D}$, then the same is true for any $t \geqq 0$.

It would be nice to have an analytic characterization of $\Sigma_d(\partial D)$. A sufficient (analytic) condition for c to be in $\Sigma_d(\partial D)$ is given below. We don't know if this condition is also necessary.

Proposition 2.6. *Assume that c satisfies*

$$\limsup_{\alpha \downarrow 0}\ \sup_{x \in \bar{D}} \int_{\partial D \cap B(x;\alpha)} G_d(x,y)\,|c(y)|\,\sigma(dy) = 0, \tag{2.13}$$

where $G_d(x,y)$ is the potential kernel for R^d (see Def. I). Then $c \in \Sigma_d(\partial D)$.

Proof. Proposition 1.1 gives

$$E^x\left\{\int_0^t |c(X_s)|\,dL_s\right\}=\frac{1}{2}\int_0^t\int_{\partial D} p(s,x,y)\,|c(y)|\,\sigma(dy)\,ds.$$

We choose some $\alpha>0$ and then we split the above integral over ∂D into two integrals I_1 and I_2, the first over $\partial D\cap B(x;\alpha)$ and the second over $\partial D\cap B(x;\alpha)^c$. As $t\downarrow 0$, $I_2\to 0$ uniformly in x, because of $P3$ of Sect. 1 and the fact that (2.13) implies that $c\in L^1(\partial D)$. Now, by $P3$ again there are constants $K, b>0$ such that, for all t sufficiently small,

$$I_1\leqq K\int_0^t\int_{\partial D\cap B(x;\alpha)} s^{-d/2}e^{-b|x-y|^2/s}|c(y)|\,\sigma(dy)\,ds.$$

The rest follows by elementary calculus (reverse the order of integration, substitute $u=b|x-y|^2 s^{-1}$ and use (2.13). □

From now on, q is taken in $K_d(D)$ and c in $\Sigma_d(\partial D)$.

3 Semigroup and gauge

The main purpose of this work is the study of the third (or mixed or Robin) boundary value problem for the time-independent Schrödinger equation, using probabilistic methods. In mathematical terminology, we want to obtain a probabilistic expression for the (*weak*) solution u of the following boundary value elliptic problem:

(3.1)
$$\frac{1}{2}\Delta u+qu=0,\quad \text{in } D;\qquad \frac{\partial u}{\partial n}+cu=-f,\quad \text{on } \partial D.$$

If $D=(a,b)\subset R^1$, then $\partial u/\partial n$ is $u'(a)$ at a and $-u'(b)$ at b.

In this section we investigate the properties of the semigroup associated with the above problem. The main result is Theorem 3.4. Then we define the so-called *gauge* for (3.1) and we prove the theorem about its finiteness (Theorem 3.6).

In the probabilistic treatment of problems like (3.1), there is a famous functional that plays a dominant role. It is, traditionally, called the *Feynman-Kac functional* and is defined as follows:

(3.2)
$$e_q(t)=\exp\left[\int_0^t q(X_s)\,ds\right].$$

For the mixed problem, we need a second functional, in addition to the above, that will play the role of $e_q(t)$ on the boundary. So, for $c\in\Sigma_d(\partial D)$, we define

(3.3)
$$\hat{e}_c(t)=\exp\left[\int_0^t c(X_s)\,dL_s\right],\qquad \text{if } d\geqq 2.$$

Notice that $\hat{e}_c(t)=e^{A_c(t)}$.

In the case $d=1$, D must be a finite open interval of R^1, say $D=(a,b)$. Then $\partial D=\{a,b\}$, so c is defined for two points only. Let's put $c(a)=c_1$ and $c(b)=c_2$. Also, let $L_t(a)$ and $L_t(b)$ be the local time of X at a and b respectively. Then (3.3) becomes

$$\hat{e}_c(t)=\exp[c_1\, L_t(a)+c_2\, L_t(b)]. \tag{3.3'}$$

The same applies to $A_c(t)$ of (1.11) of Sect. 1, i.e. $A_c(t)=c_1\, L_t(a)+c_2\, L_t(b)$.

Proposition 3.1. *Under the previous assumptions we have that, for any fixed* $t\geqq 0$,

$$F(x)=E^x\{e_q(t)\,\hat{e}_c(t)\}$$

is continuous on $\bar{D}$.

Proof. For $t>0$ sufficiently small the finiteness follows from Theorem 2.3, Proposition 2.5 and Khas'minskii's Lemma (see Appendix). The continuity is a consequence of the strong Feller property of X. The rest follows from the multiplicativity (see the definition below). □

Definition. Let M be a (right continuous) functional of a Markov process Y. We say that M is a *multiplicative functional of* Y if $M_0=1$ a.s. and for $0\leqq s\leqq t$ we have

$$M_t=M_s(M_{t-s}\circ\theta_s), \quad \text{a.s.}$$

Remarks. (a) If M and N are multiplicative functionals, then so is MN.

(b) If A is an additive functional, then e^A is a multiplicative functional.

(c) If M is a multiplicative functional of Y, then the operators

$$T_t(f)(y)=E^y\{M_t\, f(Y_t)\}, \qquad t\geqq 0,$$

from (at least formally) a semigroup.

So, $e_q(t)$, $\hat{e}_c(t)$ and $e_q(t)\,\hat{e}_c(t)$ are multiplicative functionals of X and we can define a semigroup by

$$(T_t f)(x)=E^x\{e_q(t)\,\hat{e}_c(t)\, f(X_t)\}. \tag{3.4}$$

As we will see, the above semigroup is the basis for the study of the boundary problem (3.1). We could call it the *Feynman-Kac semigroup associated to the third (or mixed) problem.* There are similar semigroups for the Dirichlet and the Neumann problem (see [C-Z] and [H2] respectively). To examine the properties of this semigroup, we construct its kernels $k(t,x,y)$ by using a standard method, similar to the one used by P. Hsu in [H2]. (For $c\in C(\partial D)$, the properties of this semigroup can be derived from the corresponding ones of the Neumann semigroup. This is demonstrated in Sect. 5).

For $n=0$ we define

$$k_0(t,x,y)=p(t,x,y) \tag{3.5a}$$

and for $n=1, 2, 3, \ldots$

$$(3.5\text{b}) \qquad k_n(t,x,y)=\int_0^t\int_D p(s,x,z)\,q(z)\,k_{n-1}(t-s,z,y)\,dz\,ds + \frac{1}{2}\int_0^t\int_{\partial D} p(s,x,z)\,c(z)\,k_{n-1}(t-s,z,y)\,\sigma(dz)\,ds.$$

Also, we set

$$(3.6) \qquad M(t)=\sup_{x\in\bar{D}} E^x\left\{\int_0^t |q(X_s)|\,ds\right\}+\sup_{x\in\bar{D}} E^x\left\{\int_0^t |c(X_s)|\,dL_s\right\}.$$

Observe that, if $q\in K_d(D)$ and $c\in\Sigma_d(\partial D)$, then $M(t)$ is finite for all t, increases with t and $\lim_{t\downarrow 0} M(t)=0$.

Theorem 3.2. *Assume that $q\in K_d(D)$ and $c\in\Sigma_d(\partial D)$. Then $k_n(t,x,y)$ is continuous on $(0,\infty)\times\bar{D}\times\bar{D}$ and symmetric in x and y. Furthermore, there is a constant $A>0$ such that*

$$(3.7) \qquad |k_n(t,x,y)|\leqq A^{n+1}\,t^{-d/2}\,M(t)^n.$$

Proof. Throughout the proof we assume, without loss of generality that $q\geqq 0$ and $c\geqq 0$.

The symmetry of $k_n(t,x,y)$ in x and y follows by expanding it in terms of $p(t,x,y)$ using (3.5b) repeatedly and the fact that $p(t,x,y)$ is symmetric in x and y by *P1* of Sect. 1.

To establish (3.7) we need the following inequality (for all $n\geqq 0$, $y\in\bar{D}$):

$$(3.8) \qquad \int_0^t\int_D q(x)\,k_n(s,x,y)\,dx\,ds+\frac{1}{2}\int_0^t\int_{\partial D} c(x)\,k_n(s,x,y)\,\sigma(dx)\,ds\leqq M(t)^{n+1}.$$

For the proof of (3.8) we use induction. If $n=0$, then (3.8) is true by the definition of $M(t)$. Also, by (3.5b) and the symmetry of k_n, we can write the left-hand side of (3.8) as

$$\begin{aligned}
&\int_0^t\int_D q(x)\left[\int_0^s\int_D p(r,y,z)\,q(z)\,k_{n-1}(s-r,z,x)\,dz\,dr\right]dx\,ds\\
&+\frac{1}{2}\int_0^t\int_D q(x)\left[\int_0^s\int_{\partial D} p(r,y,z)\,c(z)\,k_{n-1}(s-r,z,x)\,\sigma(dz)\,dr\right]dx\,ds\\
&+\frac{1}{2}\int_0^t\int_{\partial D} c(x)\left[\int_0^s\int_D p(r,y,z)\,q(z)\,k_{n-1}(s-r,z,x)\,dz\,dr\right]\sigma(dx)\,ds\\
&+\frac{1}{4}\int_0^t\int_{\partial D} c(x)\left[\int_0^s\int_{\partial D} p(r,y,z)\,c(z)\,k_{n-1}(s-r,z,x)\,\sigma(dz)\,dr\right]\sigma(dx)\,ds.
\end{aligned}$$

Now we change the order of integration. We have to be careful only when we interchange the integrals with respect to r and s. After the interchange we make the substitution $u=s-r$ in the integral with respect to s. Then, the above expression becomes

$$\int_0^t \int_D p(r,y,z)\,q(z)\,A_{n-1}(t-r,z)\,dz\,dr+\frac{1}{2}\int_0^t \int_{\partial D} p(r,y,z)\,c(z)\,A_{n-1}(t-r,z)\,\sigma(dz)\,dr,$$

where

$$A_{n-1}(t-r,z)=\int_0^{t-r} \int_D q(x)\,k_{n-1}(u,z,x)\,dx\,du+\frac{1}{2}\int_0^{t-r} \int_{\partial D} c(x)\,k_{n-1}(u,z,x)\,\sigma(dx)\,du.$$

But $A_{n-1}(t-r,z)$ is less than $M(t)^n$ by the inductive hypothesis, the symmetry of k_n and the monotonicity of $M(t)$. Then (3.8) follows immediately.

Coming back to the proof of (3.7), we first observe that it is true for $n=0$ (by $P3$ of Sect. 1) and then we use (3.5b) to express k_n in terms of k_{n-1}. Then we split the integrals with respect to s into two parts: from 0 to $t/2$ and from $t/2$ to t. Using the induction hypothesis we obtain

$$k_n(t,x,y)\leqq 2^{d/2}A^n t^{-d/2}M(t)^{n+1}+\int_{t/2}^t \int_D p(s,x,z)\,q(z)\,k_{n-1}(t-s,z,y)\,dz\,ds$$
$$+\frac{1}{2}\int_{t/2}^t \int_{\partial D} p(s,x,z)\,c(z)\,k_{n-1}(t-s,z,y)\,\sigma(dz)\,ds$$

and so, by $P3$ of Sect. 1 and (3.8)

$$k_n(t,x,y)\leqq 2^{d/2}A^n t^{-d/2}M(t)^{n+1}+K t^{-d/2}2^{d/2}M(t)^n$$
$$\leqq 2^{d/2}(A^n+K')\,t^{-d/2}M(t)^{n+1},$$

where K and K' are constants independent of n. Therefore (3.7) is established by choosing $A\geqq 2^{d/2}(1+K')$.

Finally, we show the continuity of k_n. For $n=0$ the statement is true by $P2$ of Sect. 1. Assume that k_{n-1} is continuous on $(0,\infty)\times\bar{D}\times\bar{D}$ and split the integrals with respect to s in (3.5b) into three parts: from 0 to ε, from ε to $t-\varepsilon$ and from $t-\varepsilon$ to t. The integrals from ε to $t-\varepsilon$ are continuous by the induction hypothesis. The integrals from 0 to ε tend to 0 uniformly as $\varepsilon\downarrow 0$ since they are bounded by $M(\varepsilon)\,k_{n-1}(t-\varepsilon,x,y)$. Likewise, the integrals from $t-\varepsilon$ to t tend to 0 with ε. To show that we substitute s for $t-s$ in the integrals with respect to s and then we use (3.8) and the estimate for $p(t,x,y)$ given in $P3$ of Sect. 1. □

Remark. The above properties of k_n allow us to write (3.5b) in the form (see Proposition 1.1)

$$(3.9)\quad k_n(t,x,y)=E^x\left\{\int_0^t k_{n-1}(t-s,X_s,y)\,q(X_s)\,ds+\int_0^t k_{n-1}(t-s,X_s,y)\,c(X_s)\,dL_s\right\}.$$

Inequality (3.7) has an interesting consequence:

Corollary 3.3. *(Same assumptions and notation as in Theorem 3.2.) There is a $t_0>0$ such that the series*

$$\sum_{n=0}^{\infty} k_n(t,x,y)$$

converges absolutely and uniformly on any compact subset of $(0,t_0]\times\bar D\times\bar D$.

We define

$$(3.10)\qquad k(t,x,y)=\sum_{n=0}^{\infty} k_n(t,x,y).$$

Thus k is continuous on $(0,t_0]\times\bar D\times\bar D$ and $k=O(t^{-d/2})$ as $t\downarrow 0$. Of course, we expect $k(t,x,y)$ to be the kernels of our semigroup and this can be justified in the following way:

Let $g\in L^1(D)$. We set

$$K_n(t,x;g)=\int_D k_n(t,x,y)\,g(y)\,dy.$$

By the formulas (3.5) we get

$$K_0(t,x;g)=E^x\{g(X_t)\}$$

and

$$K_n(t,x;g)=E^x\left\{\int_0^t q(X_s)\,K_{n-1}(t-s,X_s;g)\,ds+\int_0^t c(X_s)\,K_{n-1}(t-s,X_s;g)\,dL_s\right\}.$$

Then the Markov property gives

$$K_n(t,x;g)=\frac{1}{n!}E^x\left\{\left[\int_0^t q(X_s)\,ds+\int_0^t c(X_s)\,dL_s\right]^n g(X_t)\right\}$$

and so, for $0<t\leqq t_0$

$$(3.11)\qquad \int_D k(t,x,y)\,g(y)\,dy=\sum_{n=0}^{\infty} K_n(t,x;g)=(T_t\,g)(x),$$

where

$$(3.12)\qquad (T_t\,g)(x)=E^x\{e_q(t)\,\hat e_c(t)\,g(X_t)\}.$$

It follows (by Chapman-Kolmogorov) that $k(t,x,y)$ is continuous on $(0,\infty) \times \bar{D} \times \bar{D}$. Furthermore, $k(t,x,y)$ is nonnegative and $k(t,x,y) \to \delta_y(x)$ as $t \downarrow 0$. A probabilistic representation of $k(t,x,y)$ is

$$k(t,x,y) = E^x\{e_q(t)\,\hat{e}_c(t) | X_t = y\}\, p(t,x,y).$$

Therefore we have shown the following (main) theorem regarding the semigroup $\{T_t, t \geqq 0\}$.

Theorem 3.4. *For each $t>0$, T_t is a bounded (linear) operator that maps $L^1(D)$ into $C(\bar{D})$ (so, T_t maps $L^p(D)$ into itself, for any p in $[1,\infty]$) and there are positive constants K and β that depend only on D, q and c such that*

$$\sup_{x \in \bar{D}} |(T_t f)(x)| \leqq K t^{-d/2} e^{\beta t} \int_D |f(x)|\, dx, \tag{3.13}$$

i.e. $\|T_t\|_{1,\infty} \leqq K t^{-d/2} e^{\beta t}$. *Moreover:*

(a) If $f, g \in L^1(D)$ or if, f, g are just positive, then

$$\int_D f(x)(T_t g)(x)\, dx = \int_D (T_t f)(x)\, g(x)\, dx, \tag{3.14}$$

which says that T_t is symmetric;

(b) for each $p \in [1,\infty]$ and each $t>0$, T_t is a *compact* operator from $L^p(D)$ into itself with norm satisfying

$$\|T_t\|_p \leqq \|T_t\|_\infty \leqq K e^{\lambda_1 t} \qquad \text{(in the selfadjoint case: } \|T_t\|_2 = e^{\lambda_1 t}), \tag{3.15}$$

where K and λ_1 are constants. (The exponent β that appears in (3.13) can be taken equal to λ_1, since $\|T_t\|_{1,\infty} \leqq \|T_1\|_{1,\infty} \|T_{t-1}\|_1$. As we will see in Sect. 5, in the self-adjoint case, λ_1 is the largest eigenvalue of the mixed problem.) If c is real and T_t is considered acting on $L^2(D)$, it is a self-adjoint operator. In fact, it possesses a symmetric and continuous kernel $k_t(x,y)$, namely

$$(T_t f)(x) = \int_D k_t(x,y) f(y)\, dy$$

and its eigenfunctions are in $C(\bar{D})$. Therefore, T_t is a compact operator on $C(\bar{D})$ too. Finally, if $f \in C(\bar{D})$, then

$$\lim_{t \downarrow 0} (T_t f)(x) = f(x), \qquad \text{for all } x \in \bar{D},$$

which is equivalent (by a standard argument that can be found in [C1]) to the fact that T_t is *Fellerian* (i.e. T_t is an bounded operator on $C(\bar{D})$ and it converges strongly to the identity as $t \downarrow 0$).

Remarks. (a) An immediate consequence is that if we replace q by $q-\lambda$, where λ is a sufficiently large constant, then the norms $\|T_t\|_{1,\infty}$ and $\|T_t\|_p$, where $1 \leqq p \leqq \infty$, go to 0 exponentially fast, as $t \to \infty$.

(b) If $q \leqq 0$ and $c \leqq 0$ then T_t is (submarkovian and so) the transition semigroup of a diffusion in D. In this case, $-q$ and $-c$ are the killing rates in D and on ∂D respectively. If $q \equiv 0$ the process is called *elastic Brownian motion.*

(c) If q and c were *complex-valued* functions, then the estimates in (3.13) and (3.15) would still be true (by applying the theorem to the real parts of q and c). However, T_t would not be self-adjoint any more (but it would still be compact).

We continue with another property of the semigroup which complements the previous theorem, since it gives a lower bound for $(T_t f)(x)$. It plays an essential role in the proof of the Gauge Theorem (Theorem 3.6).

Proposition 3.5. *Let $f \geqq 0$ be Borel measurable on $\bar{D}$. Then, for any $t>0$, there is a constant C_t such that*

$$\|f\|_1 = \int_{\bar{D}} f(y)\,dy \leqq C_t \inf_{x\in\bar{D}} (T_t f)(x). \tag{3.16}$$

Proof. (Fatou's Lemma cannot help.) Assume first that f is integrable. Observe that, for each $t>0$, by *P1* and *P2* there is a constant $A_t>0$ such that

$$E^x\{f(X_t)\} = \int_D p(t,x,y)\, f(y)\,dy \geqq A_t \|f\|_1. \tag{3.17}$$

Now, using a nice trick we found in [C-H], we have

$$\begin{aligned} E^x\{f(X_t)\}^2 &= E^x\{e_{q/2}(t)\,\hat{e}_{c/2}(t)\, f^{\frac{1}{2}}(X_t)\, e_{-q/2}(t)\,\hat{e}_{-c/2}(t)\, f^{\frac{1}{2}}(X_t)\}^2 \\ &\leqq E^x\{e_q(t)\,\hat{e}_c(t)\, f(X_t)\}\; E^x\{e_{-q}(t)\,\hat{e}_{-c}(t)\, f(X_t)\} \\ &= (T_t f)(x)(\tilde{T}_t f)(x) \\ &\leqq B_t \|f\|_1 (T_t f)(x), \end{aligned}$$

where we first applied Schwarz's inequality, then we denoted by $\{\tilde{T}_t, t \geqq 0\}$ the semigroup that corresponds to $-q$ and $-c$ and, finally, we applied Theorem 3.4 to $\tilde{T}_t$ to get the last inequality (where $B_t>0$).

Therefore, (3.17) implies that

$$A_t^2 \|f\|_1^2 \leqq B_t \|f\|_1 (T_t f)(x)$$

and so

$$\|f\|_1 \leqq \frac{B_t}{A_t^2} (T_t f)(x), \qquad \text{for every } x\in\bar{D}.$$

If f is not integrable, we apply (3.16) to $f \wedge n$ and then we use monotone convergence. □

For the rest of this section, q and c are real-valued.

In the probabilistic treatment of the Dirichlet or the Neumann problem for the (time independent) Schrödinger equation in a bounded domain D, there is a positive function that plays an important role. It is called the *gauge* for the corresponding boundary value problem. One property of the gauge is that, if it is finite at one point of $\bar{D}$, then it is bounded (in fact continuous) on $\bar{D}$. A second property is that the gauge is finite if and only if $\lambda_1 < 0$, where λ_1 is the first eigenvalue of the corresponding boundary value problem. The gauge is not uniquely defined, but, usually, the (probabilistic) solution of the problem for boundary data $\equiv 1$ can be taken as a gauge.

We will see that all the above work for the third problem too. We define the gauge for (3.1) to be

$$G(x)=E^x\left\{\int_0^\infty \hat{e}_c(t)\, e_q(t)\, dL_t\right\}. \tag{3.18}$$

Then we have the following

Theorem 3.6. (The Gauge Theorem.) *If there exist an $x_0 \in \bar{D}$ such that $G(x_0)$ is finite, then the function $G(x)$ is continuous on $\bar{D}$ (and so bounded on $\bar{D}$).*

Proof. We can imitate the proof for the Neumann case, given in [C-H]. From (3.18) we get

$$G(x)=E^x\left\{\int_0^t \hat{e}_c(s)\, e_q(s)\, dL_s\right\}+E^x\left\{\int_t^\infty \hat{e}_c(s)\, e_q(s)\, dL_s\right\}, \tag{3.19}$$

where t is taken to be some (fixed) number in $(0, \infty)$. Let $E_1(t)$ and $E_2(t)$ be the first and the second term, respectively, of the right-hand side of (3.19). Then

$$0 \leqq E_1(t) \leqq E^x\{e_{|q|}(t)\, \hat{e}_{|c|}(t)\, L(t)\},$$

which is bounded on $\bar{D}$ by Schwarz's inequality and Proposition 3.1. In fact, the same statements imply that

$$\lim_{t\downarrow 0}\sup_{x\in\bar{D}} E_1(t)=0. \tag{3.20}$$

Next, set

$$Y=Y(\omega)=\int_0^\infty e_q(s)\, \hat{e}_c(s)\, dL_s. \tag{3.21}$$

Then

$$\begin{aligned} Y\circ\theta_t &= \int_0^\infty [e_q(s)\circ\theta_t]\,[\hat{e}_c(s)\circ\theta_t]\, d(L_s\circ\theta_t)\\ &= \int_0^\infty \exp\left[\int_t^{t+s} q(X_u)\, du\right] \exp\left[\int_t^{t+s} c(X_u)\, dL_u\right] dL_{s+t}\\ &= e_q(t)^{-1}\, \hat{e}_c(t)^{-1} \int_t^\infty e_q(s)\, \hat{e}_c(s)\, dL_s, \end{aligned}$$

where in the integral with respect to dL_{s+t}, the dummy variable is s. Hence, (3.19) gives

$$\begin{aligned} E_2(t) &= E^x\{e_q(t)\, \hat{e}_c(t)\,[Y\circ\theta_t]\}\\ &= E^x\{e_q(t)\, \hat{e}_c(t)\, E[Y\circ\theta_t \mid \mathscr{F}_t]\}\\ &= E^x\{e_q(t)\, \hat{e}_c(t)\, E^{X_t}[Y]\}, \end{aligned}$$

where the last equality follows from the Markov property. Using (3.21) and (3.18) we get

$$E_2(t) = E^x\{e_q(t)\,\hat{e}_c(t)\,G(X_t)\} = (T_t\,G)(x)$$

and so (3.19) becomes

$$G(x) = E_1(t) + (T_t\,G)(x). \tag{3.22}$$

Now we use Proposition 3.5 to get (since $G \geqq 0$)

$$\infty > G(x_0) \geqq (T_t\,G)(x_0) \geqq C_t\,\|G\|_1.$$

Therefore $G \in L^1(D)$ and so, by Theorem 3.4 we get that

$$(T_t\,G) \in C(\bar{D}).$$

Hence G is bounded on $\bar{D}$, because of (3.20) and (3.22). Furthermore, since the limit in (3.20) is uniform in x, we get that G is in $C(\bar{D})$. □

Remark. Since $\bar{D}$ is compact, we get that $G(x) \geqq G_0 > 0$, for all $x \in \bar{D}$.

Thus, we have shown that the Gauge Theorem follows, in a rather straightforward way, from Theorem 3.4 and Proposition 3.5. The theorem that follows shows that the relation between the gauge and the semigroup is really intimate.

Theorem 3.7. *The gauge is finite if and only if* $\lambda_1 < 0$, *where* $\|T_t\|_2 = e^{\lambda_1 t}$ *(in other words,* $e^{\lambda_1 t}$ *is the first eigenvalue of* T_t*).*

Proof. First we assume that $G \not\equiv \infty$. We write again the formula (3.22) that appears in the proof of the Gauge Theorem:

$$G(x) = E^x\left\{\int_0^t e_q(s)\,\hat{e}_c(s)\,dL_s\right\} + (T_t\,G)(x).$$

Letting $t \to \infty$ and using monotone convergence and the definition of the gauge we get

$$\lim_{t\to\infty} (T_t\,G)(x) = 0.$$

If G_0 is the minimum of $G(x)$ in $\bar{D}$ then

$$0 < G_0(T_t\,1)(x) \leqq (T_t\,G)(x)$$

and so

$$\lim_{t\to\infty} (T_t\,1)(x) = 0. \tag{3.23}$$

Now, by Proposition 3.5,

$$(T_t\,1)(x) = [T_1(T_{t-1}\,1)](x) \geqq C\,\|T_{t-1}\,1\|_1,$$

hence

$$\lim_{t\to\infty} \| T_t 1 \|_1 = 0.$$

Finally, by applying (3.13) to T_1 we get

$$(T_t 1)(x) \leqq K \| T_{t-1} 1 \|_1$$

and so the limit in (3.23) is uniform in x, i.e.

$$\lim_{t\to\infty} \| T_t 1 \|_\infty = \lim_{t\to\infty} \| T_t \|_\infty = 0,$$

which means that $\lambda_1 < 0$, because of (3.15).

Conversely, assume that $\lambda_1 < 0$. Using the Markov property and the definition of G, we get

$$G(x) = \sum_{n=0}^{\infty} E^x \left\{ e_q(n)\, \hat{e}_c(n)\, E^{X_n} \left[\int_0^1 e_q(t)\, \hat{e}_c(t)\, dL_t \right] \right\}.$$

But

$$E^{X_n} \left\{ \int_0^1 e_q(t)\, \hat{e}_c(t)\, dL_t \right\} \leqq \sup_{x\in\bar{D}} E^x \{ e_{|q|}(1)\, \hat{e}_{|c|}(1)\, L(1) \} = M < \infty,$$

by Schwarz's inequality etc. So,

$$G(x) \leqq M \sum_{n=0}^{\infty} E^x \{ e_q(n)\, \hat{e}_c(n) \} \leqq M \sum_{n=0}^{\infty} \| T_n \|_\infty < 0,$$

by (3.15), since $\lambda_1 < 0$. □

4 The third boundary value problem

We are now ready to give the probabilistic solution of the mixed problem (3.1). We apply the method introduced by Stroock and Varadhan for a more general set-up (see [S-V]). The same method was used by P. Hsu in [H2] for the Neumann problem. The case where q and c are smooth and negative was studied, rather briefly, by Sato and Ueno in [S-U] (also, recently, in [Fr]) with analytic methods. In our treatment the functions q and c are not necessarily smooth or real-valued.

In this section:

(a) all martingales we consider are assumed to have mean equal to zero;

(b) when we say that a process is continuous or has continuous paths, we mean that it has a version with continuous paths.

4 A Weak solution and its path integral representation

Let $u \in C^2(D) \cap C^1(\bar{D})$ be a (strong) solution of (3.1). We can then apply the multidimensional Itô formula (see [C-W]) to $u(X_t)$, since $X = \{X_t, t \geqq 0\}$ (the SRBM in D) is a semimartingale, by (1.2), and get

$$\begin{aligned} u(X_t) - u(X_0) &= \int_0^t \nabla u(X_s) \cdot dB_s + \int_0^t \nabla u(X_s) \cdot n(X_s)\, dL_s + \frac{1}{2} \int_0^t \Delta u(X_s)\, ds \\ &= \int_0^t \nabla u(X_s) \cdot dB_s - \int_0^t c(X_s)\, u(X_s)\, dL_s - \int_0^t f(X_s)\, dL_s - \int_0^t q(X_s)\, u(X_s)\, ds. \end{aligned}$$

Hence, if we define

$$(4.1) \qquad M_f^u(t) = u(X_t) - u(X_0) + \int_0^t c(X_s)\, u(X_s)\, dL_s + \int_0^t f(X_s)\, dL_s + \int_0^t q(X_s)\, u(X_s)\, ds,$$

we must also have

$$M_f^u(t) = \int_0^t \nabla u(X_s) \cdot dB_s$$

and so $M_f^u(t)$ is a continuous P^x-martingale. This computation motivates the following definition.

Definition I. A function $u \in b\mathscr{B}(\bar{D})$ is called a *weak solution of the third problem* (3.1) if, for all $x \in \bar{D}$, $M_f^u(t)$ of (4.1) is a continuous P^x-martingale.

The fact that q is in $K_d(D)$ implies that the third integral in the right-hand side of (4.1) has a continuous version, since for any fixed $t_0 > 0$, $q(X_s)$ is (as a function of s) in $L^1(0, t_0)$, P^x-a.s. for all $x \in \bar{D}$, by the remark after Theorem 2.3 and

$$\left| \int_0^t [q(X_s) - n \wedge q(X_s)]\, ds \right| \leqq \int_0^{t_0} |q(X_s) - n \wedge q(X_s)|\, ds,$$

which says that $\int_0^t q(x_s)\, ds$ is the limit of continuous processes and this limit is uniform in t on bounded intervals. Similarly, the integrals in (4.1) with respect to dL_s (which are pathwise Lebesgue-Stieltjes) they exist, having in fact continuous versions, since L is a continuous process. Thus, assuming u is bounded we get that $M_f^u(t)$ must be a continuous process. It turns out that if u is a weak solution, then it is automatically continuous (see Theorem 4.3). Furthermore, (see Appendix) it is easy to see that $M_f^u(t)$ is in $L^2(P^x)$, for all $x \in \bar{D}$ and the process is L^2-bounded if t is restricted in a finite interval (in fact, $\exp[M_f^u(t)]$ is in $L^p(P^x)$, for any $p < \infty$).

Remarks. (a) Notice that the above definition does not involve test functions.

(b) It follows that, if a strong solution of (3.1) exists, it is automatically a weak solution.

It turns out that there is an equivalent way to define the weak solutions, which, sometimes, is more convenient for this kind of calculations.

Proposition 4.1. *Let $u \in b\mathscr{B}(\bar{D})$ and assume that q, c and f are as specified at the beginning of this section. If we define*

$$\tilde{M}_f^u(t) = e_q(t)\,\hat{e}_c(t)\,u(X_t) - u(X_0) + \int_0^t e_q(s)\,\hat{e}_c(s)\,f(X_s)\,dL_s, \tag{4.2}$$

then $\tilde{M}_f^u(t)$ is a continuous P^x-martingale, for every $x \in \bar{D}$, if and only if the process $M_f^u(t)$ of (4.1) *is.*

Proof. (In this proof, the integrals with respect to martingales are Itô stochastic integrals.)

First, let's assume that $\tilde{M}_f^u(t)$ is a continuous P^x-martingale. Then (4.2) implies that $e_q(t)\,\hat{e}_c(t)\,u(X_t)$ is a continuous P^x-semimartingale and so

$$u(X_t) = e_{-q}(t)\,\hat{e}_{-c}(t)\,[e_q(t)\,\hat{e}_c(t)\,u(X_t)]$$

is also a continuous P^x-semimartingale (since $e_{-q}(t)\,\hat{e}_{-c}(t)$ is a continuous process which is locally of bounded variation). So, (4.2) gives

$$\begin{aligned} d\tilde{M}_f^u(s) = {} & e_q(s)\,\hat{e}_c(s)\,du(X_s) + e_q(s)\,\hat{e}_c(s)\,u(X_s)\,c(X_s)\,dL_s \\ & + e_q(s)\,\hat{e}_c(s)\,u(X_s)\,q(X_s)\,ds + e_q(s)\,\hat{e}_c(s)\,f(X_s)\,dL_s \end{aligned}$$

which implies that

$$e_{-q}(s)\,\hat{e}_{-c}(s)\,d\tilde{M}_f^u(s) = du(X_s) + u(X_s)\,c(X_s)\,dL_s + u(X_s)\,q(X_s)\,ds + f(X_s)\,dL_s.$$

Now, we integrate the above from 0 to t and then we use (4.1). The result is:

$$\int_0^t e_{-q}(s)\,\hat{e}_{-c}(s)\,d\tilde{M}_f^u(s) = M_f^u(t)$$

which shows that $M_f^u(t)$ is a continuous P^x-martingale. The converse follow in a similar way. □

Remark. Since (by Jensen's and then Schwarz's inequality)

$$\begin{aligned} E^x\left\{\left[\int_0^t e_q(s)\,\hat{e}_c(s)\,f(X_s)\,dL_s\right]^p\right\} & \leq \|f\|_\infty^p\, E^x\{L_t^p\, e_{p|q|}(t)\,\hat{e}_{p|c|}(t)\} \\ & \leq \|f\|_\infty^p\, E^x\{L_t^{2p}\}^{\frac{1}{2}}\, E^x\{e_{2p|q|}(t)\,\hat{e}_{2p|c|}(t)\}^{\frac{1}{2}}, \end{aligned}$$

we get (see Appendix) that $\tilde{M}_f^u(t)$ is in $L^p(P^x)$, for all $p<\infty$, $x \in \bar{D}$ and, in fact, $\tilde{M}_f^u(t)$ is L^p-bounded, if we restrict t in a finite interval. Moreover, if $u \in C(\bar{D})$, then $\tilde{M}_f^u(t)$ is a continuous process (to show that, we can use an argument similar to the one used for the continuity of M_f^u).

The next lemma consists of two simple (but useful) formulas.

Lemma 4.2. *If* $0 \leqq s \leqq t$, *then*

$$E\{M_f^u(t)|\mathscr{F}_s\} = M_f^u(s) + E^{X_s}\{M_f^u(t-s)\}$$

and

$$E\{\tilde{M}_f^u(t)|\mathscr{F}_s\} = \tilde{M}_f^u(s) + e_q(s)\, \hat{e}_c(s)\, E^{X_s}\{\tilde{M}_f^u(t-s)\},$$

where M_f^u *and* $\tilde{M}_f^u$ *are the processes defined by* (4.1) *and* (4.2) *respectively.*

Proof. We will prove only the second formula (the proof of the first formula is even easier).

By applying the Markov property to (4.2) we obtain

$$\begin{aligned} E\{\tilde{M}_f^u(t)|\mathscr{F}_s\} &= e_q(s)\, \hat{e}_c(s)\, E^{X_s}\{e_q(t-s)\, \hat{e}_c(t-s)\, u(X_{t-s})\} \\ &\quad - u(X_0) + \int_0^s e_q(r)\, \hat{e}_c(r)\, f(X_r)\, dL_r \\ &\quad + e_q(s)\, \hat{e}_c(s)\, E^{X_s}\left\{\int_0^{t-s} e_q(r)\, \hat{e}_c(r)\, f(X_r)\, dL_r\right\}. \end{aligned}$$

If we add and subtract $e_q(s)\, \hat{e}_c(s)\, u(X_s)$ in the right-hand side of the above equation, we get the desired result. □

The lemma implies that in order to prove that M_f^u or $\tilde{M}_f^u$ is a martingale, it is enough to show that $E^x\{M_f^u(t)\} = 0$ (or $E^x\{\tilde{M}_f^u(t)\} = 0$ respectively) for all $t \geqq 0$ and all $x \in \bar{D}$.

We continue with one of our main results.

Theorem 4.3. *Suppose that, for any* $t > 0$, u *satisfies the following integral equation*

$$(4.3) \qquad (I - T_t)\, u(x) = E^x\left\{\int_0^t e_q(s)\, \hat{e}_c(s)\, f(X_s)\, dL_s\right\}, \qquad \text{for all } x \in \bar{D},$$

where I *is the identity operator. Then,* u *is continuous on* $\bar{D}$ *and it is a weak solution of the mixed problem* (3.1). *Conversely, if* u *is a weak solution of* (3.1), *then it satisfies* (4.3), *for all* $t > 0$; *therefore it is continuous. If* $\lambda_n \neq 0$ *for all* n *(see part 4C for the definition of* λ_n*), then the mixed problem has at most one weak solution.*

Proof. Assume that u satisfies (4.3). Let

$$F(x) = E^x\left\{\int_0^t e_q(s)\, \hat{e}_c(s)\, f(X_s)\, dL_s\right\}.$$

This function could be called the *"truncated gauge"* of the mixed problem (3.1). If we can show that F is continuous on $\bar{D}$, then, by Theorem 3.4 and the Fredholm Alternative, u is continuous on $\bar{D}$. Here is the proof of the continuity of F:

Set

$$Y = \int_0^t e_q(s)\, \hat{e}_c(s)\, f(X_s)\, dL_s.$$

For $0 \leqq \varepsilon \leqq t$ we have

$$F(x) = E^x\{Y\} = E^x\{Y \circ \theta_\varepsilon\} + E^x\{Y - Y \circ \theta_\varepsilon\}. \tag{4.4}$$

The first term of the right-hand side of (4.4) is continuous on $\bar{D}$ because X has the strong Feller property (see Sect. 1). Also,

$$Y \circ \theta_\varepsilon = e_{-q}(\varepsilon)\, \hat{e}_{-c}(\varepsilon) \int_\varepsilon^{t+\varepsilon} e_q(s)\, \hat{e}_c(s)\, f(X_s)\, dL_s$$

and so

$$Y - Y \circ \theta_\varepsilon = \int_0^\varepsilon e_q(s)\, \hat{e}_c(s)\, f(X_s)\, dL_s + [1 - e_{-q}(\varepsilon)\, \hat{e}_{-c}(\varepsilon)] \int_\varepsilon^t e_q(s)\, \hat{e}_c(s)\, f(X_s)\, dL_s$$
$$- e_{-q}(\varepsilon)\, \hat{e}_{-c}(\varepsilon) \int_t^{t+\varepsilon} e_q(s)\, \hat{e}_c(s)\, f(X_s)\, dL_s.$$

Thus,

$$\limsup_{\varepsilon \downarrow 0\ x \in \bar{D}} E^x\{Y - Y \circ \theta_\varepsilon\} = 0,$$

by standard arguments.

Now, for any $x \in \bar{D}$ and any $t \geqq 0$, (4.3) implies

$$E^x\{\tilde{M}_f^u(t)\} = 0$$

and so $\{\tilde{M}_f^u(t)\} = 0$ is a continuous P^x-martingale, by Lemma 4.2 (second part). Hence, the same is true for $M_f^u(t)$, by Proposition 4.1. If u is a weak solution then, by Proposition 4.1 it satisfies (4.2). By taking expectations in (4.2), we arrive at (4.3).

Finally, $\lambda_n \neq 0$ implies that $e^{\lambda_n t} \neq 1$ and so, 1 is not an eigenvalue of T_t, if $t > 0$. Hence, by the Fredholm alternative, there is at most one u that can satisfy (4.3) simultaneously, for all t. □

The previous theorem has a very interesting corollary. We remind the reader that the gauge of (3.1) is defined in (3.18) to be

$$G(x) = E^x\left\{\int_0^\infty e_q(t)\, \hat{e}_c(t)\, dL_t\right\}.$$

Corollary 4.4. (The Path Integral Representation of the Weak Solution.) *Let $q \in K_d(D)$, $f \in \mathscr{B}(\partial D)$ and $c \in C(\partial D)$, where $D \subset R^d$ is a bounded domain with C^3 boundary. If, for some $x_0 \in \bar{D}$, we have that $G(x_0) < \infty$, then*

$$u(x) = E^x\left\{\int_0^\infty e_q(t)\, \hat{e}_c(t)\, f(X_t)\, dL_t\right\} \tag{4.5}$$

is the unique weak solution of (3.1). *Furthermore, u is continuous (and so bounded) on $\bar{D}$.*

Proof. In the same way that we obtained (3.22) we can get that (for any $t>0$) u of (4.5) satisfies

$$u(x)=E^x\left\{\int_0^t e_q(s)\,\hat{e}_c(s)\,f(X_s)\,dL_s\right\}+E^x\{e_q(t)\,\hat{e}_c(t)\,u(X_t)\}.$$

This is (4.3) and so, u is a weak solution.

Conversely, assume that u is a weak solution. Then, by the previous theorem

$$u(x)=E^x\{e_q(t)\,\hat{e}_c(t)\,u(X_t)\}+E^x\left\{\int_0^t e_q(s)\,\hat{e}_c(s)\,f(X_s)\,dL_s\right\}.$$

Letting $t\to\infty$ and using Theorem 3.7 and dominated convergence, we get that u satisfies (4.5) and (because of that) is unique. □

In Theorem 4.3, the requirement that (4.3) must hold for all $t>0$, in order for u to be a weak solution seems too strong, but it is not (at least for the case where 1 is not an eigenvalue of T_t), because if it holds for one $t>0$ then it holds for all t:

Theorem 4.5. *Suppose that none of the λ_n's is 0. Then, the mixed problem* (3.1) *has* always *a unique weak solution.*

Proof. Our assumption is equivalent to the fact that 1 is not an eigenvalue of T_t, for all $t>0$. So, for any fixed t, there is a unique u that satisfies (4.3). We have to show that this u is independent of t.

We start with a convenient definition.

$$[u]=\{s>0\colon u \text{ satisfies (4.3) for } t=s\}.$$

Now, suppose that $a\in[u]$ and $b\in[u]$. Then, (4.3) for $t=a$ gives

$$u(x)=E^x\left\{\int_0^a e_q(s)\,\hat{e}_c(s)\,f(X_s)\,dL_s\right\}+(T_a\,u)(x).$$

Next, we apply T_b to the above equation. The result is

$$(T_b\,u)(x)=E^x\left\{e_q(b)\,\hat{e}_c(b)\,E^{X_b}\left[\int_0^a e_q(s)\,\hat{e}_c(s)\,f(X_s)\,dL_s\right]\right\}+(T_{a+b}\,u)(x),$$

which (by the Markov property) is equivalent to

$$(T_b\,u)(x)=E^x\left\{\int_b^{a+b} e_q(s)\,\hat{e}_c(s)\,f(X_s)\,dL_s\right\}+(T_{a+b}\,u)(x).$$

But $b\in[u]$, i.e. u satisfies (4.3) for $t=b$. Using this in the above equation we get

$$a+b\in[u].$$

Now let's fix u to be the solution of (4.3) for $t=1$; in other words $1\in[u]$. Then, the above analysis implies that $n\in[u]$ for every positive integer n. Also, if v is the unique function for which

$$\frac{1}{n}\in[v],$$

then $1\in[v]$ and so $v=u$, by uniqueness. Therefore,

$$r\in[u], \qquad \text{for any positive rational } r.$$

Thus, u satisfies (4.3) for a set of t's which is dense in $(0, \infty)$. Since u is continuous on $\bar{D}$ and T_t is Fellerian (see Theorem 3.4), we get, by dominated convergence, that u satisfies (4.7) for all t. □

For the rest of the section we assume that $c\in C(\partial D)$.

4B Connection with the weak solution in the classical sense

We want to examine the relation between weak solution as was defined in the previous section and the weak solution in the classical sense. We start with some notation.

$$bC^2(D)=\{g\in C^2(D)\colon \ g \text{ has bounded second derivatives in } D\}$$

and

$$bC^2_\partial(D)=\left\{g\in bC^2(D)\cap C^1(\bar{D})\colon \ \frac{\partial g}{\partial n}=0 \ \text{ on } \ \partial D\right\}.$$

Next, we want to specify what we mean by a "classical weak solution":

Definition II. A Borel measurable function, u defined on $\bar{D}$, is called a *weak solution of* (3.1) *in the classical sense* (or a *classical weak solution*) if, for every (test function) $g\in bC^2_\partial(D)$, we have

$$\int_D u(x)\left[\frac{\Delta}{2}+q(x)\right]g(x)\,dx=-\tfrac{1}{2}\int_{\partial D} g(z)[f(z)+c(z)\,u(z)]\,\sigma(dz),$$

where, as usual, $\sigma(dz)$ is the $(d-1)$-dimensional volume element on ∂D.

(To justify the above definition, write the second Green's identity for the operator $\Delta/2+q$ on $\bar{D}$, applied to u and g, and then use the fact that u "satisfies" (3.1) and that g is in $bC^2_\partial(D)$. Notice that a strong solution is automatically weak in the classical sense.)

Remark. A slightly different but essentially equivalent definition for the classical weak solution could be used, in which the test functions satisfy $\partial g/\partial n+cg=0$ on ∂D.

We continue with a couple of technical lemmas.

Lemma 4.6. *Let* $g \in bC^2(D) \cap C^1(\bar{D})$. *Then*

$$(4.6\text{a}) \qquad (T_t g)(x) - g(x) = E^x\left\{\int_0^t e_q(s)\,\hat{e}_c(s)\left(\frac{\Delta}{2}+q\right) g(X_s)\,ds\right\} + E^x\left\{\int_0^t e_q(s)\,\hat{e}_c(s)\left(\frac{\partial}{\partial n}+c\right) g(X_s)\,dL_s\right\}$$

$$(4.6\text{b}) \qquad = \int_0^t \left[T_s\left(\frac{\Delta}{2}+q\right) g\right](x)\,ds + E^x\left\{\int_0^t e_q(s)\,\hat{e}_c(s)\left(\frac{\partial}{\partial n}+c\right) g(X_s)\,dL_s\right\}.$$

Proof. Apply the Itô formula to $V_t\, g(X_t)$, where $V_t = e_q(t)\,\hat{e}_c(t)$ is a process which is locally of bounded variation:

$$e_q(t)\,\hat{e}_c(t)\,g(X_t) - g(X_0) = \int_0^t e_q(s)\,\hat{e}_c(s)\,\nabla g(X_s)\cdot dB_s + \int_0^t e_q(s)\,\hat{e}_c(s)\left(\frac{\Delta}{2}+q\right) g(X_s)\,ds + \int_0^t e_q(s)\,\hat{e}_c(s)\left[\frac{\partial g}{\partial n}(X_s) + c(X_s)\,g(X_s)\right] dL_s.$$

Taking expectations we obtain (4.6a). To get the second formula, we just need to justify the application of Fubini's Theorem to the first term of the right-hand side of (4.6a). Let

$$\sup_{x\in D}\left|\frac{\Delta}{2} g(x)\right| + \sup_{x\in D}|g(x)| = M \quad \text{and} \quad \sup_{x\in\partial D}|c(z)| = c_0.$$

Then

$$(4.7) \qquad E^x\left\{\int_0^t \left|e_q(s)\,\hat{e}_c(s)\left(\frac{\Delta}{2}+q\right) g(X_s)\right| ds\right\} \leqq M E^x\{e^{c_0 L_t} e_{|q|}(t)\} + M E^x\left\{e^{c_0 L_t}\int_0^t e_{|q|}(s)\, q(X_s)|\,ds\right\} = M E^x\{e^{c_0 L_t} e_{|q|}(t)\} + M E^x\{e^{c_0 L_t}[e_{|q|}(t) - 1]\},$$

which is bounded uniformly in x, by the case $p=\infty$ of Theorem 3.4, since the semigroups that appear above are of the same type as the one examined in this theorem. □

Remark. Theorem 4.6 implies that $\Delta/2+q$ is the infinitesimal generator of T_t.

The next lemma is a little tedious. It is harder than its corresponding one for the Neumann case (see [H2]), because of the presence of $\hat{e}_c(t)$ which, as $t\downarrow 0$, goes to 1 slower than $e_q(t)$.

Lemma 4.7. *Let h be in $C(\bar{D})$ and k be in $b\mathscr{B}(\partial D)$. Then*

$$\lim_{t\downarrow 0}\frac{1}{t}\int_D h(x)\,E^x\left\{\int_0^t e_q(s)\,\hat{e}_c(s)\,k(X_s)\,dL_s\right\}dx=\tfrac{1}{2}\int_{\partial D} h(z)\,k(z)\,\sigma(dz). \tag{4.8}$$

Proof. First we show that

$$\lim_{t\downarrow 0}\frac{1}{t}\int_D E^x\left\{\int_0^t |\hat{e}_c(s)-1|\,dL_s\right\}dx=0. \tag{4.9}$$

Let $c_0=\|c\|_\infty$, as usual. If $c_0=0$ then (4.9) is trivially true, so let's assume that $c_0>0$. To show (4.9), it is enough to show that

$$\lim_{t\downarrow 0}\frac{1}{t}\int_D E^x\left\{\int_0^t (e^{c_0L_s}-1)\,dL_s\right\}dx=0, \tag{4.10}$$

because $|\hat{e}_c(s)-1|\leqq e^{c_0L_s}-1$.

The left-hand side of (4.10) equals

$$\frac{1}{c_0}\lim_{t\downarrow 0}\frac{1}{t}\int_D E^x\{e^{c_0L_s}-1-c_0\,L_t\}\,dx.$$

Therefore (expand $e^{c_0L_s}$ in powers of $c_0\,L_s$), to prove (4.10) it is enough to show that

$$\lim_{t\downarrow 0}\frac{1}{t}\int_D E^x\{L_s^2\}\,dx=0. \tag{4.11}$$

Now, (see (1.8) for the definition of D_δ)

$$\frac{1}{t}\int_D E^x\{L_s^2\}\,dx=\frac{1}{t}\int_{D\setminus D_\delta} E^x\{L_s^2\}\,dx+\frac{1}{t}\int_{D_\delta} E^x\{L_s^2\}\,dx.$$

The second integral of the right-hand side is less than $K\delta$, where K is a constant which depends only on D. Also

$$\begin{aligned}\frac{1}{t}\int_{D\setminus D_\delta} E^x\{L_s^2\}\,dx&=\frac{1}{t}\int_{D\setminus D_\delta} E^x\{L_s^2;\tau_D\leqq t\}\,dx\\&\leqq\frac{1}{t}\int_{D\setminus D_\delta} E^x\{L_s^4\}^{\frac{1}{2}}P^x\{\tau_D\leqq t\}^{\frac{1}{2}}\,dx,\end{aligned}$$

which goes to 0 with t (see Appendix). Thus, we have shown that the limit in (4.11) is (nonnegtive and) less than $K\delta$, where $\delta>0$ is arbitrary. Therefore, we have established (4.11) and so (4.9).

Next, we are going to show that

$$\lim_{t\downarrow 0}\frac{1}{t}\int_D E^x\left\{e^{c_0 L_t}\int_0^t [e_{|q|}(s)-1]\,dL_s\right\}dx=0. \tag{4.12}$$

The expression inside the limit, above, is equal to

$$\begin{aligned}
&\frac{1}{t}\int_D E^x\left\{e^{c_0 L_t}\int_0^t\int_0^s |q(X_r)|\,e_{|q|}(r)\,dr\,dL_s\right\}dx\\
&=\frac{1}{t}\int_D E^x\left\{e^{c_0 L_t}\int_0^t |q(X_r)|\,e_{|q|}(r)(L_t-L_r)\,dr\right\}dx\\
&=\frac{1}{t}\int_0^t\int_D E^x\{e^{c_0 L_r}|q(X_r)|\,e_{|q|}(r)\,E^{X_r}[e^{c_0 L(t-r)}L(t-r)]\}\,dx\,dr,
\end{aligned}$$

where we have apply Tonelli (easily justified, since everything is nonnegative) and then the Markov property. Now, by Schwarz's inequality, we get that, as long as t is bounded above (say by 1),

$$E^{X_r}\{e^{c_0 L(t-r)}L(t-r)\}\leqq K\sqrt{t-r}, \qquad 0\leqq r\leqq t,$$

where K is a "universal" constant. Therefore,

$$\begin{aligned}
\frac{1}{t}\int_D E^x\left\{e^{c_0 L_t}\int_0^t [e_{|q|}(s)-1]\,dL_s\right\}dx &\leqq \frac{K}{t}\int_0^t \sqrt{t-r}\int_D E^x\{e^{c_0 L_r}|q(X_r)|\,e_{|q|}(r)\}\,dx\,dr\\
&=\frac{K}{t}\int_0^t \sqrt{t-r}\int_D (T_r|q|)(x)\,dx\,dr,
\end{aligned}$$

where $(T_r f)(x)=E^x\{e^{c_0 L_r}e_{|q|}(r)\,f(X_r)\}$ is a semigroup of the form we have studied in Sect. 3. In particular, Theorem 3.4 gives

$$\|T_r f\|_1\leqq K'\|f\|_1,$$

where $\|\cdot\|_1$ is the norm of $L^1(D)$ and K is independent of r, if r is bounded (say by 1). Hence, the above formula becomes

$$\frac{1}{t}\int_D E^x\left\{e^{c_0 L_t}\int_0^t [e_{|q|}(s)-1]\,dL_s\right\}dx\leqq\frac{KK'}{t}\int_0^t\sqrt{t-r}\,dr\int_D |q|(x)|\,dx.$$

Remember that $q\in K_d(D)\subset L^1(D)$. Therefore, letting $t\downarrow 0$ above and observing that

$$\lim_{t\downarrow 0}\frac{1}{t}\int_0^t\sqrt{t-r}\,dr=0,$$

we arrive at (4.12).

Combining (4.9) and (4.12) we get

$$\lim_{t\downarrow 0}\frac{1}{t}\int_D E^x\left\{\int_0^t [e_{|q|}(s)\,\hat{e}_{|c|}(s)-1]\,dL_s\right\}dx=0. \tag{4.13}$$

(Because

$$\int_0^t [e_{|q|}(s)\,\hat{e}_{|c|}(s)-1]\,dL_s=\int_0^t [e_{|q|}(s)\,\hat{e}_{|c|}(s)-\hat{e}_{|c|}(s)]\,dL_s+\int_0^t [\hat{e}_{|c|}(s)-1]\,dL_s$$

$$\leqq e^{c_0 L_t}\int_0^t [e_{|q|}(s)-1]\,dL_s+\int_0^t [\hat{e}_{|c|}(s)-1]\,dL_s\Bigg).$$

We continue with the following computation.

$$\begin{aligned}
&\lim_{t\downarrow 0}\frac{1}{t}\int_D h(x)\,E^x\left\{\int_0^t k(X_s)\,dL_s\right\}dx \\
&=\lim_{t\downarrow 0}\frac{1}{2t}\int_D h(x)\int_0^t\int_{\partial D} k(y)\,p(s,x,y)\,\sigma(dy)\,ds\,dx \\
&=\lim_{t\downarrow 0}\int_{\partial D} k(y)\frac{1}{2t}\int_0^t (P_s h)(y)\,ds\,\sigma(dy) \\
&=\tfrac{1}{2}\int_{\partial D} k(y)\,h(y)\,\sigma(dy),
\end{aligned} \tag{4.14}$$

where Fubini is justified because everything is bounded, the first equality follows from the remark after Proposition 1.2, the fact that $p(s,\cdot,\cdot)$ is symmetric (see *P1* of Sect. 1) and $\{P_s, s\geqq 0\}$, the transition semigroup of the SRBM, is Fellerian (see Sect. 1).

Finally,

$$\left|\frac{1}{t}\int_D h(x)\,E^x\left\{\int_0^t [e_q(s)\,\hat{e}_c(s)-1]\,k(X_s)\,dL_s\right\}dx\right|$$

$$\leqq \|h\|_\infty\,\|k\|_\infty\,\frac{1}{t}\int_D E^x\left\{\int_0^t [e_{|q|}(s)\,\hat{e}_{|c|}(s)-1]\,dL_s\right\}dx$$

and so we are done, by letting $t\downarrow 0$ and using (4.13) and (4.14). □

The next theorem shows that the weak solution (as was defined in part 4A) is essentially equivalent to the classical weak solution.

Theorem 4.8. *The function u is a weak solution of* (4.1) *in the sense of Definition I if and only if it is a* continuous *weak solution in the classical sense.*

Proof. Let $g\in bC^2_\partial(D)$. Then (4.6b) becomes

$$(T_t g)(x)-g(x)=\int_0^t\left[T_s\left(\frac{\Delta}{2}+q\right)g\right](x)\,ds+E^x\left\{\int_0^t e_q(s)\,\hat{e}_c(s)\,c(X_s)\,g(X_s)\,dL_s\right\}.$$

We multiply the above equation by $u(x)$ and then we integrate over D. The result is:

$$\int_D u(x)[(T_t g)(x)-g(x)]\,dx=\int_D u(x)\int_0^t \left[T_s\left(\frac{\Delta}{2}+q\right)g\right](x)\,ds\,dx$$
$$+\int_D u(x)\,E^x\left\{\int_0^t e_q(s)\,\hat{e}_c(s)\,c(X_s)\,g(X_s)\,dL_s\right\}dx.$$

Next, we apply Fubini to the first term of the right-hand side. This is allowed because, as in (4.7), the integrand is bounded. Then, we use the symmetry of T_t, which was proved in Theorem 3.4 and so, we obtain

$$\text{(4.15)}\quad \int_D g(x)[(T_t u)(x)-u(x)]\,dx=\int_D \left(\frac{\Delta}{2}+q\right)g(x)\left[\int_0^t (T_s u)(x)\,ds\right]dx$$
$$+\int_D u(x)\,E^x\left\{\int_0^t e_q(s)\,\hat{e}_c(s)\,c(X_s)\,g(X_s)\,dL_s\right\}dx.$$

Now, assume that $M_f^u(t)$ is a continuous P^x-martingale. Then Proposition 4.1 implies

$$\text{(4.16)}\qquad (T_t u)(x)-u(x)=-E^x\left\{\int_0^t e_q(s)\,\hat{e}_c(s)\,f(X_s)\,dL_s\right\}.$$

We substitute the above in (4.15) and, after dividing by t, we get

$$\int_D \left(\frac{\Delta}{2}+q\right)g(x)\left[\frac{1}{t}\int_0^t (T_s u)(x)\,ds\right]dx=-\int_D g(x)\,E^x\left\{\frac{1}{t}\int_0^t e_q(s)\,\hat{e}_c(s)\,f(X_s)\,dL_s\right\}dx$$
$$-\int_D u(x)\,E^x\left\{\frac{1}{t}\int_0^t e_q(s)\,\hat{e}_c(s)\,g(X_s)\,c(X_s)\,dL_s\right\}dx.$$

Letting $t\downarrow 0$ and using Lemma 4.7 and the fact that

$$\lim_{t\downarrow 0}\frac{1}{t}\int_0^t (T_s u)(x)\,ds=u(x),\qquad \text{boundedly}$$

(by Theorem 3.4 applied to u, which is continuous by Theorem 4.3), we get that u is a weak solution in the classical sense.

Conversely, assume that u is a continuous classical weak solution. By (4.1) we know that $M_f^u(t)$ is a continuous process. To show that $M_f^u(t)$ is a P^x-martingale, because of the first part of Lemma 4.2, it is enough to show that, for any fixed $t\geq 0$,

$$\text{(4.17)}\qquad E^x\{M_f^u(t)\}=0,\qquad \text{for all } x\in\bar{D}.$$

We apply a trick found in [H2]:

From Sect. 1 we have that $p(t,x,\cdot)$ is in $bC^2_\partial(D)$, if $t>0$. So, we can use it in (4.5) in the place of g and get

$$\begin{aligned}\frac{d}{ds}E^x\{u(X_s)\} &= \int_D u(y)\frac{\partial}{\partial s}p(s,x,y)\,dy\\ &= \int_D u(y)\tfrac{1}{2}\Delta_y p(s,x,y)\,dy\\ &= -\int_D u(y)\,q(y)\,p(s,x,y)\,dy\\ &\quad -\tfrac{1}{2}\int_{\partial D}[f(y)+u(y)\,c(y)]\,p(s,x,y)\,\sigma(dy).\end{aligned}$$

Now, we integrate the above from r to t and we apply Fubini (justified since $q\in K_d(D)$ and u is continuous). With the help of Proposition 1.1, we get

$$\begin{aligned}E^x\{u(X_t)\}-E^x\{u(X_r)\} &= -E^x\left\{\int_r^t q(X_s)\,u(X_s)\,ds\right\}\\ &\quad -E^x\left\{\int_r^t [f(X_s)+c(X_s)\,u(X_s)]\,dL_s\right\}.\end{aligned}$$

The continuity of u and the Dominated Convergence Theorem allow us to let $r\downarrow 0$. Then, what we get, together with (4.1), which is just the definition of the process $M^u_f(t)$, imply immediately (4.17). □

Remarks. (a) If none of the λ_n's is 0, the above theorem together with Theorem 4.5 imply that there is always a *continuous* weak solution (in both senses) to the mixed problem and this solution is, also, unique.

(b) For any $\alpha\in(0,1)$ let $q\in C^{0,\alpha}(D)$ and $c,f\in C^{0,\alpha}(\partial D)$. Then (given that no eigenvalue of the problem is zero) there is a unique strong solution u, i.e. $u\in C^{2,\alpha}(D)\cap C^{0,\alpha}(\bar{D})$. This is a result of the (analytic) theory of second-order elliptic partial differential equations (see [S.L]). So, our weak solutions (in both senses) must agree with this u, by uniqueness.

Important observation. The previous results remain true if q and c are taken to be complex-valued functions such that $|q|\in K_d(D)$ and $c\in C(\partial D)$. The only exceptions are Theorem 4.5 and the uniqueness part of Theorem 4.3, since the eigenvalues of T_t may not be real and so the λ_n's are not well-defined (see Theorem 3.4). The gauge, also, becomes meaningless in this case, but we could consider instead the gauge that corresponds to the mixed problem that one obtains by replacing q and c by $\Re(q)$ and $\Re(c)$ respectively. The finiteness of this gauge again guarantees the existence of a weak solution in the form of (4.5).

Finally we would like to mentioned that the kernel

$$m(x,y)=\int_0^\infty k(t,x,y)\,dt \tag{4.18}$$

could be called the "Poisson kernel" for the third problem. It is finite for $x \neq y$ if and only if the gauge of (3.1) is finite (equivalently $\lambda_1 < 0$) and, if this is the case, then the (weak) solution of (3.1) is given by

$$u(x) = \int_{\partial D} m(x, y) f(y) \sigma(dy), \tag{4.19}$$

which is, of course, the analytic analog (4.5).

If we let $w(t, x) = (T_t f)(x)$, then w satisfies the (backward) equation

$$\frac{\partial w}{\partial t} = \frac{1}{2} \Delta_x w + q w.$$

Also, (trivially)

$$w(0, x) = f(x).$$

The above two equations are satisfied by the Dirichlet and Neumann semigroups too. It is their boundary behaviours that distinguish them. In our case we have that, if $z \in \partial D$, then (in a certain sense)

$$\frac{\partial w}{\partial n}(t, z) + c(z) w(t, z) = 0, \qquad \text{for all } t > 0.$$

5 Connection with the Neumann, the Dirichlet and the general mixed case

In this section we show that the semigroup of (3.1) is a uniform limit of Neumann semigroups and, also, that the Dirichlet semigroup is a uniform limit of semigroups that correspond to third boundary value problems. As an application of the second statement, we give the probabilistic solution of the general mixed problem.

5 A The connection with the Neumann problem

In what follows c is in $C(\partial D)$, but, without loss of generality it can be considered being in $C(\bar{D})$ and δ is a positive number not bigger than some fixed $\delta_0 > 0$ (this δ_0 must satisfy $P5$ of Sect. 1). We set

$$q_\delta(x) = q(x) + \frac{1}{2\delta} c(x) I_{D_\delta}(x) \tag{5.1}$$

and

$$(T_t^\delta f)(x) = E^x\{e_{q_\delta}(t) f(X_t)\}. \tag{5.2}$$

Notice that $q_\delta \in K_\delta(D)$ and for each $\delta > 0$, $\{T_t^\delta, t \geqq 0\}$ is a semigroup associated to a Neumann problem.

Theorem 5.1. *Let $f \in L^p(D)$, where $p > 1$, and fix a $t > 0$. Then*

$$\lim_{\delta \downarrow 0} (T_t^\delta f)(x) = (T_t f)(x), \qquad \text{uniformly on } \bar{D}. \tag{5.3}$$

In particular, for any fixed $t \geqq 0$, T_t is the uniform limit of T_t^δ, as $\delta \downarrow 0$, in $C(\bar{D})$ and in $L^p(D)$ for $1 < p \leqq \infty$.

Proof. Let $\|f\|_p$ be the norm of f in $L^p(D)$. Using Holder's inequality we get (r is taken such that $1/p + 1/r = 1$):

$$\begin{aligned}\sup_{x\in\bar{D}} |(T_t f)(x) - (T_t^\delta f)(x)| &= \sup_{x\in\bar{D}} E^x\{|e^{A_c(t)} - e^{A_c^\delta(t)}|\, e_q(t)\, f(X_t)\} \\ &\leqq \sup_{x\in\bar{D}} E^x\{|e^{A_c(t)} - e^{A_c^\delta(t)}|^r e_q(t)^r\}^{\frac{1}{r}} \sup_{x\in\bar{D}} E^x\{|f(X_t)|^p\}^{\frac{1}{p}} \\ &\leqq C_t \|f\|_p \sup_{x\in\bar{D}} E^x\{|e^{A_{rc}(t)} - e^{A_{rc}^\delta(t)}|\, e_{rq}(t)\}^{\frac{1}{r}},\end{aligned}$$

where $C_t > 0$ is a constant that depends only on t. To bound $\sup_{x\in\bar{D}} E^x\{|f(X_t)|^p\}^{\frac{1}{p}}$ by $C_t \|f\|_p$, we have used $P5$ of Sect. 1. We have also used the trivial inequality:

$$|x-y|^r \leqq |x^r - y^r|, \qquad \text{if } x, y \geqq 0 \quad \text{and} \quad r \geqq 1.$$

So,

$$\begin{aligned}&\sup_{x\in\bar{D}} |(T_t f)(x) - (T_t^\delta f)(x)| \\ &\quad \leqq C_t \|f\|_p \sup_{x\in\bar{D}} E^x\{|e^{A_{rc}(t)} - e^{A_{rc}^\delta(t)}|^2\}^{\frac{1}{2r}} \sup_{x\in\bar{D}} E^x\{e_{rq}(t)^2\}^{\frac{1}{2r}},\end{aligned}$$

which goes to 0, (see Appendix). □

Remarks. (a) The condition $p > 1$ in the hypothesis of the theorem enabled us to use Holder's inequality. Clearly, the proof fails for $p = 1$ (although the statement may still be correct).

(b) The theorem suggests that the problem (3.1) can be viewed (since $T_t^\delta \to T_t$, as $\delta \downarrow 0$) a limiting case of the Neumann problems

$$\begin{aligned}\tfrac{1}{2}\Delta u + q_\delta u &= 0, \qquad \text{in } D; \\ \frac{\partial u}{\partial n} &= -f, \quad \text{on } \partial D.\end{aligned}$$

This is, in fact, an alternative way to approach (3.1).

5B The Dirichlet and the general mixed problem

Let $B = \{B_t, t \geqq 0\}$ be the standard BM in R^d with $B_0 = x \in \bar{D}$. We set

$$\tau = \inf\{t > 0 : B_t \in D^c\}.$$

Then, as we have already pointed out in the remark after Theorem 1.3,

(5.4) $$\tau = \inf\{t>0: X_t \in \partial D\} = \inf\{t>0: L_t>0\}, \qquad P^x\text{-a.s.}$$

Consider the Dirichlet problem

(5.5) $$\begin{aligned} \tfrac{1}{2}\Delta u + qu &= 0, \quad \text{in } D; \\ u &= f, \quad \text{on } \partial D. \end{aligned}$$

The probabilistic solution of (5.5) is well known (see [C-R1] or [C-Z]). The corresponding semigroup $\{S_t, t \geqq 0\}$ is given by (since $B_s = X_s$, if $s \leqq r$)

(5.6) $$(S_t g)(x) = E^x\{1_{\{t \leqq r\}} e_q(t)\, g(X_t)\},$$

where $e_q(t)$ can be taken to be the one defined in (3.2). Next, let's consider the mixed problem

(5.7) $$\begin{aligned} \frac{1}{2}\Delta u + qu &= 0, \quad \text{in } D; \\ \frac{\partial u}{\partial n} - Nu &= -f, \quad \text{on } \partial D, \end{aligned}$$

where $N \geqq 0$ is a constant. Its corresponding semigroup $\{T_t^N, t \geqq 0\}$ is defined by (according to (3.4), where $c(x) \equiv -N$)

(5.8) $$(T_t^N g)(x) = E^x\{e_q(t)\, e^{-NL_t} g(X_t)\}.$$

Theorem 5.2. *Let T_t^N and S_t be as in formulas* (5.8) *and* (5.6) *respectively. Then for any fixed t we have*

(5.9) $$\lim_{N \uparrow \infty} \sup_{\|g\|_2 = 1} \|T_t^N g - S_t g\|_\infty = 0.$$

Proof. If $t=0$ there is nothing to prove, so let's assume $t>0$. Then

$$\begin{aligned} |(T_T^N g)(x) - (S_t g)(x)| &\leqq E^x\{e_q(t)|\, e^{-NL_t} - 1_{[t<\tau]}|\, |g(X_t)|\} \\ &\leqq E^x\{|e^{-NL_t} - 1_{[t<\tau]}|^2\}^{1/2} E^x\{e_{2q}(t)|g(X_t)|^2\}^{1/2}. \end{aligned}$$

The second factor above is bounded by some constant, say A_t, uniformly in x, by Theorem 3.4 (since $\|g^2\|_1 = \|g\|_2^2 = 1$). Also, $L_t = 0$ if $t<\tau$, so

$$|(T_T^N g)(x) - (S_t g)(x)| \leqq A_t\, E^x\{e^{-2NL_t} 1_{[t \geqq \tau]}\}^{1/2} = A_t\, E^x\{e^{-2NL_t} 1_{[t>\tau]}\}^{1/2},$$

which, by dominated convergence, decreases to 0 as $N \uparrow 0$, since $L_t>0$, P^x-a.s. if $t>\tau$. So we are done by Dini's Theorem. □

Remarks. (a) The functions g above are taken in $L^2(D)$, but we could take them in any $L^p(D)$ for $p>1$. The proof is essentially the same in this case, but if $p=1$ we don't know if the result is still valid.

(b) Notice that the above theorem implies that, for each fixed t, T_t^N converges to S_t (considered as an operator on $C(\bar{D})$ or $L^p(D)$, $1<p \leqq \infty$) in the uniform operator topology, and since T_t^N is compact and self-adjoint we can conclude

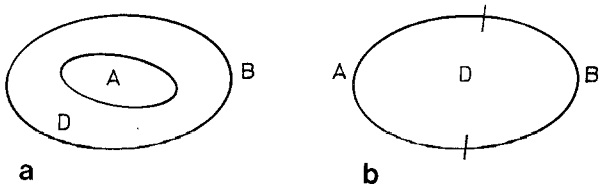

Fig. 1. a first case, **b** second case

that the spectrum of S_t consists of the limit points, as $N\uparrow\infty$, of the spectrum of T_t^N. In fact we get that

$$\lambda_k(N)\downarrow\mu_k,$$

where $\lambda_1(N)\geqq\lambda_2(N)\geqq\ldots$ are the eigenvalues of (5.7) and $\mu_1\geqq\mu_2\geqq\ldots$ are the eigenvalues of (5.5). This last statement is stated (without proof) in Vol. 1, Ch. IV, Sect. 2 of [Co-H], namely the *Methods of Mathematical Physics* by R. Courant and D. Hilbert.

Now consider again a bounded domain D in R^d, $d\geqq 1$, with C^3 boundary ∂D. Let's assume that $\partial D=\bar{A}\cup\bar{B}$, where A and B are disjoint *open* portions of ∂D (i.e. they are open relative to ∂D) and connected. (Connectivity is not really necessary but we assume it here in order to avoid wild cases.) We also require $\bar{A}\cap\bar{B}$, which, in general, is a $(d-2)$-dimensional manifold, to be at least C^1 (it may be empty or just two points). The possible cases are shown in Fig. 1.

Let $f\in b\mathscr{B}(A)$ and $g\in b\mathscr{B}(B)$. We consider the following general mixed problem:

(5.10)
$$\begin{aligned} \tfrac{1}{2}\Delta u+qu&=0, && \text{in } D;\\ \frac{\partial u}{\partial n}+cu&=-f, && \text{on } A;\\ u&=g, && \text{on } B. \end{aligned}$$

(n is, as usual, the inward unit normal vector on ∂D.)

This problem can be considered as a limiting case (as $N\uparrow\infty$) of the family of problems

(5.11)
$$\begin{aligned} \tfrac{1}{2}\Delta u_N+qu_N&=0, && \text{in } D;\\ \frac{\partial u_N}{\partial n}+c_N u_N&=-f_N, && \text{on } \partial D \end{aligned}$$

where

$$c_N=c\,1_A-N\,1_B$$

and

$$f_N=f\,1_A+Ng\,1_B.$$

The semigroup that corresponds to (5.11) is given by (3.4):

(5.12)
$$(T_t^N h)(x)=E^x\left\{e_q(t)\exp\left[\int_0^t c(X_s)\,1_A(X_s)\,dL_s\right]\cdot\exp\left[-N\int_0^t 1_B(X_s)\,dL_s\right]h(X_t)\right\}.$$

Letting $N\uparrow\infty$ we obtain (as in the Dirichlet case) that, for each fixed $t\geqq 0$, T_t^N converges in the sense of Theorem 5.2 to T_t, where

$$(5.13)\qquad (T_t h)(x)=E^x\{e_q(t)\,\hat{e}_c(t)\,1_{[t<T_B]}\,h(X_t)\}.$$

Notice that this T_t has all the properties listed in Theorem 3.4, since T_t^N has them and the convergence happens in a very strong sense.

By Theorem 4.3, the weak solution u_N of (5.11) must satisfy

$$(5.14)\qquad (I-T_t^N)\,u_N(x)=E^x\left\{\int_0^t e_q(s)\,\hat{e}_{c_N}(s)\,f_N(X_s)\,dL_s\right\}.$$

Assuming that 1 is not an eigenvalue of T_t, for some $t>0$, we can take limits as $N\uparrow\infty$ in (5.14) and conclude that $u_N(x)\to u(x)$ where

$$(5.15)\quad (I-T_t)\,u(x)=E^x\left\{\int_0^{t\wedge T_B} e_q(s)\,\hat{e}_c(s)\,f(X_s)\,dL_s+1_{[t\geqq T_B]}\,e_q(T_B)\,g(X_{T_B})\right\}.$$

(T_B is, of course, the first time X_t hits B.)

Imitating the steps of Sect. 4 we can show that the above u is the unique weak solution (in the martingale sense) of (5.10). Furthermore, if all the eigenvalues of T_t are strictly less than 1, then (5.15) implies (by letting $t\uparrow\infty$)

$$u(x)=E^x\left\{\int_0^{T_B} e_q(t)\,\hat{e}_c(t)\,f(X_t)\,dL_t+e_q(T_B)\,g(X_{T_B})\right\}.$$

6 The drift term

In this section we examine the problem

$$(6.1)\qquad \begin{aligned} &\tfrac{1}{2}\Delta u+b\cdot\nabla u+qu=0, &&\text{in } D;\\ &\frac{\partial u}{\partial n}+cu=-f, &&\text{on } \partial D,\end{aligned}$$

where $D\subset R^d$ is, as usual, a bounded domain with C^3 boundary, $b=(b_1,\ldots,b_d)$ is a R^d-valued function of class $C^{1,\alpha}$ defined on $\bar{D}$, q is bounded and Borel measurable on $\bar{D}$ and c and f are continuous on ∂D. (We take q bounded in order to be sure that the semigroup associated to (6.1) – see (6.8) later in this section – is Fellerian.)

It is well-known (see [L-S] or [S.Y]) that there exists a (normally) reflected diffusion $Y=\{Y_t,t\geqq 0\}$ in D (similar to the standard reflected Brownian motion – SRBM) whose paths satisfy the Skorokhod equation

$$(6.2)\qquad Y_t=B_t+\int_0^t b(Y_s)\,ds+\int_0^t n(Y_s)\,dL_s,$$

where $n(z)$ is, as usual, the inward unit normal vector of ∂D at $z\in\partial D$, $(B_t,\mathscr{F}_t,P_B^x)$ is a Brownian motion in R^d with B_0 in $\bar{D}$ and $L=\{L_t,t\geqq 0\}$, $L_0=0$, is the boundary local time of Y. The process L has similar properties to those of the corresponding process for the SRBM as given in Sect. 1; for example, it is nondecreasing and it increases only when Y_t is on ∂D (for more details look at the references mentioned above). Furthermore, we notice that (6.2) can be written as a stochastic differential equation

$$dY_t = dB_t + b(Y_t)\,dt + n(Y_t)\,dL_t. \tag{6.2'}$$

Now let A be the operator

$$A = \frac{\Delta}{2} + b \cdot \nabla. \tag{6.3}$$

(The part $b \cdot \nabla$ is the drift term.) We consider the initial-boundary value (parabolic) problem (compare with (1.3))

$$\begin{aligned} &\frac{\partial}{\partial t} r(t, x, y) = A_x r(t, x, y), && (t, x, y) \in (0, \infty) \times D \times D; \\ &\lim_{t \downarrow 0} r(t, x, y) = \delta_y(x), && (x, y) \in \bar{D} \times \bar{D}; \\ &\frac{\partial}{\partial n_x} r(t, x, y) = 0, && (t, x, y) \in (0, \infty) \times \partial D \times \bar{D}, \end{aligned} \tag{6.4}$$

where A_x is the operator A of (6.3) acting on the x variables and $\partial/\partial n_x = n(x) \cdot \nabla_x$. It is known (see [I]) that the above problem has a unique (fundamental) solution $r(t, x, y)$ which is strictly positive. It belongs to $C^2(\bar{D} \times \bar{D})$, as a function of (x, y) it is continuous in (t, x, y) on $(0, \infty) \times \bar{D} \times \bar{D}$, its integral with respect to y on D (and on $\bar{D}$, of course) is equal to 1 and it satisfies the Chapman-Kolmogorov equation

$$r(s+t, x, y) = \int_D r(s, x, u)\, r(t, u, y)\, du.$$

(Compare with the properties of $p(t, x, y)$ which are given in Sect. 1.) From the same reference we also get that $r(t, x, y)$ is not necessarily symmetric in x and y, but as a function of (t, y) it solves an initial-boundary value parabolic problem which is the adjoint of (6.4), namely

$$\begin{aligned} &\frac{\partial}{\partial t} r(t, x, y) = A_y^* r(t, x, y), && (t, x, y) \in (0, \infty) \times D \times D; \\ &\lim_{t \downarrow 0} r(t, x, y) = \delta_y(x), && (x, y) \in \bar{D} \times \bar{D}; \\ &\frac{\partial}{\partial n_y} r(t, x, y) - 2(b \cdot n)(y)\, r(t, x, y) = 0, && (t, x, y) \in (0, \infty) \times \bar{D} \times \partial D, \end{aligned} \tag{6.5}$$

where A_y^* is the (formal) adjoint operator A^* of A acting on the y variables. Integration by parts gives

$$A^* v = \frac{\Delta}{2} v - \nabla \cdot (v b) = \frac{\Delta}{2} v - \sum_{j=1}^{d} \frac{\partial}{\partial x_j} (v b_j).$$

The above properties imply that there is a Markov process $\tilde{Y}=\{\tilde{Y}_t, \mathscr{G}_t, t\geqq 0\}$ with transition densities $r(t,x,y)$ and, in fact, that $\tilde{Y}$ has the strong Feller property. In fact, $\tilde{Y}$ is also a Feller process (see [S-U]). It is reasonable to expect that, as in the case of the SRBM, the processes $\tilde{Y}$ and Y of (6.2) are essentially the same. Such a result is not mentioned in our references, so we give it below as a theorem.

Theorem 6.1. *The processes Y and $\tilde{Y}$ have the same law.*

Proof. (Following the method of the proof of Theorem 3.2 of [H1].) If we apply the Itô formula to $g(Y_t)$, where $g\in C^2(\bar{D})$ we get

$$g(Y_t)-g(Y_0)=\int_0^t \nabla g(Y_s)\cdot dB_s+\int_0^t \frac{\partial}{\partial n} g(Y_s)\, dL_s+\int_0^t (Ag)(Y_s)\, ds,$$

thus, if we set

$$F(g;Y_t)=g(Y_t)-g(Y_0)-\int_0^t (Ag)(Y_s)\,ds,$$

we have that $F(g;Y_t)$ is a submartingale whenever $\partial g/\partial n\geqq 0$ on ∂D. But then Y must be unique in law (see [S-V]), so we just have to show that $F(g;\tilde{Y}_t)$ is a $\mathscr{G}_t$-submartingale whenever $\partial g/\partial n\geqq 0$ on ∂D. Now, since $F(g;\tilde{Y}_t)$ is an additive functional (not of bounded variation, of course), the Markov property gives

$$E\{F(g;\tilde{Y}_t)|\mathscr{G}_s\}=E^{\tilde{Y}_s}\{F(g;\tilde{Y}_{t-s})\}+F(g;\tilde{Y}_s).$$

Therefore, to finish the proof we need to show that

$$E^x\{F(g;\tilde{Y}_t)\}\geqq 0$$

for any (t,x) in $(0,\infty)\times\bar{D}$. But the properties of $r(t,x,y)$ – especially (6.5) – give

$$\begin{aligned} E^x\{F(g;\tilde{Y}_t)\} &= E^x\{g(\tilde{Y}_t)\}-g(x)-E^x\left\{\int_0^t (Ag)(\tilde{Y}_s)\,ds\right\}\\ &=\int_D r(t,x,y)\,g(y)\,dy-g(x)-\int_0^t\int_D r(s,x,y)(Ag)(y)\,dy\,ds\\ &=\int_0^t\int_D g(y)(A_y^* r)(s,x,y)\,dy\,ds-\int_0^t\int_D r(s,x,y)(Ag)(y)\,dy\,ds\\ &=\frac{1}{2}\int_0^t\int_{\partial D} r(s,x,y)\frac{\partial}{\partial n} g(y)\,\sigma(dy)\,ds, \end{aligned}$$

which is nonnegative. To obtain the last equation we made use of the boundary condition of (6.5) and the standard multi-dimensional formula for integration by parts. □

We can now define the weak (probabilistic) solution of (6.1) in the same way as we did in Sect. 4 for the problem (3.1), namely

Definition. A function $u \in b\mathscr{B}(\bar{D})$ is called a *weak solution of the problem* (6.1), if, for all $x \in \bar{D}$, $M_f^u(t)$ is a continuous P_B^x-martingale, where

$$(6.6)\quad M_f^u(t) = u(Y_t) - u(Y_0) + \int_0^t c(Y_s)\, u(Y_s)\, dL_s + \int_0^t f(Y_s)\, dL_s + \int_0^t q(Y_s)\, u(Y_s)\, ds.$$

This definition is justified by the fact that if u is a strong solution of (6.1) and we apply the Itô formula to $u(Y_t)$, we will find that the right hand side of (6.5) is equal to a stochastic integral with respect to B.

As in the case without the drift term we set

$$e_q(t) = \exp\left[\int_0^t q(Y_s)\, ds\right] \quad \text{and} \quad \hat{e}_c(t) = \exp\left[\int_0^t c(Y_s)\, dL_s\right].$$

Then we have the following propositions

Lemma 6.2. *The process $M_f^u(t)$ of (6.5) is a continuous P_B^x-martingale if and only if $N_f^u(t)$ is where*

$$(6.7)\qquad N_f^u(t) = e_q(t)\, \hat{e}_c(t)\, u(Y_t) - u(Y_0) + \int_0^t e_q(s)\, \hat{e}_c(s)\, f(Y_s)\, dL_s.$$

Proof. The proof is identical to that of Proposition 4.1 □

Theorem 6.3. *For $g \in L^2(D)$ let's define the semigroup*

$$(6.8)\qquad (T_t\, g)(x) = E_B^x\{e_q(t)\, \hat{e}_c(t)\, g(Y_t)\}.$$

Then u is a weak solution of (6.1) if and only if there is a $t > 0$ such that

$$(6.9)\qquad (I - T_t)\, u(x) = E_B^x\left\{\int_0^t e_q(s)\, \hat{e}_c(s)\, f(Y_s)\, dL_s\right\}.$$

Proof. Again we can repeat (essentially without any change) the proofs of the corresponding statements for the case with $b \equiv 0$ (see the proofs of Lemma 4.2, Theorem 4.3 and Theorem 4.5). □

Remark. If the norm of T_t is strictly less than 1, then we can let $t \to \infty$ in (6.9) and get the representation

$$u(x) = E_B^x\left\{\int_0^\infty e_q(t)\, \hat{e}_c(t)\, f(Y_t)\, dL_t\right\}.$$

The above statements reduce problem (6.1) to the study of the semigroup T_t, defined in (6.9). One way to do this analysis is by using estimates for $r(t, x, y)$ that can be found in [I] and [S-U]. This way is the analog of what we did in the previous chapters for the case without drift. Here we prefer to follow a different approach, based on the Cameron-Martin-Girsanov (C-M-G) transformation.

In (6.2) we set

(6.10) $$W_t = B_t + \int_0^t b(Y_s)\,ds.$$

This can be also written as

(6.10′) $$dB_t = -b(Y_t)\,dt + dW_t.$$

By the C-M-G transformation (see [Ø] or [I-W]), if we assume that $(W_t, \mathscr{F}_t, P_W^x)$ is a Brownian motion in R^d, then $(B_t, \mathscr{F}_t, P_B^x)$ is also a Brownian motion, provided that for any $Z \in \mathscr{F}_t$, we *define*

(6.11) $$E_B^x\{Z\} = E_W^x\{M_t Z\},$$

where

(6.19) $$M_t = \exp\left\{-\frac{1}{2}\int_0^t |b(Y_s)|^2\,ds + \int_0^t b(Y_s)\cdot dW_s\right\}.$$

(If $b = \nabla\phi$, we can use the Itô formula to get rid of the stochastic integral that appears in the above formula.) Thus, (going backwards) since we have already taken B to be a Brownian motion, we can, in addition, take W also to be a Brownian motion and still be consistent if we assume that (6.11) is always valid. Now, by using (6.10) in (6.2) we get

$$Y_t = W_t + \int_0^t n(Y_s)\,dL_s.$$

In other words (Y_t, P_W^x) becomes a SRBM! Then, because of (6.11), (6.8) becomes

(6.13) $$(T_t g)(x) = E_W^x\{M_t\, e_q(t)\, \hat{e}_c(t)\, g(X_t)\}.$$

Theorem 6.4. *For each $t > 0$ the operator T_t is compact on $L^p(D)$, for $p \in (1, \infty]$ and on $C(\bar{D})$. If $g \in L^p(D)$, then $T_t g \in C(\bar{D})$ and there is a constant C_t (depending only on t) such that*

(6.14) $$\| T_t g \|_\infty \leqq C_t \| g \|_p.$$

Also, if $g \geqq 0$ is Borel measurable and $p > 1$, then there is another constant C_t' such that

(6.15) $$\| g \|_p \leqq C_t' \inf_{x \in \bar{D}} (T_t g)(x).$$

Proof. Let p' be such that $1/p + 1/p' = 1$. We apply Hölder's inequality to (6.13) and get

(6.16) $$|(T_t g)(x)| \leqq E_W^x\{M_t^{p'}\}^{1/p'}\, E_W^x\{e_{pq}(t)\, \hat{e}_{pc}(t)\, g(X_t)^p\}^{1/p}.$$

Since $M_t^{p'}$ has the same form as M_t, by the stochastic process version of the John-Nirenberg inequality (see [D-M]) there is a constant K_t such that

$$E_W^x\{M_t^{p'}\}^{1/p'} \leqq K_t.$$

Hence, if we define

$$(T_t^0 g)(x) = E_W^x\{e_{pq}(t)\, \hat{e}_{pc}(t)\, g(Y_t)\},$$

then (6.16) becomes

$$|(T_t g)(x)| \leqq K_t |(T_t^0 g^p)(x)|^{1/p'}$$

or

$$\text{(6.17)} \qquad \| T_t g \|_\infty \leqq K_t \| T_t^0 g^p \|_\infty^{1/p}.$$

But T_t^0 is a semigroup of the type we studied in Sect. 3 (remember that (Y_t, P_W^x) is a SRBM). Thus, by Theorem 3.4, (6.17) becomes

$$\| T_t g \|_\infty \leqq C_t \| g^p \|_1^{1/p},$$

which is (6.14). The continuity of $T_t g$ follows from the Feller and the strong Feller property of Y (see [C2]). Since the inclusion operator from $L^\infty(D)$ to $L^p(D)$, $p \in (1, \infty)$, is compact (6.14) implies that T_t is compact on $L^p(D)$ for $p \in (1, \infty)$. Also, for $0 < r < s < t$, we have that $T_t = T_{t-s} T_{s-r} T_r$ and we can suppose that T_r maps $L^\infty(D)$ or $C(\bar{D})$ into $L^p(D)$, where $p \in (1, \infty)$, that T_{s-r} maps $L^p(D)$ into itself and that T_{t-s} maps $L^p(D)$ into $L^\infty(D)$ or $C(\bar{D})$ respectively, by (6.14). Since T_{s-r} is compact on $L^p(D)$ we can conclude (since the product of a compact and a bounded operator is compact) that T_t is compact on $L^\infty(D)$ and on $C(\bar{D})$.

To prove (6.15), we follow the method of Proposition 3.5. First we assume that g is bounded. The positivity and continuity of $r(t, x, y)$ imply

$$E_B^x\{g(Y_t)\} = \int_D r(t, x, y)\, g(y)\, dy \geqq A_t' \| g \|_1.$$

Therefore, for each $p > 1$ there is a constant A_t (independent of g and x) for which

$$\text{(6.18)} \qquad A_t \| g \|_p \leqq E_B^x\{g(Y_t)\}.$$

Now

$$\begin{aligned} E^x\{g(Y_t)\}^2 &= E_B^x\{e_{q/2}(t)\, \hat{e}_{c/2}(t)\, g^{\frac{1}{2}}(Y_t)\, e_{-q/2}(t)\, \hat{e}_{-c/2}(t)\, g^{\frac{1}{2}}(Y_t)\}^2 \\ &\leqq E_B^x\{e_q(t)\, \hat{e}_c(t)\, g(Y_t)\}\; E_B^x\{e_{-q}(t)\, \hat{e}_{-c}(t)\, g(Y_t)\} \\ &= (T_t g)(x) (\tilde{T}_t g)(x) \\ &\leqq C_t \| g \|_p (T_t g)(x), \end{aligned}$$

where we first applied Schwarz's inequality, then we denoted by $\{\tilde{T}_t, t \geqq 0\}$ the semigroup that corresponds to $-q$ and $-c$ and, finally, we applied (6.14) to $\tilde{T}_t$ to get the last inequality (where $C_t > 0$).

Hence, (6.18) implies

$$A_t^2 \| g \|_p^2 \leqq C_t \| g \|_p (T_t g)(x)$$

and so

$$\|g\|_p \le \frac{C_t}{A_t^2}(T_t f)(x), \qquad \text{for every } x \in \bar{D}.$$

This is (6.15) for bounded g. If g is not bounded, we apply (6.15) to $g \wedge n$ and then we use monotone convergence. □

Remark. We don't know whether Theorem 6.4 remains valid for $p=1$.

Corollary 6.5. *If* 1 *is not an eigenvalue of* T_t, *then problem* (6.1) *has a unique weak (probabilistic) solution which is continuous on* $\bar{D}$.

Proof. The result follows from (6.9), Theorem 6.4 and the Fredholm Alternative, provided that the function

$$F(x) = E_B^x \left\{ \int_0^t e_q(s)\, \hat{e}_c(s)\, f(Y_s)\, dL_s \right\},$$

of the right-hand side of (6.9), is in $C(\bar{D})$. This can be shown in exactly the same way as in the case without drift (see the proof of Theorem 4.3). We need the strong Feller property of Y, (6.14) and an estimate for $r(t,x,y)$ that can be found in [S-U], namely

$$E_B^x\{L_t\} = \frac{1}{2} \int_0^t \int_{\partial D} r(s,x,y)\, \sigma(dy)\, ds \le C\sqrt{t}. \quad \square$$

Corollary 6.6. *If u is a strong (analytic) solution of* (6.1) *and* 1 *is not an eigenvalue of* T_t, *then u is also the weak (probabilistic) solution of* (6.1).

Proof. This follows immediately by applying the Itô formula to u and the uniqueness of the weak solution (Corollary 6.5). □

Finally, we define the gauge of (6.1) to be

$$G(x) = E_B^x \left\{ \int_0^\infty e_q(t)\, \hat{e}_c(t)\, dL_t \right\}. \tag{6.19}$$

Notice that monotone convergence gives

$$G(x) = \lim_{t\to\infty} E_B^x \left\{ \int_0^t e_q(s)\, \hat{e}_c(s)\, dL_s \right\} = \lim_{t\to\infty} E_W^x \left\{ M_t \int_0^t e_q(s)\, \hat{e}_c(s)\, dL_s \right\}. \tag{6.19'}$$

We have to write the expressions for finite t first and then take limits because, if we just put $t=\infty$, we get the factor M_∞ which usually doesn't make sense. Then we have the following theorem which is similar to Theorem 3.6.

Theorem 6.7. (The Gauge Theorem.) *If there is an $x_0 \in \bar{D}$ for which $G(x_0)$ is finite, then G is continuous on $\bar{D}$.*

Proof. (The proof is a slight modification of the proof of Theorem 3.6.)

From (6.19) we get

$$G(x)=E_B^x\left\{\int_0^t \hat{e}_c(s)\,e_q(s)\,dL_s\right\}+E_B^x\left\{\int_t^\infty \hat{e}_c(s)\,e_q(s)\,dL_s\right\}, \tag{6.20}$$

where t is taken to be some (fixed) number in $(0, \infty)$. Let $E_1(t)$ and $E_2(t)$ be the first and the second term, respectively, of the right-hand side of (6.20). Then

$$0\leqq E_1(t)\leqq E_B^x\{e_{|q|}(t)\,\hat{e}_{|c|}(t)\,L(t)\},$$

which is bounded on $\bar{D}$ by Schwarz's inequality and the estimate for $E_B^x\{L_t\}$ mentioned in the proof of Corollary 6.5. In fact, the same estimate implies that

$$\limsup_{t\downarrow 0}\ \sup_{x\in\bar{D}} E_1(t)=0. \tag{6.21}$$

Next we set

$$Z=Z(\omega)=\int_0^\infty e_q(s)\,\hat{e}_c(s)\,dL_s. \tag{6.22}$$

Then

$$\begin{aligned}Z\circ\theta_t&=\int_0^\infty [e_q(s)\circ\theta_t][\hat{e}_c(s)\circ\theta_t]\,d(L_s\circ\theta_t)\\&=\int_0^\infty \exp\left[\int_t^{t+s} q(Y_u)\,du\right]\exp\left[\int_t^{t+s} c(Y_u)\,dL_u\right]dL_{s+t}\\&=e_q(t)^{-1}\,\hat{e}_c(t)^{-1}\int_t^\infty e_q(s)\,\hat{e}_c(s)\,dL_s,\end{aligned}$$

where in the integral with respect to dL_{s+t}, the dummy variable is s. Hence, (6.20) gives

$$\begin{aligned}E_2(t)&=E_B^x\{e_q(t)\,\hat{e}_c(t)[Z\circ\theta_t]\}\\&=E_B^x\{e_q(t)\,\hat{e}_c(t)\,E_B[Z\circ\theta_t|\mathscr{F}_t]\}\\&=E_B^x\{e_q(t)\,\hat{e}_c(t)\,E_B^{Y_t}[Z]\},\end{aligned}$$

where the last equality follows from the Markov property. Using (6.22) and (6.19) we get

$$E_2(t)=E_B^x\{e_q(t)\,\hat{e}_c(t)\,G(Y_t)\}=(T_t\,G)(x)$$

and so (6.20) becomes

$$G(x)=E_1(t)+(T_t\,G)(x). \tag{6.23}$$

Now we use (6.15) of Theorem 6.4 to get (since $G\geqq 0$)

$$\infty>G(x_0)\geqq(T_t\,G)(x_0)\geqq C_t'\,\|G\|_p,\qquad \text{for any } p>1.$$

Therefore $G \in L^p(D)$ and so, by Theorem 6.4 we get that

$$(T_t G) \in C(\bar{D}).$$

Hence G is bounded on $\bar{D}$, because of (6.21) and (6.23). Furthermore, since the limit in (6.21) is uniform in x, we get that G is in $C(\bar{D})$. □

Appendix

First we mention a (slightly generalized version of a) result (known as Khas'minskii's Lemma) that plays a dominant role in the probabilistic study of the Schrödinger equation, when the potential function is not negative. The original statement can be found in [Kh].

Proposition A1. *Let $A^{(j)}, j=1, \ldots, n$, be (nonnegative) additive functionals of the Markov process Y with state space E. If for a fixed $t>0$ there are α_j's such that*

$$\sup_{y \in E} E^y\{A_t^{(j)}\} \leqq \alpha_j, \qquad j=1, \ldots, n,$$

then

$$\sup_{y \in E} E^y\left\{\prod_{j=1}^n A_t^{(j)}\right\} \leqq n! \prod_{j=1}^n \alpha_j.$$

Remark. Instead of the deterministic time t we could have a *terminal* time T. An optional time T is terminal if $T = \lim_{s \downarrow 0}(s + T \circ \theta_s)$ and $T = s + T \circ \theta_s$, on the set $\{\omega : s < T(\omega)\}$; for example, every hitting time is terminal).

Corollary A2. *If*

$$\sup_{y \in E} E^y\{A_t\} \leqq \alpha < 1,$$

then

$$\sup_{y \in E} E^y\{e^{A_t}\} \leqq \frac{1}{1-\alpha}.$$

Remark. If there is a $t>0$ such that $\sup_{y \in E} E^y\{e^{A_t}\} < \infty$, then the same is true for all $t \in R^+$. This is because

$$E^y\{e^{A_{2t}}\} = E^y\{e^{A_t}(e^{A_t} \circ \theta_t)\} = E^y\{e^{A_t} E^{Y_t}[e^{A_t}]\} \leqq E^y\{e^{A_t}\} \sup_{y \in E} E^y\{e^{A_t}\} \qquad \text{etc.}$$

(In fact, this argument works for any multiplicative functional. For the definition of a multiplicative functional see Sect. 3.)

Proposition A3. *Let A and B be additive functionals of Y such that*

$$\sup_{y \in E} E^y\{A_t\} \leqq \alpha_t \quad \textit{and} \quad \sup_{y \in E} E^y\{B_t\} \leqq \alpha_t,$$

where $\lim_{t \downarrow 0} \alpha_t = 0$. *Then,*

$$\limsup_{t \downarrow 0} \sup_{y \in E} E^y\{|e^{A_t} - e^{B_t}|^p\} = 0,$$

for any $p \geqq 1$.

Sketch of proof. Let k be a positive integer. Using the binomial expansion of $(e^{A_t} - 1)^k$ and the previous corollary we get

$$\limsup_{t \downarrow 0} \sup_{y \in E} E^y\{(e^{A_t} - 1)^k\} = 0.$$

But,

$$E^y\{|e^{A_t} - e^{B_t}|^p\}^{1/p} \leqq E^y\{(e^{A_t} - 1)^p\}^{1/p} + E^y\{(e^{B_t} - 1)^p\}^{1/p}. \quad \square$$

From now on we assume that $c \in C(\bar{D})$. The additive functionals $A_c(t)$ and $A_c^\delta(t)$ are as in (1.11) and (1.12) respectively.

Proposition A4. *Let* n *be a positive integer. If* $0 < \delta \leqq \delta_0$ (see $P5$ of Sect. 1), *then*

$$\text{(A1)} \qquad E^x\{A_c^\delta(t)^n\} = \frac{n!}{(2\delta)^n} \int_0^t \int_0^{t-s_n} \cdots \int_0^{t-s_n-\dots-s_2} \int_{D_\delta} \cdots \int_{D_\delta} p(s_1, x, y_1) \dots$$
$$p(s_{n-1}, y_{n-2}, y_{n-1})\, p(s_n, y_{n-1}, y_n)\, c(y_1) \dots c(y_n)\, dy_1 \dots dy_n\, ds_1 \dots ds_{n-1}\, ds_n.$$

Also,

$$\text{(A2)} \qquad E^x\{A_c(t)^n\} = \frac{n!}{2^n} \int_0^t \int_0^{t-s_n} \cdots \int_0^{t-s_n-\dots-s_2} \int_{\partial D} \cdots \int_{\partial D} p(s_1, x, y_1) \dots$$
$$p(s_{n-1}, y_{n-2}, y_{n-1})\, p(s_n, y_{n-1}, y_n)\, c(y_1) \dots c(y_n)\, \sigma(dy_1) \dots \sigma(dy_n)\, ds_1 \dots ds_{n-1}\, ds_n$$

and, in fact, (A2) is true for any $c \in \Sigma_d(\partial D)$.

Proof. The proof is a direct extension of the proof of Proposition 1.1. $\square$

Corollary A5. *There is a* $\delta_0 > 0$ *and a* $K = K(\delta_0) > 0$ *such that, for all* $t \in [0, \infty)$, *all* $\delta \in (0, \delta_0]$ *and all integers* $n > 0$ *we have*

$$\sup_{x \in \bar{D}} E^x\{|A_c^\delta(t)|^n\} \leqq K^n (t + \sqrt{t})^n \quad \text{and} \quad \sup_{x \in \bar{D}} E^x\{|A_c(t)|^n\} \leqq K^n (t + \sqrt{t})^n.$$

Notice that the above formulas imply immediately that

$$\sup_{x \in \bar{D}} E^x\{e^{A_c^\delta(t)}\} \leqq e^{K(t + \sqrt{t})} \leqq K' e^{Kt} \quad \text{and} \quad \sup_{x \in \bar{D}} E^x\{e^{A_c(t)}\} \leqq e^{K(t + \sqrt{t})} \leqq K' e^{Kt},$$

where $K' \geqq 1$ is some constant.

Proof. Just use $P5$ of Sect. 1 and the previous proposition. $\square$

Lemma A6. *Let* δ_0 *and* D_δ *be as in* $P5$ *of Sect. 1. Also, assume that* $c \in C^2(\bar{D})$ *i.e.* c *is twice continuously differentiable in* D *and its derivatives extend continuously on* $\bar{D}$. *Then there is a family of functions* $\{g_\delta, 0 \leqq \delta \leqq \delta_0\}$ *satisfying the following conditions:*

(a) $g_\delta \in C^2(\bar{D})$, $g_\delta = O(\delta^2)$ *and* $\operatorname{supp}(g_\delta) \subset \bar{D}_\delta$;
(b) $\nabla g_\delta = O(\delta)\, I_{D_\delta}$ *and* $\nabla g_\delta(z) = -\delta\, c(z)\, n(z) + O(\delta^2)$, *for* $z \in \partial D$;
(c) $\Delta g_\delta = [c(x) + O(\delta)]\, I_{D_\delta}$.

The notation $f = O(\delta)$ *means, as usual, that there is a positive constant* K *such that* $|f(x)| \leqq K\delta$, *for all* $x \in \bar{D}$, $\delta \in [0, \delta_0]$.

(Remember that D *is a bounded domain with* C^3 *boundary and* $n(z)$ *is the inward unit normal at* $z \in \partial D$.*)*

Proof. If $\delta \in [0, \delta_0]$, every $x \in \bar{D}_\delta$ can be written (uniquely) as

$$x = z + \alpha n(z),$$

where z is the (unique) point on ∂D which is closest to x and $\alpha = d(x, \partial D)$. Now, let

$$f_\delta(x) = \begin{cases} \frac{1}{2}(\delta - \alpha)^2, & \text{if } x = z + \alpha n(z) \in \bar{D}\delta; \\ 0, & \text{if } x \in \bar{D} \setminus \bar{D}_\delta. \end{cases}$$

It is easy to prove (see [H1] for the details) that the family $\{f_\delta, 0 \leqq \delta \leqq \delta_0\}$ satisfies the following:
(a′) $f_\delta \in C^2(\bar{D})$, $f_\delta = O(\delta^2)$ and $\operatorname{supp}(f_\delta) \subset \bar{D}_\delta$;
(b′) $\nabla f_\delta = O(\delta)\, I_{D_\delta}$ and $\nabla f_\delta(z) = -\delta\, n(z)$ for $z \in \partial D$;
(c′) $\Delta f_\delta = [1 + O(\delta)]\, I_{D_\delta}$.

So, if we take $g_\delta = c\, f_\delta$, we are done since:

$$\nabla g_\delta = c \nabla f_\delta + f_\delta \nabla c = c O(\delta)\, I_{D_\delta} + O(\delta^2)\, I_{D_\delta} \nabla c = O(\delta)\, I_{D_\delta}.$$

Similarly,

$$\Delta g_\delta = c \Delta f_\delta + 2 \nabla c \cdot \nabla f_\delta + f_\delta \Delta c = [c + O(\delta)]\, I_{D_\delta}$$

and, in exactly the same way, we can evaluate $\nabla g_\delta(z)$ for z in ∂D. □

Proposition A7. *Let* $c \in C(\bar{D})$ *and* $T > 0$. *Then*

$$\sup_{x \in \bar{D}} E^x \{ \sup_{0 \leqq t \leqq T} [A_c^\delta(t) - A_c(t)]^2 \} \to 0, \qquad \text{as } \delta \downarrow 0.$$

(i.e. A_c^δ *converges to* A_c *in the* $L^2(P^x)$ *sense and the convergence is uniform in* x *and in* t *on compact sets.)*

Proof. First take $c \in C^2(\bar{D})$. Let's write Itô formula for g_δ (of the previous lemma) and X (the SRBM which is a semimartingale). Using (1.2) we get

$$g_\delta(X_t) = g_\delta(X_0) + \int_0^t \nabla g_\delta(X_s) \cdot dB_s + \int_0^t \frac{\partial g_\delta}{\partial n}(X_s)\, L(ds) + \frac{1}{2} \int_0^t \Delta g_\delta(X_s)\, ds.$$

Now, we divide through by δ and apply (a), (b) and (c) of Lemma A6. Notice that (b) implies that $(\partial g_\delta / \partial n)(z) = -\delta\, c(z) + O(\delta^2)$, if $z \in \partial D$. The result is:

$$\begin{aligned} \int_0^t c(X_s)\, L(ds) - \frac{1}{2\delta} \int_0^t c(X_s)\, I_{D_\delta}(X_s)\, ds = {} & O(\delta) + \int_0^t O(\delta)\, L(ds) \\ & + O(1) \int_0^t I_{D_\delta}(X_s)\, ds \\ & + \frac{1}{\delta} \int_0^t \nabla g_\delta(X_s) \cdot dB_s \end{aligned}$$

or, equivalently,

$$A_c(t)-A_c^\delta(t)=O(\delta)+O(\delta)\,L(t)+O(\delta)\,L^\delta(t)+\frac{1}{\delta}\int_0^t \nabla g_\delta(X_s)\cdot dB_s.$$

Next, we take the supremum in t over $[0, T]$ and then $L^2(P^x)$-norms. Trivially, the norm of the right-hand side is less or equal than the sum of the norms of its terms. The first three terms have norms that go to 0 with δ by Corollary A5 (applied to L and L^δ). To finish this case, we need to estimate the norm of the stochastic integral.

By Doob's inequality (see [C-W]) we have

$$E^x\left\{\sup_{0\leqq t\leqq T}\left(\frac{1}{\delta}\int_0^t \nabla g_\delta(X_s)\cdot dB_s\right)^2\right\}\leqq 4E^x\left\{\left(\frac{1}{\delta}\int_0^t \nabla g_\delta(X_s)\cdot dB_s\right)^2\right\}.$$

Using the independence of the components of B, the isometry for the stochastic integrals and the fact that the stochastic integral is a zero mean martingale, we get that the last term above is equal to

$$E^x\left\{\frac{1}{\delta^2}\int_0^T |\nabla g_\delta|^2\, ds\right\}$$

and by condition (b) of Lemma A6, this equals

$$E^x\left\{\frac{1}{\delta^2}\int_0^T O(\delta)^2\, I_{D_\delta(X_s)}\, ds\right\}=O(\delta)\,E^x\{L^\delta(T)\},$$

which tends to 0 as $\delta\downarrow 0$. So, we have proved the proposition for $c\in C^2(\bar{D})$.

Finally, if $c\in C(\bar{D})$, for any (given) $\varepsilon>0$ there is a $\bar{c}\in C^2(\bar{D})$ such that

$$\sup_{x\in\bar{D}}|c(x)-\bar{c}(x)|<\varepsilon$$

(by the Stone-Weierstrass Theorem). Therefore,

$$\begin{aligned}E^x\{\sup_{t\in[0,T]}|A_c(t)-A_c^\delta(t)|^2\}^{\frac{1}{2}}\leqq{}& E^x\{\sup_{t\in[0,T]}|A_c(t)-A_{\bar{c}}(t)|^2\}^{\frac{1}{2}}\\ &+E^x\{\sup_{t\in[0,T]}|A_{\bar{c}}(t)-A_{\bar{c}}^\delta(t)|^2\}^{\frac{1}{2}}\\ &+E^x\{\sup_{t\in[0,T]}|A_{\bar{c}}^\delta(t)-A_c^\delta(t)|^2\}^{\frac{1}{2}}\\ \leqq{}& \varepsilon E^x\{L(T)^2\}^{\frac{1}{2}}+E^x\{\sup_{t\in[0,T]}|A_{\bar{c}}(t)-A_{\bar{c}}^\delta(t)|^2\}^{\frac{1}{2}}\\ &+\varepsilon E^x\{L^\delta(T)^2\}^{\frac{1}{2}}.\end{aligned}$$

Now, take suprema over $\bar{D}$ and then let $\delta \downarrow 0$. Since $\bar{c} \in C^2(\bar{D})$, we can use the previous analysis for the second term above and conclude that it goes to 0. Also, we can apply Corollary A5 to the other two terms and get

$$\limsup_{\delta \downarrow 0} \sup_{x \in \bar{D}} E^x\{\sup_{t \in [0,T]} |A_c(t) - A_c^\delta(t)|^2\}^{\frac{1}{2}} \leqq \varepsilon K_T,$$

where K_T is a constant that depends on T only. So we are done, since ε was arbitrary. □

Theorem A8. *For all $t \geqq 0$ we have*

$$\lim_{\delta \downarrow 0} e^{A_c^\delta(t)} = e^{A_c(t)} \qquad \text{in } L^1(P^x)$$

and the convergence is uniform in x (on $\bar{D}$) and in t on compact sets.
In particular,

$$\lim_{\delta \downarrow 0} e^{L^\delta(t)} = e^{L(t)} \qquad \text{in } L^1(P^x).$$

Proof. If $\delta \downarrow 0$, then (see the proof of Proposition 1.1)

$$A_c^\delta(t) \to A_c(t) \qquad \text{for a.e. } \omega.$$

Next, we notice that

$$\text{(A3)} \qquad \frac{e^x - 1}{\sqrt{x}} \leqq e^x, \quad \text{if } x > 0 \quad \text{and} \quad |e^x - 1| \leqq e^{|x|} - 1, \qquad \text{for all } x \in R^1.$$

Now,

$$\text{(A4)} \qquad E^x\{|e^{A_c^\delta(t)} - e^{A_c(t)}|\} = E^x\{e^{A_c(t)} |e^{A_c^\delta(t) - A_c(t)} - 1|\}$$
$$\leqq E^x\{e^{2A_c(t)}\}^{\frac{1}{2}} E^x\{[e^{|A_c^\delta(t) - A_c(t)|} - 1]^2\}^{\frac{1}{2}},$$

by Schwarz's inequality and (A3). But $E^x\{e^{2A_c(t)}\} = E^x\{e^{A_{2c}(t)}\}$ and so it is bounded by some bound K_t^2 that depends only on t, because of Corollary A5. So, (A4) becomes

$$E^x\{|e^{A_c^\delta(t)} - e^{A_c(t)}|\} \leqq K_t E^x\left\{|A_c^\delta(t) - A_c(t)| \left[\frac{e^{|A_c^\delta(t) - A_c(t)|} - 1}{|A_c^\delta(t) - A_c(t)|^{\frac{1}{2}}}\right]^2\right\}^{\frac{1}{2}}$$
$$\leqq K_t E^x\{|A_c^\delta(t) - A_c(t)|^2\}^{\frac{1}{4}} E^x\{e^{2|A_c^\delta(t) - A_c(t)|}\}^{\frac{1}{4}},$$

by Schwarz and (A3). But

$$E^x\{e^{2|A_c^\delta(t) - A_c(t)|}\} \leqq E^x\{e^{A_{2|c|}^\delta(t) + A_{2|c|}(t)}\}$$

and so it is bounded, again by Schwarz and Corollary A5. So we are done by using Proposition A7. □

Remark. An immediate corollary of the theorem is that if $\delta \downarrow 0$, then

$$A_c^\delta(t) \to A_c(t) \qquad \text{in } L^p(P^x)$$

uniformly in x on $\bar{D}$ and in t on compact sets.

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