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A stochastic Hopf bifurcation

Peter H. Baxendale *

Department of Mathematics, University of Southern California,
Los Angeles, CA 90089-1113, USA

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Summary. Let $\{x_t : t \geqq 0\}$ be the solution of a stochastic differential equation (SDE) in $\mathbb{R}^d$ which fixes 0, and let λ denote the Lyapunov exponent for the linear SDE obtained by linearizing the original SDE at 0. It is known that, under appropriate conditions, the sign of λ controls the stability/instability of 0 and the transience/recurrence of $\{x_t : t \geqq 0\}$ on $\mathbb{R}^d \setminus \{0\}$. In particular if the coefficients in the SDE depend on some parameter z which is varied in such a way that the corresponding Lyapunov exponent λ^z changes sign from negative to positive the (almost-surely) stable fixed point at 0 is replaced by an (almost-surely) unstable fixed point at 0 together with an attracting invariant probability measure μ^z on $\mathbb{R}^d \setminus \{0\}$. In this paper we investigate the limiting behavior of μ^z as λ^z converges to 0 from above. The main result is that the rescaled measures $(1/\lambda^z)\,\mu^z$ converge (in an appropriate weak sense) to a non-trivial σ-finite measure on $\mathbb{R}^d \setminus \{0\}$.

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1 Introduction

Consider the (Stratonovich) stochastic differential equation in $\mathbb{R}^d$

$$dx_t = V_0(x_t)\,dt + \sum_{\alpha=1}^{r} V_\alpha(x_t) \circ dW_t^\alpha \tag{1.1}$$

where $V_0, V_1, \ldots, V_r$ are smooth vector fields on $\mathbb{R}^d$ satisfying

$$V_0(0) = V_1(0) = \ldots = V_r(0) = 0 \tag{1.2}$$

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and $\{(W_t^1, \ldots, W_t^r): t \geqq 0\}$ is a standard $\mathbb{R}^r$-valued Brownian motion on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$. The resulting (possibly explosive) diffusion process $\{x_t: t \geqq 0\}$ in $\mathbb{R}^d$ has 0 as a fixed point and the non-trivial behavior of the diffusion takes place in $\mathbb{R}^d \setminus \{0\}$. In a previous paper [Ba3] we studied the relationship between the stability of the linearized system at 0 and the recurrence or transience of the diffusion process $\{x_t: t \geqq 0\}$ on $\mathbb{R}^d \setminus \{0\}$. More precisely consider the linear stochastic differential equation

$$dv_t = A_0 v_t \, dt + \sum_{\alpha=1}^{r} A_\alpha v_t \circ dW_t^\alpha \tag{1.3}$$

where $A_\alpha = DV_\alpha(0) \in L(\mathbb{R}^d)$ for $0 \leqq \alpha \leqq r$, and define the Lyapunov exponent

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log \|v_t\| \quad \text{w.p.1.} \tag{1.4}$$

Under suitable non-degeneracy conditions λ is well-defined (i.e. the limit exists and is independent of $v_0 \neq 0$) and then it is clear that the sign of λ controls the almost sure stability properties of the linearized process $\{v_t: t \geqq 0\}$. Returning to the original process, if we impose conditions which ensure that $\mathbf{P}\{\|x_t\| \to \infty \text{ as } t \to \infty\} = 0$ then the process $\{x_t: t \geqq 0\}$ on $\mathbb{R}^d \setminus \{0\}$ is transient, or null recurrent, or positive recurrent according as $\lambda < 0$, or $\lambda = 0$, or $\lambda > 0$. For full details and proofs see [Ba3]. A slightly more general version is given here in Theorem 2.8.

In this paper we consider the situation where the vector fields in the stochastic differential equation (1.1) depend on some parameter z, say. Then the Lyapunov exponent λ and the invariant probability measure μ on $\mathbb{R}^d \setminus \{0\}$ (defined when $\lambda > 0$) will also depend upon the parameter. We write λ^z and μ^z. Theorem 2.12 asserts the continuous dependence of μ^z upon z in and up to the boundary of the region where $\lambda^z > 0$. Our main result, Theorem 2.13, describes the rate of convergence of the invariant probability measures μ^w as the parameter w is varied in such a way that $w \to z$ and λ^w converges to $\lambda^z = 0$ from above. It asserts that, under suitable conditions, the rescaled measures $(1/\lambda^w)\mu^w$ converge to an invariant measure $\bar{\mu}$, say, on $\mathbb{R}^d \setminus \{0\}$ for the system with parameter z. Moreover the measure $\bar{\mu}$ assigns finite mass to sets of the form $\mathbb{R}^d \setminus B(0, r)$, and we identify the finite positive limit of $\bar{\mu}(\mathbb{R}^d \setminus B(0, r))/|\log r|$ as $r \to 0$. Thus our result gives a rate for the weak convergence (as probability measures in $\mathbb{R}^d$) of μ^w to the unit mass at 0.

The reason for our title is as follows. Suppose that the parameter z is varied in such a way that λ^z passes from negative values through zero into positive values. Then the (almost-surely) stable fixed point at 0 is replaced by an (almost-surely) unstable fixed point at 0 together with an attracting invariant probability measure μ^z on $\mathbb{R}^d \setminus \{0\}$. Moreover for small positive λ^z the corresponding measure μ^z has most of its mass near 0;

our Theorem 2.13 gives a quantitative version of this assertion. Thus our findings may be viewed as a stochastic version of the Hopf bifurcation for deterministic dynamical systems (see e.g. [GH]). Notice however in the deterministic Hopf bifurcation the limit cycle is at distance $O(\sqrt{\lambda})$ from 0; this is a very different sort of scaling from that involved in the stochastic case, see Corollary 2.14.

While discussing stochastic bifurcation theory we should mention the work of Arnold and Boxler and Xu Kedai on bifurcation theory for random dynamical systems using multiplicative ergodic theory and stochastic stable, unstable and center manifolds, see [AB1], [AB2], [AK4], [Bo]. There is also recent work by Arnold and Xu Kedai on normal forms for random dynamical systems, see [AK1–3].

The plan of this paper is as follows. Section 2 describes the setting, makes precise the assumptions needed at various stages, and has the statements of the three theorems 2.8, 2.12 and 2.13. Section 3 deals with the behavior of $\{x_t: t \geqq 0\}$ away from 0. Section 4 is concerned with constructing suitable Lyapunov style functions in a neighborhood of 0; these are then used in sub- and supermartingale inequalities in Sect. 5 to obtain some very detailed estimates of occupation times for the process $\{x_t: t \geqq 0\}$ near 0. Section 6 deals with the construction of the invariant measures μ^z on $\mathbb{R}^d \setminus \{0\}$ whenever the Lyapunov exponent $\lambda^z \geqq 0$. The results of Sects. 3 through 6 are put together in Sect. 7 to obtain the proofs of Theorems 2.8, 2.12 and 2.13. Finally in Sect. 8 we give three examples of applications of our results to bifurcation scenarios.

2 Statement of results

Throughout this paper z will denote a parameter which can vary smoothly in some fixed parameter space N. The precise structure of N will not be important except that we wish to be able to talk about smooth dependence on z. Since our results are local in z, we may without loss of generality assume that N is an open set in some Euclidean space. In particular z can be multidimensional. Consider for any $z \in N$ the (Stratonovich) stochastic differential equation in $\mathbb{R}^d$

$$dx_t = V_0(x_t, z)\,dt + \sum_{\alpha=1}^{r} V_\alpha(x_t, z) \circ dW_t^\alpha \tag{2.1}$$

where $V_0, V_1, \ldots, V_r$ are smooth functions from $\mathbb{R}^d \times N$ to $\mathbb{R}^d$ satisfying

$$V_0(0, z) = V_1(0, z) = \ldots = V_r(0, z) = 0 \tag{2.2}$$

for all $z \in N$. For each fixed z and α we regard the mapping $x \mapsto V_\alpha(x, v)$ as a vector field V_α^z on $\mathbb{R}^d$. Thus we may rewrite (2.1) as

$$dx_t = V_0^z(x_t)\,dt + \sum_{\alpha=1}^{r} V_\alpha^z(x_t) \circ dW_t^\alpha.$$

Where appropriate we shall denote the dependence on z of the resulting (possibly explosive) diffusion process by writing $\{x_t^z: t \geqq 0\}$. For $z \in N$ the process $\{x_t^z: t \geqq 0\}$ has generator

$$L^z = \tfrac{1}{2} \sum_{\alpha=1}^{r} (V_\alpha^z)^2 + V_0^z \tag{2.3}$$

and we write $P^z(t, x, A)$ for the corresponding transition probability.

Linearizing (2.1) at 0 we obtain the linear stochastic differential equation

$$dv_t = A_0^z v_t \, dt + \sum_{\alpha=1}^{r} A_\alpha^z v_t \circ dW_t^\alpha \tag{2.4}$$

where $A_\alpha^z = DV_\alpha^z(0) \in L(\mathbb{R}^d)$ for $0 \leqq \alpha \leqq r$. Define $\theta_t = v_t / \|v_t\| \in S^{d-1}$. Then applying Itô's formula to (2.4) we obtain

$$d\theta_t = \tilde{A}_0^z(\theta_t) \, dt + \sum_{\alpha=1}^{r} \tilde{A}_\alpha^z(\theta_t) \circ dW_t^\alpha \tag{2.5}$$

where the vector fields $\tilde{A}_\alpha^z$ on S^{d-1} are defined by $\tilde{A}_\alpha^z(\theta) = A_\alpha^z \theta - \langle A_\alpha^z \theta, \theta \rangle \theta$ for $\alpha \geqq 0$ and $\theta \in S^{d-1}$.

We are almost ready to define some of the hypotheses which we shall require at various places in this paper. Let $S(x, r) = \{y \in \mathbb{R}^d: \|y - x\| = r\}$, $B(x, r) = \{y \in \mathbb{R}^d: \|y - x\| < r\}$, and $B'(x, r) = \{y \in \mathbb{R}^d: 0 < \|y - x\| < r\}$. For $T > 0$ let $\mathcal{U}_T = C([0, T]; \mathbb{R}^r)$. For $u \in \mathcal{U}_T$ let $\{\xi^z(t, x; u): 0 \leqq t \leqq T\}$ denote the solution of the control problem in $\mathbb{R}^d$ associated with (2.1)

$$\frac{\partial \xi^z}{\partial t}(t, x; u) = V_0^z(\xi^z(t, x; u)) + \sum_{\alpha=1}^{r} V_\alpha^z(\xi^z(t, x; u)) \, u_\alpha(t)$$

with $\xi^z(0, x; u) = x$. Similarly let $\{\eta^z(t, \theta; u): 0 \leqq t \leqq T\}$ denote the solution of the control problem in $\mathbf{S}^{d-1}$ associated with (2.5)

$$\frac{\partial \eta^z}{\partial t}(t, x; u) = \tilde{A}_0^z(\eta^z(t, \theta; u)) + \sum_{\alpha=1}^{r} \tilde{A}_\alpha^z(\eta^z(t, \theta; u)) \, u_\alpha(t)$$

with $\eta^z(0, \theta; u) = \theta$.

For any $z \in N$ consider the following assumptions.

H1(z) There exist functions $f \in C(\mathbb{R}^d)$ and $g \in C(\mathbb{R}^d)$ with $g \geqq 1$, positive constants c and R_1 and a neighborhood W of z in N such that for each $w \in W$ the process $\{x_t^w: t \geqq 0\}$ is non-explosive and there exists $f^w \in C^2(\mathbb{R}^d)$ satisfying $0 \leqq f^w \leqq f$, $L^w f^w + g \leqq c$, and $L^w f^w(x) + g(x) \leqq 0$ for $\|x\| \geqq R_1$.

H2(z) For all $r > 0$ and $x \neq 0$ there exists $T < \infty$ such that $P^z(T, x, B(0, r)) > 0$.

H3(z) (i) $\mathrm{Lie}(A_0^z, A_1^z, \ldots, A_r^z)(v) = \mathbb{R}^d$ for all $v \neq 0$, and if $d=2$ the linear mappings $A_1^z, \ldots, A_r^z$ are not all multiples of I.
(ii) $\{\eta^z(T, \theta; u): T>0, u \in \mathscr{U}_T\}$ is dense in S^{d-1} for all $\theta \in S^{d-1}$.
(iii) For all sufficiently small $R>0$ there exist $r_0 \in (0, R)$ and a neighborhood W of z in N such that

$$\{\xi^w(t, x; u): t>0, u \in \mathscr{U}_t, \|\xi^w(s, x; u)\| < R \text{ for all } s \leqq t\} \cap S(0, r_0)$$

in dense in $S(0, r_0)$ for all $x \in S(0, r_0)$ whenever $w \in W$.

Under the assumption H3(z) the Lyapunov exponent

$$\lambda^z = \lim_{t \to \infty} \frac{1}{t} \log \|v_t^z\| \quad \text{(almost-surely)} \tag{2.6}$$

and the Lyapunov moment function

$$\Lambda^z(p) = \lim_{t \to \infty} \frac{1}{t} \log \mathbf{E}(\|v_t^z\|^p) \qquad \text{for } p \in \mathbb{R} \tag{2.7}$$

are well-defined, i.e. the limits exist and do not depend on $v_0^z \neq 0$. Clearly the values of λ^z and $\Lambda^z(p)$ control the almost-sure stability and p^{th}-moment stability of the linearized process (2.4). For more details see [Arn], [AOP], [Ba1] and [Str]. They are related by the formula $\lambda^z = \frac{d}{dp}(\Lambda^z)(0)$. The function $\Lambda^z(p)$ is analytic and convex in p, so we may define

$$V^z = \frac{d^2}{dp^2}(\Lambda^z)(0).$$

The convexity of Λ^z implies that $V^z \geqq 0$.

Let us now assume H1(z), H2(z), H3(z) for some fixed $z \in N$. Our first result is essentially a restatement of Theorems 2.12, 2.13, 2.14 of [Ba3], except that the assertions are valid for all parameter values w throughout some neighborhood of z. We write $P^{x,w}$ to denote probabilities associated with the process $\{x_t: t \geqq 0\}$ started at $x_0 = x \in \mathbb{R}^d$ and run with fixed parameter value $w \in N$.

(2.8) Theorem. *Suppose that* H1(z), H2(z), H3(z) *are satisfied for some* $z \in N$. *In the case that* $\lambda^z > 0$ *assume also that* $\Lambda^z(p) > 0$ *for some* $p < 0$. *Then there exists a neighborhood* W *of* z *in* N *such that for all* $w \in W$ *the Lyapunov exponent* λ^w *and the Lyapunov moment function* $\Lambda^w(p)$ *are well defined in* (2.6) *and* (2.7) (*and vary continuously in* w) *and the following are true.*

(i) *If* $\lambda^w < 0$ *then*

$$\mathbf{P}^{x,w}\left\{\limsup_{t \to \infty} \frac{1}{t} \log \|x_t\| \leqq \lambda^w\right\} = 1$$

for all $x \neq 0$.

(ii) *If* $\lambda^w > 0$ *then there exists a unique probability measure* μ^w *on* $\mathbb{R}^d \setminus \{0\}$ *such that*

$$\mathbf{P}^{x,w}\left\{\frac{1}{t}\int_0^t \phi(x_s)\,ds \to \int \phi\,d\mu^w \ \text{as } t\to\infty\right\}=1 \tag{2.9}$$

for all bounded measurable $\phi\colon \mathbb{R}^d\setminus\{0\}\to\mathbb{R}$ *and all* $x\neq 0$. *In particular* μ^w *is the unique invariant measure for* $\{x_t^w\colon t\geqq 0\}$ *on* $\mathbb{R}^d\setminus\{0\}$. *Moreover there exist* $\gamma^{(w)}>0$, $\delta^{(w)}>0$ *and* $K^{(w)}<\infty$ *such that* $\Lambda^w(-\gamma^{(w)})=0$ *and*

$$\frac{1}{K^{(w)}}\,r^{\gamma^{(w)}} \leqq \mu^w(B'(0,r)) \leqq K^{(w)}\,r^{\gamma^{(w)}} \tag{2.10}$$

for $0<r<\delta^{(w)}$; *and*

$$\int g\,d\mu^w<\infty.$$

(iii) *If* $\lambda^w=0$ *then there exists a* σ*-finite measure* μ^w *on* $\mathbb{R}^d\setminus\{0\}$, *unique up to a multiplicative constant, such that*

$$\mathbf{P}^{x,w}\left\{\frac{\int_0^t \phi(x_s)\,ds}{\int_0^t \psi(x_s)\,ds}\to\frac{\int\phi\,d\mu^w}{\int\psi\,d\mu^w}\ \text{as } t\to\infty\right\}=1$$

for all bounded measurable μ^w*-integrable* $\phi,\psi\colon\mathbb{R}^d\setminus\{0\}\to\mathbb{R}$ *with* $\int\psi\,d\mu^w\neq 0$ *and all* $x\neq 0$. *In particular* μ^w *is the unique, up to multiplicative constant, invariant measure for* $\{x_t^w\colon t\geqq 0\}$ *on* $\mathbb{R}^d\setminus\{0\}$. *Moreover there exists* $a\in(0,\infty)$ *such that*

$$\frac{\mu^w(\mathbb{R}^d\setminus B(0,\varepsilon))}{|\log\varepsilon|}\to a \quad \text{as } \varepsilon\to 0; \tag{2.11}$$

and

$$\int_{\mathbb{R}^d\setminus B(0,\varepsilon)} g\,d\mu^w<\infty \quad \text{for all } \varepsilon>0.$$

We now consider how the invariant measure μ^w on $\mathbb{R}^d\setminus\{0\}$ changes as w is varied. The next result considers the μ^w as probability measures on $\mathbb{R}^d$. For notational convenience we define $\mu^w:=\delta(0)$, the unit mass at 0, whenever $\lambda^w\leqq 0$.

(2.12) Theorem. *Suppose that* H1(z), H2(z), H3(z) *are satisfied for some* $z\in N$ *with* $\lambda^z\geqq 0$, *and that the function* g *in* H1(z) *satisfies* $g(x)\to\infty$ *as* $\|x\|\to\infty$. *There exists a neighborhood* W *of* z *in* N *such that the family* $\{\mu^w\colon w\in W\}$ *of probability measures in* $\mathbb{R}^d$ *is tight. Moreover the mapping* $w\mapsto\mu^w$ *is contin-*

uous at z (with respect to the topology of weak convergence of probability measures in $\mathbb{R}^d$).

Finally we give our main result, which considers the rate at which the probability measures μ^w converge to $\delta(0)$ when $w \to z$ with λ^w converging to 0 from above.

(2.13) Theorem. *Suppose that* H1(z), H2(z), H3(z) *are satisfied for some* $z \in N$ *such that* $\lambda^z = 0$. *Then* $V^z > 0$ *and we denote by* $\bar{\mu}$ *the unique* σ*-finite invariant measure for* $\{x_t^z : t \geqq 0\}$ *on* $\mathbb{R}^d \setminus \{0\}$ *satisfying*

$$\frac{\bar{\mu}(\mathbb{R}^d \setminus B(0, \varepsilon))}{|\log \varepsilon|} \to \frac{2}{V^z} \quad \textit{as } \varepsilon \to 0.$$

There exists a neighborhood W of z in N such that the following are true.

(i) *The mapping* $w \mapsto \lambda^w$ *is continuous on W.*

(ii) *As* $w \to z$ *through* $W^+ \equiv \{w \in W : \lambda^w > 0\}$ *the rescaled measures* $(1/\lambda^w)\,\mu^w$ *converge to* $\bar{\mu}$ *in the sense that*

$$\frac{1}{\lambda^w} \int \phi(x)\, d\mu^w(x) \to \int \phi(x)\, d\bar{\mu}(x)$$

for all continuous $\phi : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ *satisfying* $\phi(x)/g(x) \to 0$ *as* $\|x\| \to \infty$ *and* $\phi(x)/\|x\|^p \to 0$ *as* $x \to 0$ *for some* $p > 0$.

(iii) *As* $w \to z$ *through* W^+ *the exponents* $\gamma^{(w)}$ *in* (2.10) *converge to* 0.

(2.14) Corollary. *Under the assumptions of Theorem* 2.13, *for each* $p > 0$ *such that* $\|x\|^p / g(x) \to 0$ *as* $\|x\| \to \infty$ *there exists* $0 < C_p < \infty$ *such that*

$$\int \|x\|^p\, d\mu^w(x) \sim C_p \lambda^w \quad \textit{as } w \to z \textit{ through } W^+.$$

The estimate of Corollary 2.14 should be compared with the deterministic Hopf bifurcation where the nearby attracting limit cycle consists of points at distance $O(|\lambda|^{1/2})$.

(2.15) Remark. The three assumptions H1(z), H2(z), H3(z) concern the behavior of the process $\{x_t^w : t \geqq 0\}$ near infinity, between infinity and 0, and near 0 respectively. They correspond more or less to the three conditions in [Ba3], except that they are now designed to have implications for $\{x_t^w : t \geqq 0\}$ when w lies in some small neighborhood of z.

The assumption H3(z) is a weaker version of the simpler condition

H4(z) $\mathrm{Lie}(A_1^z, \ldots, A_r^z)(v) = \mathbb{R}^d$ for all $v \neq 0$.

It is clear that H4(z) implies H3(z)(i) and H3(z)(ii), and the fact that it implies H3(z)(iii) follows easily from a version of Lemma 6.1 applied to H4(z) instead of to H3(z)(i). However H4(z) is too strong for many interesting examples, see e.g. Example 8.6.

The condition H3(z) deals with behavior near 0 and in particular it contains assumptions about the linearized process $\{v_t: t \geqq 0\}$. In Sects. 4 and 5 we will use just that part of H3(z) which we need to obtain results about $\{v_t: t \geqq 0\}$. Accordingly we define the condition

H5(z) (i) $\mathrm{Lie}(\tilde{A}_0^z, \tilde{A}_1^z, \ldots, \tilde{A}_r^z)(\theta) = T_\theta S^{d-1}$ for all $\theta \in S^{d-1}$, and if $d=2$ the vector fields $\tilde{A}_1^z, \ldots, \tilde{A}_r^z$ are not all identically 0.
(ii) $\{\eta^z(T, \theta; u): T>0, u \in \mathscr{U}_T\}$ is dense in S^{d-1} for all $\theta \in S^{d-1}$.

It easy to verify that H3(z) implies H5(z).

(2.16) Remark. In Theorem 2.8 in the case $\lambda^w > 0$ there will exist $p<0$ with $\Lambda^z(p)>0$ unless there exists $Q \in GL(d, \mathbb{R})$ such that $QA_1^z Q^{-1}$, $QA_2^z Q^{-1}, \ldots, QA_r^z Q^{-1}$ are all skewsymmetric and $\inf\{\langle QA_0^z Q^{-1} v, v\rangle: \|v\|=1\} \geqq 0$ (see [Ba1, Thm 4.2] and [AOP, Prop 4.1]). In particular such a p exists under the stronger assumption H4(z). Even if $\lambda^z>0$ and $\Lambda^z(p) \leqq 0$ for all $p<0$ then all of Theorem 2.8 except for the estimate (2.10) remains valid.

(2.17) Remark. We have stated all of our results with a state space $\mathbb{R}^d$ of general dimension $d \geqq 1$. The results remain valid in the special case $d=1$ subject to the following comments.

(i) The process $\{x_t: t \geqq 0\}$ on $\mathbb{R} \setminus \{0\}$ decomposes into two separate processes on the two components $\{x \in \mathbb{R}: x>0\}$ and $\{x \in \mathbb{R}: x<0\}$. Any statements involving ergodicity of the process $\{x_t: t \geqq 0\}$ on $\mathbb{R}^d \setminus \{0\}$ or uniqueness of the invariant measure on $\mathbb{R}^d \setminus \{0\}$ should be replaced when $d=1$ by the corresponding statements for the process restricted to one of the components $\{x \in \mathbb{R}: x>0\}$ or $\{x \in \mathbb{R}: x<0\}$. For $d=1$ our result should be called a stochastic pitchfork bifurcation instead of a stochastic Hopf bifurcation.
(ii) The assumptions H3(z), H4(z) and H5(z) and most of the calculations in Sect. 4 involve the behavior of the projection $\{\theta_t: t \geqq 0\}$ of the linearized process $\{v_t: t \geqq 0\}$ onto the unit sphere S^{d-1} in $\mathbb{R}^d$. When $d=1$ the unit sphere S^{d-1} reduces to the two point set $\{-1, +1\}$ and $\{\theta_t: t \geqq 0\}$ is constant. We can replace the assumption H3(z) by the simple requirement that at least one of the linear mappings $A_0^z, A_1^z, \ldots, A_r^z$ is non-zero.

The fact that S^0 is disconnected is obviously related to (i) above, and it makes $d=1$ a special case. On the other hand we observe that the projection of $\{v_t: t \geqq 0\}$ onto S^{d-1} can equally well be done onto the projective space P^{d-1}, and now for example the controllability condition H3(z)(ii) is trivially satisfied on the singleton set P^0; thus the case $d=1$ really does fit into the general setting.

Of course if we were only concerned with the one-dimensional case the proofs in this paper would be very different. The theory for one-dimensional diffusion processes is very well developed. In this paper we have to do some hard work in Sect. 4 to obtain Lyapunov like functions for the processes $\{x_t^w: t \geqq 0\}$. Using these functions we can estimate the behavior of the one-dimensional process $\{\|x_t^w\|: t \geqq 0\}$. For $d \geqq 2$ the process

$\{\|x_t^w\|: t \geqq 0\}$ is not in general a diffusion process since we are ignoring the angular part $\{x_t^w/\|x_t^w\|: t \geqq 0\}$. The usefulness of the Lyapunov exponent and the Lyapunov moment function comes about because they contain information about the growth of the linearized process $\{\|v_t^w\|: t \geqq 0\}$ where the dependence on the angular part $\{\theta_t^w: t \geqq 0\}$ is averaged out in exactly the right way. They play a crucial role when one wishes to apply one-dimensional estimates to multi-dimensional diffusion processes.

3 Behavior away from 0

This section contains 'locally uniform' versions of results in Sects. 3 and 4 of [Ba3]. For any $a>0$ we define $\tau_a = \inf\{t>0: \|x_t\| = a\}$.

(3.1) Lemma. *Assume* H2(z) *and fix* $r>0$. *For every* $x \neq 0$ *there exist* $T>0$, $\varepsilon>0$, *and neighborhoods* $U^{(x)}$ *of* x *in* $\mathbb{R}^d$ *and* $W^{(x)}$ *of* z *in* N *such that* $P^w(T, y, B(0, r)) > \varepsilon$ *whenever* $y \in U^{(x)}$ *and* $w \in W^{(x)}$.

Proof. Use Lemma 4.5 of [Ba3] applied to the process (x_t^w, w) in $\mathbb{R}^d \times N$, vector fields $(V_\alpha^w(x), 0)$, and truncated vector fields of the form $\phi_n(x, z)(V_\alpha^w(x), 0)$ where ϕ_n has compact support and $\phi_n \equiv 1$ on $B(0, n) \times W$ for some neighborhood W of z. □

(3.2) Proposition. *Assume* H2(z) *and fix* $0<r<R_2<\infty$. *Then there exist* $T>0$, $\varepsilon>0$, *and a neighborhood* W *of* z *in* N *such that*

$$\mathbf{P}^{x, w}\{\tau_r < T\} \geqq \varepsilon$$

whenever $r < \|x\| \leqq R_2$ *and* $w \in W$.

Proof. This follows from Lemma 3.1 using exactly the same method as in the proof of Prop 4.6 in [Ba3]. □

(3.3) Corollary. *Assume* H2(z) *and fix* $0<r<R_2<\infty$. *Then there exist a neighborhood* W *of* z *in* N *and* $K<\infty$ *such that*

$$\mathbf{E}^{x, w}\left(\int_0^{\tau_r} \chi_{(0, R_2]}(\|x_s\|)\, ds\right) \leqq K$$

whenever $\|x\| \geqq r$ *and* $w \in W$.

Proof. Use the proof of Theorem 4.7 of [Ba3]. Notice that W is the same as in Prop 3.2 and that we can take $K = T/\varepsilon$. □

(3.4) Proposition. *Assume* H1(z) *and* H2(z). *Then for each* $r>0$ *there exist a neighborhood* W_r *of* z *in* N *and* $K_r<\infty$ *such that*

$$\mathbf{E}^{x, w}\left(\int_0^{\tau_r} g(x_s)\, ds\right) \leqq K_r + f(x)$$

whenever $\|x\| \geqq r$ *and* $w \in W_r$. *In particular* $\mathbf{E}^{x,w}(\tau_r) < \infty$ *whenever* $\|x\| \geqq r$ *and* $w \in W_r$.

Proof. Fix $r > 0$ and take R_2 to be the R_1 of H1(z). Then let W_r be the intersection of the neighborhoods W of H1(z) and W of Corollary 3.3. The proof of Corollary 4.8 of [Ba3] now gives

$$\mathbf{E}^{x,w}\left(\int_0^{\tau_r} g(x_s)\,ds\right) \leqq K \sup\{L^w f^w(x) + g(x) : x \in \mathbb{R}^d\} + f^w(x)$$

whenever $\|x\| \geqq r$ and $w \in W_r$, where K is the constant in Corollary 3.3. The result now follows from H1(z). □

4 Lyapunov style functions for the behavior near 0

In this section we construct some functions which will be used in sub- and super-martingale estimates for the process $\{x_t^z : t \geqq 0\}$ when $\|x\|$ is small. Throughout this section we will assume just H5(z).

We establish some notation. Define $\tilde{L}^z = \tilde{A}_0^z + \sum_{\alpha=1}^{r} (\tilde{A}_\alpha^z)^2$. Define functions $q_\alpha^z(\theta) = \langle A_\alpha^z \theta, \theta \rangle$, $Q^z(\theta) = q_0^z(\theta) + \frac{1}{2} \sum_{\alpha=1}^{r} (\tilde{A}_\alpha^z q_\alpha^z)(\theta)$, $R^z(\theta) = \sum_{\alpha=1}^{r} (q_\alpha^z(\theta))^2$ and the vector field $X^z(\theta) = \sum_{\alpha=1}^{r} q_\alpha^z(\theta)\, \tilde{A}_\alpha^z(\theta)$. Finally write $\tilde{L}_p^z = \tilde{L}^z + pX^z + pQ^z + \frac{p^2}{2} R^z$ for $p \in \mathbb{R}$.

(4.1) Proposition. *Assume* H5(z). *Then for each bounded interval* $[a, b]$ *there exists a constant* $K < \infty$ *such that the following assertions hold.*

(i) *For each* $p \in [a, b]$ *there exists a smooth function* $\phi_p^z : S^{d-1} \to \mathbb{R}$ *satisfying* $\tilde{L}_p^z \phi_p^z = \Lambda^z(p)\, \phi_p^z$, $\frac{1}{K} \leqq \phi_p^z(\theta) \leqq K$ *for all* $\theta \in S^{d-1}$, *and* $\|\phi_p^z\|_{C^2} \leqq K$.

(ii) *There exists a smooth function* $\psi^z : S^{d-1} \to \mathbb{R}$ *satisfying* $\tilde{L}^z \psi^z = \lambda^z - Q^z$ *and* $\|\psi^z\|_{C^2} \leqq K$.

(iii) *There exists a smooth function* $\eta^z : S^{d-1} \to \mathbb{R}$ *satisfying* $\tilde{L}^z \eta^z + 2(X^z + Q^z - \lambda^z)\, \psi^z + R^z = V^z$ *and* $\|\eta^z\|_{C^2} \leqq K$.

Proof. These results appear in the preliminaries to Theorem 3.18 of [BS] and Proposition 5.2 of [Ba2]; the original ideas are contained in Arnold et al. [AOP]. We briefly sketch the proof. The Perron-Frobenius theorem (together with our assumption H5(z)) yields the existence of positive eigenfunctions ϕ_p^z corresponding to the eigenvalue problem $\tilde{L}_p^z \phi_p^z = \Lambda^z(p)\, \phi_p^z$ on S^{d-1}. In particular we can take $\phi_0^z \equiv 1$. Since for fixed z the operator $\tilde{L}_p^z$ is an analytic (in p) perturbation of $\tilde{L}^z$ then it follows that ϕ_p^z can be chosen so as to depend analytically on p. Parts (ii) and (iii) now follow by differentiating the eigenvalue equation $\tilde{L}^z_p \phi^z_p = \Lambda^z(p)\,\phi^z_p$ with respect to p and putting $p=0$. In particular we obtain $\psi^z = \frac{\partial}{\partial p}\phi^z_p|_{p=0}$ and $\eta^z = \frac{\partial^2}{\partial p^2}\phi^z_p|_{p=0}$. □

The result above deals just with the fixed parameter value z, and so in particular the constant K may depend upon z. Since we are concerned with behavior when the parameter changes we would like a version of Proposition 4.1 where the constant K is locally bounded as a function of z. The following result is a strengthening of part (ii) of the Proposition above, and will suffice for our purposes in this paper.

(4.2) Proposition. *Assume* H5(z). *Then there exist a neighborhood W of z in N and a constant $K<\infty$ such that for each $w \in W$ there exists a smooth function $\psi^w\colon S^{d-1} \to \mathbb{R}$ satisfying $\tilde{L}^w \psi^w = \lambda^w - Q^w$ and $\|\psi^w\|_{C^2} \leqq K$. Moreover the ψ^w may be chosen so that the mapping $w \mapsto \psi^w$ of W to $C^2(S^{d-1})$ is continuous.*

Notice that although the perturbation $\tilde{L}^z_p$ of $\tilde{L}^z$ in p is a lower order perturbation (and analytic perturbation theory can be used), the dependence of $\tilde{L}^z_p$ upon the parameter z can occur in the coefficients of the top order terms. Moreover we do not assume that the operator $\tilde{L}^z$ is elliptic, and so we cannot claim that $\tilde{L}^w - \tilde{L}^z$ is bounded relative to $\tilde{L}^z$. Accordingly the method of proof of Proposition 4.2 is very different from the method used in 4.1 above.

Before giving the proof of Proposition 4.2 we give some preliminary results. Let $\tilde{P}^z(t,\theta,B) = \mathbf{P}\{\theta^z_t \in B \mid \theta^z_0 = \theta\}$ denote the transition probability for the process $\{\theta^z_t : t \geqq 0\}$ on S^{d-1}, and let $\tilde{P}^z_t$ denote the corresponding operator acting on functions on S^{d-1}. We will denote by $\|\tilde{P}^z_t\|_{C^2,C^2}$ and $\|\tilde{P}^z_t\|_{C^0,C^2}$ the operator norms of $\tilde{P}^z_t$ when acting as an operator from $C^2(S^{d-1})$ to $C^2(S^{d-1})$ and from $C^0(S^{d-1})$ to $C^2(S^{d-1})$ respectively.

(4.3) Lemma. *Assume* H5(z). *Then there exists a neighborhood W of z in N such that for all $w \in W$ and all $t>0$ the transition probability $\tilde{P}^w(t,\theta,d\xi)$ has a smooth density $\tilde{p}^w(t,\theta,\xi)$ with respect to the uniform probability measure on S^{d-1}. Moreover for each $T_0>0$ there exists $K_0<\infty$ such that*

$$|D^i_\theta D^j_\xi \tilde{p}^w(T_0,\theta,\xi)| \leqq K_0$$

for all $i+j \leqq 2$, $\theta, \xi \in S^{d-1}$, and $w \in W$.

Proof. The proof is based on results of Kusuoka and Stroock [KS]. By assumption we have

$$\mathrm{Lie}(\tilde{A}^z_0, \tilde{A}^z_1, \ldots, \tilde{A}^z_r)(\theta) = T_\theta S^{d-1}$$

for all $\theta \in S^{d-1}$. Since each $\tilde{A}^z_\alpha$ is an analytic vector field on S^{d-1} we deduce by Nagano's theorem that

$$\mathscr{I}_{\tilde{A}^z_0}(\tilde{A}^z_1, \ldots, \tilde{A}^z_r)(\theta) = T_\theta S^{d-1}$$

for all $\theta \in S^{d-1}$, where $\mathscr{I}_{\tilde{A}_0^z}(\tilde{A}_1^z, \ldots, \tilde{A}_r^z)$ denotes the smallest Lie algebra of vector fields containing $\tilde{A}_1^z, \ldots, \tilde{A}_r^z$ and closed under Lie multiplication by $\tilde{A}_0^z$. (Here and elsewhere in this proof we use the notation of [KS].) If $d \geqq 3$ this follows since S^{d-1} is not a product, and if $d=2$ then otherwise we would get $\tilde{A}_1^z \equiv \ldots \equiv \tilde{A}_r^z \equiv 0$; for details of this argument see [IK, Thms 2* and 2**]. It follows that for each $\theta \in S^{d-1}$ there exists $L \geqq 1$ such that

$$\mathscr{V}_L^z(\theta, \eta) := \sum_{\alpha=1}^{r} \sum_{\|\beta\| \leqq L-1} ((\tilde{A}_\alpha^z)_{(\beta)}(\theta), \eta)^2 > 0$$

for all $\eta \in T_\theta S^{d-1}$, $\eta \neq 0$. Here each $(\tilde{A}_\alpha^z)_{(\beta)}$ is a vector field obtained by applying a finite number of Lie bracket operations involving the vector fields $\tilde{A}_0^z, \tilde{A}_1^z, \ldots, \tilde{A}_r^z$ to the vector field $\tilde{A}_\alpha^z$. For complete details of the notation see [KS]. A simple compactness argument shows that there exists $L \geqq 1$ and $\varepsilon > 0$ such that

$$\mathscr{V}_L^z(\theta, \eta) \geqq \varepsilon \|\eta\|^2$$

for all $\theta \in S^{d-1}$ and $\eta \in T_\theta S^{d-1}$. Now we vary the parameter. Since each vector field $\tilde{A}_\alpha^w$ depends smoothly upon w, the same is true of each of the finitely many vector fields $(\tilde{A}_\alpha^w)_{(\beta)}$, $1 \leqq \alpha \leqq r$, $\|\beta\| \leqq L-1$. So there exists a neighborhood W of z in N such that

$$\mathscr{V}_L^w(\theta, \eta) \geqq \frac{\varepsilon}{2} \|\eta\|^2$$

for all $\theta \in S^{d-1}$, $\eta \in T_\theta S^{d-1}$, and $w \in W$. We may now apply Corollary 3.25 of [KS] to the stochastic differential equation (2.5) for the process $\{\theta_t^w : t \geqq 0\}$ on S^{d-1}, noting that in this case since S^{d-1} is compact then automatically $\gamma_m \equiv 0$. (The result of Kusuoka and Stroock is given with state space $\mathbb{R}^d$, but it is clear that the result remains true when $\mathbb{R}^d$ is replaced by any smooth compact manifold.)

(4.4) Lemma. *There exists a neighborhood W of z in N and a constant $K_1 < \infty$ such that*

$$\sup\{\|\tilde{P}_t^w\|_{C^2, C^2} : 0 \leqq t \leqq 1\} \leqq K_1$$

for all $w \in W$, and the mapping $(w, t) \mapsto \tilde{P}_t^w$ from $W \times [0, 1]$ to $L(C^2(S^{d-1}))$ is continuous.

Proof. This is merely a uniform version of Theorem 3.14 of [KS]. More precisely it comes from applying [KS, Thm 3.14] to the process $\{(\theta_t^w, w) : t \geqq 0\}$ on $S^{d-1} \times N$. $\square$

(4.5) Lemma. *Assume* H5(z). *Then there exist a neighborhood W of z in N and constants $T>0$ and $\varepsilon>0$ such that*

$$\tilde{p}^w(T, \theta, \xi) \geqq \varepsilon \tag{4.6}$$

for all $\theta, \xi \in S^{d-1}$ and $w \in W$.

Proof. By Lemma 4.3 there exists a neighborhood W of z such that for all $t>0$ and $w \in W$ the density $\tilde{p}^w(t, \theta, \xi)$ exists and is a smooth function of (t, θ, ξ).

First we prove the estimate (4.6) for $w=z$. For each $\xi \in S^{d-1}$ the open set $U_\xi = \{\zeta \in S^{d-1}: \tilde{p}^z(1, \zeta, \xi) > 0\}$ is non-empty (since otherwise $\tilde{p}^z(1, \cdot, \xi) \equiv 0$ which is impossible). For each $\theta \in S^{d-1}$ there exists $t>0$ and a control path $u \in \mathscr{U}_t$ such that $\eta^z(t, \theta; u) \in U_\xi$. It follows from the support theorem for diffusion processes (see [SV]) that $\tilde{P}^z(t, \theta, U_\xi) > 0$ and therefore $\tilde{p}^z(t+1, \theta, \xi) > 0$. Moreover using the continuity of $\tilde{p}^z(t, \theta, \xi)$, we deduce that for each $\theta, \xi \in S^{d-1}$ there exist an open interval $A \subset (0, \infty)$ and neighborhoods B and C of θ and ξ respectively such that $\tilde{p}^z > 0$ on $A \times B \times C$. An elementary combinatorial argument can now be used to prove the existence of $T>0$ and $\delta>0$ such that

$$\tilde{p}^z(T, \theta, \xi) \geqq \delta \quad \text{for all } \theta, \xi \in S^{d-1}. \tag{4.7}$$

Now we wish to vary the parameter w. We observe that Lemma 4.3 yields the following Lipschitz estimate: there exists a constant $K_0 < \infty$ such that

$$|\tilde{p}^w(T, \theta, \xi) - \tilde{p}^w(T, \theta, \bar{\xi})| \leqq K_0 \rho(\xi, \bar{\xi}) \tag{4.8}$$

for all $\theta, \xi, \bar{\xi} \in S^{d-1}$ and all $w \in W$. (Here ρ denotes the geodesic distance on S^{d-1}.) Moreover the Feller property for the process $\{(\theta_t^w, w): t \geqq 0\}$ in $S^{d-1} \times N$ implies that the mapping $(\theta, w) \mapsto \int_{S^{d-1}} \tilde{p}^w(T, \theta, \xi) f(\xi)\, d\xi$ is continuous for each $f \in C^0(S^{d-1})$.

For $\gamma > 0$ choose $\xi_1, \ldots, \xi_N$ so that the balls $B(\xi_i, \gamma)$ cover S^{d-1}. We note that all of these balls have the same volume B_γ. For each i choose a continuous $f_i: S^{d-1} \to [0, 1]$ such that $\operatorname{supp}(f_i) \subset \mathrm{B}(\xi_i, \gamma)$ and $\int_{S^{d-1}} f_i(\xi)\, d\xi \geqq \frac{1}{2} B_\gamma$. Then by (4.7) we have

$$\tilde{P}_T^z f_i(\theta) \geqq \delta \int_{S^{d-1}} f_i(\xi)\, d\xi \geqq \frac{\delta}{2} B_\gamma$$

for all $i = 1, \ldots, N$ and $\theta \in S^{d-1}$. The Feller property implies the existence of a neighborhood W_1 of z in N such that

$$\tilde{P}_T^w f_i(\theta) \geqq \frac{\delta}{4} B_\gamma$$

for all $i=1, \ldots, N$, $\theta \in S^{d-1}$, and $w \in W_1$. Now for $\theta, \xi \in S^{d-1}$ choose i so that $\xi \in B(\xi_i, \gamma)$. Since $\rho(\xi, \bar{\xi}) < 2\gamma$ for all $\bar{\xi} \in \operatorname{supp}(f_i)$ we obtain

$$\begin{aligned}\frac{\delta}{4} B_\gamma &\leqq \int_{S^{d-1}} f_i(\bar{\xi})\, \tilde{p}^w(T, \theta, \bar{\xi})\, d\bar{\xi} \\ &\leqq [2\gamma K_0 + \tilde{p}^w(T, \theta, \xi)] \int_{S^{d-1}} f_i(\bar{\xi})\, d\bar{\xi} \\ &\leqq [2\gamma K_0 + \tilde{p}^w(T, \theta, \xi)]\, B_\gamma\end{aligned}$$

for all $w \in W \cap W_1$, where the second inequality uses (4.8). Therefore $\tilde{p}^w(T, \theta, \xi) \geqq \delta/4 - 2\gamma K_0$ and the result follows by choosing $\gamma < \delta/8 K_0$. □

(4.9) Lemma. *Assume* H5(z). *Then there exist a neighborhood W of z in N such that the following are true.*
(i) H5(w) *is valid for all $w \in W$.*
(ii) *The mapping $w \to \lambda^w$ is continuous on W.*
(iii) *For each $p \in \mathbb{R}$ the mapping $w \to \Lambda^w(p)$ is continuous on W.*

Proof. The fact that H5(w)(i) is valid in some neighborhood follows easily from the fact that each mapping $w \mapsto \tilde{A}_\alpha^w$ is continuous in w. The validity of the controllability condition H5(w)(ii) in a neighborhood follows immediately from Lemma 4.5 (using the Stroock-Varadhan support theorem for diffusion processes, see [SV]).

For W as in Lemma 4.5 the lower bound on the transition density implies that for $w \in W$ the process $\{\theta_t^w : t \geqq 0\}$ on S^{d-1} has a unique stationary measure π^w, say. Then Khas'minskii's formula (see [Kh1]) gives $\lambda^w = \int Q^w\, d\pi^w$. Now for $w \in W$ there exists ψ^w such that $\lambda^w = \tilde{L}^w \psi^w + Q^w$. So if $u \in W$ is close to w then $\tilde{L}^u \psi^w + Q^u$ is uniformly close to λ^w and consequently $\lambda^u = \int (\tilde{L}^u \psi^w + Q^u)\, d\pi^u$ is close to λ^w, proving (ii).

To prove (iii) let $w \in W$, $p \in \mathbb{R}$ and let ϕ_p^w satisfy $\tilde{L}_p^w \phi_p^w = \Lambda^w(p)\, \phi_p^w$. Given $\varepsilon > 0$ there exists a neighborhood U of w inside W such that $(\Lambda^w(p) - \varepsilon)\, \phi_p^w \leqq \tilde{L}_p^u \phi_p^w \leqq (\Lambda^w(p) + \varepsilon)\, \phi_p^w$ for $u \in U$. It follows (arguing as in [Bal, Cor 2.3]) that $\Lambda^w(p) - \varepsilon \leqq \Lambda^u(p) \leqq \Lambda^w(p) + \varepsilon$, and we are done. □

Proof of Proposition 4.2 Let W, T and ε be as in Lemma 4.5. Then a standard argument in the theory of Markov processes (see e.g. Doob [Doo]) shows that for the stationary measure π^w we have

$$|\tilde{P}_t^w f(\theta) - \int_{S^{d-1}} f\, d\pi^w| \leqq 2\|f\|_{C^0} (1-\varepsilon)^{t/T - 1}$$

for all $f \in C^0(S^{d-1})$, all $t \geqq 0$, all $\theta \in S^{d-1}$ and all $w \in W$. Moreover we may assume (by Lemmas 4.3 and 4.4) the existence of $K < \infty$ such that $\|\tilde{P}_1^w\|_{C^0, C^2} \leqq K$ and $\|\tilde{P}_t^w\|_{C^2, C^2} \leqq K$ for all $t \in [0, 1]$ whenever $w \in W$. Define for $n \geqq 1$ the function $\psi_n^w \in C^2(S^{d-1})$ by

$$\psi_n^w(\theta) = \int_0^n [(\tilde{P}_t^w Q^w)(\theta) - \int_{S^{d-1}} Q^w\, d\pi^w]\, dt.$$

For $m>n\geqq 1$ we have

$$\|\psi_m^w-\psi_n^w\|_{C^2}=\left\|\tilde{P}_1^w\left(\int_{n-1}^{m-1}[\tilde{P}_t^w Q^w-\int_{S^{d-1}} Q^w\,d\pi^w]\,dt\right)\right\|_{C^2}$$

$$\leqq \|\tilde{P}_1^w\|_{C^0,C^2}\left\|\int_{n-1}^{m-1}[\tilde{P}_t^w Q^w-\int_{S^{d-1}} Q^w\,d\pi^w]\,dt\right\|_{C^0}$$

$$\leqq 2K\|Q^w\|_{C^0}\int_{n-1}^{m-1}(1-\varepsilon)^{t/T-1}\,dt$$

$$\leqq \frac{2KT\|Q^w\|_{C^0}}{\log\frac{1}{1-\varepsilon}}(1-\varepsilon)^{(n-1)/T-1}.$$

Therefore $\psi_n^w\to\psi^w$, say, in $C^2(S^{d-1})$ as $n\to\infty$. Now

$$\tilde{L}^w\psi^w=\lim_{n\to\infty}\tilde{L}^w\psi_n^w=\lim_{n\to\infty}\tilde{P}_n^w Q^w-Q^w=\int_{S^{d-1}} Q^w\,d\pi^w-Q^w=\lambda^w-Q^w$$

so that ψ^w satisfies the required equation. To obtain the uniform C^2 estimate on ψ^w notice that

$$\|\psi^w\|_{C^2}=\lim_{n\to\infty}\|\psi_n^w-\psi_1^w\|_{C^2}+\left\|\int_0^1[\tilde{P}_t^w Q^w-\int_{S^{d-1}} Q^w\,d\pi^w]\,dt\right\|_{C^2}$$

$$\leqq \frac{2KT\|Q^w\|_{C^0}}{\log\frac{1}{1-\varepsilon}}(1-\varepsilon)^{-1}+\sup_{0\leqq t\leqq 1}\{\|\tilde{P}_t^w\|_{C^2,C^2}\}\|Q^w-\lambda^w\|_{C^2}$$

$$\leqq\left(\frac{2KT}{\log\frac{1}{1-\varepsilon}}(1-\varepsilon)^{-1}+K\right)\|Q^w\|_{C^2}$$

and that $Q^w(\theta)$ depends smoothly on θ and w. Finally it follows from the continuity with respect to w of $\tilde{P}_t^w$ and $\int_{S^{d-1}} Q^w\,d\pi^w=\lambda^w$ (see Lemmas 4.4 and 4.9) that the mapping $w\mapsto\psi_n^w$ is continuous from W to $C^2(S^{d-1})$ for each n. The estimate above on $\|\psi_m^w-\psi_n^w\|_{C^2}$ implies that the continuity is preserved in the limit as $n\to\infty$, i.e. the mapping $w\mapsto\psi^w$ is continuous from W to $C^2(S^{d-1})$. □

The reason for our interest in the functions in Proposition 4.1 lies in the following lemma. Let TL^z denote the generator of the $\{v_t^z\colon t\geqq 0\}$ process in $\mathbb{R}^d$ given by (2.4), so that

$$TL^z=A_0^z+\sum_{\alpha=1}^{r}(A_\alpha^z)^2$$

where we regard each of the linear mappings A^z_α, $0 \leqq \alpha \leqq r$, as a vector field $v \mapsto A^z_\alpha v$ on $\mathbb{R}^d$.

(4.10) Lemma. *Let ϕ^z_p, ψ^z and η^z be the functions in Proposition* 4.1. *Then*

$$TL^z\left(\|v\|^p \phi^z_p\left(\frac{v}{\|v\|}\right)\right) = \Lambda^z(p)\,\|v\|^p \phi^z_p\left(\frac{v}{\|v\|}\right),$$

$$TL^z\left(\psi^z\left(\frac{v}{\|v\|}\right) + \log\|v\|\right) = \lambda^z,$$

and if $\lambda^z = 0$

$$TL^z\left((\log\|v\|)^2 + 2\psi^z\left(\frac{v}{\|v\|}\right)\log\|v\| + \eta^z\left(\frac{v}{\|v\|}\right)\right) = V^z.$$

Proof. This is a direct calculation based on the fact that in polar coordinates (r, θ) where $r = \|v\|$ and $\theta = v/\|v\|$ we have $A^z_\alpha = r q^z_\alpha \dfrac{\partial}{\partial r} + \tilde{A}^z_\alpha$. For details see the preliminaries to Theorem 3.18 of [BS] and Proposition 5.2 of [Ba2]. □

It is clear that the equations above are very useful when combined with martingale inequalities in describing the growth properties of the linearized process $\{v^z_t : t \geqq 0\}$. Our next step is to obtain some similar results for the original process $\{x^z_t : t \geqq 0\}$ near to 0. We adapt here the method used in [BS] to obtain some estimates which are locally uniform in the parameter w.

(4.11) Lemma. *For any $z \in N$ and any $0 < \delta < 1$ there exist a neighborhood W of z and a constant $K < \infty$ such that*

$$(4.12)\quad |(L^w - TL^w)(f \otimes g)(r\theta)| \leqq K(r|f(r)| + r^2|f'(r)| + r^3|f''(r)|)\,\|g\|_{C^2}$$

for all $f \in C^2(0, \delta)$, all $g \in C^2(S^{d-1})$, all $(r, \theta) \in (0, \delta) \times S^{d-1}$ and all $w \in W$. Here $f \otimes g$ denotes the function $x \mapsto f(\|x\|)\,g(x/\|x\|)$. Moreover in the special case when $g \equiv 1$ the term $r|f(r)|$ may be omitted from the right hand side of (4.12).

Proof. We mimic the calculations of [BS, Cor 3.10]. Define

$$H^w_\alpha(r, \theta) = (V^w_\alpha(r\theta) - A^w_\alpha(r\theta))/r^2$$

for $0 < r < \delta$ and $\theta \in S^{d-1}$. Then Taylor's theorem gives

$$H^w_\alpha(r, \theta) = \int_0^1 (1-s)\,D^2 V^w_\alpha(sr\theta)(\theta, \theta)\,ds.$$

Since for each α the vector field V^w_α together with its first three derivatives depend continuously on w, it follows that there is a neighborhood W on which all the H^w_α $(\alpha \geqq 0)$ and their first covariant derivatives (in the directions r and θ) are bounded. Moreover (so long as W is precompact) all the matrices

A_α^w will be bounded on the same neighborhood. The proof in [BS] now gives the desired result; the constant K depends only on the bounds on the H_α^w and their first derivatives and the bounds on the A_α^w. □

(4.13) Proposition. *Assume* H5(z) *and* $\Lambda^z(p) \neq 0$.
(i) *There exist* $\delta > 0$, $K < \infty$, *and a neighborhood* W *of* z *in* N *such that for every* $w \in W$ *there exist smooth functions* $\psi^{w\pm}: B'(0, \delta) \to \mathbb{R}$ *satisfying*

$$L^w \psi^{w+}(x) \geqq \lambda^w \geqq L^w \psi^{w-}(x)$$

and

$$|\psi^{w\pm}(x) - \log \|x\|| \leqq K$$

whenever $0 < \|x\| < \delta$.
(ii) *For each* $p \neq 0$ *and each* $\varepsilon_1 > 0$ *there exist* $\delta > 0$, $K < \infty$, *a neighborhood* W *of* z *in* N *and a smooth function* $\phi_p^z: S^{d-1} \to (0, \infty)$ *such that the function* $\bar{\phi}_p^z(x) \equiv \|x\|^p \phi_p^z\left(\frac{x}{\|x\|}\right)$ *satisfies*

$$(\Lambda^z(p) + \varepsilon_1)\, \bar{\phi}_p^z(x) \geqq L^w \bar{\phi}_p^z(x) \geqq (\Lambda^z(p) - \varepsilon_1)\, \bar{\phi}_p^z(x)$$

for all $0 < \|x\| < \delta$ *and* $w \in W$, *and*

$$\frac{1}{K} \|x\|^p \leqq \bar{\phi}_p^z(x) \leqq K \|x\|^p$$

for all $x \neq 0$.
(iii) *If* $\lambda^z = 0$ *then for all* $\varepsilon_1 > 0$ *there exist* $\delta > 0$, $K < \infty$, *a neighborhood* W *of* z *and for each* $w \in W$ *satisfying* $\lambda^w = 0$ *a smooth function* $\bar{\eta}^w: B'(0, \delta) \to \mathbb{R}$ *such that*

$$V^z + \varepsilon_1 \geqq L^w \bar{\eta}^w(x) \geqq V^z - \varepsilon_1$$

and

$$|\bar{\eta}^w(x) - (\log \|x\|)^2| \leqq K |\log \|x\||$$

whenever $0 < \|x\| < \delta$. *Moreover there exist* $\delta > 0$, $K < \infty$, *and smooth functions* $\eta^{z\pm}: B'(0, \delta) \to \mathbb{R}$ *such that*

$$L^z \eta^{z+}(x) \geqq V^z \geqq L^z \eta^{z-}$$

and

$$|\eta^{z\pm}(x) - (\log \|x\|)^2| \leqq K |\log \|x\||$$

whenever $0 < \|x\| < \delta$.

Proof. We remark first that for the parameter w fixed at z the results are contained in [BS, Thm 3.18] and [Ba2, Prop 5.2]. In particular the final assertion of part (iii) is contained in these references. To prove the proposition we will use essentially the same construction as in [BS] and verify that the constants δ and K can be chosen to be locally uniform in w.

To prove (i), choose $q\in[\frac{1}{2},\frac{3}{4}]$ such that $\Lambda^z(q)\neq 0$, and define

$$\psi^{w\pm}(x)=\log\|x\|+\psi^w\left(\frac{x}{\|x\|}\right)\pm k\|x\|^q\phi_q^z\left(\frac{x}{\|x\|}\right)$$

where k is a constant to be determined later. Then

$$\begin{aligned}L^w\psi^{w\pm}(x)&=TL^w\psi^{w\pm}(x)+(L^w-TL^w)\psi^{w\pm}(x)\\&=\lambda^w\pm k\|x\|^q\tilde{L}_q^w\phi_q^z\left(\frac{x}{\|x\|}\right)+R_1\end{aligned}$$

where the remainder term R_1 can be estimated by applying Lemma 4.11 to the functions $\log\|x\|+\psi^w(x/\|x\|)$ and $\|x\|^q\phi_q^z(x/\|x\|)$. Since $\phi_q^z\in C^2(S^{d-1})$ and the coefficients of $\tilde{L}_q^w$ are continuous in w it follows that $\tilde{L}_q^w\phi_q^z(\theta)$ has the same sign as $\Lambda^z(q)$ and satisfies

$$|\tilde{L}_q^w\phi_q^z(\theta)|\geqq\left|\frac{\Lambda^z(q)}{2}\right|\phi_q^z(\theta)$$

for all w sufficiently close to z. The rest of the proof of (i) follows as in [BS, Thm 3.18]. To prove (ii), let $\phi_p^z(\theta)$ be the function given by Proposition 4.1. The result follows from an application of Lemma 4.11 to the function $\|x\|^p\phi_p^z(x/\|x\|)$ together with the estimate

$$|(\tilde{L}_p^w-\tilde{L}_p^z)\phi_p^z(\theta)|\leqq\frac{\varepsilon_1}{2}\phi_p^z(\theta)$$

for all w sufficiently close to z.

To prove (iii) define

$$\tilde{\eta}^w(x)=(\log\|x\|)^2+2\psi^w\left(\frac{x}{\|x\|}\right)\log\|x\|+\eta^z\left(\frac{x}{\|x\|}\right).$$

By a calculation similar to those in Lemma 4.10 we obtain, whenever $\lambda^w=0$,

$$TL^w\tilde{\eta}^w(v)=(R^w+2(X^w+Q^w)\psi^w+\tilde{L}^w\eta^z)\left(\frac{v}{\|v\|}\right).$$

Now as $w\to z$ the right side above converges to V^z, so that

$$V^z-\frac{\varepsilon_1}{2}\leqq TL^w\tilde{\eta}^w(v)\leqq V^z+\frac{\varepsilon_1}{2}$$

whenever $\lambda^w=0$ and w is sufficiently close to z. The result now follows from another application of Lemma 4.11.

Remark. Henceforth in this paper δ and K will be used solely to refer to the constants in the appropriate part of Proposition 4.13.

5 Expected occupation times near 0

Recall $\tau_a = \inf\{t > 0\colon \|x_t\| = a\}$. In this section we obtain some estimates on the random variables τ_ε and τ_R when $0 < \varepsilon < \|x\| < R$ and R is sufficiently small. We use the functions provided by Proposition 4.13 to get uniform versions of estimates in [BS], [Ba2] and [Ba3].

(5.1) Lemma. *Assume* H5(z).
(i) *There exist* $\delta > 0$, $K < \infty$ *and a neighborhood* W *of* z *in* N *such* $\mathbf{P}^{x,w}\{\tau_\varepsilon \wedge \tau_R < \infty\} = 1$ *whenever* $0 < \varepsilon < \|x\| < R < \delta$ *and* $w \in W$.
(ii) *Moreover if there exists* $p < 0$ *such that* $\Lambda^z(p) > 0$ *then there exists also* $k < 1$ *such that* $\mathbf{P}^{x,w}\{\tau_\varepsilon < \tau_R\} > 0$ *whenever* $0 < \varepsilon < \|x\| < kR < k\delta$ *and* $w \in W$.

Proof. Write $\sigma = \tau_\varepsilon \wedge \tau_R$. We consider the three cases $\lambda^z > 0$, $\lambda^z < 0$ and $\lambda^z = 0$ separately. First suppose $\lambda^z > 0$. We take δ, K and W so that the assertions of Lemma 4.9 and Proposition 4.13(i) are valid. Moreover we may assume that W is sufficiently small that $\lambda^w > 0$ for all $w \in W$. The assertion (i) now follows from the fact, from Proposition 4.13(i), that $\psi^{w+}(x_{t\wedge\sigma}) - \lambda^w(t \wedge \sigma)$ is a $\mathbf{P}^{x,w}$ submartingale. To prove the second assertion we may now assume that δ, K and W are chosen so that additionally the assertions of Proposition 4.13(ii) with $p < 0$ so that $\Lambda^z(p) > 0$ and $\varepsilon_1 = \Lambda^z(p)/2$ are valid. Then Proposition 4.13(ii) implies that $\|x_{t\wedge\sigma}\|^p \phi_p^w(x_{t\wedge\sigma}/\|x_{t\wedge\sigma}\|)$ is a $\mathbf{P}^{x,w}$ submartingale, which yields the inequality

$$\mathbf{P}^{x,w}\{\tau_\varepsilon < \tau_R\} \geqq \frac{K^{-2}\|x\|^p - R^p}{\varepsilon^p - R^p}.$$

So we take $k = K^{2/p}$ and the case $\lambda^z > 0$ is complete.

Now suppose $\lambda^z < 0$. (Notice that in this case automatically $\Lambda^z(p) > 0$ for all $p < 0$.) We take δ, K and W so that the assertions of Lemma 4.9 and Proposition 4.13(i) are valid and so that $\lambda^w < 0$ for all $w \in W$. The proof of both assertions now follows easily from the fact that $\psi^{w-}(x_{t\wedge\sigma}) - \lambda^w(t \wedge \sigma)$ is a $\mathbf{P}^{x,w}$ supermartingale. In particular we obtain the inequality

$$\mathbf{P}^{x,w}\{\tau_\varepsilon < \tau_R\} \geqq \left[\log\frac{R}{\|x\|} - 2K\right] \bigg/ \left[\log\frac{R}{\varepsilon}\right]. \tag{5.2}$$

Finally we consider the case $\lambda^z = 0$. Here the existence of $p < 0$ such that $\Lambda^z(p) > 0$ is equivalent to either of the statements $\Lambda^z(p) \not\equiv 0$ or $V^z > 0$. We take δ, K and W so that the assertions of Lemma 4.9, Proposition 4.13(i), Proposition 4.13(ii) (with $p < 0$ and $\varepsilon_1 = \Lambda^z(p)/2$) and Proposition 4.13(iii) (with $\varepsilon_1 = V^z/2$) are valid. If $\lambda^w \neq 0$ the assertions (i) and (ii) follow as above. If $\lambda^w = 0$ the assertion (i) follows from the fact that $\bar{\eta}^w(x_{t\wedge\sigma}) - V^z(t \wedge \sigma)/2$ is a $\mathbf{P}^{x,w}$ submartingale, and the assertion (ii) from the fact that $\psi^{w-}(x_{t\wedge\sigma})$ is a $\mathbf{P}^{x,w}$ supermartingale. In fact we obtain the inequality (5.2) in this case also. $\square$

(5.3) Proposition. *Assume* H5(z). *There exist* $\delta > 0$, $K < \infty$ *and a neighborhood* W *of* z *in* N *such the following hold.*

(i) *If* $w \in W$ *and* $\lambda^w > 0$ *and* $0 < \|x\| < R < \delta$ *then* $\mathbf{P}^{x,w}\{\tau_R < \infty\} = 1$ *and*

$$\frac{1}{\lambda^w}\left[\log\frac{R}{\|x\|} - 2K\right] \leqq \mathbf{E}^{x,w}(\tau_R) \leqq \frac{1}{\lambda^w}\left[\log\frac{R}{\|x\|} + 2K\right].$$

(ii) *If* $w \in W$ *and* $\lambda^w = 0$ *and* $0 < \|x\| < R < \delta$ *then* $\mathbf{P}^{x,w}\{\tau_R < \infty\} = 1$. *If in addition* $\|x\| < e^{-2K} R$ *then* $\mathbf{E}^{x,w}(\tau_R) = \infty$.

Proof. We use the same notation as in Lemma 5.1. Suppose first that $\lambda^w > 0$. Using the facts that $\psi^{w+}(x_{t\wedge\sigma}) - \lambda^w(t \wedge \sigma)$ is a $\mathbf{P}^{x,w}$ submartingale and that $\psi^{w-}(x_{t\wedge\sigma}) - \lambda^w(t\wedge\sigma)$ is a $\mathbf{P}^{x,w}$ supermartingale, we obtain

$$\frac{1}{\lambda^w}[\mathbf{E}^{x,w}(\psi^{w-}(x_\sigma)) - \psi^{w-}(x)] \leqq \mathbf{E}^{x,w}(\sigma) \leqq \frac{1}{\lambda^w}[\mathbf{E}^{x,w}(\psi^{w+}(x_\sigma)) - \psi^{w+}(x)].$$

The estimates on $\psi^{w\pm}$ imply

$$\begin{aligned} &\frac{1}{\lambda^w}[\mathbf{E}^{x,w}\log\|x_\sigma\| - \log\|x\| - 2\mathrm{K}] \\ &\leqq \mathbf{E}^{x,w}(\sigma) \\ &\leqq \frac{1}{\lambda^w}[\mathbf{E}^{x,w}\log\|x_\sigma\| - \log\|x\| + 2K]. \end{aligned} \tag{5.4}$$

The upper bound on $\mathbf{E}^{x,w}$ follows immediately by letting $\varepsilon \to 0$, and the lower bound will follow similarly as soon as we can prove that

$$(\log\varepsilon)\,\mathbf{P}^{x,w}\{\tau_\varepsilon < \tau_R\} \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.5}$$

Notice that the right hand inequality in (5.4) implies that $(\log\varepsilon)\,\mathbf{P}^{x,w}\{\tau_\varepsilon < \tau_R\}$ is bounded below as $\varepsilon \to 0$. In particular for fixed $0 < r < R < \delta$ and fixed $w \in W$ with $\lambda^w > 0$ we obtain

$$\sup_{\|x\|=r} \mathbf{P}^{x,w}\{\tau_\varepsilon < \tau_R\} \to 0 \quad \text{as } \varepsilon \to 0.$$

The condition H5(w) together with $\lambda^w > 0$ implies that $\Lambda^w(-\gamma) < 0$ for some $\gamma > 0$. It follows that there exist $\delta_0 > 0$, $K_0 < \infty$ and a function $\bar{\phi}^w_{-\gamma}$: $B'(0, \delta_0) \to (0, \infty)$ such that

$$\frac{1}{K_0}\|x\|^{-\gamma} \leqq \bar{\phi}^w_{-\gamma}(x) \leqq K\|x\|^{-\gamma}$$

and $\bar{\phi}^w_{-\gamma}(x_{t\wedge\tau_\varepsilon\wedge\tau_{\delta_0}})$ is a $\mathbf{P}^{x,w}$ supermartingale for $0 < \varepsilon < \|x\| < \delta_0$. (This follows from Proposition 4.13(ii) with z replaced by w, p by $-\gamma$ and $\varepsilon_1 = -\Lambda^w(\gamma)$.) It follows that

$$\mathbf{P}^{y,w}\{\tau_\varepsilon < \tau_{\delta_0}\} \leqq K_0^2\left(\frac{\varepsilon}{\|y\|}\right)^\gamma$$

whenever $0 < \varepsilon < \|y\| < \delta_0$. Here the constants δ_0, γ and K_0 may depend upon w. In particular δ_0 may be very much smaller than R so we cannot deduce (5.5) immediately. Instead we proceed as follows. We need only prove (5.5) for each fixed w and x so without loss of generality we may assume $\delta_0 \leqq \|x\|$, and then we may choose $\delta_1 < \delta_0$ such that $\mathbf{P}^{y,w}\{\tau_{\delta_1} < \tau_R\} \leqq \frac{1}{2}$ whenever $\|y\| = \delta_0$. Then we have

$$\mathbf{P}^{x,w}\{\tau_\varepsilon < \tau_R\} \leqq \tfrac{1}{2} \sup_{\|y\|=\delta_1} \mathbf{P}^{y,w}\{\tau_\varepsilon < \tau_R\}$$

and for any y with $\|y\| = \delta_1$ we have

$$\mathbf{P}^{y,w}\{\tau_\varepsilon < \tau_R\} \leqq \mathbf{P}^{y,w}\{\tau_\varepsilon < \tau_{\delta_0}\} + \tfrac{1}{2} \sup_{\|z\|=\delta_1} \mathbf{P}^{z,w}\{\tau_\varepsilon < \tau_R\}.$$

Together we obtain

$$\mathbf{P}^{x,w}\{\tau_\varepsilon < \tau_R\} \leqq \sup_{\|y\|=\delta_1} \mathbf{P}^{y,w}\{\tau_\varepsilon < \tau_{\delta_0}\} \leqq K_0^2 \left(\frac{\varepsilon}{\delta_1}\right)^\gamma$$

which implies (5.5) and the proof of (i) is complete.

To prove (ii) observe that for $\lambda^w = 0$ the process $\psi^{w+}(x_{t\wedge\sigma})$ is a $\mathbf{P}^{x,w}$ submartingale, which gives the inequality

$$\mathbf{P}^{x,w}\{\tau_\varepsilon < \tau_R\} \leqq \left[\log \frac{R}{\|x\|} + 2K\right] \Big/ \left[\log \frac{R}{\varepsilon}\right]. \tag{5.6}$$

Letting $\varepsilon \to 0$ in (5.6) we obtain $\mathbf{P}^{x,w}\{\tau_R < \infty\} = 1$. Using the $\mathbf{P}^{x,w}$ supermartingale $\bar{\eta}^w(x_{t\wedge\sigma}) - 3V^z(t\wedge\sigma)/2$ together with the inequality (5.2) we obtain

$$\begin{aligned} &\frac{3V^z}{2} \mathbf{E}^{x,w}(\tau_\varepsilon \wedge \tau_R) \\ &\quad \geqq [(\log\varepsilon)^2 - (\log R)^2 - K|\log\varepsilon| - K|\log R|]\left[\log\frac{R}{\|x\|} - 2K\right]\left[\log\frac{R}{\varepsilon}\right]^{-1} \\ &\qquad + (\log R)^2 - (\log\|x\|)^2 - K|\log R| - K|\log\|x\||. \end{aligned}$$

Now let $\varepsilon \to 0$ and we are done. □

The previous result gave some estimates on $\mathbf{E}^{x,w}(\tau_\varepsilon \wedge \tau_R)$ when $\lambda^w = 0$. We now go on to obtain estimates on the same quantity which are valid in a neighborhood of some z with $\lambda^z = 0$. The extra strength is that the estimates are valid for all w sufficiently close to z; this is paid for by the fact that in Proposition 5.7 the neighborhood W_1 depends on ε.

(5.7) Proposition. *Assume* H5(z) *for some* z *with* $\lambda^z = 0$ *and* $\Lambda^z(p) \not\equiv 0$. *Let* δ, K *and* W *be as in the proof of Lemma* 5.1. *Fix* $0 < \varepsilon < R < \delta$ *with* $\varepsilon R < 1$. *Then for each* $\beta \in (0, V^z)$ *there exists a neighborhood* W_1 *of* z *with* $W_1 \subset W$ *such that for all* $w \in W_1$ *and* $\varepsilon < \|x\| < R$ *the following assertions hold.*

(i)
$$\mathbf{E}^{x,w}(\tau_\varepsilon \wedge \tau_R) \leqq \frac{|\log\varepsilon|^2 + 2K|\log\varepsilon|}{V^z - \beta}.$$

(ii)
$$\frac{\log\frac{R}{\|x\|} - (2K + \beta\mathbf{E}^{x,w}(\tau_\varepsilon \wedge \tau_R))}{\log\frac{R}{\varepsilon}} \leqq \mathbf{P}^{x,w}\{\tau_\varepsilon < \tau_R\}$$

$$\leqq \frac{\log\frac{R}{\|x\|} + (2K + \beta\mathbf{E}^{x,w}(\tau_\varepsilon \wedge \tau_R))}{\log\frac{R}{\varepsilon}}.$$

(iii) $$\frac{1}{V^z+\beta}\left(\log\frac{R}{\|x\|}\log\frac{\|x\|}{\varepsilon}-[6K+2\beta\mathbf{E}^{x,w}(\tau_\varepsilon\wedge\tau_R)]\,|\log\varepsilon|\right)$$
$$\leqq \mathbf{E}^{x,w}(\tau_\varepsilon\wedge\tau_R)$$
$$\leqq\frac{1}{V^z-\beta}\left(\log\frac{R}{\|x\|}\log\frac{\|x\|}{\varepsilon}+[6K+2\beta\mathbf{E}^{x,w}(\tau_\varepsilon\wedge\tau_R)]\,|\log\varepsilon|\right).$$

Proof. Let $\psi^{z\pm}$ and $\eta^{z\pm}$ denote the functions (corresponding to parameter z) in Proposition 4.13. Then we have $L^z\psi^{z+}\geqq 0$, $L^z\psi^{z-}\leqq 0$, $L^z\eta^{z+}\geqq V^z$, and $L^z\eta^{z-}\leqq V^z$ in the compact set $\{x\in\mathbb{R}^d\colon \varepsilon\leqq\|x\|\leqq R\}$. Since the coefficients of L^w are continuous in w, it follows that for each $\beta>0$ there exists a neighborhood W_1 of w with $W_1\subset W$ such that for all $w\in W_1$ and $\varepsilon\leqq\|x\|\leqq R$ we have $L^w\psi^{z+}(x)\geqq-\beta$, $L^w\psi^{z-}(x)\leqq\beta$, $L^w\eta^{z+}(x)\geqq V^z-\beta$, and $L^w\eta^{z-}(x)\leqq V^z+\beta$.

Write $\sigma=\tau_\varepsilon\wedge\tau_R$ as above. To prove (i) we use the fact that $M_t\equiv\eta^{z+}(x_{t\wedge\sigma})-(V^z-\beta)(t\wedge\sigma)$ is a $\mathbf{P}^{x,w}$ submartingale. To prove (ii) we use the fact that $\psi^{z+}(x_{t\wedge\sigma})+\beta(t\wedge\sigma)$ is a $\mathbf{P}^{x,w}$ submartingale to obtain

$$\mathbf{E}^{x,w}(\log\|x_\sigma\|)+K+\beta\mathbf{E}^{x,w}(\sigma)\geqq\log\|x\|-K$$

which yields right hand inequality in (ii). The left hand inequality follows in a similar manner from the $\mathbf{P}^{x,w}$ supermartingale $\psi^{z-}(x_{t\wedge\sigma})-\beta(t\wedge\sigma)$. Finally to prove (iii) we return to the submartingale M_t above to obtain the estimate

$$\mathbf{E}^{x,w}(\sigma)\leqq\frac{1}{V^z-\beta}(\mathbf{E}^{x,w}(\eta^{z+}(x_\sigma)-\eta^{z+}(x))$$
$$\leqq\frac{1}{V^z-\beta}(\mathbf{E}^{x,w}(\log\|x_\sigma\|)^2-(\log\|x\|)^2+2K|\log\varepsilon|)$$
$$\leqq\frac{1}{V^z-\beta}((\log\varepsilon)^2\mathbf{P}^{x,w}\{\tau_\varepsilon<\tau_R\}+(\log R)^2(1-\mathbf{P}^{x,w}\{\tau_\varepsilon<\tau_R\})$$
$$-(\log\|x\|)^2+2K|\log\varepsilon|)$$
$$\leqq\frac{1}{V^z-\beta}\left(\log\frac{R}{\|x\|}\log\frac{\|x\|}{\varepsilon}+2K|\log\varepsilon|\right.$$
$$\left.+[2K+\beta\mathbf{E}^{x,w}(\tau_\varepsilon\wedge\tau_R)]\log\frac{1}{\varepsilon R}\right),$$

where the last inequality used the upper bound in (ii) and the fact that $|\log\varepsilon|>|\log R|$. The right hand inequality in (iii) now follows immediately, and the left hand inequality follows similarly from lower bound in (ii) together with the fact that $\eta^{z-}(x_{t\wedge\sigma})-(V^z+\beta)(t\wedge\sigma)$ is a $\mathbf{P}^{x,w}$ supermartingale. □

Define for $0<\varepsilon<R$ and $0<\|x\|<R$

$$G^w_{\varepsilon,R}(x)=\mathbf{E}^{x,w}\left(\int_0^{\tau_R}1_{[\varepsilon,\infty)}(\|x_t\|)\,dt\right).$$

We write $G^{w,+}_{\varepsilon,R}(r)=\sup_{\|x\|=r} G^w_{\varepsilon,R}(x)$ and $G^{w,-}_{\varepsilon,R}(r)=\inf_{\|x\|=r} G^w_{\varepsilon,R}(x)$.

(5.8) Corollary. *With the assumptions and notation of Proposition 5.7 the following are true.*
(i) *For each $0<\varepsilon<R<\delta$ there exist a neighborhood W_1 of z and a constant $K_{\varepsilon,R}$ (depending only on ε, R, V^z and K) such that $G^w_{\varepsilon,R}(x)\leqq K_{\varepsilon,R}$ whenever $w\in W_1$ and $0<\|x\|<R$.*
(ii) *For each $0<r<R<\delta$*

$$\frac{2}{V^z}\left[\log\frac{R}{r}-4K\right]\leqq \liminf_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\liminf_{w\to z} G^{w,-}_{\varepsilon,R}(r)$$
$$\leqq \limsup_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\limsup_{w\to z} G^{w,+}_{\varepsilon,R}(r)$$
$$\leqq \frac{2}{V^z}\left[\log\frac{R}{r}+4K\right].$$

Proof. Without loss of generality we may assume that $\varepsilon R<1$. Notice first that for $\|x\|<\varepsilon$ we have $G^w_{\varepsilon,R}(x)\leqq G^{w,+}_{\varepsilon,R}(\varepsilon)$. For $\|x\|>\varepsilon$ we may argue as in [Ba2, Prop 5.6] to obtain the estimates

$$G^w_{\varepsilon,R}(x)\leqq\frac{\mathbf{P}^{x,w}\{\tau_\varepsilon<\tau_R\}\sup_{\|y\|=\varepsilon}\mathbf{E}^{y,w}(\tau_b\wedge\tau_R)}{1-\sup_{\|y\|=\varepsilon}\mathbf{P}^{y,w}\{\tau_b<\tau_R\}}+\mathbf{E}^{x,w}(\tau_\varepsilon\wedge\tau_R)$$

and

$$G^w_{\varepsilon,R}(x)\geqq\frac{\mathbf{P}^{x,w}\{\tau_\varepsilon<\tau_R\}\inf_{\|y\|=a}\mathbf{E}^{y,w}(\tau_\varepsilon\wedge\tau_R)}{1-\inf_{\|y\|=a}\mathbf{P}^{y,w}\{\tau_\varepsilon<\tau_R\}}+\mathbf{E}^{x,w}(\tau_\varepsilon\wedge\tau_R)$$

for any $0<b<\varepsilon<a<R$ and $\varepsilon\leqq\|x\|<R$. With the obvious modification these estimates are valid also when $\|x\|=\varepsilon$. Now the results of Proposition 5.7 (with ε replaced by b in places) give the required estimates. In particular the method of proof in [Ba2, Prop 5.6] for $G^z_{\varepsilon,R}(x)$ is equally valid for estimates on $\liminf_{w\to z} G^{w,-}_{\varepsilon,R}(r)$ and $\limsup_{w\to z} G^{w,+}_{\varepsilon,R}(r)$.

(5.9) Proposition. *Assume* H5(z) *for some z with $\lambda^z=0$ and $\Lambda^z(p)\neq 0$. Let δ be as in the proof of Lemma* 5.1. *Then for each $0<R<\delta$ and $p\in(0,1]$ there exist a neighborhood $W^{(R,p)}$ of z and $K^{(R,p)}<\infty$ such that*

$$\mathbf{E}^{x,w}\int_0^{\tau_R}\|x_t\|^p\,dt\leqq K^{(R,p)}$$

whenever $w\in W^{(R,p)}$ and $0<\|x\|<R$.

Proof. For $p \in (0, 1]$ we have $\Lambda^z(p) > 0$ and so by Proposition 4.1.3(ii) there exist a neighborhood $W^{(p)}$ of z, $\delta_p > 0$ and $K^{(p)} < \infty$ such that

$$L^w \bar{\phi}_p^z(x) \geqq \frac{\Lambda^z(p)}{2} \bar{\phi}_p^z(x)$$

for all $0 < \|x\| < \delta_p$ and $w \in W^{(p)}$, and

$$\frac{1}{K^{(p)}} \|x\|^p \leqq \bar{\phi}_p^z(x) \leqq K^{(p)} \|x\|^p$$

for all $x \neq 0$. Since the coefficients of L^w are continuous functions of w there exists $c_{R,p} < \infty$ such that $L^w \bar{\phi}_p^z(x) - \Lambda^z(p) \bar{\phi}_p^z(x)/2 \geqq -c_{R,p}$ on the compact set $\delta_p \leqq \|x\| \leqq R$ for all $w \in W^{(p)}$. (Here we may have to replace $W^{(p)}$ by a smaller neighborhood which we write $W^{(R,p)}$.) Then

$$\begin{aligned}
\mathbf{E}^{x,w}(\bar{\phi}_p^z(x_{t \wedge \tau_R})) &= \bar{\phi}_p^z(x) + \mathbf{E}^{x,w} \int_0^{t \wedge \tau_R} L^w \bar{\phi}_p^z(x_s)\, ds \\
&\geqq \mathbf{E}^{x,w} \int_0^{t \wedge \tau_R} \left(\frac{\Lambda^z(p)}{2} \bar{\phi}_p^z(x_s) - c_{R,p} 1_{[\delta_p, \infty)}(x_s) \right) ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{E}^{x,w} \int_0^{t \wedge \tau_R} \|x_s\|^p\, ds &\leqq \frac{2K^{(p)}}{\Lambda^z(p)} \mathbf{E}^{x,w} \left(\bar{\phi}_p^z(x_{t \wedge \tau_R}) + \int_0^{t \wedge \tau_R} c_{R,p} 1_{[\delta_p, \infty)}(x_s)\, ds \right) \\
&\leqq \frac{2K^{(p)}}{\Lambda^z(p)} (K^{(p)} R^p + c_{R,p} G^w_{\delta_p, R}(x))
\end{aligned}$$

and the result now follows from Corollary 5.8. □

6 Construction of μ^w for $\lambda^w \geqq 0$

(6.1) Lemma. *Assume $H3(z)(i)$. There exists a neighborhood W of z in N and $\delta > 0$ such that*

$$\operatorname{Lie}(V_0^w, V_1^w, \ldots, V_r^w)(x) = \mathbb{R}^d$$

whenever $0 < \|x\| < \delta$ and $w \in W$.

Proof. The proof is based on that of [BS, Lemma 4.4]. By assumption there exist $\varepsilon > 0$ and linear mappings $A^z_{(\beta)}$, $1 \leqq \beta \leqq M$, each one obtained by a finite number of Lie bracket operations involving the linear mappings $A_0^z, \ldots, A_r^z$, such that

$$\sum_{\beta=1}^{M} \langle A^z_{(\beta)} x, u \rangle^2 \geqq \varepsilon \|x\|^2$$

whenever $x, u \in \mathbb{R}^d$ with $x \neq 0$ and $\|u\|=1$. Let $A^w_{(\beta)}$, $1 \leqq \beta \leqq M$ be linear mappings obtained from the linear mappings $A^w_0, \ldots, A^w_r$ and let $V^w_{(\beta)}$, $1 \leqq \beta \leqq M$ be vector fields obtained from the vector fields $V^w_0, \ldots, V^w_r$ using the same sequences of Lie bracket operations. Then with appropriate sign conventions for the two sorts of Lie brackets (namely on linear mappings and on vector fields), we obtain $DV^w_{(\beta)}(0)=A^w_{(\beta)}$ for all w and β. For the finite number of vector fields $V^w_{(1)}, \ldots, V^m_{(M)}$ there exists a neighborhood W of z and $\delta>0$ such that

$$\|V^w_{(\beta)}(x)-A^z_{(\beta)}x\|<\sqrt{\frac{\varepsilon}{M}}\,\|x\|$$

whenever $0<\|x\|<\delta$, $w\in W$ and $1\leqq\beta\leqq M$. It follows that

$$\sum_{\beta=1}^{M}\langle V^w_{(\beta)}(x), u\rangle^2>0$$

whenever $0<\|x\|<\delta$, $\|u\|=1$ and $w\in W$, and we are done. □

Now we assume H1(z), H2(z), H3(z) and $\lambda^z \geqq 0$. Notice that H3(z) implies that $A^z(p) \neq 0$. Choose $\delta>0$, a neighborhood W of z in N, and $K<\infty$, $k<1$ so that the assertions of Lemma 4.9, Proposition 5.3 and Lemma 6.1 are valid. Now fix $R\in(0,\delta)$. Replacing W by a smaller neighborhood if necessary we may assume that with this value of R the assertion of H3(z)(iii) is valid for all $w\in W$. Finally we take $r\in(0, r_0]$ where r_0 is as in H3(z)(iii) so that $r<Re^{-2K}$, and let W_r and K_r be as in Proposition 3.4.

For $0<r<R<\infty$ as above define random times inductively

$$\begin{aligned}\sigma_0&=\inf\{t\geqq 0\colon \|x_t\|=r\}\\ \sigma'_n&=\inf\{t\geqq \sigma_n\colon \|x_t\|=R\}\\ \sigma_{n+1}&=\inf\{t\geqq \sigma'_n\colon \|x_t\|=r\}.\end{aligned}$$

Propositions 3.4 and 5.3 imply that

$$\mathbf{P}^{x,w}\{\sigma_n<\infty \text{ for all } n\geqq 0\}=1$$

for all $x\neq 0$ and $w\in W\cap W_r$ such that $\lambda^w\geqq 0$. Consider the induced Markov chain $\{Z_n\colon n\geqq 0\}$ on $S(0,r)$ given by $Z_n=x_{\sigma_n}$. It has transition probability $\Pi^w(x,A)=\mathbf{P}^{x,w}\{x_{\sigma_1}\in A\}$ for $x\in S(0,r)$, $A\in\mathscr{B}(S(0,r))$. The following lemma is based on [BS, Lemma 4.4].

(6.2) Lemma. *Assume* H1(z), H2(z), H3(z) *and* $\lambda^z\geqq 0$. *Let* $0<r<R<\delta$, W *and* W_r *satisfy the conditions above. Then for all* $w\in W\cap W_r$ *with* $\lambda^w\geqq 0$ *the Markov chain* $\{Z_n\colon n\geqq 0\}$ *on* $S(0,r)$ *has a unique invariant probability measure* v^w, *say, and* $\Pi^w(x,\cdot)$ *is equivalent to* v^w *for all* $x\in S(0,r)$ *and* $A\in\mathscr{B}(S(0,r))$.

Proof. Let $w\in W\cap W_r$ with $\lambda^w\geqq 0$. The assertion will follow from standard Markov chain theory once we know that for each fixed $A\in\mathscr{B}(S(0,r))$ then
(i) $x\mapsto \Pi^w(x,A)$ is a continuous function on $S(0,r)$; and

(ii) either $\Pi^w(\cdot, A) \equiv 0$ or else $\Pi^w(x, A) > 0$ for all $x \in S(0, r)$.
This is because (i) and (ii) imply that $\{Z_n: n \geqq 0\}$ satisfies Doeblin's condition, see for example [MT].

So fix $A \in \mathscr{B}(S(0, r))$ and define $h: B'(0, R) \to [0, 1]$ by

$$h(x) = 1 - \int_{\|y\| = R} \mathbf{P}^{y, w}\{x_{\tau_r} \in A\} \, \mathbf{P}^{x, w}\{x_{\tau_R} \in dy\}.$$

Then $L^w h = 0$ and $h|_{S(0,r)} = 1 - \Pi^w(\cdot, A)$. Lemma 6.1 implies that L^w is hypoelliptic on $B'(0, R)$ so we obtain (i). Moreover if $h(x) = 1$ for some $x \in S(0, r)$ then the Stroock-Varadhan maximum principle [SV] implies first that $h(y) = 1$ for some $y \in S(0, r_0)$ (since $\mathbf{P}^{x, w}\{\tau_{r_0} < \infty\} = 1$) and then $h(y) = 1$ for all $y \in S(0, r_0)$ (using the assumption H3(z)(iii)). But $h|_{S(0, r_0)} \equiv 1$ implies $h|_{S(0, r)} \equiv 1$ for all $r \in (0, r_0]$ and we have (ii). □

We may now define for each $w \in W \cap W_r$ with $\lambda^w > 0$ a probability measure μ^w on $\mathbb{R}^d \setminus \{0\}$ as follows. For each $A \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$ define

$$\mu^w(A) = C^w \int_{S(0,r)} \mathbf{E}^{x, w}\left(\int_0^{\sigma_1} 1_A(x_s) \, ds\right) d\nu^w(x) \tag{6.3}$$

where the normalizing constant C^w is given by

$$(C^w)^{-1} = \int_{S(0,r)} \mathbf{E}^{x, w}(\sigma_1) \, d\nu^w(x). \tag{6.4}$$

This construction is due to Khasminskii [Kh2] and Maruyama and Tanaka [MT]. The proof of the fact that μ^w is the unique invariant probability measure for $\{x_t^w: t \geqq 0\}$ on $\mathbb{R}^d \setminus \{0\}$ is given in these references.

A similar construction can be made of the invariant measure for $\{x_t^w: t \geqq 0\}$ when $\lambda^w = 0$ except that in this case $\mathbf{E}^{x, w}(\sigma_1) = \infty$ for all $x \in S(0, r)$ so that the normalizing constant C^w in (6.3) is omitted and the resulting measure has infinite total mass, see [Ba3] for more details.

We conclude this section with some of the estimates which we shall use in the proof of Theorem 2.13 in the next section. Throughout we continue to assume H1(z), H2(z), H3(z) and $\lambda^z \geqq 0$, and $0 < r < R < \delta$, W, W_r, K and K_r satisfy the conditions above.

(6.5) Lemma. *For all $w \in W \cap W_r$ with $\lambda^w > 0$*

$$\frac{1}{\lambda^w}\left[\log \frac{R}{r} - 2K\right] \leqq (C^w)^{-1} \leqq \frac{1}{\lambda^w}\left[\log \frac{R}{r} + 2K\right] + K_r + f(R).$$

Proof. This is a combination of the estimates of Proposition 5.3 for the first half cycle $0 \leqq t \leqq \sigma_0'$ and Proposition 3.4 for the second half cycle $\sigma_0' \leqq t \leqq \sigma_1$. □

(6.6) Lemma. *For all $w \in W^+ \cap W_r$ with $\lambda^w > 0$*

$$\int_{\|x\| \geqq R} g(x) \, d\mu^w(x) \leqq C^w(K_r + f(R)).$$

Proof. This uses Proposition 3.4. $\square$

(6.7) Lemma. *Assume in addition that* $\lambda^z = 0$. *For all* $p > 0$ *there exists a neighborhood* $W^{(p)}$ *of* z *in* N *and a constant* $K^{(p)} < \infty$ *such that for all* $w \in W \cap W_r \cap W^{(p)}$ *with* $\lambda^w > 0$

$$\int_{\|x\| \leqq R} \|x\|^p \, d\mu^w(x) \leqq C^w (K^{(p)} + R^p (K_r + f(R))).$$

Proof. Clearly there is no loss in generality in assuming $p \leqq 1$. During the first half cycle we get a contribution at most

$$C^w \sup_{\|x\| = r} \mathbf{E}^{x,w} \left(\int_0^{\tau_R} \|x_s\|^p \, ds \right)$$

and we use Proposition 5.9. During the second half cycle we get at most $C^w \sup_{\|x\| = R} \mathbf{E}^{x,w}(R^p \tau_r)$ and we use Proposition 3.4. $\square$

(6.8) Lemma. *Assume in addition that* $\lambda^z = 0$. *For* $0 < \varepsilon < r < R < S < \infty$ *write* $U_{\varepsilon,S} = \{x \in \mathbb{R}^d : \varepsilon \leqq \|x\| \leqq S\}$. *For each* $S \in (R, \infty)$

$$\begin{aligned}
\frac{2}{V} \frac{\left[\log \frac{R}{r} - 4K\right]}{\left[\log \frac{R}{r} + 2K\right]} &\leqq \liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \liminf_{w \to z} \frac{\mu^w(U_{\varepsilon,S})}{\lambda^w} \\
&\leqq \limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \limsup_{w \to z} \frac{\mu^w(U_{\varepsilon,S})}{\lambda^w} \\
&\leqq \frac{2}{V} \frac{\left[\log \frac{R}{r} + 4K\right]}{\left[\log \frac{R}{r} - 2K\right]}
\end{aligned}$$

where the $\liminf_{w \to z}$ *and* $\limsup_{w \to z}$ *are taken as* w *tends to* z *through values where* $\lambda^w > 0$.

Proof. Observe that for $w \in W \cap W_r$ with $\lambda^w > 0$

$$\frac{C^w}{\lambda^w} G^{w,-}_{\varepsilon,R}(r) \leqq \frac{\mu^w(U_{\varepsilon,S})}{\lambda^w} \leqq \frac{C^w}{\lambda^w} (G^{w,+}_{\varepsilon,R}(r) + K_r + f(R)).$$

The result now follows from Corollary 5.8 and Lemma 6.5. $\square$

7 Proofs of the theorems

Proof of Theorem 2.8 It suffices to choose a neighborhood W of z in N such that the conditions for Theorems 2.12, 2.13, 2.14 of [Ba3] are satisfied whenever $w \in W$.

We note first that the condition that $\Lambda^z(p)>0$ for some $p<0$ is automatically satisfied whenever $\lambda^z \leqq 0$. Let us fix such a p. Choose W so that the assertions of H1(z), Lemma 4.9 and Lemma 5.1 are valid, and so that $\Lambda^w(p)>0$ whenever $w \in W$. If $\lambda^w=0$ we deduce immediately that $V^w>0$. If $\lambda^w>0$ then the convexity of $\Lambda^w(\cdot)$ implies the existence of a unique $\gamma^{(w)}>0$ such that $\Lambda^w(-\gamma^{(w)})=0$.

Let δ, K and k be as in Lemma 5.1. Choose r and R so that $0<r<kR<k\delta$, and consider the neighborhood W_r as in Proposition 3.4. Clearly the assumption (2.3) of [Ba3] is satisfied for $w \in W$. Combining Proposition 3.4 and Lemma 5.1 (ii) we see that $\mathbf{P}^{x,w}\{\tau_\varepsilon<\infty\}>0$ whenever $0<\varepsilon<r, \|x\| \geqq r$ and $w \in W \cap W_r$, which implies condition (2.4) of [Ba3] for all $w \in W \cap W_r$.

If $\lambda^z<0$ we are done at this point, see [Ba3, Thm 2.12]. If $\lambda^z \geqq 0$ we continue to shrink our neighborhood by assuming that W is small enough so as to satisfy the requirements in the preamble to Lemma 6.2. Now Lemmas 4.9 and 6.2 together with the assertions about $\gamma^{(w)}$ and V^w above imply that the conditions of Remark (6.4) of [Ba3] are satisfied whenever $w \in W \cap W_r$ satisfies $\lambda^w \geqq 0$. □

Proof of Theorem 2.12 It follows from Lemmas 6.5 and 6.6, together with the continuity of $w \to \lambda^w$, that there exists $R>0$ and a neighborhood W of z such that

$$\sup_{w \in W} \int_{\|x\| \geqq R} g(x)\, d\mu^w(x) < \infty .$$

The first assertion follows immediately by Markov's inequality. Now suppose that $w_n \to z$ and μ^{w_n} converges weakly to a probability measure ν, say. It suffices to show that $\nu=\mu^z$. Since the coefficients of the generator L^w depend continuously on w, and since each μ^{w_n} is invariant for the corresponding diffusion process, it follows easily that $\int L^z \phi\, d\nu=0$ for all C^2 functions $\phi: \mathbb{R}^d \to \mathbb{R}$ with compact support. Therefore ν is an invariant probability measure for the process $\{x_t^z: t \geqq 0\}$ on $\mathbb{R}^d$. If $\lambda^z=0$ we are done since $\mu^z=\delta(0)$ is the unique such measure. However if $\lambda^z>0$ there are two possibilities, namely μ^z and $\delta(0)$. In this case choose $p<0$ such that $\Lambda^z(p)<0$. We take W, δ, R and r as in Sect. 6 except that in addition we wish the conclusion of Proposition 4.13 (ii) with $\varepsilon_1=-\Lambda^z(p)/2$ to be valid, and also we choose W sufficiently small that λ^w is bounded away from 0 and ∞ on W. Then for $w \in W$ and $\varepsilon<r$ we obtain

$$\begin{aligned}
\mu^w(B(0,\varepsilon)) &\leqq C^w\left(\sup_{\|x\|=r} \mathbf{E}^{x,w} \int_0^{\tau_R} 1_{(0,\varepsilon)}(\|x_s\|)\, ds\right) \\
&\leqq C^w \sup_{\|x\|=r} \mathbf{P}^{x,w}\{\tau_\varepsilon<\tau_R\} \sup_{\|y\|=\varepsilon} \mathbf{E}^{y,w}(\tau_R) \\
&\leqq C^w K^2 \left(\frac{\varepsilon}{r}\right)^{-p} \frac{1}{\lambda^w}\left[\log\frac{R}{\varepsilon}+2K\right]
\end{aligned}$$

where the upper bound on $\mathbf{P}^{x,w}\{\tau_\varepsilon<\tau_R\}$ follows directly from the fact that $\bar{\Phi}_p^z(x_{t\wedge\sigma})$ is a $\mathbf{P}^{x,w}$ supermartingale together with the bounds on $\bar{\Phi}_p^z$. It follows from Lemma 6.5 that $\nu(\{0\})=0$ so that $\nu=\mu^z$ and we are done. □

Proof of Theorem 2.13 Notice first that the simultaneous vanishing $\lambda^z=0=V^z$ would imply that $\Lambda^z(p)\equiv 0$ and hence that $A_0^z, A_1^z, \ldots, A_r^z$ can be simultaneously conjugated into skew-symmetric matrices; this would contradict H3(z) (see [Ba1, Thm 3.1] and [AOP, Thm 3.2]). We choose the neighborhood W and constants δ, K, R and r as in Sect. 6. Lemma 4.9 gives assertion (i) immediately. We now commence the proof of assertion (ii). For $0<p\leqq 1$ choose a continuous function $h\colon \mathbb{R}^d\setminus\{0\}\to(0,\infty)$ so that $h(x)=\|x\|^p$ for $0<\|x\|\leqq R$, $h(x)=g(x)$ for $\|x\|\geqq 1$ and $h(x)\leqq g(x)$ for all $x\in\mathbb{R}^d\setminus\{0\}$. Here g is the function in assumption H1(z). Let $\{w_n\colon n\geqq 1\}$ be any sequence converging to z through W^+. For convenience of notation we write $(1/\lambda^{w_n})\,\mu^{w_n}=\mu_n$.

(7.1) Lemma. *There exists a σ-finite measure γ, say, on the Borel sets of $\mathbb{R}^d\setminus\{0\}$ and a subsequence $\{n_k\colon k\geqq 1\}$ such that*

$$\int_{\mathbb{R}^d\setminus\{0\}} \phi\, d\mu_{n_k}\to \int_{\mathbb{R}^d\setminus\{0\}} \phi\, d\gamma$$

as $k\to\infty$ for all continuous $\phi\colon\mathbb{R}^d\setminus\{0\}\to\mathbb{R}$ satisfying $\phi(x)/h(x)\to 0$ as $\|x\|\to 0$ and as $\|x\|\to\infty$.

Proof. We may assume $w_n\in W^+\cap W_r\cap W^{(p)}$ for all $n\geqq 1$. By Lemmas 6.5, 6.6 and 6.7 we obtain

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\{0\}} h\, d\mu_n &\leqq \frac{[K(p)+(1+R^p)(K_r+f(R)]}{\left[\log\dfrac{R}{r}-2K\right]}\\ &=K_1,\ \text{say}.\end{aligned}$$

Define M to be the one-point compactification of $\mathbb{R}^d\setminus\{0\}$, so that $M=(\mathbb{R}^d\setminus\{0\})\cup\{*\}$, say, and let $\mathscr{P}(M)$ denote the space of Borel probability measures on M with the weak topology. Define $\bar\mu_n\in\mathscr{P}(M)$ by

$$\bar\mu_n(A)=\frac{1}{K_1}\int_A h\, d\mu_n$$

if $A\in\mathscr{B}(\mathbb{R}^d\setminus\{0\})$, and

$$\bar\mu_n(\{*\})=1-\frac{1}{K_1}\int_{\mathbb{R}^d\setminus\{0\}} h\, d\mu_n.$$

Since M is a compact metrizable space, then so is $\mathscr{P}(M)$ and so there exists $\bar\gamma\in\mathscr{P}(M)$ and a subsequence $\{n_k\colon k\geqq 1\}$ such that $\bar\mu_{n_k}\to\bar\gamma$ weakly in $\mathscr{P}(M)$ as $k\to\infty$. Now define the σ-finite Borel measure γ on $\mathbb{R}^d\setminus\{0\}$ by

$$\gamma(A)=K_1\int_A (1/h)\, d\bar\gamma.$$

If ϕ is as in the statement of the lemma, then ϕ/h has a unique extension to a continuous function $\bar{\phi}$, say, on M with $\bar{\phi}(*)=0$. Then

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\{0\}} \phi\, d\mu_{n_k} &= K_1 \int_M \bar{\phi}\, d\bar{\mu}_{n_k}\\ &\to K_1 \int_M \bar{\phi}\, d\bar{\gamma}\\ &= \int_{\mathbb{R}^d\setminus\{0\}} \phi\, d\gamma\end{aligned}$$

as required. □

(7.2) Lemma. *Let γ be as in Lemma 7.1. Then $\gamma=\bar{\mu}$.*

Proof. Observe first that γ is an invariant measure for L^z. This follows as in the proof of Theorem 2.12 above, except now we test on C^2 functions with compact support in $\mathbb{R}^d\setminus\{0\}$. Therefore by Theorem 2.8(iii) γ is unique up to a multiplicative constant. In order to complete the proof it suffices to show that for any sufficiently large S

$$\frac{\gamma(U_{\varepsilon,S})}{|\log\varepsilon|} \to \frac{2}{V^z} \quad \text{as } \varepsilon\to 0. \tag{7.3}$$

Now if $\delta<S_1<S<S_2$ then

$$\limsup_{k\to\infty} \mu_{n_k}(U_{\varepsilon/2,S_1}) \leqq \gamma(U_{\varepsilon,S}) \leqq \liminf_{k\to\infty} \mu_{n_k}(U_{2\varepsilon,S_2}).$$

It follows from Lemma 6.8 that

$$\frac{2}{V^z}\,\frac{\left[\log\frac{R}{r}-4K\right]}{\left[\log\frac{R}{r}+2K\right]} \leqq \lim_{\varepsilon\to 0} \frac{\gamma(U_{\varepsilon,S})}{|\log\varepsilon|} \leqq \frac{2}{V^z}\,\frac{\left[\log\frac{R}{r}+4K\right]}{\left[\log\frac{R}{r}-2K\right]}. \tag{7.4}$$

The existence of the limit in the central term of (7.4) is guaranteed by Theorem 2.8(iii). Now the invariant probability measures μ^w are unchanged if we replace r by some smaller positive value (even though the size of the neighborhood W_r used in Lemmas 6.5 through 6.7 may change with r). In particular the measure γ obtained as a limit in Lemma 7.1 does not depend on the particular choice of r. As a result we may let $r\to 0$ in (7.4), thus obtaining (7.3), and we are done. □

We have done the hard work now and can quickly complete the proof of assertion (ii). The fact that the limit γ in Lemma 7.1 is unique implies that for any sequence $\{w_n\colon n\geqq 1\}$ converging to z through W^+ we have

$$\frac{1}{\lambda^{w_n}} \int_{\mathbb{R}^d\setminus\{0\}} \phi\, d\mu^{w_n} \to \int_{\mathbb{R}^d\setminus\{0\}} \phi\, d\bar{\mu}$$

as $n\to\infty$ for all continuous $\phi\colon \mathbb{R}^d\setminus\{0\}\to\mathbb{R}$ satisfying $\phi(x)/h(x)\to 0$ as $\|x\|\to 0$ and as $\|x\|\to\infty$. Since this result holds for all $p\in(0,1]$ and since obviously $\bar{\mu}$ does not depend on the value of p then we obtain the assertion (ii) of Theorem 2.12.

Finally we prove assertion (iii). Since $\lambda^z=0$ and $V^z>0$ then $\Lambda^z(p)>0$ for all $p\neq 0$. For any $\varepsilon>0$ we have $\Lambda^z(-\varepsilon)>0$ so by Lemma 4.9 there is a neighborhood U say of z in which $\Lambda^w(-\varepsilon)>0$. For $w\in W^+\cap U$ we also have $\lambda^w>0$ so that there must exist $\gamma^{(w)}\in(0,\varepsilon)$ such that $\Lambda^w(-\gamma^{(w)})=0$, and we are done. □

8 Examples

(8.1) *Example.* Stochastic pitchfork bifurcation. Consider the Itô stochastic differential equation on $\mathbb{R}$ given by

$$dx_t=(ax_t-bx_t^3)\,dt+\sigma x_t\,dW_t \tag{8.2}$$

where $a\in\mathbb{R}$, $b>0$ and $\sigma\geqq 0$. (The law of the process is unchanged if we replace σ by $-\sigma$ so there is no loss in generality in assuming $\sigma\geqq 0$.) In Stratonovich form we have

$$dx_t=((a-\tfrac{1}{2}\sigma^2)\,x_t-bx_t^3)\,dt+\sigma x_t\circ dW_t.$$

This is equation is also studied by Arnold and Boxler [AB2]. The linearized system is

$$dv_t=(a-\tfrac{1}{2}\sigma^2)\,v_t\,dt+\sigma v_t\circ dW_t$$

which has solution

$$v_t=v_0\exp\{(a-\tfrac{1}{2}\sigma^2)\,t+\sigma W_t\}.$$

We obtain $\lambda=\lambda^{(a,b,\sigma)}=a-\sigma^2/2$ and $\Lambda(p)=\Lambda^{(a,b,\sigma)}=(a-\sigma^2/2)\,p+\sigma^2p^2/2$. For any (a,b,σ) with $b>0$ and $\sigma>0$, we may easily check that the conditions $\mathrm{H1}(a,b,\sigma)$, $\mathrm{H2}(a,b,\sigma)$, $\mathrm{H3}(a,b,\sigma)$ are satisfied. In $\mathrm{H1}(a,b,\sigma)$ we may take $f(x)=g(x)=\exp\{Ax^2\}$ for any $A<b/\sigma^2$.

In this one dimensional example we can explicitly solve the stationary Fokker-Planck equation to find the density $\rho=\rho^{(a,b,\sigma)}$ of the invariant measure $\mu=\mu^{(a,b,\sigma)}$ on $(0,\infty)$ whenever $\lambda\geqq 0$. We obtain

$$\rho(x)=Cx^{2a/\sigma^2-2}\exp\{-bx^2/\sigma^2\} \tag{8.3}$$

for some constant C. Notice that $\rho\in L^1(0,\infty)$ if and only if $2a/\sigma^2-2>-1$ which happens if and only if $\lambda>0$; in this case $C=C^{(a,b,\sigma)}$ can be chosen so that ρ is the density of a probability measure. Moreover if $\lambda>0$, for any $r>0$ we have

$$\frac{C\exp\{-br^2/\sigma^2\}}{2a/\sigma^2-1}\,r^{2a/\sigma^2-1}\leqq\mu(0,r)\leqq\frac{C}{2a/\sigma^2-1}\,r^{2a/\sigma^2-1}$$

and $A(1-2a/\sigma^2)=0$, verifying (2.10). Now suppose $(a, b, \sigma)\to(\bar{a}, \bar{b}, \bar{\sigma})$ with $\bar{b}>0$, $\bar{\sigma}\neq 0$ and $\bar{a}-\bar{\sigma}^2/2=0$, through values where $a-\sigma^2/2>0$. Then $C=C^{(a,b,\sigma)}$ satisfies

$$\frac{C^{(a,b,\sigma)}}{2a/\sigma^2-1}\to 1$$

and we obtain

$$\begin{aligned}\frac{1}{\lambda^{(a,b,\sigma)}}\rho^{(a,b,\sigma)}(x)&=\frac{C^{(a,b,\sigma)}}{a-\sigma^2/2}x^{2a/\sigma^2-2}\exp\{-bx^2/\sigma^2\}\\&\to\frac{2}{\bar{\sigma}^2}\frac{1}{x}\exp\{-\bar{b}x^2/\bar{\sigma}^2\}.\end{aligned}$$

It is easy to check that the limit is the density of an invariant measure for the process with parameters $(\bar{a}, \bar{b}, \bar{\sigma})$, and that

$$\lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\int_{\varepsilon}^{\infty}\frac{2}{\bar{\sigma}^2}\frac{1}{x}\exp\{-\bar{b}x^2/\bar{\sigma}^2\}\,dx=\frac{2}{\bar{\sigma}^2},$$

thus verifying Theorem 2.12(ii). Notice that in this case we obtain a stronger form of convergence, namely convergence of densities rather than the particular version of weak convergence given in the Theorem.

(8.4) *Example.* Stochastic Hopf bifurcation. Consider the Itô stochastic differential equation on $\mathbb{R}^2$ given by

$$\begin{aligned}dx_t&=[-y_t+(a-b(x_t^2+y_t^2))\,x_t]\,dt+\sigma x_t\,dW_t\\dy_t&=[x_t+(a-b(x_t^2+y_t^2))\,y_t]\,dt+\sigma y_t\,dW_t\end{aligned}\tag{8.5}$$

with parameters $a\in\mathbb{R}$, $b>0$ and $\sigma\geqq 0$. Notice that if the noise intensity σ is zero then the equations reduce to a normal form of the (deterministic) Hopf bifurcation, see for example Guckenheimer and Holmes [GH]. If $\sigma=0$ and $a<0$ then $(0,0)$ is an attracting fixed point, while if $\sigma=0$ and $a>0$ then $(0,0)$ is a repelling fixed point and the circle of radius $\sqrt{a/b}$ is an attracting limit cycle. To study the stochastic version with $\sigma>0$ we pass to polar coordinates (r,θ) and obtain by Itô's formula

$$\begin{aligned}dr_t&=(ar_t-br_t^3)\,dt+\sigma r_t\,dW_t\\d\theta_t&=dt.\end{aligned}\tag{8.6}$$

The fact that there is rotational symmetry in the system (8.5) causes the equations (8.6) to be uncoupled, and this in turn allows us to do explicit calculations here. The equation for r_t in (8.6) is the same as equation (8.2) for x_t in example 8.1, and we may use the computations above. The Lyapunov exponent and Lyapunov moment function are given by the same formulae as in Example 8.1. The bifurcation occurs when $a=\sigma^2/2$. (This shift in the bifurcation point is a typical phenomenon when noise is added to a deterministic system). When $a-\sigma^2/2>0$ the process $\{(x_t, y_t): t\geqq 0\}$ has an 'attracting' invariant probability measure $\mu=\mu^{(a,b,\sigma)}$ on $\mathbb{R}^2\setminus\{(0,0)\}$

which has density $\rho(x, y)=\rho^{(a,b,\sigma)}(x, y)$ with respect to Lebesgue measure given by

$$\rho(x, y)=\frac{C}{2\pi}(x^2+y^2)^{a/\sigma^2-3/2}\exp\{-b(x^2+y^2)/\sigma^2\}$$

where $C=C^{(a,b,\sigma)}$ is the same constant as in (8.3). Notice that μ does not look like some blurry version of the deterministic limit cycle $x^2+y^2=\text{constant}$; at this point similarities with the deterministic scenario break down.

(8.6) *Example.* The noisy non-linear harmonic oscillator. Consider the following equation for a non-linear harmonic oscillator with a parametric white noise excitation.

$$\ddot{x}_t+2a\dot{x}_t+(1+bx_t^2+\sigma\dot{W}_t)x_t=0. \tag{8.7}$$

This equation is studied by Wedig [Wed], where it is used to model the amplitude of the first mode of vibration of a flexible beam under axial excitations. The constants $a\in\mathbb{R}$, $b\geqq 0$, and $\sigma\geqq 0$ represent respectively external viscous damping, the cubic rigidity of the beam, and the intensity of the white noise excitation. See also Ariaratnam and Xie [AX]. We rewrite (8.7) as the two dimensional stochastic differential equation

$$\begin{aligned} dx_1&=y_t\,dt\\ dy_t&=(-x_t-bx_t^3-2ay_t)\,dt-\sigma x_t\,dW_t. \end{aligned} \tag{8.8}$$

Note that for this equation the Itô and Stratonovich versions coincide.

Before studying the stochastic case let us consider briefly the deterministic case $\sigma=0$. If $a=0$ the system is conservative with closed orbits $x^2+bx^4/2+y^2=\text{constant}$. If $a>0$ then $(0, 0)$ is an attracting fixed point with $\mathbb{R}^2$ as its basin of attraction, while if $a<0$ then $(0, 0)$ is a repelling fixed point and all orbits go to infinity. (These statements can all be easily verified using the Lyapunov function $x^2+bx^4/2+y^2$.) Notice that in the deterministic setting there is no qualitative difference between the cases $b=0$ and $b>0$. This fact will change in the stochastic setting.

Now we consider the case $\sigma>0$. The linearized version of (8.8) is obtained by simply setting $b=0$. This linear stochastic differential equation was studied by Kozin and Prodromou [KP], who obtained conditions (involving numerical integration) which characterize the regions in (a, σ) space where the Lyapunov exponent λ is positive or negative. In particular they show that if $\sigma>0$ and $a\leqq 0$ then $\lambda>0$. Notice that, for the linearized system, as soon as $\lambda>0$ all trajectories go to infinity exponentially fast (almost-surely) so there is no invariant probability for the linearized system on $\mathbb{R}^2\setminus\{(0, 0)\}$. It is at this point that the strict positivity of b becomes important.

Henceforth we restrict to the region in parameter space where $a>0$, $b>0$ and $\sigma>0$. We check the conditions H1(a, b, σ), H2(a, b, σ), H3(a, b, σ). It is easy to verify H1(a, b, σ) with functions f and g each of the form Ax^4+By^2 and

$$f^{(\tilde{a},\tilde{b},\tilde{\sigma})}(x,y)=x^2+\tilde{b}x^4/2+y^2+\tilde{c}xy$$

where $\tilde{c}=\min\{2,\tilde{a}\}$ for $(\tilde{a},\tilde{b},\tilde{\sigma})$ near (a,b,σ). With somewhat more work it can be shown that we may also take

$$f^{(\tilde{a},\tilde{b},\tilde{\sigma})}(x,y)=\exp\{\beta\sqrt{x^2+\tilde{b}x^4+y^2+\tilde{c}xy}\}$$

for sufficiently small $\beta>0$, and then f and g can be taken to be of the form $\exp\{\beta\sqrt{(x^4+y^2)}\}$ (with possibly a different β). This extra work can be justified in view of the information it gives us about the integrability properties of the measure $\mu^{(a,b,\sigma)}$. The validity of H2(a,b,σ) follows easily from the support theorem for diffusion processes. The linearized system may be written

$$dv_t=\begin{bmatrix}0 & 1\\ -1 & -2a\end{bmatrix}v_t\,dt+\begin{bmatrix}0 & 0\\ -\sigma & 0\end{bmatrix}v_t\circ dW_t$$

so that

$$A_0^{(a,b,\sigma)}=\begin{bmatrix}0 & 1\\ -1 & -2a\end{bmatrix}\quad\text{and}\quad A_1^{(a,b,\sigma)}=\begin{bmatrix}0 & 0\\ -\sigma & 0\end{bmatrix}.$$

Thus H3(a,b,σ)(i) is satisfied although the stronger condition H4(a,b,σ) fails in this example. If we parametrize $S^1=\{(\cos s,\sin s)\colon s\in\mathbb{R}\}$ then we obtain

$$\tilde{A}_0^{(a,b,\sigma)}(s)=(-1-2a\cos s\sin s)\frac{d}{ds}\quad\text{and}\quad \tilde{A}_1^{(a,b,\sigma)}(s)=(-\sigma\cos^2 s)\frac{d}{ds}.$$

The condition H3(a,b,σ)(ii) can now be verified. Condition H3(a,b,σ)(iii) is easily checked using elementary phase portrait sketching techniques.

Therefore we may apply the Theorems 2.8, 2.12 and 2.14 to the Eq. (8.8) whenever the parameters satisfy $a>0$, $b>0$ and $\sigma>0$. Let us observe just one of the consequences. Fix parameters $(\bar{a},\bar{b},\bar{\sigma})$ so that $\lambda^{(\bar{a},\bar{b},\bar{\sigma})}=0$ and suppose that (a,b,σ) converges to $(\bar{a},\bar{b},\bar{\sigma})$ through the region where $\lambda^{(a,b,\sigma)}>0$. Corresponding to (a,b,σ) the process $\{(x_t,y_t)\colon t\geq 0\}$ is positive recurrent on $\mathbb{R}^2\setminus\{(0,0)\}$ with invariant probability $\mu^{(a,b,\sigma)}$. Theorem 2.12 implies that for any polynomial function $\phi\colon\mathbb{R}^2\to\mathbb{R}$ satisfying $\phi(0,0)=0$ we have

$$\frac{1}{\lambda^{(a,b,\sigma)}}\int\phi(x,y)\,d\mu^{(a,b,\sigma)}\to\int\phi(x,y)\,d\bar{\mu}(x,y)$$

as (a,b,σ) converges to $(\bar{a},\bar{b},\bar{\sigma})$, for some σ-finite measure $\bar{\mu}$, and the right side is finite. In particular for any $p>0$ the mean p^{th} power amplitude $\mathbf{E}^{(a,b,\sigma)}(x_t^2+y_t^2)^{p/2}$ for the stationary version of the process with parameters (a,b,σ) satisfies

$$\mathbf{E}^{(a,b,\sigma)}(x_t^2+y_t^2)^{p/2}\sim\lambda^{(a,b,\sigma)}C_p$$

as (a, b, σ) converges to $(\bar{a}, \bar{b}, \bar{\sigma})$ where the finite positive constants C_p depend only on p and the law of the Eq. (8.8) at parameter values $(\bar{a}, \bar{b}, \bar{\sigma})$. This gives a theoretical explanation of part of the simulation results shown in [Wed, Fig. 2].

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