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Probability Theory and Related Fields

Lim inf results for the Wiener process and its increments under the L_2-norm

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Summary. Let $W(t)$ be a Wiener process. The lim inf behavior of the L_2-norm of $W(t)$ on the interval $[T-a(T), T]$ and of $|W(t+\theta T)-W(t)|$ on the interval $[\alpha T, \beta T]$ is given under suitable conditions.

1 Introduction

Let $\{W(t), t \geqq 0\}$ be a standard Wiener process. There are various types of limiting results for $W(t)$ and its increments. For an account on the subject and references, see, for example, Grill [9] for the increments of $W(t)$, Li [13] for $W(t)$ itself.

In this paper, we consider the lim inf of the Wiener process and its increments on certain intervals under the L_2-norm. On the interval $[0, T]$, Donsker and Varadhan [7] showed by using their functional law of iterated logarithm for local times that

$$\varliminf_{T\to\infty} \frac{\log\log T}{T^2} \int_0^T W^2(t)\,\mathrm{d}t = \frac{1}{8} \quad \text{a.s.} \tag{1.1}$$

What happens on the interval $[T-a(T), T]$ for $a(T) \geqq 0$? We have the following results.

Theorem 1 *Let $a(T)$ satisfy the conditions*
(i) $0 < a(T) \leqq T$, *$a(T)$ is a non-decreasing function of T, for* $0 < T < \infty$;
(ii) *$a(T)/T$ is non-increasing as $T \to \infty$; or*
(ii)′ $\lim_{T\to\infty} a(T)/T = \rho$, $0 < \rho \leqq 1$.
If $\lim_{T\to\infty} \log(T/a(T)) \cdot (\log\log T)^{-1} = \infty$, *then*

$$\varliminf_{T\to\infty} \frac{\log(T/a(T))}{a^2(T)} \int_{T-a(T)}^{T} W^2(t)\,\mathrm{d}t = \frac{1}{4} \quad \text{a.s.}\,. \tag{1.2}$$

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If $\lim_{T\to\infty} \log(T/a(T))\cdot(\log\log T)^{-1} < \infty$ *and* $\overline{\lim}_{T\to\infty} a(\gamma T)/a(T) < \infty$ *for some* $\gamma > 1$, *then*

$$\underline{\lim}_{T\to\infty} \phi(T) \int_{T-a(T)}^{T} W^2(t)\,dt = \frac{1}{4} \quad \text{a.s.} \tag{1.3}$$

where

$$\phi(T) = (\log(T/a(T)) + 2\log\log T)/a^2(T)\,.$$

To illustrate what Theorem 1 tells us, we give here the following examples.

Example 1. For $x \geqq 0$, let $a(T) = (1+x)^{-1}T$, then (1.3) tells us by the change of variable that

$$\underline{\lim}_{T\to\infty} \frac{\log\log T}{T^2} \int_{xT}^{(x+1)T} W^2(t)\,dt = \frac{1}{8} \quad \text{a.s.}\,. \tag{1.4}$$

If $x = 0$, (1.4) becomes (1.1). It is somewhat strange that (1.4) is true no matter what $x \geqq 0$ is. One might expect (1.4) has something to do with the zeros of $W(t)$. In fact, for almost all $\omega \in \Omega$, there exist $T_k(\omega)$ such that

$$W(xT_k(\omega)) = 0 \quad k = 1, 2, \ldots, \quad \lim_{k\to\infty} T_k(\omega) = \infty\,.$$

Hence we can see in a very rough sense (we use $\simeq$), for $\psi(T) = T^{-2}\log\log T$,

$$\underline{\lim}_{T\to\infty} \psi(T) \int_{xT}^{(x+1)T} W^2(t)\,dt \simeq \underline{\lim}_{k\to\infty} \psi(T_k(\omega)) \int_{xT_k(\omega)}^{(x+1)T_k(\omega)} W^2(t)\,dt$$

$$\simeq \underline{\lim}_{k\to\infty} \psi(T_k(\omega)) \int_{0}^{T_k(\omega)} W^2(t)\,dt \simeq \frac{1}{8}\,.$$

The problem, however, is to make this precise.

Example 2. Let $a_1(T) = c$, $a_2(T) = c\sqrt{\log T}$, $a_3(T) = cT^\alpha$ where $0 < \alpha < 1$ and $c > 0$ is a constant. Then (1.2) says that

$$\underline{\lim}_{T\to\infty} \log T \int_{T-c}^{T} W^2(t)\,dt = \frac{c^2}{4} \quad \text{a.s.}\,;$$

$$\underline{\lim}_{T\to\infty} \int_{T-c\sqrt{\log T}}^{T} W^2(t)\,dt = \frac{c^2}{4} \quad \text{a.s.}\,;$$

$$\underline{\lim}_{T\to\infty} \frac{\log T}{T^{2\alpha}} \int_{T-cT^\alpha}^{T} W^2(t)\,dt = \frac{c^2}{4(1-\alpha)} \quad \text{a.s.}\,.$$

Hence we see from (1.2) that $a(T) = c\sqrt{\log T}$ is the critical function, i.e. under our conditions (i) and (ii),

$$\varliminf_{T\to\infty} \int_{T-a(T)}^{T} W^2(t)\,dt = \begin{cases} 0 & \text{a.s.} \quad \text{if } \lim_{T\to\infty} a(T)/\sqrt{\log T} = 0 \\ c^2/4 & \text{a.s.} \quad \text{if } \lim_{T\to\infty} a(T)/\sqrt{\log T} = c \\ \infty & \text{a.s.} \quad \text{if } \lim_{T\to\infty} a(T)/\sqrt{\log T} = \infty\,. \end{cases}$$

Now we turn to the lim inf of the increments $|W(t+\theta T) - W(t)|$ on the interval $[\alpha T, \beta T]$ under the L_2-norm.

Theorem 2 *If* $\theta \geqq \beta - \alpha > 0$ *and* $\alpha \geqq 0$, *then*

$$\varliminf_{T\to\infty} \frac{\log\log T}{T^2} \int_{\alpha T}^{\beta T} |W(t+\theta T) - W(t)|^2\,dt = \frac{(\beta-\alpha)^2}{4} \quad \text{a.s.} \tag{1.5}$$

If $0 < \theta < \beta - \alpha$ *and* $\alpha \geqq 0$, *then*

$$\frac{\theta^2}{4} \leqq \varliminf_{T\to\infty} \frac{\log\log T}{T^2} \int_{\alpha T}^{\beta T} |W(t+\theta T) - W(t)|^2\,dt \leqq \frac{(\beta-\alpha)(\beta+\theta)\pi^2}{2} \quad \text{a.s.} \tag{1.6}$$

An interesting thing about Theorem 2 is that as long as $\theta \geqq \beta - \alpha > 0$, the limiting constant does not depend on θ which is not intuitively clear. For the case $\beta - \alpha > \theta > 0$ in Theorem 2, our proof for (1.5) will work in principle. However, due to the complexity of an eigenvalue computation, we could not obtain the desired small deviation estimates and hence the exact constant. The difficulties come in because when we consider

$$\int_{\alpha T}^{\beta T} |W(t+\theta T) - W(t)|^2\,dt$$

for $\beta - \alpha > \theta > 0$, both $t + \theta T$ and t can lie inside the interval $[\alpha T, \beta T]$ and, as a result, the computations become too involved. We also remark that the lim sup results similar to Theorem 1 and Theorem 2 are given in Li [12].

We list some necessary lemmas in Sect. 2. Our Lemma 13, Lemma 14 and Lemma 18 provide the necessary lower tail estimates that are new and can be viewed as an application of the comparison results given in Li [11] (see Lemma 1 and Lemma 2 in this paper). Our Lemma 10 and Lemma 16 are the useful probability inequalities for the Wiener process, which have independent interest and are also true for the sup-norm, L_p-norm and some other norms. We give the proof of Theorem 1 in Sect. 3 and the proof of Theorem 2 in Sect. 4.

Now we need some notation for the next three sections. Let ε stand for a small positive number given arbitrarily, and C denote various positive constants independent of k and n, whose values might change from line to line. $f(\varepsilon) \sim g(\varepsilon)$ as $\varepsilon \to 0$ means $\lim_{\varepsilon\to 0} f(\varepsilon)/g(\varepsilon) = 1$.

2 Lemmas

Let ξ_n, $n \geqq 1$ be independent and normally distributed with mean zero and variance 1. Lemma 1 and Lemma 2 below are the comparison theorems in Li [11].

Lemma 1 *Let* $a_n > 0, \sum_{n \geqq 1} a_n < \infty$ *and* $b_n > 0, \sum_{n \geqq 1} b_n < \infty$. *If* $\sum_{n \geqq 1} |1 - a_n/b_n| < \infty$, *then*

$$P\left(\sum_{n \geqq 1} a_n \xi_n^2 \leqq \varepsilon\right) \sim \left(\prod b_n/a_n\right)^{1/2} P\left(\sum_{n \geqq 1} b_n \xi_n^2 \leqq \varepsilon\right) \quad as \quad \varepsilon \to 0 .$$

Lemma 2 *For positive integer N and* $\sum_{n \geqq 1} a_n < \infty$, $a_n > 0$, *we have*

$$P\left(\sum_{n \geqq 1} a_n \xi_n^2 \leqq \varepsilon\right) \sim \left(\prod_{n=1}^{N-1} 2a_n\right)^{-1/2} \tau_N^{-(N-1)/2} P\left(\sum_{n \geqq N} a_n \xi_n^2 \leqq \varepsilon\right) \quad as \quad \varepsilon \to 0$$

where $\tau_N = \tau_N(\varepsilon)$, *for* $\varepsilon > 0$ *small enough, satisfies the equation*

$$\varepsilon = \sum_{n \geqq N} \frac{a_n}{1 + 2a_n \tau_N} .$$

The following lemma was first given by Anderson and Darling [1].

Lemma 3 *Let* $\{B(t): 0 \leqq t \leqq 1\}$ *be a Brownian bridge. Then as* $\varepsilon \to 0$,

$$P\left(\int_0^1 B^2(t)\,\mathrm{d}t < \varepsilon\right) = P\left(\sum_{n \geqq 1} \frac{1}{\pi^2 n^2} \xi_n^2 \leqq \varepsilon\right) \sim \frac{4}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{8\varepsilon}\right).$$

The lemma below was given by Cameron and Martin [3].

Lemma 4 *As* $\varepsilon \to 0$,

$$P\left(\int_0^1 W^2(t)\,\mathrm{d}t < \varepsilon\right) = P\left(\sum_{n \geqq 1} \frac{1}{\pi^2 (n - 1/2)^2} \xi_n^2 \leqq \varepsilon\right) \sim \frac{4\sqrt{\varepsilon}}{\sqrt{\pi}} \cdot \exp\left(-\frac{1}{8\varepsilon}\right).$$

Lemma 5 *For* $a_1 > 0$, *we have as* $\varepsilon \to 0$

$$P\left(a_1 \xi_1^2 + \sum_{n \geqq 1} \frac{1}{\pi^2 n^2} \xi_{n+1}^2 \leqq \varepsilon\right) \sim \frac{4\sqrt{2}}{\sqrt{a_1 \pi}} \cdot \varepsilon \cdot \exp\left(-\frac{1}{8\varepsilon}\right); \tag{2.1}$$

$$P\left(\sum_{n \geqq 1} \frac{1}{\pi^2 (n + 1/2)^2} \xi_n^2 \leqq \varepsilon\right) \sim 4\pi^{-3/2} \cdot \frac{1}{\sqrt{\varepsilon}} \cdot \exp\left(-\frac{1}{8\varepsilon}\right). \tag{2.2}$$

Proof. By Lemma 2 and Lemma 3, we see that as $\varepsilon \to 0$

$$\begin{aligned} &P\left(a_1 \xi_1^2 + \sum_{n \geqq 1} \frac{1}{\pi^2 n^2} \xi_{n+1}^2 \leqq \varepsilon\right) \\ &\sim (2a_1)^{-1/2} \cdot \tau^{-1/2} \cdot P\left(\sum_{n \geqq 1} \frac{1}{\pi^2 n^2} \xi_{n+1}^2 \leqq \varepsilon\right) \\ &\sim \frac{2}{\sqrt{a_1 \pi}} \cdot \tau^{-1/2} \cdot \exp\left(-\frac{1}{8\varepsilon}\right) \end{aligned} \tag{2.3}$$

where $\tau = \tau(\varepsilon)$, for $\varepsilon > 0$ small enough, satisfies

$$\varepsilon = \sum_{n \geqq 1} \frac{1}{\pi^2 n^2 + 2\tau} = \frac{1}{2\sqrt{2\tau}} \left(\frac{1 + \exp(-2\sqrt{2\tau})}{1 - \exp(-2\sqrt{2\tau})} - \frac{1}{\sqrt{2\tau}} \right). \tag{2.4}$$

The last equality above can be found in Gradshteyn and Ryzhik [8]. Hence we obtain (2.1) by substituting $\tau^{-1/2} \sim 2\sqrt{2} \cdot \varepsilon$ as $\varepsilon \to 0$ from (2.4) into (2.3).
Similarly, by Lemma 2 and Lemma 4, we have that as $\varepsilon \to 0$

$$\begin{aligned} P\left(\sum_{n \geqq 1} \frac{1}{\pi^2 (n + 1/2)^2} \xi_n^2 \leqq \varepsilon \right) &= P\left(\sum_{n \geqq 2} \frac{1}{\pi^2 (n - 1/2)^2} \xi_n^2 \leqq \varepsilon \right) \\ &\sim \frac{2\sqrt{2}}{\pi} \cdot \gamma^{1/2} \cdot P\left(\sum_{n \geqq 1} \frac{1}{\pi^2 (n - 1/2)^2} \xi_n^2 \leqq \varepsilon \right) \\ &\sim 8\pi^{-3/2} \cdot (2\varepsilon)^{1/2} \cdot \gamma^{1/2} \cdot \exp\left(-\frac{1}{8\varepsilon} \right), \end{aligned} \tag{2.5}$$

where $\gamma = \gamma(\varepsilon)$, for $\varepsilon > 0$ small enough, satisfies the equation

$$\varepsilon = \sum_{n \geqq 2} \frac{1}{\pi^2 (n - 1/2)^2 + 2\gamma} = \sum_{n \geqq 1} \frac{4}{\pi^2 n^2 + 8\gamma} - \sum_{n \geqq 1} \frac{1}{\pi^2 n^2 + 2\gamma} - \frac{4}{\pi^2 + 8\gamma}.$$

Hence by using the identity in (2.4), we have $\gamma^{1/2} \sim (2\sqrt{2}\varepsilon)^{-1}$ as $\varepsilon \to 0$. Therefore we obtain (2.2) by substituting $\gamma^{1/2} \sim (2\sqrt{2}\varepsilon)^{-1}$ into (2.5). This finishes the proof.

By the Karhunen-Loève expansion, we have the following lemma. The detailed calculations can be found in Li [12] and are similar to the calculation in our Lemma 18.

Lemma 6 *For any $b > a \geqq 0$ and $s > 0$*

$$P\left(\int_a^b W^2(t)\,\mathrm{d}t \leqq s \right) = P\left(\sum_{n \geqq 1} \lambda_n(a, b) \xi_n^2 \leqq s \right)$$

where $\lambda_n(a, b)$ is the n^{th} solution of the equation in decreasing order

$$a \cdot \sin \frac{b - a}{\sqrt{x}} = \sqrt{x} \cdot \cos \frac{b - a}{\sqrt{x}}. \tag{2.6}$$

Now we list some of the properties of $\lambda_n(a, b)$ defined in (2.6).

Lemma 7 *Let $\lambda_n(a, b)$ $(n \geqq 1)$ be defined as in (2.6) and $b > a \geqq d > 0$. Then*

$$(b - a)^2 (n - 1/2)^{-2} \pi^{-2} < \lambda_n(a, b) < (b - a)^2 (n - 1)^{-2} \pi^{-2} \quad \textit{for} \quad n \geqq 2; \tag{2.7}$$

$$a(b - a) < \lambda_1(a, b) < (b^2 - a^2)/2; \tag{2.8}$$

$$\lambda_n(a - d, b - d) < \lambda_n(a, b) < a^2 (a - d)^{-2} \lambda_n(a - d, b - d). \tag{2.9}$$

$$(\lambda_n(a, b))^{-1/2} = (b - a)^{-1} (n - 1)\pi + O(1/n). \tag{2.10}$$

Proof. Let $\rho_n = (\lambda_n(a, b))^{-1/2}$. Then $\tan(b-a)\rho_n = (a\rho_n)^{-1}$. It is easy to see by looking at the graph of the function $\tan x$ and $(ax)^{-1}$ that $(n-1)\pi < (b-a)\rho_n < (n-1/2)\pi$ for $n \geqq 1$. This gives (2.7).

By using the inequality $\tan x > x$ on $(0, \pi/2)$, we have

$$(a\rho_1)^{-1} = \tan(b-a)\rho_1 > (b-a)\rho_1$$

which gives our lower bound in (2.8). Turn to the upper bound in (2.8). We need to show $\rho_1 > (2/(b^2-a^2))^{1/2}$. If $(b-a)\rho_1 \geqq \sqrt{2}$, then $\rho_1 \geqq \sqrt{2}/(b-a) > (2/(b^2-a^2))^{1/2}$. If $(b-a)\rho_1 < \sqrt{2}$, then by using the inequality $\tan x < 2x/(2-x^2)$ on $(0, \sqrt{2})$, we have

$$(a\rho_1)^{-1} = \tan(b-a)\rho_1 < 2(b-a)\rho_1/(2-(b-a)^2\rho_1^2)$$

which is $\rho_1 > (2/(b^2-a^2))^{1/2}$. Hence (2.8) holds.

For (2.9), it is easy to see it holds by the lower half of (2.7) when $d = a$. Let $\rho_n' = (\lambda_n(a-d, b-d))^{-1/2}$ and $a > d > 0$. Then

$$\rho_n' \tan(b-a)\rho_n' = (a-d)^{-1} > a^{-1} = \rho_n \tan(b-a)\rho_n\,.$$

Hence $\rho_n' > \rho_n$ which is the lower half of (2.9). The upper half follows from

$$(\rho_n'(a-d))^{-1} = \tan(b-a)\rho_n' > \tan(b-a)\rho_n = (\rho_n a)^{-1}\,.$$

Now turn to (2.10). By the inequality $\tan x > x$ on $(0, \pi/2)$, we have

$$\begin{aligned}(a\pi(n-1))^{-1} > (a\rho_n)^{-1} &= \tan(b-a)\rho_n \\ &= \tan((b-a)\rho_n - (n-1)\pi) > (b-a)\rho_n - (n-1)\pi > 0\end{aligned}$$

which gives (2.10). Thus we finish the proof.

Our next lemma is a particular case of Theorem 2.1 of Hoffmann-Jørgensen et al. [10] which is a well known fact about the measure of the translated ball.

Lemma 8 *For any $b > a \geqq 0$, $\varepsilon > 0$ and $x \in R$*

$$P\left(\int_a^b (W(t)+x)^2\,dt \leqq \varepsilon\right) \leqq P\left(\int_a^b W^2(t)\,dt \leqq \varepsilon\right)$$

Lemma 9 *For any $b > a \geqq d \geqq 0$, $s > 0$ and $x \in R$*

$$P\left(\int_a^b W^2(t)\,dt \leqq s \mid W(d) = x\right) \leqq P\left(\int_{a-d}^{b-d} W^2(t)\,dt \leqq s\right)$$

Proof. By using Lemma 8 and the fact that the Wiener process has independent and stationary increments, we have

$$\begin{aligned}&P\left(\int_a^b W^2(t)\,dt \leqq s \mid W(d) = x\right) \\ &= P\left(\int_a^b (W(t) - W(d) + x)^2\,dt \leqq s \mid W(d) = x\right)\end{aligned}$$

$$= P\left(\int_a^b (W(t) - W(d) + x)^2 \, dt \leqq s\right)$$

$$= P\left(\int_{a-d}^{b-d} (W(t) + x)^2 \, dt \leqq s\right)$$

$$\leqq P\left(\int_{a-d}^{b-d} W^2(t) dt \leqq s\right)$$

where the second equality is by the vector form of Corollary 4.38 in Breiman [2].

Lemma 10 *For any* $b > a \geqq d \geqq 0$ *and* $s > 0$

$$P\left(\int_a^b W^2(t)\,dt \leqq s\right) \leqq P\left(\int_{a-d}^{b-d} W^2(t)\,dt \leqq s\right) \leqq \left(\frac{a}{a-d}\right)^{1/2} P\left(\int_a^b W^2(t)\,dt \leqq s\right).$$

In particular, if $d = a$,

$$P\left(\int_a^b W^2(t)\,dt \leqq s\right) \leqq P\left(\int_0^{b-a} W^2(t)\,dt \leqq s\right) = P\left(\int_0^1 W^2(t)\,dt \leqq \frac{s}{(b-a)^2}\right).$$

Proof. The first part can be easily seen by integrating the inequality in Lemma 9. For the other part, we have by the basic properties of the Wiener process

$$P\left(\int_{a-d}^{b-d} W^2(t)\,dt \leqq s\right)$$

$$= \int_{-\infty}^{\infty} P\left(\int_0^{b-a} (W(t) + x)^2\,dt \leqq s\right) dP(W(a-d) < x)$$

$$\leqq \left(\frac{a}{a-d}\right)^{1/2} \int_{-\infty}^{\infty} P\left(\int_0^{b-a} (W(t) + x)^2\,dt \leqq s\right) dP(W(a) < x)$$

$$= \left(\frac{a}{a-d}\right)^{1/2} P\left(\int_a^b W^2(t)\,dt \leqq s\right)$$

where the first equality is by similar argument as those in the proof of Lemma 9 and the last equality follows from the first equality backward.

Lemma 11 *Let* $a_n > 0$ *and* $\sum_{n \geqq 1} a_n < \infty$. *Then for any* $s > 0$,

$$P\left(a_1 \xi_1^2 + \sum_{n \geqq 2} a_n \xi_n^2 \leqq s\right) \leqq \sqrt{s/a_1} \cdot P\left(\sum_{n \geqq 2} a_n \xi_n^2 \leqq s\right).$$

Proof. By conditioning on $\{\xi_1 = x\}$, we have

$$P\left(a_1 \xi_1^2 + \sum_{n \geqq 2} a_n \xi_n^2 \leqq s\right) = \int_{a_1 x^2 \leqq s} P\left(a_1 x^2 + \sum_{n \geqq 2} a_n \xi_n^2 \leqq s\right) dP(\xi_1 < x)$$

$$\leqq P\left(\sum_{n \geqq 2} a_n \xi_n^2 \leqq s\right) \cdot \int_{a_1 x^2 \leqq s} dP(\xi_1 \leqq x) \leqq \sqrt{s/a_1} \cdot P\left(\sum_{n \geqq 2} a_n \xi_n^2 \leqq s\right).$$

Lemma 12 *Let $a_n > 0$ and $\sum_{n \geqq 1} a_n < \infty$. Then for any $s > 0$ and $0 < \delta < 1$,*

$$P\left(a_1 \xi_1^2 + \sum_{n \geqq 2} a_n \xi_n^2 \leqq s \right)$$

$$\geqq (2\pi^{-1})^{1/2} \cdot (\delta s a_1^{-1})^{1/2} \cdot \exp\left(-\frac{1}{2} \delta s a_1^{-1} \right) \cdot P\left(\sum_{n \geqq 2} a_n \xi_n^2 \leqq (1-\delta)s \right).$$

Proof. By conditioning on $\{\xi_1 = x\}$ as we did in the proof of Lemma 11 and restricting $a_1 x^2 \leqq \delta s$, we have

$$P\left(a_1 \xi_1^2 + \sum_{n \geqq 2} a_n \xi_n^2 \leqq s \right)$$

$$\geqq \int_{a_1 x^2 \leqq \delta s} P\left(a_1 x^2 + \sum_{n \geqq 2} a_n \xi_n^2 \leqq s \right) \mathrm{d}P(\xi_1 < x)$$

$$\geqq \int_{a_1 x^2 \leqq \delta s} P\left(\delta s + \sum_{n \geqq 2} a_n \xi_n^2 \leqq s \right) \mathrm{d}P(\xi_1 < x)$$

$$\geqq (2\pi^{-1})^{1/2} \cdot (\delta s a_1^{-1})^{1/2} \cdot \exp\left(-\frac{1}{2} \delta s a_1^{-1} \right) \cdot P\left(\sum_{n \geqq 2} a_n \xi_n^2 \leqq (1-\delta)s \right).$$

Lemma 13 *If $s/(b-a)^2$ small enough, then*

$$P\left(\int_a^b W^2(t)\,\mathrm{d}t \leqq s \right) \leqq K_2 \cdot ((b-a)/a)^{1/2} \cdot \exp\left(-\frac{1}{8} \cdot \frac{(b-a)^2}{s} \right),$$

where $K_2 > 0$ is a constant independent of a, b and s.

Proof. By using Lemma 6, Lemma 7, Lemma 11 and Lemma 5, we have

$$P\left(\int_a^b W^2(t)\,\mathrm{d}t \leqq s \right) = P\left(\sum_{n \geqq 1} \lambda_n(a,b) \xi_n^2 \leqq s \right)$$

$$\leqq P\left(a(b-a)\xi_1^2 + \sum_{n \geqq 2} \frac{(b-a)^2}{(n-1/2)^2 \pi^2} \xi_n^2 \leqq s \right)$$

$$\leqq (s/a(b-a))^{1/2} \cdot P\left(\sum_{n \geqq 1} \frac{1}{(n+1/2)^2 \pi^2} \xi_n^2 \leqq \frac{s}{(b-a)^2} \right)$$

$$\leqq K_2 \cdot ((b-a)/a)^{1/2} \cdot \exp\left(-\frac{1}{8} \cdot \frac{(b-a)^2}{s} \right).$$

Lemma 14 *If $s/(b-a)^2$ small enough, then for $0 < \delta < 1$*

$$P\left(\int_a^b W^2(t)\,\mathrm{d}t \leqq s \right) \geqq K_1 \cdot (\delta s/(b^2 - a^2))^{1/2} \cdot \exp\left(-\frac{1}{8} \cdot \frac{(b-a)^2}{(1-\delta)s} \right)$$

where $K_1 > 0$ is a constant independent of a, b, δ and s.

Proof. By using Lemmas 6 and 7, Lemma 12 and Lemma 3, we have

$$P\left(\int_a^b W^2(t)\,\mathrm{d}t \leqq s\right) = P\left(\sum_{n\geqq 1} \lambda_n(a,b)\xi_n^2 \leqq s\right)$$

$$\geqq P\left(\frac{b^2-a^2}{2}\xi_1^2 + \sum_{n\geqq 2} \frac{(b-a)^2}{(n-1)^2\pi^2}\xi_n^2 \leqq s\right)$$

$$\geqq (2\pi^{-1})^{1/2}\cdot(2\delta s/(b^2-a^2))^{1/2}\cdot\exp(-\delta s/(b^2-a^2))$$

$$\cdot P\left(\sum_{n\geqq 1}\frac{1}{n^2\pi^2}\xi_n^2 \leqq \frac{(1-\delta)s}{(b-a)^2}\right)$$

$$\geqq K_1\cdot(\delta s/(b^2-a^2))^{1/2}\cdot\exp\left(-\frac{1}{8}\cdot\frac{(b-a)^2}{(1-\delta)s}\right).$$

The following is a well known version of the Borel-Cantelli lemma.

Lemma 15 *If A_k are events such that $\sum_{k\geqq 1} P(A_k) = \infty$ and*

$$\varliminf_{n\to\infty} \frac{\sum_{k=1}^{n}\sum_{l=1}^{n} P(A_k A_l)}{\sum_{k=1}^{n}\sum_{l=1}^{n} P(A_k)P(A_l)} \leqq 1\,,$$

then $P(A_k \text{ i.o.}) = 1$.

Lemma 16 *For any $b' > a' \geqq b > a \geqq 0$ and $s' > 0$, $s > 0$, we have*

$$P\left(\int_a^b W^2(t)\,\mathrm{d}t \leqq s,\quad \int_{a'}^{b'} W^2(t)\mathrm{d}t \leqq s'\right)$$

$$\leqq P\left(\int_a^b W^2(t)\,\mathrm{d}t \leqq s\right)\cdot P\left(\int_{a'-b}^{b'-b} W^2(t)\,\mathrm{d}t \leqq s'\right)$$

$$\leqq \left(\frac{a'}{a'-b}\right)^{1/2}\cdot P\left(\int_a^b W^2(t)\,\mathrm{d}t \leqq s\right)\cdot P\left(\int_{a'}^{b'} W^2(t)\,\mathrm{d}t \leqq s'\right).$$

Proof. By Lemma 9, Lemma 10 and the basic properties of the Wiener process, we have

$$P\left(\int_a^b W^2(t)\,\mathrm{d}t \leqq s, \int_{a'}^{b'} W^2(t)\,\mathrm{d}t \leqq s'\right)$$

$$= \int_{-\infty}^{\infty} P\left(\int_a^b W^2(t)\,\mathrm{d}t \leqq s, \int_{a'}^{b'} W^2(t)\,\mathrm{d}t \leqq s' \mid W(b) = x\right)\mathrm{d}P(W(b) < x)$$

$$= \int_{-\infty}^{\infty} P\left(\int_a^b W^2(t)\,\mathrm{d}t \leqq s \mid W(b) = x\right)\cdot P\left(\int_{a'}^{b'} W^2(t)\,\mathrm{d}t \leqq s' \mid W(b) = x\right)\mathrm{d}P(W(b) < x)$$

$$\leqq \int_{-\infty}^{\infty} P\left(\int_a^b W^2(t)\,\mathrm{d}t \leqq s \mid W(b) = x\right)\cdot P\left(\int_{a'-b}^{b'-b} W^2(t)\,\mathrm{d}t \leqq s'\right)\mathrm{d}P(W(b) < x)$$

$$= P\left(\int_a^b W^2(t)\,\mathrm{d}t \leqq s\right)\cdot P\left(\int_{a'-b}^{b'-b} W^2(t)\,\mathrm{d}t \leqq s'\right)$$

$$\leqq \left(\frac{a'}{a'-b}\right)^{1/2}\cdot P\left(\int_a^b W^2(t)\,t \leqq s\right)\cdot P\left(\int_{a'}^{b'} W^2(t)\,\mathrm{d}t \leqq s'\right).$$

where the second equation is the fact that the past and the future are conditionally independent given the present (see Theorem 9.2.4 in Chung [5]).

Lemma 17 *Let $\beta > 1$ and $M > 1$. If $k^{1-\beta} < 1/2$ for $k \geqq k_0$, then*

$$\sup_{k \geqq k_0} \int_{k+1}^{Mk} \frac{\mathrm{d}x}{(x^\beta - 1 - k^\beta)^{1/2}\cdot x^{1-\beta/2}} \leqq (M-1)^{1/2}.$$

Proof. Observing $(k/x)^\beta < k/x$ and $x^{-\beta} < 1/(2x)$ for $x > k > k_0$, we have

$$\begin{aligned}\sup_{k \geqq k_0} \int_{k+1}^{Mk} \frac{\mathrm{d}x}{(x^\beta - 1 - k^\beta)^{1/2}\cdot x^{1-\beta/2}} &= \sup_{k \geqq k_0} \int_{k+1}^{Mk} \frac{\mathrm{d}x}{(1 - x^{-\beta} - (k/x)^\beta)^{1/2}\cdot x} \\ &\leqq \sup_{k \geqq k_0} \int_{k+1}^{Mk} \frac{\mathrm{d}x}{(1 - 1/(2x) - k/x)^{1/2}\cdot x} \\ &\leqq \sup_{k \geqq k_0} \frac{1}{k^{1/2}}\cdot \int_{k+1}^{Mk} \frac{\mathrm{d}x}{(x - k - 1/2)^{1/2}} \\ &\leqq (M-1)^{1/2}.\end{aligned}$$

The following two lemmas are basic for the proof of our Theorem 2.

Lemma 18 *If $a \geqq 1$, then*

$$P\left(\int_0^1 |W(t+a) - W(t)|^2\,\mathrm{d}t < \varepsilon\right) \sim K(a)\cdot\varepsilon\cdot\exp\left(-\frac{1}{4\varepsilon}\right) \quad as \quad \varepsilon \to 0$$

where $K(a)$ is a positive constant.

Proof. Let $X(t) = W(t+a) - W(t)$, $t \geqq 0$ and $a \geqq 1$. Then $\{X(t): 0 \leqq t \leqq 1\}$ is a Gaussian process with mean zero and covariance function

$$r(s,t) = EX(s)X(t) = \max(0, a - |s-t|) \quad \text{for} \quad s, t \in [0,1].$$

Hence we have in distribution

$$\int_0^1 |W(t+a) - W(t)|^2\,\mathrm{d}t = \sum_{n \geqq 1} \lambda_n^{(a)} \xi_n^2, \quad \lambda_n^{(a)} > 0,$$

by the Karhunen-Loève expansion. Here, in decreasing order, $\lambda_n^{(a)} > 0$, $n \geqq 1$, are the eigenvalues of the equation

$$\lambda f(t) = \int_0^1 r(s,t) f(s)\,\mathrm{d}s \quad 0 \leqq t \leqq 1.$$

We need to find $\lambda_n^{(a)}$. For $a \geqq 1$, the above equation can be written as

$$\lambda f(t) = \int_0^t (a - t + s) f(s)\, ds + \int_t^1 (a + t - s) f(s)\, ds, \quad 0 \leqq t \leqq 1 . \tag{2.11}$$

We may differentiate (2.11) with respect to t to obtain

$$\lambda f'(t) = -\int_0^t f(s)\, ds + \int_t^1 f(s)\, ds . \tag{2.12}$$

Differentiate again to obtain $\lambda f''(t) = -2f(t)$. Hence

$$f(t) = c_1 \sin \sqrt{2\lambda^{-1}} t + c_2 \cos \sqrt{2\lambda^{-1}} t . \tag{2.13}$$

Setting $t = 0$ in (2.11) and (2.12), we obtain boundary conditions

$$\lambda f(0) = \int_0^1 (a - s) f(s)\, ds \quad \text{and} \quad \lambda f'(0) = \int_0^1 f(s)\, ds . \tag{2.14}$$

Substituting (2.13) into (2.14) and simplifying yields

$$\left(a + (1-a)\cos\sqrt{\frac{2}{\lambda}} - \sqrt{\frac{\lambda}{2}} \sin\sqrt{\frac{2}{\lambda}}\right) c_1$$

$$+ \left((a-1)\sin\sqrt{\frac{2}{\lambda}} - \sqrt{\frac{\lambda}{2}}\left(1 + \cos\sqrt{\frac{2}{\lambda}}\right)\right) c_2 = 0$$

and

$$(1 + \cos\sqrt{2\lambda^{-1}}) c_1 + (\sin\sqrt{2\lambda^{-1}}) c_2 = 0 .$$

In order that there are non-zero choices for c_1 and c_2, the determinant of the above two equations has to be zero. We obtain after some simplification

$$\left((2a-1)\sin\frac{1}{\sqrt{2\lambda}} - \sqrt{2\lambda}\cos\frac{1}{\sqrt{2\lambda}}\right)\cos\frac{1}{\sqrt{2\lambda}} = 0 . \tag{2.15}$$

Hence from (2.15) we have for $n \geqq 1$, $(2\lambda_{2n}^{(a)})^{-1/2} = (n-1)\pi + \pi/2$ and $(2\lambda_{2n-1}^{(a)})^{-1/2}$ are the only solutions of the equation

$$(2a-1)\tan x = x^{-1}, \quad a \geqq 1, \quad \text{on } [(n-1)\pi, (n-1)\pi + \pi/2) . \tag{2.16}$$

Using the inequality $\tan x > x$ on $(0, \pi/2)$ and (2.16), we have

$$(2a-1)^{-1}(2\lambda_{2n-1}^{(a)})^{1/2} = \tan(2\lambda_{2n-1}^{(a)})^{-1/2}$$

$$= \tan((2\lambda_{2n-1}^{(a)})^{-1/2} - (n-1)\pi) > (2\lambda_{2n-1}^{(a)})^{-1/2} - (n-1)\pi \geqq 0$$

which gives us $(2\lambda_{2n-1})^{-1/2} = (n-1)\pi + O(1/n)$. Hence

$$\sum_{n \geqq 1} \left| (2^{-1}\lambda_{n+1}^{(a)}) \cdot \left(\frac{1}{n^2\pi^2}\right)^{-1} - 1 \right| < \infty .$$

Thus by Lemma 1 and the first part of Lemma 5, we obtain

$$P\left(\int_0^1 |W(t+a)-W(t)|^2\,dt \leq \varepsilon\right)$$

$$= P\left(\sum_{n\geq 1} \lambda_n^{(a)}\xi_n^2 \leq \varepsilon\right)$$

$$= P\left(2^{-1}\lambda_1^{(a)}\xi_1^2 + \sum_{n\geq 1} 2^{-1}\lambda_{n+1}^{(a)}\xi_{n+1}^2 \leq 2^{-1}\varepsilon\right)$$

$$\sim C(a)\cdot P\left(2^{-1}\lambda_1^{(a)}\xi_1^2 + \sum_{n\geq 1}\frac{1}{n^2\pi^2}\xi_{n+1}^2 \leq 2^{-1}\varepsilon\right)$$

$$\sim K(a)\cdot\varepsilon\cdot\exp\left(-\frac{1}{4\varepsilon}\right) \quad \text{as} \quad \varepsilon\to 0$$

where $C(a)$ and $K(a)$ are positive constants. This finishes the proof of Lemma 18.

As mentioned in the introduction, when $0<a<1$ we are unable to find an expression similar to (2.15) for $\lambda_n^{(a)}$.

Lemma 19 *If $b \geq a \geq 1$ and $s > 0$, then*

$$P\left(\int_0^1 |W(t+b)-W(t)|^2\,dt < s\right) \leq P\left(\int_0^1 |W(t+a)-W(t)|^2\,dt < s\right).$$

Proof. Let $\lambda_n^{(a)}$, $n \geq 1$, be defined as in the proof of Lemma 18. Then for $b \geq a \geq 1$,

$$\lambda_{2n}^{(a)} = \lambda_{2n}^{(b)} \quad \text{and} \quad \lambda_{2n-1}^{(a)} \leq \lambda_{2n-1}^{(b)}$$

since the function $x\tan x$ is increasing function on $[(n-1)\pi,\ (n-1)\pi+\pi/2)$. Hence

$$P\left(\int_0^1 |W(t+b)-W(t)|^2\,dt < s\right) = P\left(\sum_{n\geq 1}\lambda_n^{(b)}\xi_n^2 < s\right) \leq P\left(\sum_{n\geq 1}\lambda_n^{(a)}\xi_n^2 < s\right)$$

$$= P\left(\int_0^1 |W(t+a)-W(t)|^2\,dt < s\right)$$

which concludes the proof.

Finally, we mention two results for further reference. They relate to lemmas we give early and are not used in the proof of our theorem. First, by Lemma 6 and (2.9) in Lemma 7, we have for any $b > a \geq d \geq 0$ and $s > 0$,

$$P\left(\int_{a-d}^{b-d} W^2(t)\,dt \leq \frac{(a-d)^2}{a^2}\cdot s\right) \leq P\left(\int_a^b W^2(t)\,dt \leq s\right) \leq P\left(\int_{a-d}^{b-d} W^2(t)\,dt \leq s\right).$$

A little bit stronger form of the second part of the above inequality is given in Lemma 9. Second, we have from (2.10) in Lemma 7,

$$\sum_{n\geq 1} |\lambda_{n+1}(a,b)\cdot(b-a)^{-2}\pi^2 n^2 - 1| < \infty\,. \tag{2.17}$$

Hence for $0 < \theta < 1$, we obtain as $\varepsilon \to 0$,

$$P\left(\int_{\theta}^{1} W^2(t)\,dt \leqq \varepsilon\right) = P\left(\sum_{n \geqq 1} \lambda_n(\theta, 1)\xi_n^2 \leqq \varepsilon\right) \tag{2.18}$$

$$= P\left(\lambda_1(\theta, 1)\xi_1^2 + \sum_{n \geqq 1} \lambda_{n+1}(\theta, 1)\xi_{n+1}^2 \leqq \varepsilon\right)$$

$$\sim C(\theta) \cdot P\left(\lambda_1(\theta, 1)\xi_1^2 + \sum_{n \geqq 1} \frac{(1-\theta)^2}{\pi^2 n^2}\xi_{n+1}^2 \leqq \varepsilon\right)$$

$$\sim K(\theta) \cdot \varepsilon \cdot \exp\left(-\frac{(1-\theta)^2}{8} \cdot \frac{1}{\varepsilon}\right)$$

where $C(\theta)$ and $K(\theta)$ are constants. For the above estimates, the first equality is Lemma 6, the first $\sim$ follows from Lemma 1 together with (2.17), and the second $\sim$ holds by (2.1) in Lemma 5.

3 Proof of Theorem 1

Let us note that under our conditions (i) and (ii)′, our theorem becomes

$$\varliminf_{T \to \infty} \frac{\log \log T}{T^2} \int_{T-a(T)}^{T} W^2(t)\,dt = \frac{\rho^2}{8} \quad \text{a.s.,} \quad 0 < \rho \leqq 1 .$$

This can be easily derived as follows if our theorem holds under our conditions (i) and (ii). For $0 < \rho < 1$ and $\varepsilon > 0$ small, we have $0 < T - (\rho + \varepsilon)T \leqq T - a(T) \leqq T - (\rho - \varepsilon)T < T$ if T is large and thus

$$\varliminf_{T \to \infty} \frac{\log \log T}{T^2} \int_{T-(\rho+\varepsilon)T}^{T} W^2(t)\,dt = \frac{(\rho+\varepsilon)^2}{8} \quad \text{a.s.}$$

$$\geqq \varliminf_{T \to \infty} \frac{\log \log T}{T^2} \int_{T-a(T)}^{T} W^2(t)\,dt$$

$$\geqq \varliminf_{T \to \infty} \frac{\log \log T}{T^2} \int_{T-(\rho-\varepsilon)T}^{T} W^2(t)\,dt = \frac{(\rho-\varepsilon)^2}{8} \quad \text{a.s.}$$

For $\rho = 1$, the above argument also works by using (1.4) as the upper bound. So we only need to show our theorem under conditions (i) and (ii).

Under conditions (i) and (ii), $\lim_{T \to \infty} a(T)/T = \rho \leqq 1$ and when $\rho = 1$ we actually have $a(T) = T$. In this case the result follows immediately from (1.4). Hence, for the rest of this section, we assume conditions (i) and (ii) hold and $\lim_{T \to \infty} a(T)/T = \rho < 1$. Now we formulate the following three statements which together imply our theorem.

$$\varliminf_{T \to \infty} \phi(T) \int_{T-a(T)}^{T} W^2(t)\,dt \geqq \frac{1}{4} \quad \text{a.s.} \tag{I}$$

$$\varliminf_{T \to \infty} \frac{\log(T/a(T))}{a^2(T)} \int_{T-a(T)}^{T} W^2(t)\,dt \leqq \frac{1}{4} \quad \text{a.s. .} \tag{II}$$

If $\lim_{T\to\infty} \log(T/a(T))\cdot(\log\log T)^{-1} < \infty$ and $\overline{\lim}_{T\to\infty} a(\gamma T)/a(T) < \infty$ for some $\gamma > 1$, then

$$\underline{\lim}_{T\to\infty} \phi(T) \int_{T-a(T)}^{T} W^2(t)\,dt \leqq \frac{1}{4} \quad \text{a.s.}\,. \tag{III}$$

Let us first show (I). Define

$$T_1 = 1, \quad T_{k+1} - \varepsilon_1 a(T_{k+1}) = T_k \tag{3.1}$$

where $\varepsilon_1 = 1 - (1-\varepsilon^2)^{1/2}$ and $0 < \varepsilon < 1$. Note that $T - \varepsilon_1 a(T)$ is a strictly increasing and continuous function by our conditions (i) and (ii). Hence T_k in (3.1) is well defined and $T_{k+1} > T_k$, $\lim_{k\to\infty} T_k = \infty$. Since

$$\phi(T) \geqq (\log(T_k/a(T_k)) + 2\log\log T_k)/a^2(T_{k+1}) \text{ and } T - a(T) \leqq T_{k+1} - a(T_{k+1})$$

for $T_{k+1} > T \geqq T_k$, it is sufficient to show

$$\underline{\lim}_{k\to\infty} \frac{\log(T_k/a(T_k)) + 2\log\log T_k}{a^2(T_{k+1})} \int_{T_{k+1}-a(T_{k+1})}^{T_k} W^2(t)\,dt \geqq \frac{1-\varepsilon}{4} \quad \text{a.s.}\,. \tag{3.2}$$

Note that for k large, $T_k - a(T_k) \geqq (1-\rho)/2$. Thus by Lemma 13, we have for k large

$$\begin{aligned}
&P\left(\frac{\log(T_k/a(T_k)) + 2\log\log T_k}{a^2(T_{k+1})} \int_{T_{k+1}-a(T_{k+1})}^{T_k} W^2(t)\,dt \leqq \frac{1-\varepsilon}{4}\right) \\
&= P\left(\int_{T_{k+1}-a(T_{k+1})}^{T_k} W^2(t)\,dt \leqq \frac{1-\varepsilon}{4}\cdot\frac{a^2(T_{k+1})}{\log(T_k/a(T_k)) + 2\log\log T_k}\right) \\
&\leqq C\cdot\left(\frac{T_k - T_{k+1} + a(T_{k+1})}{T_{k+1} - a(T_{k+1})}\right)^{1/2} \\
&\quad \cdot\exp\left(-\frac{\log(T_k/a(T_k)) + 2\log\log T_k}{2(1-\varepsilon)}\cdot\frac{(T_k - T_{k+1} + a(T_{k+1}))^2}{a^2(T_{k+1})}\right) \\
&= C\cdot\left(\frac{(1-\varepsilon_1)a(T_{k+1})}{T_{k+1} - a(T_{k+1})}\right)^{1/2}\cdot\exp\left(-\frac{\log(T_k/a(T_k)) + 2\log\log T_k}{2(1-\varepsilon)}\cdot(1-\varepsilon_1)^2\right) \\
&\leqq C\cdot\left(\frac{a(T_{k+1})}{T_{k+1} - a(T_{k+1})}\right)^{1/2}\cdot\left(\frac{a(T_k)}{T_k}\right)^{(1+\varepsilon)/2}\cdot\left(\frac{1}{\log T_k}\right)^{1+\varepsilon} \\
&\leqq C\cdot\left(\frac{a(T_k)}{T_k - a(T_k)}\right)^{1/2}\cdot\left(\frac{a(T_k)}{T_k}\right)^{1/2}\cdot\left(\frac{1}{\log T_k}\right)^{1+\varepsilon} \\
&\leqq C\cdot\frac{T_k - T_{k-1}}{(T_k - a(T_k))^{1/2}\,T_k^{1/2}\cdot(\log T_k)^{1+\varepsilon}} \\
&\leqq C\cdot\frac{T_k - T_{k-1}}{T_k(\log T_k)^{1+\varepsilon}} \leqq C\cdot\int_{T_{k-1}}^{T_k}\frac{dx}{x(\log x)^{1+\varepsilon}}\,.
\end{aligned}$$

Hence by the Borel-Cantelli lemma, we obtain (3.2) which shows (I).
Now turn to the proof of (II). Let T_k be the unique solution of the equation

$$x/a(x) = k^{\beta} \quad \text{where} \quad \beta = 2(1+\varepsilon)/(2+\varepsilon) > 1 . \tag{3.3}$$

Then $T_{k+1} > T_k$ and $\lim_{k\to\infty} T_k = \infty$. Define the events

$$A_k = \left\{ \int_{T_k - a(T_k)}^{T_k} W^2(t)\,dt \leqq \frac{(1+\varepsilon)a^2(T_k)}{4\log(T_k/a(T_k))} \right\}.$$

We then show $P(A_k \text{ i.o.}) = 1$ by Lemma 15 which in turn gives us (II). Let $\delta_k = (\log k)^{-1}$. Note that $T_k/a(T_k) = k^{\beta}$. Hence by Lemma 14 and the choice of β in (3.3), we have for k large

$$P(A_k) \geqq C\cdot\left(\frac{\delta_k\cdot a(T_k)(1+\varepsilon)}{4(2T_k - a(T_k))\log(T_k/a(T_k))}\right)^{1/2}\cdot\exp\left(-\frac{\log(T_k/a(T_k))}{2(1+\varepsilon)(1-\delta_k)}\right)$$

$$\geqq C\cdot\left(\frac{\delta_k}{\log(T_k/a(T_k))}\right)^{1/2}\cdot\left(\frac{a(T_k)}{T_k}\right)^{1/2}$$

$$\cdot\exp\left(-\frac{\log(T_k/a(T_k))}{2(1+\varepsilon)} - \delta_k\log(T_k/a(T_k))\right) \geqq C\cdot(k\log k)^{-1}$$

which shows $\sum_{k\geqq 1} P(A_k) = \infty$.
For given $\delta > 0$ small, define k_0 large such that for $l > k > k_0$, we have by Lemma 17

$$\sum_{k<l<(\delta^{-1}+1)k} (l^{\beta} - 1 - k^{\beta})^{-1/2}\cdot l^{-1+\beta/2} \tag{3.4}$$

$$\leqq 2\int_{k+1}^{(\delta^{-1}+1)k} (x^{\beta} - 1 - k^{\beta})^{-1/2}\cdot x^{-1+\beta/2}\,dx \leqq C .$$

Note that for $l > k$ and $\beta > 1$,

$$T_l - a(T_l) \geqq T_{k+1} - a(T_{k+1}) = (k+1)^{\beta} a(T_{k+1}) - a(T_{k+1}) \geqq k^{\beta} a(T_k) = T_k .$$

Hence for given k, $k_0 < k \leqq n$, we can split the set $\{l: k_0 < k < l \leqq n\}$ into two parts,

$$L_1 = \{l: k_0 < k < l, \quad T_l - a(T_l) \geqq T_k > \delta(T_l - a(T_l))\} ;$$

$$L_2 = \{l: k_0 < k < l, \delta(T_l - a(T_l)) \geqq T_k\} .$$

If $l \in L_2$, then by Lemma 16,

$$P(A_k A_l) \leqq \left(\frac{T_l - a(T_l)}{T_l - a(T_l) - T_k}\right)^{1/2} P(A_k)P(A_l) \leqq (1-\delta)^{-1/2} P(A_k)P(A_l) . \tag{3.5}$$

Note that by Lemma 16, we also have for $k_0 < k < l$

$$P(A_k A_l) \leqq P(A_k)\cdot P\left(\int_{T_l - a(T_l) - T_k}^{T_l - T_k} W^2(t)\,dt \leqq \frac{a^2(T_l)(1+\varepsilon)}{4\log(T_l/a(T_l))}\right). \tag{3.6}$$

If $l \in L_1$, then

$$\delta^{-1} k^{\beta} a(T_k) = \delta^{-1} T_k \geqq T_l - a(T_l) = (l^{\beta} - 1) a(T_l) \geqq (l^{\beta} - 1) a(T_k)$$

which gives $k < l < (\delta^{-1} + 1)k$. Now for $l \in L_1$, we have by Lemma 13,

$$P\left(\int_{T_l - a(T_l) - T_k}^{T_l - T_k} W^2(t)\,dt \leqq \frac{a^2(T_l)(1+\varepsilon)}{4 \log(T_l/a(T_l))} \right) \tag{3.7}$$

$$\leqq C \cdot \left(\frac{a(T_l)}{T_l - a(T_l) - T_k} \right)^{1/2} \cdot \exp\left(- \frac{\log(T_l/a(T_l))}{2(1+\varepsilon)} \right)$$

$$\leqq C \cdot (T_l/a(T_l) - 1 - T_k/a(T_k))^{-1/2} \cdot (T_l/a(T_l))^{-1/(2+2\varepsilon)}$$

$$= C \cdot (l^{\beta} - 1 - k^{\beta})^{-1/2} \cdot l^{-1+\beta/2} .$$

Hence we have by combining (3.4), (3.6) and (3.7)

$$\sum_{k_0 < k \leqq n} \sum_{l \in L_1} P(A_k A_l) \tag{3.8}$$

$$\leqq \sum_{k_0 < k \leqq n} \left(C \cdot P(A_k) \sum_{k < l < (\delta^{-1}+1)k} (l^{\beta} - 1 - k^{\beta})^{-1/2} \cdot l^{-1+\beta/2} \right)$$

$$\leqq C \sum_{k_0 < k \leqq n} P(A_k) .$$

Now by Lemma 15, (II) follows from $\lim_{n \to \infty} \sum_{k=1}^{n} P(A_k) = \infty$ and the estimates

$$\sum_{k=1}^{n} \sum_{l=1}^{n} P(A_k A_l) = \sum_{k=1}^{n} P(A_k) + 2 \sum_{1 \leqq k < l \leqq n} P(A_k A_l)$$

$$= \sum_{k=1}^{n} P(A_k) + 2 \sum_{k=1}^{k_0} \sum_{l=k+1}^{n} P(A_k A_l) + 2 \sum_{k=k_0+1}^{n} \sum_{l \in L_1} P(A_k A_l)$$

$$+ 2 \sum_{k=k_0+1}^{n} \sum_{l \in L_2} P(A_k A_l)$$

$$\leqq (1 + 2k_0 + 2C) \sum_{k=1}^{n} P(A_k) + (1-\delta)^{-1/2} \sum_{k=1}^{n} \sum_{l=1}^{n} P(A_k) P(A_l) .$$

Now turn to the proof of (III). Define

$$T_1 = 1, \quad T_{k+1} - a(T_{k+1}) = T_k . \tag{3.9}$$

Then for k large,

$$\lim_{k \to \infty} T_k = \infty \quad \text{and} \quad 1 > T_k/T_{k+1} > (1-\rho)/2 > 0 . \tag{3.10}$$

Thus

$$T_k \cdot (\log T_k) \cdot (\log \log T_k)^{1/2} \leqq C \cdot T_{k-1} \cdot (\log T_{k-1}) \cdot (\log \log T_{k-1})^{1/2} . \tag{3.11}$$

Define the events

$$B_k = \left\{ \int_{T_k - a(T_k)}^{T_k} W^2(t)\,dt \leqq \frac{(1+\varepsilon) a^2(T_k)}{4(\log(T_k/a(T_k)) + 2 \log\log T_k)} \right\} .$$

We then show $P(B_k \text{ } i.o.) = 1$ by Lemma 16. By our assumptions for case (III) and the fact that if $\varepsilon > 0$ is small enough, we have

$$\log(T/a(T)) \leqq 2\varepsilon^{-1} \cdot \log\log T . \tag{3.12}$$

Hence by Lemma 14 (choose $\delta > 0$ such that $(1+\varepsilon)(1-\delta) = 1$), (3.12) and (3.11), we have for k large

$$\begin{aligned} P(B_k) &\geqq C \cdot \left(\frac{\delta \cdot a(T_k)}{(2T_k - a(T_k))(\log(T_k/a(T_k)) + 2\log\log T_k)} \right)^{1/2} \\ &\quad \cdot \exp\left(- \frac{\log(T_k/a(T_k)) + 2\log\log T_k}{2(1+\varepsilon)(1-\delta)} \right) \\ &\geqq C \cdot \left(\frac{a(T_k)}{T_k} \right)^{1/2} \cdot \left(\frac{1}{(2\varepsilon^{-1} + 2)\log\log T_k} \right)^{1/2} \\ &\quad \cdot \exp\left(-\frac{1}{2}\log(T_k/a(T_k)) - \log\log T_k \right) \\ &\geqq C \cdot \frac{a(T_k)}{T_k \cdot \log T_k \cdot (\log\log T_k)^{1/2}} \\ &\geqq C \cdot \frac{T_k - T_{k-1}}{T_{k-1} \cdot \log T_{k-1} \cdot (\log\log T_{k-1})^{1/2}} \\ &\geqq C \cdot \int_{T_{k-1}}^{T_k} \frac{dx}{x \cdot \log x \cdot (\log\log x)^{1/2}} \end{aligned}$$

which shows $\sum_{k \geqq 1} P(B_k) = \infty$.

Since $a(T)$ is non-decreasing, we observe that $a(2(1-\rho)^{-1}T)/a(T) \leqq C$ for T large by iterating $\underline{\lim}_{T\to\infty} a(\gamma T)/a(T) < \infty$ for some $\gamma > 1$ if necessary. Hence by (3.10), we can define k_0 large such that for $l > k_0$,

$$a(T_l) \leqq C \cdot a(T_{l-1}) . \tag{3.13}$$

Note that for $l > k+1$, $T_l - a(T_l) > T_{k+1} - a(T_{k+1}) = T_k$. Hence for given $\delta > 0$ small and $k_0 < k \leqq n$, we can split the set $\{l : k_0 + 1 < k + 1 < l \leqq n\}$ into two parts,

$$\begin{aligned} L_1 &= \{l : k_0 + 1 < k + 1 < l,\ T_l - a(T_l) > T_k > \delta(T_l - a(T_l))\} ; \\ L_2 &= \{l : k_0 + 1 < k + 1 < l,\ \delta(T_l - a(T_l)) \geqq T_k\} . \end{aligned}$$

If $l \in L_2$, then by Lemma 16,

$$P(B_k B_l) \leqq \left(\frac{T_l - a(T_l)}{T_l - a(T_l) - T_k} \right)^{1/2} P(B_k)P(B_l) \leqq (1-\delta)^{-1/2} P(B_k)P(B_l) . \tag{3.14}$$

Note that by Lemma 16, we also have for $k_0 < k < l$

$$P(B_k B_l) \leqq P(B_k) \cdot P\left(\int_{T_l - a(T_l) - T_k}^{T_l - T_k} W^2(t)\, dt \leqq \frac{a^2(T_l)(1+\varepsilon)}{4(\log(T_l/a(T_l)) + 2\log\log T_l)} \right) . \tag{3.15}$$

If $l \in L_1$, then $T_k > \delta(T_l - a(T_l)) = \delta T_{l-1}$ which gives $T_k \leqq T_{l-1} < \delta^{-1} T_k$. Now for $l \in L_1$, we have by Lemma 13, (3.12) and (3.13),

$$P\left(\int_{T_l - a(T_l) - T_k}^{T_l - T_k} W^2(t)\,dt \leqq \frac{a^2(T_l)(1+\varepsilon)}{4(\log(T_l/a(T_l)) + 2\log\log T_l)}\right) \tag{3.16}$$

$$\leqq C\cdot\left(\frac{a(T_l)}{T_l - a(T_l) - T_k}\right)^{1/2}\cdot\exp\left(-\frac{\log(T_l/a(T_l)) + 2\log\log T_l}{2(1+\varepsilon)}\right)$$

$$= C\cdot\frac{a(T_l)}{(T_l - a(T_l) - T_k)^{1/2}\cdot T_l^{1/2}}\cdot\frac{(T_l/a(T_l))^{\varepsilon/(2+2\varepsilon)}}{(\log T_l)^{1/(1+\varepsilon)}}$$

$$\leqq C\cdot\frac{a(T_l)}{(T_l - a(T_l) - T_k)^{1/2}\cdot T_l^{1/2}}$$

$$\leqq C\cdot\frac{a(T_{l-1})}{(T_l - a(T_l) - T_k)^{1/2}\cdot T_l^{1/2}}$$

$$= C\cdot\frac{T_{l-1} - T_{l-2}}{(T_{l-1} - T_k)^{1/2}\cdot T_l^{1/2}}$$

$$\leqq C\cdot\int_{T_{l-2}}^{T_{l-1}}\frac{dx}{(x - T_k)^{1/2}\cdot x^{1/2}}\,.$$

Hence we have by combining (3.15) and (3.16),

$$\sum_{k_0 < k \leqq n}\sum_{l \in L_1} P(B_k B_l) \leqq \sum_{k_0 < k \leqq n}\left(C\cdot P(B_k)\sum_{T_k < T_{l-1} < (\delta^{-1}+1)T_k}\int_{T_{l-2}}^{T_{l-1}}\frac{dx}{(x - T_k)^{1/2}\cdot x^{1/2}}\right)$$

$$\leqq \sum_{k_0 < k \leqq n}\left(C\cdot P(B_k)\cdot\int_{T_k}^{(\delta^{-1}+1)T_k}\frac{dx}{(x - T_k)^{1/2}\cdot x^{1/2}}\right)$$

$$\leqq C\sum_{k_0 < k \leqq n} P(B_k)\,. \tag{3.17}$$

Now similarly to what we did at the end of the proof of (II), $P(B_k\ i.o.) = 1$ follows from (3.14), (3.17) and $\lim_{n\to\infty}\sum_{k=1}^{n} P(B_k) = \infty$. Thus we complete the proof of (III) and hence finish the proof of our Theorem 1.

4 Proof of Theorem 2

Note that (1.6) follows from (1.5), Chung's law of iterated logarithm [4]

$$\varliminf_{T\to\infty}\left(\frac{\log\log T}{T}\right)^{1/2}\sup_{0\leqq t\leqq T}|W(t)| = \frac{\pi}{\sqrt{8}} \quad \text{a.s.}$$

and the simple estimation

$$\int_{\alpha T}^{(\alpha+\theta)T}|W(t+\theta T) - W(t)|^2\,dt \leqq \int_{\alpha T}^{\beta T}|W(t+\theta T) - W(t)|^2\,dt$$

$$\leqq 4(\beta - \alpha)T\sup_{0\leqq t\leqq(\beta+\theta)T} W^2(t)$$

if $0 < \theta < \beta - \alpha$. Hence we only need to show (1.5).

Define $T_{k+1} = (1 + f(T_k))T_k$, $T_1 = 2e$ and $f(x) = (\log_2 x)^{-5}$. Here and throughout this section, $\log_2 x = \log\log(\max\{x, 2e\})$. Let us first show that for $\theta \geqq \beta - \alpha > 0, \alpha \geqq 0$

$$\varliminf_{T\to\infty} \frac{\log_2 T_k}{T_k^2} \int_{\alpha T_{k+1}}^{\beta T_k} |W(t + \theta T_k) - W(t)|^2 \, dt \geqq \frac{(\beta - \alpha)^2}{4} \quad \text{a.s.} . \tag{4.1}$$

For any $\varepsilon > 0$ and k large, we can pick $\delta > 0$ such that

$$T_{k+1}/T_k < 1 + \delta \text{ and } \varepsilon' = ((\beta - \alpha - \alpha\delta)/(\beta - \alpha))^2 \cdot (1 - \varepsilon)^{-1} - 1 > 0 .$$

Thus we have for k large

$$P\left(\frac{\log_2 T_k}{T_k^2} \int_{\alpha T_{k+1}}^{\beta T_k} |W(t + \theta T_k) - W(t)|^2 \, dt \leqq \frac{(\beta - \alpha)^2}{4} \cdot (1 - \varepsilon)\right) \tag{4.2}$$

$$= P\left(\int_{\alpha T_{k+1}/T_k}^{\beta} |W(t + \theta) - W(t)|^2 \, dt \leqq \frac{(\beta - \alpha)^2}{4 \log_2 T_k} \cdot (1 - \varepsilon)\right)$$

$$\leqq P\left(\int_{\alpha(1+\delta)}^{\beta} |W(t + \theta) - W(t)|^2 \, dt \leqq \frac{(\beta - \alpha)^2}{4 \log_2 T_k} \cdot (1 - \varepsilon)\right)$$

$$= P\left(\int_{0}^{\beta - \alpha(1+\delta)} |W(t + \theta) - W(t)|^2 \, dt \leqq \frac{(\beta - \alpha)^2}{4 \log_2 T_k} \cdot (1 - \varepsilon)\right)$$

$$= P\left(\int_0^1 \left| W\left(t + \frac{\theta}{\beta - \alpha - \alpha\delta}\right) - W(t)\right|^2 dt \leqq \left(\frac{\beta - \alpha}{\beta - \alpha - \alpha\delta}\right)^2 \cdot \frac{1 - \varepsilon}{4 \log_2 T_k}\right)$$

$$\leqq C \cdot \exp(-(1 + \varepsilon') \cdot \log_2 T_k) = C \cdot (\log T_k)^{-(1+\varepsilon')}$$

where the first and the third equality hold by the scaling properties of the Wiener process, the second equality holds since the Wiener process has stationary increment, and the last inequality follows from Lemma 18. Now by $\lim_{k\to\infty} T_k/T_{k+1} = 1$,

$$\begin{aligned}\sum_{k \geqq 1} (\log T_k)^{-(1+\varepsilon')} &= \sum_{k \geqq 1} \frac{(\log T_k)^{-(1+\varepsilon')}}{T_k f(T_k)} \cdot (T_{k+1} - T_k) \\ &\leqq C \sum_{k \geqq 1} \frac{(\log T_{k+1})^{-(1+\varepsilon')}}{T_{k+1} f(T_k)} \cdot (T_{k+1} - T_k) \\ &\leqq C \int_{2e}^{\infty} \frac{(\log\log x)^5}{x(\log x)^{1+\varepsilon'}} \, dx < \infty .\end{aligned}$$

We conclude (4.1) by the Borel-Cantelli Lemma and (4.2).

Now consider $T_k \leqq T < T_{k+1}$, and note that we have

$$\lim_{T\to\infty} T_k/T = 1 \quad \text{and} \quad \lim_{T\to\infty} T/T_{k+1} = 1 .$$

Define

$$X(t) = |W(t + \theta T_k) - W(t)| ;$$

$$Y(t) = \sup_{0 \leqq s \leqq \theta(T - T_k)} |W(t + \theta T_k + s) - W(t + \theta T_k)| ;$$

$$Z(t) = |W(t + \theta T) - W(t)| .$$

Then $Z(t) \geqq X(t) - Y(t)$ and therefore

$$Z^2(t) \geqq X^2(t) - (X(t) + Z(t))Y(t) .$$

Hence

$$\underline{\lim}_{T\to\infty} \frac{\log_2 T}{T^2} \int_{\alpha T}^{\beta T} Z^2(t)\,dt \geqq \underline{\lim}_{T\to\infty} \frac{\log_2 T}{T^2} \int_{\alpha T}^{\beta T} X^2(t)\,dt \tag{4.3}$$

$$- \overline{\lim}_{T\to\infty} \frac{\log_2 T}{T^2} \int_{\alpha T}^{\beta T} (X(t) + Z(t))Y(t)\,dt .$$

From (4.1) and $\lim_{T\to\infty} T_k/T_{k+1} = 1$, we have

$$\underline{\lim}_{T\to\infty} \frac{\log_2 T}{T^2} \int_{\alpha T}^{\beta T} X^2(t)\,dt \geqq \underline{\lim}_{T\to\infty} \frac{\log_2 T_k}{T_{k+1}^2} \int_{\alpha T_{k+1}}^{\beta T_k} |W(t + \theta T_k) - W(t)|^2\,dt$$

$$\geqq \underline{\lim}_{T\to\infty} \frac{\log_2 T_k}{T_k^2} \int_{\alpha T_{k+1}}^{\beta T_k} |W(t + \theta T_k) - W(t)|^2\,dt$$

$$\geqq \frac{(\beta - \alpha)^2}{4} \quad \text{a.s.} . \tag{4.4}$$

By Theorem 1.2.1 in Csörgő and Révész [6], we obtain

$$\overline{\lim}_{T\to\infty} (T \log_2 T)^{-1/2} \sup_{\alpha T \leqq t \leqq \beta T} X(t) \tag{4.5}$$

$$\leqq \overline{\lim}_{T\to\infty} (T \log_2 T)^{-1/2} \sup_{0 \leqq t \leqq (\beta+\theta)T - \theta T_k} |W(t + \theta T_k) - W(t)|$$

$$\leqq \sqrt{2\theta} \quad \text{a.s.}$$

and

$$\overline{\lim}_{T\to\infty} (T \log_2 T)^{-1/2} \sup_{\alpha T \leqq t \leqq \beta T} Z(t) \tag{4.6}$$

$$\leqq \overline{\lim}_{T\to\infty} (T \log_2 T)^{-1/2} \sup_{0 \leqq t \leqq (\beta+\theta)T - \theta T} |W(t + \theta T) - W(t)|$$

$$\leqq \sqrt{2\theta} \quad \text{a.s.} .$$

Note that

$$\sup_{\alpha T \leqq t \leqq \beta T} Y(t) \leqq \sup_{0 \leqq t \leqq \beta T} \sup_{0 \leqq s \leqq \theta(T_{k+1} - T_k)} |W(t + \theta T_k + s) - W(t + \theta T_k)|$$

$$\leqq \sup_{0 \leqq t \leqq (\beta+\theta)T} \sup_{0 \leqq s \leqq \theta f(T_k) T_k} |W(t + s) - W(t)| .$$

We have by Theorem 1.2.1 in Csörgő and Révész [6]

$$\begin{aligned}
&\overline{\lim_{T\to\infty}}\,(f(T)T\log_2 T)^{-1/2} \sup_{\alpha T \leqq t \leqq \beta T} Y(t) \\
&\leqq \overline{\lim_{T\to\infty}}\,(f(T)T\log_2 T)^{-1/2} \sup_{0\leqq t\leqq(\beta+\theta)T}\ \sup_{0\leqq s\leqq \theta f(T_k)T_k} |W(t+s)-W(t)| \\
&\leqq C \quad \text{a.s.}\,.
\end{aligned}$$

Hence

$$\overline{\lim_{T\to\infty}}\,(T^{-1}(\log_2 T)^3)^{1/2} \sup_{\alpha T\leqq t\leqq \beta T} Y(t) = 0 \quad \text{a.s.}\,. \tag{4.7}$$

Combining (4.5), (4.6) and (4.7), it follows that

$$\begin{aligned}
&\overline{\lim_{T\to\infty}}\,\frac{\log_2 T}{T^2}\int_{\alpha T}^{\beta T}(X(t)+Z(t))Y(t)\,\mathrm{d}t \qquad (4.8)\\
&\leqq(\beta-\alpha)\cdot \overline{\lim_{T\to\infty}}\,(T\log_2 T)^{-1/2}\left(\sup_{\alpha T\leqq t\leqq\beta T} X(t)+\sup_{\alpha T\leqq t\leqq \beta T} Z(t)\right) \\
&\quad\cdot \overline{\lim_{T\to\infty}}\,(T^{-1}\log_2 T)^3)^{1/2}\sup_{\alpha T\leqq t\leqq\beta T} Y(t) \\
&=0 \quad \text{a.s.}\,.
\end{aligned}$$

Therefore we obtain our lower bound of Theorem 1 by (4.3), (4.4) and (4.8). Now let us show that for $\theta \geqq \beta-\alpha>0$, $\alpha\geqq 0$ and any $\varepsilon>0$

$$\underline{\lim_{T\to\infty}}\,\frac{\log_2 T}{T^2}\int_{\alpha T}^{\beta T}|W(t+\theta T)-W(t)|^2\,\mathrm{d}t \leqq \frac{(\beta-\alpha)^2}{4}\cdot(1+\varepsilon) \quad \text{a.s.}\,. \tag{4.9}$$

Case (*I*): $\alpha>0$. Let $T_k=b^k$ for $b>1$, $\alpha b>\beta+\theta$ and define the events

$$B_k=\left\{\frac{\log_2 T_k}{T_k^2}\int_{\alpha T_k}^{\beta T_k}|W(t+\theta T_k)-W(t)|^2\,\mathrm{d}t\leqq\frac{(\beta-\alpha)^2}{4}\cdot(1+\varepsilon)\right\}.$$

Note that $\alpha T_{k+1}>\beta T_k+\theta T_k$ by the choice of b. Thus the events B_k are independent since $W(t)$ has independent increments. By using Lemma 18 and looking at what we did in (4.2),

$$\begin{aligned}
P(B_k)&=P\left(\int_0^1\left|W\left(t+\frac{\theta}{\beta-\alpha}\right)-W(t)\right|^2\mathrm{d}t\leqq\frac{1+\varepsilon}{4}\cdot(\log_2 b^k)^{-1}\right)\\
&\geqq C\cdot(\log_2 b^k)^{-1}\cdot\exp\left(-\frac{1}{1+\varepsilon}\log_2 b^k\right)
\end{aligned}$$

which shows $\sum_{k\geqq 1}P(B_k)=\infty$. Hence by the Borel-Cantelli lemma, we conclude (4.9).

Case (*II*): $\alpha=0$. Let $T_k=(\log k)^{3k}$ and the events

$$\begin{aligned}
A_k&=\left\{\frac{\log_2 T_k}{T_k^2}\int_{(\beta+\theta)T_{k-1}}^{\beta T_k}|W(t+\theta T_k)-W(t)|^2\,\mathrm{d}t\leqq\frac{\beta^2}{4}\cdot(1+\varepsilon)\right\}\\
&\supseteq\left\{\int_0^{\beta T_k}|W(t+\theta T_k)-W(t)|^2\,\mathrm{d}t\leqq\frac{\beta^2(1+\varepsilon)}{4}\cdot\frac{T_k^2}{\log_2 T_k}\right\}.
\end{aligned}$$

By using the scaling property of the Wiener process and Lemma 18, we obtain

$$P(A_k) \geqq P\left(\int_0^1 \left|W\left(t+\frac{\theta}{\beta}\right) - W(t)\right|^2 dt \leqq \frac{1+\varepsilon}{4}\cdot(\log(3k\log_2 k))^{-1}\right)$$

$$\geqq C\cdot(\log(3k\log_2 k))^{-1}\cdot\exp\left(-\frac{1}{1+\varepsilon}\log(3k\log_2 k)\right)$$

which shows $\sum_{k\geqq 1} P(A_k) = \infty$. Since the A_k are independent, we have $P(A_k$ *i.o.*$) = 1$. Note that by the law of iterated logarithm,

$$\frac{\log_2 T_k}{T_k^2}\int_0^{(\beta+\theta)T_{k-1}} |W(t+\theta T_k) - W(t)|^2\, dt = O\left(\frac{T_{k-1}}{T_k}\cdot(\log_2 T_k)^2\right) \to 0 \quad \text{a.s.} \tag{4.10}$$

as $k\to\infty$. (4.9) follows from (4.10) and $P(A_k$ *i.o.*$) = 1$.

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