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A characterization of *h*-Brownian motion by its exit distributions

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Summary. Let X^h be an h-Brownian motion in the unit ball $D \subset \mathbf{R}^d$ with h harmonic, such that the representing measure of h is not singular with respect to the surface measure on ∂D. If Y is a continuous strong Markov process in D with the same killing distributions as X^h, then Y is a time change of X^h. Similar results hold in simply connected domains in $\mathbf{C}$ provided with either the Martin or the Euclidean boundary.

1 Introduction

Question of recognizing two stochastic processes as a time change one of another is a repeating theme in the theory of stochastic processes. The most general result in the Markov process theory is the celebrated Blumenthal-Getoor-McKean theorem (from now on referred to as BGM). Loosely speaking, the result is that two processes with equal hitting distributions for compact sets have same geometrical trajectories. They can only run with different speed. We state the theorem for further reference in generality we will need (see [1], V-5.1).

Theorem 1.1 *Let (X_t, P_x) and (Y_t, Q_x) be standard processes with the same locally compact second countable state space $(E, \mathscr{E})$ and cemetery point Δ. Let $E_\Delta = E \cup \{\Delta\}$ and $\mathscr{E}_\Delta = \mathscr{E} \vee \{\Delta\}$. Suppose that the hitting distribution of X and Y satisfy*

$$P_K(x, \cdot) = Q_K(x, \cdot) \tag{1.1}$$

for all $x \in E$ and all compact subsets K of E_Δ. Then there exists a continuous additive functional $A = (A_t)$ of X, which is strictly increasing and finite on $[0, \zeta)$, such that if $\tau = (\tau_t)$ is the right continuous inverse of A, then (X_{τ_t}, P_x) and (Y_t, Q_x) have same joint distributions.

We will say that Y is a time change of X whenever the conclusion of the theorem holds.

The BGM theorem has been generalized in several directions. Glover showed in [5] that if X and Y are transient processes with identical hitting probabilities, then the conclusion of Theorem 1.1 still holds (see also [6] and [3]). In the other direction, it was shown in [4] that the state space can be a Radon space and X, Y right processes.

The strength of the BGM theorem is in the fact that the state space E and processes X and Y are as general as they can be. On the other hand, knowing nothing specific about X and Y seems to require information from all over the state space: one needs to know that $P_K(x, \cdot) = Q_K(x, \cdot)$ for *all* compact subsets of E_Δ. It is conceivable that if the process X is specified (which automatically determines the state space), one should be able to recognize Y as its time change by requiring a seemingly weaker condition than (1.1). The easiest candidate to start with is Brownian motion in some open connected subset of $\mathbf{R}^d$, $d \geqq 2$. Since a subset of $\mathbf{R}^d$ comes equipped with its (Euclidean) boundary ∂D, a natural question arises: Can we recognize Brownian motion in D by knowing only how it hits the boundary? For nice domains the answer is yes. Here is the precise statement (see [11]).

Theorem 1.2 *Let (X_t, P_x) be a Brownian motion in a bounded Lipschitz domain $D \subset \mathbf{R}^d$, killed while exiting D, ζ the lifetime of X, and (Y_t, Q_x) a normal strong Markov process in D with continuous paths up to its lifetime $\tilde{\zeta}$. Assume that $Y_{\tilde{\zeta}-}$ exists and*

$$P_x(X_{\zeta-} \in C) = Q_x(Y_{\tilde{\zeta}-} \in C) \tag{1.2}$$

for all Borel subsets C of ∂D and all $x \in D$. Then Y is a time change of X.

Remark 1.1. It is clear that the result is not valid if Brownian motion is replaced by an arbitrary continuous Markov process. For example, let X^h denote an h-Brownian motion in D with minimal harmonic h representing the boundary point z. Then X^h exits D at z. Let Y be any other diffusion conditioned to exit D at z. Obviously (1.2) is satisfied, yet X^h and Y can be very different.

In this paper Theorem 1.2 is extended to a certain class of h-Brownian motions in the unit ball. Minimal harmonic functions must be a priori ruled out due to the remark above: The available information from the boundary is far from being sufficient to say anything about the process inside. Therefore we restrict ourselves to harmonic functions h such that the representing measure of h is not singular with respect to the surface measure on ∂D. With this assumption a result similar to Theorem 1.2 is proved. Moreover, the method we use shows that one can characterize Brownian motion up to a time change by knowing the exit distributions only on an arbitrary Borel subset of the boundary with positive surface measure.

The paper is organized as follows. In Sect. 2 we introduce the notation and give a precise setting. Then we prove the main result for h-Brownian motion in the unit ball. Methods and results given in this section are central for the paper. In Sect. 3 we apply the Riemann mapping theorem to transfer results to a simply connected domain in $\mathbf{R}^2$ provided with the Martin boundary or the Euclidean boundary. Theorem 3.2 is a generalization of Theorem 1.2 for planar simply connected domains with much less regular boundaries than those of Lipschitz domains.

2 Brownian motion in the unit ball

Let D be the unit ball in $\mathbf{R}^d$, $d \geqq 2$, with the Euclidean boundary ∂D. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P_x)$ be a Brownian motion in D, killed while exiting D, with lifetime ζ. By $p(t, x, y)$ we denote the transition density of (X_t, P_x). For a positive harmonic function h, let (X_t^h, P_x^h) denote Doob's h-transformed Brownian motion in D (see [2]). The transition density of X^h is given by

$$p^h(t, x, y) = h(x)^{-1} p(t, x, y) h(y) .$$

Let $K(x, z) = (1 - |x|^2)/|z - x|^d$, $x \in D$, $z \in \partial D$, denote the Poisson kernel. Without loss of generality we shall assume that $h(0) = 1$. There is a unique probability measure μ on ∂D such that

$$h(x) = \int_{\partial D} K(x, z) \mu(dz), \quad \text{for all } x \in D . \tag{2.1}$$

Let σ denote the normalized surface measure on ∂D and let $\mu = \mu_s + \mu_a$ be the Lebesgue decomposition of μ in the singular part μ_s and the absolutely continuous part μ_a (with respect to σ). The measure μ_a can be written as $\mu_a(dz) = g(z)\sigma(dz)$ where g is a nonnegative Borel function on ∂D. Having in mind Remark 1.1, we will assume that the harmonic function h satisfies the following hypothesis:

(H) The measure μ is not singular with respect to the surface measure σ. With this hypothesis, the set $\{g > 0\}$ has strictly positive surface measure. Moreover, σ and μ_a are equivalent on $\{g > 0\}$. Let V be any Borel subset of $\{g > 0\} \cap \operatorname{supp}(\mu_a)$ with $\sigma(V) > 0$ (here $\operatorname{supp}(\mu_a)$ denotes the support of the measure μ_a). Such set V exists since $\sigma(\{g > 0\} \cap \operatorname{supp}(\mu_a)) > 0$. Then $\mu_a(V) > 0$ and, in particular, $\mu(V) > 0$. The set V will remain fixed throughout this section.

Remark 2.1. Let $\omega_x(dz)$ denote the harmonic measure at $x \in D$. Then $\sigma(dz) = \omega_0(dz)$ where 0 denotes the origin. Since $\omega_x(dz) = K(x, z)\sigma(dz)$, the hypothesis **(H)** is equivalent to the hypothesis that μ is not singular with respect to the harmonic measure.

Let $(\Omega, \mathscr{G}, \mathscr{G}_t, Y_t, \theta_t, Q_x)$ be a normal strong Markov process in D with continuous paths up to its lifetime $\tilde{\zeta}$. For a Borel subset B of D, let

$$T_B = \inf\{t > 0 \colon X_t^h \in B\}, \ \tilde{T}_B = \inf\{t > 0 \colon Y_t \in B\} \tag{2.2}$$

be the hitting times of B for X^h and Y respectively. Let

$$P_B^h f(x) = P_x^h[f(X^h(T_B))], \quad Q_B f(x) = Q_x[f(Y(\tilde{T}_B))] \tag{2.3}$$

(f positive Borel function on D), be the hitting operators of B for X^h and Y. For $\omega \in \Omega$, let $\Gamma(\omega)$ denote the set of accumulation points of $Y_t(\omega)$ as $t \uparrow \tilde{\zeta}(\omega)$. We assume:

$$\text{If } \Gamma(\omega) \cap V \neq \emptyset, \quad \text{then } \Gamma(\omega) \text{ is a singleton, } Q_x \text{ a.s.} \tag{2.4}$$

This assumption justifies an expression like $\{\omega \in \Omega \colon Y_{\tilde{\zeta}-}(\omega) \in A\}$ for $A \subset V$. The goal of this section is to prove the following theorem.

Theorem 2.1 *Let (X_t^h, P_x^h) be an h-Brownian motion in D with h satisfying* **(H)**, *(Y_t, Q_x) a continuous normal strong Markov process in D satisfying* (2.4). *Assume*

$$P_x^h(X_{\zeta-}^h \in A) = Q_x(Y_{\tilde{\zeta}-} \in A) \tag{2.5}$$

for all Borel subsets A of V and for all $x \in D$. Then Y is a time change of X^h.

Before proving the theorem, we give an example.

Example. Fix $z \in \partial D$ and let h be the function $x \mapsto 1 + K(x, z)$ normalized so that $h(0) = 1$. The corresponding h-process X^h is a mixing of Brownian motion in D and Brownian motion conditioned to exit D at z. Consequently, X^h will exit D at z with positive probability, and z is the only such point. Suppose that V is an open subset of ∂D such that $z \notin V$. If (Y_t, Q_x) satisfies (2.5), then it is a time change of X^h. A curious thing is that by observing Y only on V, one can detect exactly the point $z \in \partial D$ at which Y exits D with positive probability.

The proof of the theorem follows the ideas from [11]. By using analytical methods, we show that X^h and Y have identical exit distributions from relatively compact open balls in D. Then we quote a probabilistic result of Øksendal and Stroock ([8]) to show that (1.1) holds. The BGM theorem finishes the proof. We start off with several lemmas.

Lemma 2.1 *Let K be a compact subset of D, ν a probability measure on K, $L(u, x) = 1/|u - x|^d$, and g a strictly positive continuous function on K. Then $G: \partial D \to \mathbf{R}$ defined by*

$$G(x) = \int_K g(u)(1 - |u|^2) L(u, x) \nu(du) \tag{2.6}$$

is real analytic on the sphere ∂D.

Proof. It is enough to show that G is real analytic on some open subset Ξ of $\mathbf{R}^d$ containing ∂D and disjoint with K. Then its restriction to the real analytic manifold ∂D will be also real analytic. Let $\tilde{\Xi} = \Xi \times B(0, \delta)$ where $\delta < \operatorname{dist}(\Xi, K)$. Consider $\tilde{\Xi}$ as a subset of $\mathbf{C}^d$. Then $L(u, \cdot)$ can be extended as a holomorphic (complex) function on $\tilde{\Xi}$. Moreover, it is easily seen that $M(u, x, y) = g(u)(1 - |u|^2) L(u, x, y)$, $u \in K$, $(x, y) \in \tilde{\Xi}$, is holomorphic in $\tilde{\Xi}$ for each $u \in K$, and jointly continuous and uniformly bounded on $K \times \tilde{\Xi}$. Similarly, partial derivatives $\partial M/\partial x_j$, $\partial M/\partial y_j$ are jointly continuous and uniformly bounded on $K \times \tilde{\Xi}$. Let $G(x, y) = \int_K M(u, x, y) \nu(du)$. Since $M(u, \cdot, \cdot)$ satisfies the Cauchy-Riemann equations, and differentiation under the integral sign is permitted, the same holds for G. Hence, $G(x, y)$ is holomorphic on $\Xi \times B(0, \delta)$. Since $G(x) = \Re G(x, 0)$, G is real analytic on Ξ. □

Let $\omega_x(dz)$ and $\omega_x^h(dz)$ denote harmonic and h-harmonic measure at x respectively. Then

$$\omega_x^h(dz) = \frac{K(x, z)}{h(x)} \mu(dz) \tag{2.7}$$

(see [2] p. 119). Note that $\omega_x^h(A) > 0$ for every $A \subset V$ with positive surface measure. If $z \in D$ and $\Delta_n = B(z, 2^{-n}) \cap \partial D$, then it is well known that

$$\lim_{n \to \infty} \frac{\omega_x(\Delta_n)}{\omega_0(\Delta_n)} = K(x, z) \tag{2.8}$$

uniformly on compact subsets of D (here 0 denotes the origin).

Lemma 2.2 *For every $z \in \operatorname{supp}(\mu)$,*

$$\lim_{n \to \infty} \frac{\omega_x^h(\Delta_n)}{\omega_0^h(\Delta_n)} = \frac{K(x, z)}{h(x)} \tag{2.9}$$

uniformly on compact subsets of D.

Proof. Let $z \in \operatorname{supp}(\mu)$. For every $n \in \mathbf{N}$, let $h_n(x) = \int_{\Delta_n} K(x, \zeta)\mu(d\zeta)$. Then h_n is harmonic and $\omega_x^h(\Delta_n) = h_n(x)/h(x)$. Since $h(0) = 1$ and $K(0, \cdot) = 1$, it follows that $\omega_0^h(\Delta_n) = \mu(\Delta_n)$, which is strictly positive for every $n \in \mathbf{N}$. By continuity of the Poisson kernel in the second variable, it follows that

$$\lim_{n \to \infty} \frac{h_n(x)}{\mu(\Delta_n)} = \lim_{n \to \infty} \frac{\int_{\Delta_n} K(x, \zeta)\mu(d\zeta)}{\mu(\Delta_n)} = K(x, z),$$

pointwise for every $x \in D$. But a pointwise limit of positive harmonic functions is locally uniform. Since h is bounded away from zero on compact sets, it follows that

$$\lim_{n \to \infty} \frac{\omega_x^h(\Delta_n)}{\omega_0^h(\Delta_n)} = \lim_{n \to \infty} \frac{1}{h(x)} \frac{h_n(x)}{\mu(\Delta_n)} = \frac{K(x, z)}{h(x)}$$

locally uniformly. □

A measurable function u on D is said to be harmonic for Y if for every compact subset K of D and all $x \in D$, $Q_{K^c} u(x) = u(x)$.

Proof of Theorem 2.1. Let $z \in V$ and $\Delta_n = B(z, 2^{-n}) \cap \partial D$. Since V is contained in the support of μ, Lemma 2.2, the assumption (2.5), and the well-known fact that $P_x^h(X_{\zeta-}^h \in \Delta_n) = \omega_x^h(\Delta_n)$ imply that

$$\lim_{n \to \infty} \frac{Q_x(Y_{\zeta-} \in \Delta_n)}{Q_0(Y_{\zeta-} \in \Delta_n)} = \lim_{n \to \infty} \frac{\omega_x^h(\Delta_n)}{\omega_0^h(\Delta_n)} = \frac{K(x, z)}{h(x)} \tag{2.10}$$

uniformly on compacts. A locally uniform limit of harmonic functions for Y is again harmonic. Since $x \mapsto Q_x(Y_{\zeta-} \in \Delta_n)$ is harmonic for Y (by the strong Markov property), it follows from (2.10) that $x \mapsto K(x, z)/h(x)$ is also harmonic for Y, and this holds for every $z \in V$. For $x_0 \in D$ let $B = B(x_0, r)$ denote a relatively compact open ball in D with the boundary $S(x_0, r)$. Then for every $x \in B$,

$$\int_{S(x_0, r)} \frac{K(y, z)}{h(y)} P_{B^c}^h(x, dy) = \frac{K(x, z)}{h(x)} = \int_{S(x_0, r)} \frac{K(y, z)}{h(y)} Q_{B^c}(x, dy), \tag{2.11}$$

for all $z \in V$. By Lemma 2.1 the functions $z \mapsto \int_{S(x_0, r)} (K(y, z)/h(y)) P_{B^c}^h(x, dy)$ and $z \mapsto \int_{S(x_0, r)} (K(y, z)/h(y)) Q_{B^c}(x, dy)$ are real analytic on ∂D. By (2.11) they are equal on the Borel set V with $\sigma(V) > 0$ and, by the standard result on (real) analytic functions, they are equal on the whole boundary ∂D. Hence (2.11) is valid for every $z \in \partial D$. The linear span of functions $K(\cdot, z)$, $z \in \partial D$, restricted to $S(x_0, r)$ is dense in the space $\mathscr{C}(S(x_0, r))$ of all continuous functions on $S(x_0, r)$. This can be proved by uniformly approximating a continuous function on $\mathscr{C}(S(x_0, r))$ by a sequence of harmonic polynomials on $\mathbf{R}^d$ (for an alternative proof see [11], Lemma 3.2). Since h is bounded on $S(x_0, r)$ and also bounded away from zero, it easily follows that the linear span of functions $K(\cdot, z)/h(\cdot)$, $z \in \partial D$, restricted to $S(x_0, r)$ is also dense in $\mathscr{C}(S(x_0, r))$. Therefore, the measures $P_{B^c}^h(x, dy)$ and $Q_{B^c}(x, dy)$ are equal for every $x \in B$, i.e., X^h and Y have identical exit distributions from relatively compact open balls. By Theorem 2 of [8] it follows that (1.1) holds for X^h and Y (see also [11], Proposition 2.1). The BGM theorem finishes the proof. □

By using the same method one can prove a result similar to Theorem 2.1. We retain the same notation, simplify conditions on Y and in addition assume that the set V is open.

Theorem 2.2 *Let* (X_t^h, P_x^h) *be an* h*-Brownian motion in* D *with* h *satisfying* **(H)**, (Y_t, Q_x) *a continuous normal strong Markov process in* D *such that* $Y_{\zeta-} = \lim_{t\to\zeta} Y_t$ *exists in* ∂D. *Let* $F\colon \partial D \to \partial D$ *be a homeomorphism. Assume*

$$P_x^h(X_{\zeta-}^h \in F^{-1}(A)) = Q_x(Y_{\zeta-} \in A) \tag{2.12}$$

for all Borel subsets A *of* ∂D *such that* $F^{-1}(A) \subset V$ *and for all* $x \in D$. *Then* Y *is a time change of* X^h, *and, consequently,* F *is the identity on* V.

Proof. Let us denote $Q_x(Y_{\zeta-} \in A)$ by $\tilde{\omega}_x(A)$. Then $x \mapsto \tilde{\omega}_x(A)$ is harmonic for Y. Hence, if $B = B(x_0, r)$ is a relatively compact open ball in D with the boundary $S(x_0, r)$, then

$$\int_{S(x_0,r)} \tilde{\omega}_y(A) Q_{B^c}(x, dy) = \tilde{\omega}_x(A) \tag{2.13}$$

for every $x \in B$. By assumption (2.12), if $F^{-1}(A) \subset V$, this can be written as

$$\int_{S(x_0,r)} \omega_x^h(F^{-1}(A)) Q_{B^c}(x, dy) = \omega_x^h(F^{-1}(A)), \tag{2.14}$$

for every $x \in B$. On the other hand, h-harmonicity of $x \mapsto \omega_x^h(F^{-1}(A))$ yields

$$\int_{S(x_0,r)} \omega_x^h(F^{-1}(A)) P_{B^c}^h(x, dy) = \omega_x^h(F^{-1}(A)). \tag{2.15}$$

From (2.14) and (2.15) it follows that

$$\int_{S(x_0,r)} \omega_x^h(C) P_{B^c}^h(x, dy) = \int_{S(x_0,r)} \omega_x^h(C) Q_{B^c}(x, dy) \tag{2.16}$$

for every $x \in B$ and every Borel subset C of V. If $z \in V$ and $\Delta_n = B(z, 2^{-n}) \cap \partial D$, then $\omega_0^h(\Delta_n) > 0$. Let $C = \Delta_n$ in (2.16) and divide the equation by $\omega_0^h(\Delta_n)$. By letting $n \to \infty$ and using Lemma 2.2, it follows

$$\int_{S(x_0,r)} \frac{K(y,z)}{h(y)} P_{B^c}^h(x, dy) = \int_{S(x_0,r)} \frac{K(y,z)}{h(y)} Q_{B^c}(x, dy), \tag{2.17}$$

and this holds for all $z \in V$. Now the same argument as in the proof of Theorem 2.1, shows that Y is a time change of X^h. Hence, $Q_x(Y_{\zeta-} \in A) = P_x^h(X_{\zeta-}^h \in A)$ and (2.12) reads

$$P_x^h(X_{\zeta-}^h \in A) = P_x^h(X_{\zeta-}^h \in F^{-1}(A)), \tag{2.18}$$

for all Borel subsets A of ∂D such that $F^{-1}(A) \subset V$ and all $x \in D$. Since (2.18) can be written as

$$\frac{1}{h(x)} \int_A K(x,z)\mu(dz) = \frac{1}{h(x)} \int_{F^{-1}(A)} K(x,z)\mu(dz),$$

it follows easily that $F(z) = z$ for all $z \in \operatorname{supp}(\mu) \cap V = V$. □

3 Brownian motion in a simply connected domain

In this section we use the Riemann mapping theorem to transfer the result for the unit disk to a simply connected domain. If the domain is provided with the Martin boundary, the situation is essentially the same as for the disk. In the case of the Euclidean boundary, the obtained result, though less complete, seems to be more interesting. We begin with the Martin boundary case.

Let E be a simply connected domain in $\mathbf{R}^2$ with at least two boundary points and let D be the unit disk. We denote the Green function for E and D by G_E and G_D respectively. Let x_0 be a fixed point in E and for $x, y \in E$ let

$$K_E(x, y) = \frac{G_E(x, y)}{G_E(x_0, y)} \tag{3.1}$$

be the Martin function (with respect to the base point x_0). Then there exists a compactification $\bar{E}_M$ of E such that the Martin function extends to a continuous function from $E \times \bar{E}_M$ into $(0, \infty]$. The Martin boundary of E is $\partial_M E = \bar{E}_M \setminus E$. For $z \in \partial_M E$, the function $x \mapsto K_E(x, z)$ is harmonic in E. The boundary point z is called minimal, if the function $K_E(\cdot, z)$ is minimal.

Let $f\colon E \to D$ map E conformally onto D. We assume that the point $x_0 \in E$ is chosen so that $f(x_0) = 0$. Being conformal, f preserves the Green function: $G_E(x, y) = G_D(f(x), f(y))$. Thus, f also preserves the Martin function, and hence extends to a homeomorphism of $\bar{E}_M$ and the Martin compactification of D. The latter being homeomorphic to the Euclidean closure $\bar{D}$, we have that $f\colon \bar{E}_M \to \bar{D}$ is a homeomorphism. Moreover, f takes minimal points to minimal points. Since ∂D consists only of minimal points, the same is valid for $\partial_M E$.

Let h be a positive harmonic function in E such that $h(x_0) = 1$. Then there is a unique probability measure μ on Borel subsets of $\partial_M E$ such that

$$h(x) = \int_{\partial_M E} K_E(x, z)\mu(\mathrm{d}z)\,. \tag{3.2}$$

Let $\omega_{x_0}(\mathrm{d}z)$ denote the harmonic measure at x_0. We assume that h satisfies a condition analog to (**H**): ($\mathbf{H}_\mathrm{M}$) The measure μ is not singular with respect to the harmonic measure ω_{x_0}.

Let g denote the density of the absolutely continuous part μ_a of μ. Then there exists a Borel subset W of $\partial_M E$ such that $W \subset \{g > 0\} \cap \operatorname{supp}(\mu_a)$ and $\omega_{x_0}(W) > 0$. The function $h \circ f^{-1}$ is harmonic in D and its representing measure is the image measure $\tilde{\mu} = \mu \circ f^{-1}$. It easily follows that $h \circ f^{-1}$ satisfies the condition (**H**). Moreover, if $V = f(W)$, then $\tilde{\mu}(V) > 0$ and $\sigma(V) > 0$.

Let (X_t^h, P_x^h) be an h-Brownian motion in E with lifetime ζ. Then the limit $\lim_{t \uparrow \zeta} X_t^h = X_{\zeta-}^h$ exists almost surely in the Martin topology, and $X_{\zeta-}^h \in \partial_M E$ (see [2]). Let $\tilde{X}_t = f(X_t^h)$ and $\tilde{P}_x = P^h_{f^{-1}(x)}$ for $x \in D$. Since f is 1-1, $(\tilde{X}_t, \tilde{P}_x)$ is a strong Markov process in D.

Lemma 3.1 *$(\tilde{X}_t, \tilde{P}_x)$ is a time change of $h \circ f^{-1}$-Brownian motion in D.*

Proof. Let $\mathscr{S}$ and $\tilde{\mathscr{S}}$ denote the cones of excessive functions for (X_t^h, P_x^h) and $(\tilde{X}_t, \tilde{P}_x)$ respectively. A simple use of Dynkin's theorem (e.g. [10], p. 58) shows that

$F: \mathscr{S} - \tilde{\mathscr{S}}$ defined by $F(u) = u \circ f^{-1}$ maps $\mathscr{S}$ bijectively onto $\tilde{\mathscr{S}}$. Excessive functions for X^h are of the form v/h where v is excessive for Brownian motion in E. The latter are precisely positive superharmonic functions in E. Since conformal mapping preserves superharmonicity and harmonicity (e.g. [9], p. 6.19.), $v \circ f^{-1}$ $(h \circ f^{-1})$ is superharmonic (harmonic) in D. Hence, $F(u) = v \circ f^{-1}/h \circ f^{-1}$ is an $h \circ f^{-1}$-superharmonic function in D. This shows that $(\tilde{X}_t, \tilde{P}_x)$ and $h \circ f^{-1}$-Brownian motion have the same excessive functions. Now Hunt's theorem and the BGM theorem show that the lemma is true. □

Let (Y_t, Q_x) be a normal strong Markov process in E with continuous paths up to its lifetime $\tilde{\zeta}$. We shall need a condition analog to (2.4) for the process (Y_t, Q_x). We assume

$$\text{If } \Gamma(\omega) \cap W \neq \emptyset, \text{ then } \Gamma(\omega) \text{ is a singleton, } Q_x \text{ a.s.} \tag{3.3}$$

The set of accumulation points Γ is taken in the Martin topology. Let $\tilde{Y}_t = f(Y_t)$ and $\tilde{Q}_x = Q_{f^{-1}(x)}$ for $x \in D$. Then $(\tilde{Y}_t, \tilde{Q}_x)$ is a strong Markov process in D with continuous paths up to the lifetime $\tilde{\zeta}$. Moreover, continuity of f on $\bar{E}_M$ gives that condition (2.4) holds for $\tilde{Y}$ on the Borel subset $V = f(W)$ of ∂D.

Theorem 3.1 *Let (X_t^h, P_x^h) be an h-Brownian motion on the Martin space $\bar{E}_M$ with h satisfying* ($\mathbf{H}_\mathrm{M}$), *let (Y_t, Q_x) be a continuous strong Markov process in E satisfying* (3.3). *Suppose*

$$P_x^h(X_{\zeta-}^h \in A) = Q_x(Y_{\tilde{\zeta}-} \in A) \tag{3.4}$$

for all Borel subsets A of W and all $x \in E$. Then Y is a time change of X.

Proof. Let $\tilde{X}$ and $\tilde{Y}$ be as above. Continuity of f on E_M, and the discussion preceding the theorem imply that

$$\tilde{P}_x(\tilde{X}_{\zeta-} \in f(A)) = \tilde{Q}_x(\tilde{Y}_{\tilde{\zeta}-} \in f(A)) \tag{3.5}$$

for all Borel subsets A of W and all $x \in D$. Now Lemma 3.1 and Theorem 2.1 give that for all compact sets $K \subset D$ and all $x \in D$

$$\tilde{P}_x(\tilde{X}_{T(K)} \in B) = \tilde{Q}_x(\tilde{Y}_{\tilde{T}(K)} \in B)\,, \tag{3.6}$$

for all Borel subsets B of D. Since $\inf\{t > 0: \tilde{X}_t \in K\} = \inf\{t > 0: X_t^h \in f^{-1}(K)\}$ and similarly for Y, we get

$$P_x(X^h_{T(f^{-1}(K))} \in f^{-1}(B)) = Q_x(Y_{\tilde{T}(f^{-1}(K))} \in f^{-1}B))\,. \tag{3.7}$$

Hence, X^h and Y have identical hitting distributions for all compact subsets of E. Once again we use the BGM theorem to finish the proof. □

While the Martin space is appropriate for Brownian motion, it seems rather unnatural to consider other Markov processes in that setting. In the sequel we replace the Martin boundary of the domain with its Euclidean boundary ∂E and consider only Brownian motion. Notation will remain the same.

The conformal mapping $f: E \to D$ extends automatically to a homeomorphism from $\bar{E}_M$ to $\bar{D}$. Since this may not be so in the Euclidean topology, we need some kind of extension of f to the Euclidean boundary ∂E. We assume that E satisfies the following extension condition:

(E) There exists an open subset U of ∂E such that f can be continuously extended to U and $f(U)$ is open in ∂D.

The extension is denoted by the same letter f. We discuss later when (E) is satisfied.

Theorem 3.2 *Let E be a bounded simply connected domain in $\mathbf{R}^2$ such that* (E) *holds. Let (X_t, P_x) be a Brownian motion in E and (Y_t, Q_x) a continuous normal strong Markov process in E satisfying* (2.4) *for some open subset $V \subset U$. Assume*

$$P_x(X_{\zeta-} \in A) = Q_x(Y_{\tilde{\zeta}-} \in A) \tag{3.8}$$

for all Borel subsets A of V and for all $x \in E$. Then Y is a time change of X.

Proof. Let $(\tilde{X}_t, \tilde{P}_x)$ and $(\tilde{Y}_t, \tilde{Q}_x)$ be as above. The extension condition (E) makes it possible to transfer equality (3.8) into D to obtain

$$\tilde{P}_x(\tilde{X}_{\zeta-} \in f(A)) = \tilde{Q}_x(\tilde{Y}_{\tilde{\zeta}-} \in f(A)) \tag{3.9}$$

for all Borel subsets $f(A)$ of $f(V)$ and all $x \in D$. Now the proof follows word by word the proof of Theorem 3.1. □

Now we come back to the condition (E). It is well known that if E is a Jordan domain, then f can be extended to a homeomorphism from $\bar{E}$ onto $\bar{D}$. This is, for example, proved in [7]. We refer the reader to the same source for the unexplained terminology about prime ends that follows. Suppose there is an open subset U of ∂E such that: (1) two different points in U are principal points of different prime ends, and (2) each point of U is a principal point of exactly one prime end. Then it can be proved that f permits a continuous extension to U (see [7]). Instead of giving a strict proof of that statement, we rather make some comments on above conditions. The first condition guarantees that Brownian motion will hit U with positive probability. The second condition says that "U does not have two sides".

To see what can go wrong when the second condition is not satisfied, consider the following example. Let $S_+ = \{(x_1, x_2) \in D: x_1 \geqq 0, x_2 = 0\}$, $S_- = \{(x_1, x_2) \in D: x_1 < 0, x_2 = 0\}$ and let $E = D \setminus S_+$ be the unit disk slit along the positive x_1-axis. We construct the strong Markov process Y in E in the following way: if the process starts at the point $x \in E$ with $x_2 \geqq 0$, it behaves like a reflected Brownian motion in the upper half-plane until it hits the boundary of E when it is killed. If Y starts at $x \in E$ with $x_2 < 0$, it behaves like Brownian motion in E until it hits S_- (if not killed before). After hitting S_-, it acts as described above. Then for any subset $A \subset S_+$, and every $x \in E$, it holds $P_x(X_{\zeta-} \in A) = Q_x(Y_{\tilde{\zeta}-} \in A)$, but the statement of Theorem 3.2 is obviously false. Fortunately enough, we may wisely choose to observe the process Y on any open subset of the circle and realize that it is not a time change of Brownian motion.

The problem becomes more acute in domains such that for every open subset U of the boundary, Brownian motion either never hits U, or it can approach U from different sides. An example of such domain is the upper half-plane with countably many segments deleted, one at each rational point of the x-axis. The length of the segment at $(m/n, 0)$ is $1/n$. In this case, no conclusion can be derived from Theorem 3.2 and it is not clear to us if the result is true.

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