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Properties of holomorphic Wiener functions – skeleton, contraction, and local Taylor expansion

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Summary. We show that each holomorphic Wiener function has a skeleton which is intrinsic from several viewpoints. In particular, we study the topological aspects of the skeletons by using the local Taylor expansion for holomorphic Wiener functions.

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1 Introduction

Holomorphic functions of infinitely many complex variables have been discussed by many authors, among others by Shigekawa [7], who first constructed them on the complex Wiener space using the techniques of the Malliavin calculus. He noticed several properties of his holomorphic functions (let us call them *holomorphic Wiener functions*), such as the Itô–Wiener expansion in the L^p-sense. In this paper, we will investigate some other properties of holomorphic Wiener functions.

We use Shigekawa's framework and notation [7]. Let (B, H, μ, J) be an *almost complex abstract Wiener space*, i.e., B is a real separable Banach space, H is a real separable Hilbert space continuously and densely imbedded in B, μ is a Gaussian measure satisfying

$$\int_B \exp(\sqrt{-1}\langle \varphi, z\rangle)\mu(dz)=\exp(-\tfrac{1}{4}\|\varphi\|_{H^*}^2), \quad \varphi\in B^*\subset H^*,$$

and $J: B\to B$, the *almost complex structure*, is an isometric mapping such that $J^2=-\mathrm{id}$ and $J|_H: H\to H$ is also isometric (see [7]).

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In this paper, we will first show that from several viewpoints, we can endow each holomorphic function on (B, H, μ, J) with an intrinsic and unique *skeleton*, i.e., a function defined on H which is considered to be the restriction of the original Wiener function. Note that, of course, since $\mu(H)=0$, it makes no measure-theoretical sense to endow general Wiener functions with skeletons.

Let $F: B \to \mathbf{C}$ be L^{1+}-holomorphic (for precise definition, see Definition 2.1 below). Note however that F may not be continuous. Taking into account the mean value theorem for usual holomorphic functions on $\mathbf{C}^n$ and the rotation invariance of the Gaussian measure μ, we may guess that the skeleton of F should be

$$F(h) = \int_B F(z+h)\mu(dz), \quad h \in H.$$

On the other hand, if we can give an intrinsic meaning to a function $F(\alpha z)$ for $\alpha > 0$, which will be done for $0 < \alpha < 1$ and will be called the *contraction operation*, then the skeleton of F should be

$$F(h) = \lim_{\alpha \to 0} F(\alpha z + h) \text{ in probability}, \quad h \in H.$$

As expected, we can show that these two are consistent (Theorem 2.8).

Skeletons are expected to appear in the theory of large deviations as rate functions [1]. In infinite dimensional spaces, the theory of large deviations necessarily involves the topology of the spaces. Consequently, to investigate skeletons of Wiener functions, it is important to look at their fluctuations in small balls centered at each $h \in H$. In this context, we will show that

$$\lim_{r \to 0} \frac{1}{\mu(B_r)} \int_{B_r} |F(z+h) - F(h)|^2 \mu(dz) = 0, \quad h \in H, \tag{1}$$

for each L^{2+}-holomorphic function F, where B_r denotes the centered $\|\cdot\|_B$-ball with radius $r > 0$ (Theorem 4.1). But there, we must require the norm $\|\cdot\|_B$ of B to have some good property which is fitted to the almost complex structure J.

In proving our theorems, we will make use of the rotation invariance of the Gaussian measure μ. In fact, some results, such as the theorem of *local Taylor expansion* for holomorphic Wiener functions (Theorem 3.6), will be proved only by means of the rotation invariance.

2 Mean value theorem and contraction operation

First we define holomorphic functions on the almost complex abstract Wiener space (B, H, μ, J). We will review [7] briefly.

Let B^* be the topological dual space of B and let $B^{*\mathbf{C}}$ be its complexification, i.e., $B^{*\mathbf{C}} := B^* \oplus \sqrt{-1}B^*$. Defining

$$B^{*(1,0)} := \{\varphi \in B^{*\mathbf{C}} \mid J^*\varphi = \sqrt{-1}\varphi\},$$

$$B^{*(0,1)} := \{\varphi \in B^{*\mathbf{C}} \mid J^*\varphi = -\sqrt{-1}\varphi\},$$

we see that

$$B^{*\mathbf{C}}=B^{*(1,0)}\oplus B^{*(0,1)}.$$

The Hilbert spaces $H^{*\mathbf{C}}$, $H^{*(1,0)}$ and $H^{*(0,1)}$ are defined similarly.

A function $F:B\to\mathbf{C}$ is called a *holomorphic polynomial*, if it is expressed in the form

$$F(z)=f(\langle\varphi_1, z\rangle, \ldots, \langle\varphi_n, z\rangle),$$

where $n\in\mathbf{N}, f:\mathbf{C}^n\to\mathbf{C}$ is a polynomial with complex coefficients, and $\varphi_1, \ldots, \varphi_n\in B^{*(1,0)}$. The class of holomorphic polynomials is denoted by $\mathscr{P}_h$.

Definition 2.1 *We define the space $\mathscr{H}^p(B,\mu)$ of L^p-holomorphic Wiener functions as the $L^p(B,\mu)$-closure of $\mathscr{P}_h$:*

$$\mathscr{H}^p(B,\mu):=\bar{\mathscr{P}}_h^{L^p(B,\mu)}, \quad 1\leqq p<\infty.$$

Further we define the auxiliary spaces by

$$\mathscr{H}^{p+}(B,\mu):=\bigcup_{p<p'}\mathscr{H}^{p'}(B,\mu), \quad 1\leqq p<\infty.$$

Note that the above definition of $\mathscr{H}^p(B,\mu)$ is equivalent to Shigekawa's definition (see [7, Proposition 4.2).

Since the Cameron–Martin density (see Definition 2.5 below) has all moments, it is easy to see that, for each $F\in\mathscr{H}^{p+}(B,\mu)$, the translated function $F(z+h)$ with $h\in H$ also belongs to $\mathscr{H}^{p+}(B,\mu)$. Note that there are many L^p-holomorphic functions which are not continuous with respect to any measurable norm (see [10]; on measurable norms, see [2, 3]).

Since we are assuming the almost complex structure J, the measurable norm $\|\cdot\|_B$ is naturally required to have the following property.

Assumption 2.2 (Rotation invariance of the norm) *We assume that, for arbitrary $a, b\in\mathbf{R}$,*

$$\|(a+bJ)z\|_B=|a+\sqrt{-1}b|\,\|z\|_B, \quad z\in B.$$

In particular, we assume

$$\|e^{J\theta}z\|_B=\|z\|_B, \quad \theta\in\mathbf{R},$$

where $e^{J\theta}z$ is an abbreviation for $(\cos\theta+(\sin\theta)J)z$.

Example 2.3 Let $(W(\mathbf{R}^2), H(\mathbf{R}^2), P^W)$ be the two-dimensional Wiener space, i.e.,

$$W(\mathbf{R}^2):=\{w=(w^1,w^2)\in C([0,1]\to\mathbf{R}^2)\mid w(0)=0\in\mathbf{R}^2\},$$

$$H(\mathbf{R}^2):=\{h=(h^1,h^2)\in W\mid h(t) \text{ is absolutely continuous and } dh/dt\in L^2([0,1]\to\mathbf{R}^2, dt)\},$$

$$P^W:=\text{the standard two-dimensional Wiener measure.}$$

Then the abstract Wiener space $(W(\mathbf{R}^2), H(\mathbf{R}^2), P^W)$ is naturally equipped with an almost complex structure J, by the identification $\mathbf{R}^2 \cong \mathbf{C}$. Namely, we define $J: W(\mathbf{R}^2) \to W(\mathbf{R}^2)$ by

$$Jw := (-w^2, w^1), \quad w = (w^1, w^2) \in W(\mathbf{R}^2).$$

Under this almost complex structure J, the following measurable norms are typical examples of rotation invariant ones:

$$\|w\|_\infty := \max_{0 \leqq t \leqq 1} |w(t)|_{\mathbf{R}^2},$$

$$\|w\|_p := \left(\int_0^1 |w(t)|_{\mathbf{R}^2}^p \, dt \right)^{1/p}, \quad 1 \leqq p < \infty,$$

$$\|w\|_{(\alpha)} := \sup_{0 \leqq s < t \leqq 1} \frac{|w(t) - w(s)|_{\mathbf{R}^2}}{|t-s|^\alpha}, \quad 0 < \alpha < \tfrac{1}{2},$$

where $|\cdot|_{\mathbf{R}^2}$ denotes the Euclid norm in $\mathbf{R}^2$; $|(x, y)|_{\mathbf{R}^2} := \sqrt{x^2 + y^2}$.

Lemma 2.4 (Mean value theorem) *Let B_r be the centered $\|\cdot\|_B$-ball with radius $r > 0$,*

$$B_r := \{ z \in B \mid \|z\|_B < r \}.$$

Then for each $F \in \mathscr{K}^1(B, \mu)$ and each $r > 0$,

$$\frac{1}{\mu(B_r)} \int_{B^r} F(z) \mu(dz) = \int_B F(z) \mu(dz).$$

Proof. It is shown by Shigekawa [7, Theorem 4.1] that

$$\int_B F(z) \mu(dz) = \frac{1}{2\pi} \int_0^{2\pi} (U_\theta F)(z) \mu(d\theta), \quad \mu\text{-a.e. } z \in B,$$

where $(U_\theta F)(z) := F(e^{J\theta} z)$, which is well-defined on account of the rotation invariance of μ. The distribution of $U_\theta F$ is therefore just the same as that of F for each $\theta \in \mathbf{R}$. It then follows from Fubini's theorem and the rotation invariance of B_r that

$$\begin{aligned}
\mu(B_r) \times \int_B F(z) \mu(dz) &= \int_{B_r} \mu(dz) \cdot \frac{1}{2\pi} \int_0^{2\pi} (U_\theta F)(z) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{B_r} (U_\theta F)(z) \mu(dz) \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{B_r} F(z) \mu(dz) \\
&= \int_{B_r} F(z) \mu(dz),
\end{aligned}$$

which completes the proof. □

When $F \in \mathscr{H}^1(B, \mu)$ is continuous in $\|\cdot\|_B$, we have

$$F(0) = \lim_{r \to 0} \frac{1}{\mu(B_r)} \int_{B_r} F(z)\mu(dz) = \int_B F(z)\mu(dz).$$

Hence we present the following definition, which will be justified in the sequel.

Definition 2.5 *For each $F \in \mathscr{H}^{1+}(B, \mu)$, we define its* **skeleton** *by*

$$F(h) := \int_B F(z+h)\mu(dz) = \int_B e^{2\langle h, z\rangle - |h|_H^2} F(z)\mu(dz), \quad h \in H.$$

Here note that the multiplier $e^{2\langle h, z\rangle - |h|_H^2}$ is the **Cameron–Martin density.**

Lemma 2.6 *Let $\alpha \in \mathbf{R}$ be such that $0 < \alpha < 1$, and $F \in \mathscr{H}^p(B, \mu)$. If a sequence $\{F_n\}$, $F_n \in \mathscr{P}_h$, converges to F in $\mathscr{H}^p(B, \mu)$, then $\{F_n(\alpha z)\}$ is also convergent in $\mathscr{H}^p(B, \mu)$. The limit does not depend on the choice of approximating sequence $\{F_n\}$ of holomorphic polynomials.*

Proof. Since each holomorphic Fourier–Hermite function is in fact a monomial ([4, 7] and expression (3) in the next section), we note that the *Ornstein–Uhlenbeck semigroup* $\{T_t\}_{t \geqq 0}$ (for details see [9][1]) operates on $G \in \mathscr{P}_h$ as a family of contractions. Namely,

$$(T_t G)(z) = G(e^{-t/2} z), \quad z \in B \text{ and } t \geqq 0.$$

Consequently, $F_n(\alpha z) = T_{-2 \log \alpha} F_n(z)$. Thus it is clear that $\{F_n(\alpha z)\}$ is convergent in $L^p(B, \mu)$, hence in $\mathscr{H}^p(B, \mu)$, and that the limit is $T_{-2 \log \alpha} F(z)$, which does not depend on the choice of approximating sequence. □

By the above lemma, we may present the following definition of the *contraction operation.*

Definition 2.7 *Let $\alpha \in \mathbf{R}$ be such that $0 < \alpha < 1$, and $F \in \mathscr{H}^{p+}(B, \mu)$. Then we define*

$$F(\alpha z) := T_{-2 \log \alpha} F(z) \in \mathscr{H}^p(B, \mu).$$

Let us show first that Definitions 2.5 and 2.7 are consistent.

Theorem 2.8 *For any $F \in \mathscr{H}^{p+}(B, \mu)$,*

$$\lim_{\alpha \to 0} F(\alpha z + h) = F(h) \quad \text{in } \mathscr{H}^{p+}(B, \mu) \text{ for each } h \in H.$$

Proof. For any $G \in L^p(B, \mu)$, we have

$$\lim_{t \to \infty} \| T_t G - \int_B G(z)\mu(dz) \|_{L^p} = 0,$$

which proves the claim. □

[1]Our T_t here corresponds to $T_{t/2}$ in [9]

3 Local Taylor expansion

In this section, we will consider a localization of holomorphic Wiener functions.

We recall the Itô-Wiener expansion for L^2-holomorphic Wiener functions. According to [4, 7], the Hilbert space $\mathscr{H}^2(B, \mu)$ is decomposed into an orthogonal infinite direct sum as follows:

$$\mathscr{H}^2(B, \mu) = \bigoplus_{n=0}^{\infty} C_{(n,0)}, \tag{2}$$

where $C_{(n,0)}$ is the space of holomorphic n-fold Wiener integrals, i.e., the L^2-closure of the space of all the finite linear combinations of holomorphic Fourier–Hermite functions of degree n.

Recall that each holomorphic Fourier–Hermite function of degree n is expressed in the form

$$\prod_{k=1}^{K} \langle \varphi_k, z \rangle^{m_k}, \tag{3}$$

where $\{\varphi_k\}_{k=1}^{K}$ is an orthonormal system of $H^{*(1,0)}$, while $m_1, \ldots, m_K$ are positive integers such that $\sum_{k=1}^{K} m_k = n$. Therefore we may call it a *holomorphic monomial*, and we may call the Itô–Wiener expansion (2) the *global Taylor expansion* in the L^2-sense.

In this section, we will present a local version of this expansion. As before, let B_r be the centered $\|\cdot\|_B$-ball with radius $r > 0$.

Lemma 3.1 *Let G_1 and G_2 be two holomorphic monomials of different degrees. Then $G_1|_{B_r}$ and $G_2|_{B_r}$ are mutually orthogonal in $L^2(B_r, \mu)$.*

Proof. Assume G_1 is of degree m and G_2 is of degree n, where $m \neq n$. Then we obviously have

$$U_\theta G_1 = e^{\sqrt{-1} m\theta} G_1,$$
$$U_\theta G_2 = e^{\sqrt{-1} n\theta} G_2,$$

for $\theta \in \mathbf{R}$. Since $U_\theta B_r = B_r$ and the measure μ is U_θ-invariant, we have

$$\begin{aligned}\int_{B_r} G_1 \bar{G}_2 \, d\mu &= \int_{B_r} U_\theta (G_1 \bar{G}_2) \, d\mu \\ &= \int_{B_r} e^{\sqrt{-1} m\theta} e^{-\sqrt{-1} n\theta} G_1 \bar{G}_2 \, d\mu \\ &= e^{\sqrt{-1}(m-n)\theta} \int_{B_r} G_1 \bar{G}_2 \, d\mu,\end{aligned}$$

for arbitrary $\theta \in \mathbf{R}$. Thus we see that

$$\int_{B_r} G_1 \bar{G}_2 \, d\mu = 0. \quad \square$$

Remark 3.2 Since μ is rotation invariant while the real part and the imaginary part of each holomorphic function are harmonic, Lemma 3.1 is essentially a consequence of the following fact: Two homogeneous harmonic polynomials on $\mathbf{R}^n$ of different degrees are mutually orthogonal with respect to the uniform measure on the sphere S^{n-1} [8, p. 69].

In $\mathscr{H}^2(B, \mu)$, two holomorphic monomials of a same degree are mutually orthogonal, if they differ in one of the exponents m_k in the expression (3). Now what about $\mathscr{H}^2(B_r, \mu)$? To answer this question, we need a class of good measurable norms.

First we introduce the notion of *splitting* of our almost complex abstract Wiener space (see, for example, [7] for details). Suppose that $\varphi \in H^{*(1,0)}$. Then decompose $\varphi = \psi + \sqrt{-1}\psi'$ where $\psi, \psi' \in H^* \cong H$. Let H_φ be a two-dimensional subspace of H spanned by ψ and ψ', let $H_\varphi^\perp$ be its orthogonal complement, and let $B_\varphi^\perp$ be the closure of $H_\varphi^\perp$ in B. Accordingly, μ is split into $\mu = \mu_\varphi \otimes \mu_\varphi^\perp$, where μ_φ is a Gaussian measure on $H_\varphi \cong \mathbf{C}$, while $\mu_\varphi^\perp$ is also a Gaussian measure on $B_\varphi^\perp$. Now we have two almost complex abstract Wiener spaces $(H_\varphi, H_\varphi, \mu_\varphi, J|_{H_\varphi})$ and $(B_\varphi^\perp, H_\varphi^\perp, \mu_\varphi^\perp, J|_{B_\varphi^\perp})$, and the following natural identification:

$$(H_\varphi \oplus B_\varphi^\perp, H_\varphi \oplus H_\varphi^\perp, \mu_\varphi \otimes \mu_\varphi^\perp, J|_{H_\varphi} \oplus J|_{B_\varphi^\perp}) \cong (B, H, \mu, J).$$

Using this notation, we give a definition of the best fitted norms to the complex structure.

Definition 3.3 *The norm* $\|\cdot\|_B$ *of* B *is said to be* **completely rotation invariant**, *if it is rotation invariant and there exists a complete orthonormal system* $\{\varphi_k\}$ *of* $H^{*(1,0)}$ *such that for any* $\varphi := \varphi_k$,

$$\|(e^{J\theta}z_1) \oplus z_2\|_B = \|z_1 \oplus z_2\|_B, \quad z_1 \in H_\varphi, \quad z_2 \in B_\varphi^\perp, \quad \theta \in \mathbf{R}.$$

Example 3.4 Let $\{\varphi_k\}$ be a complete orthonormal system of $H^{*(1,0)}$. Put

$$\|z\| := \left(\sum_k a_k |\langle \varphi_k, z\rangle|^2\right)^{1/2}, \tag{4}$$

where $a_k > 0$, $\sum_k a_k < \infty$. Then $\|\cdot\|$ is measurable and completely rotation invariant. In the case of the two-dimensional Wiener space with almost complex structure J defined in Example 2.3, the L^2-norm

$$\|w\|_2 = \left(\int_0^1 |w(t)|_{\mathbf{R}^2}^2 \, dt\right)^{1/2}$$

has the above expression (4) and hence is completely rotation invariant. (In proving this fact, use the trigonometric series expansion for each w.)

Lemma 3.5 *Let* $\|\cdot\|_B$ *be completely rotation invariant. Then two holomorphic monomials of a same degree are mutually orthogonal in* $L^2(B_r, \mu)$, *if they differ in one of the exponents* m_k *in the expression* (3).

Proof. Let G_1 and G_2 be holomorphic monomials of degree n, expressed as follows:

$$G_1(z) = \prod_{k=1}^{K} \langle \varphi_k, z\rangle^{m_k}, \quad m_k \geqq 0, \quad \sum_{k=1}^{K} m_k = n,$$

$$G_2(z) = \prod_{k=1}^{K} \langle \varphi_k, z\rangle^{m'_k}, \quad m'_k \geqq 0, \quad \sum_{k=1}^{K} m'_k = n,$$

where $m_1 \neq m'_1$. Then

$$G_1(z)\overline{G_2(z)} = F_1(z) F_2(z),$$

where F_1 and F_2 are given by

$$F_1(z) := \langle \varphi_1, z\rangle^{m_1} \overline{\langle \varphi_1, z\rangle}^{m'_1},$$

$$F_2(z) := \prod_{k=2}^{K} \langle \varphi_k, z\rangle^{m_k} \overline{\langle \varphi_k, z\rangle}^{m'_k}.$$

Putting $\varphi := \varphi_1$, we have

$$\int_{B_r} G_1(z)\overline{G_2(z)}\mu(dz) = \int_{H_\varphi} \mu_\varphi(dz_1) F_1(z_1) \int_{B_r(z_1)} F_2(z_2)\mu_\varphi^\perp(dz_2), \tag{5}$$

where $B_r(z_1) := \{z_2 \in B_\varphi^\perp \mid z_1 \oplus z_2 \in B_r\}$, $z_1 \in H_\varphi$. For $\theta \in \mathbf{R}$, note that

$$F_1(e^{J\theta} z_1) = e^{\sqrt{-1}(m_1 - m'_1)\theta} F_1(z_1), \quad z_1 \in H_\varphi.$$

Hereafter we identify H_φ with $\mathbf{C}$. Then since the complete rotation invariance of the norm implies $B_r(e^{J\theta} z_1) = B_r(z_1)$, we have

$$\begin{aligned}
&\int_{\arg z_1 \in [\theta, \theta+\eta)} \mu_\varphi(dz_1) F_1(z_1) \int_{B_r(z_1)} F_2(z_2)\mu_\varphi^\perp(dz_2) \\
&= \int_{\arg z_1 \in [0, \eta)} \mu_\varphi(dz_1) F_1(e^{J\theta} z_1) \int_{B_r(e^{J\theta} z_1)} F_2(z_2)\mu_\varphi^\perp(dz_2) \\
&= \int_{\arg z_1 \in [0, \eta)} \mu_\varphi(dz_1) e^{\sqrt{-1}(m_1 - m'_1)\theta} F_1(z_1) \int_{B_r(z_1)} F_2(z_2)\mu_\varphi^\perp(dz_2) \\
&= e^{\sqrt{-1}(m_1 - m'_1)\theta} \int_{\arg z_1 \in [0, \eta)} \mu_\varphi(dz_1) F_1(z_1) \int_{B_r(z_1)} F_2(z_2)\mu_\varphi^\perp(dz_2).
\end{aligned}$$

Consequently, putting $\eta = 2\pi$, we get

$$\int_{B_r} G_1(z)\overline{G_2(z)}\mu(dz) = e^{\sqrt{-1}(m_1 - m'_1)\theta} \int_{B_r} G_1(z)\overline{G_2(z)}\mu(dz)$$

for every θ, which shows this value should be equal to zero. □

We define the space of L^p-holomorphic functions defined on B_r as follows.

Definition 3.6 *By $\mathscr{H}^p(B_r, \mu)$, we denote the $L^p(B_r, \mu)$-closure of $\mathscr{P}_h|_{B_r}$:*

$$\mathscr{H}^p(B_r, \mu) := \overline{\mathscr{P}_h|_{B_r}}^{L^p(B_r, \mu)}, \quad 1 \leqq p < \infty,$$

and we define the auxiliary spaces by

$$\mathscr{H}^{p+}(B_r, \mu) := \bigcup_{p<p'} \mathscr{H}^{p'}(B_r, \mu), \quad 1 \leqq p < \infty.$$

We have thus defined L^p-local holomorphic functions for each p ($1 \leqq p < \infty$), but we will exclusively deal with the case $p = 2$. The next theorem readily follows from Lemmas 3.1 and 3.5.

Theorem 3.7 *The Hilbert space $\mathscr{H}^2(B_r, \mu)$ is decomposed into an infinite orthogonal direct sum*

$$\mathscr{H}^2(B_r, \mu) = \bigoplus_{n=0}^{\infty} C_{(n,0)}(B_r), \tag{6}$$

where $C_{(n,0)}(B_r)$ is the $L^2(B_r, \mu)$-closure of all the finite linear combinations of holomorphic monomials of degree n. If, in addition, the given measurable norm is completely rotation invariant, each $F \in C_{(n,0)}(B_r)$ is orthogonally decomposed into an infinite linear combination of holomorphic monomials of degree n.

By Shigekawa's uniqueness theorem [7, Theorem 4.3], a mapping

$$\mathscr{H}^2(B, \mu) \ni F \mapsto F|_{B_r} \in \mathscr{H}^2(B_r, \mu)$$

is injective, so $\mathscr{H}^2(B, \mu)$ is continuously imbedded in $\mathscr{H}^2(B_r, \mu)$. Of course, there exist many functions in $\mathscr{H}^2(B_r, \mu)$ which cannot be extended to the whole space B as elements of $\mathscr{H}^2(B, \mu)$. Since (6) is considered as a local Taylor expansion, we may say that $\mathscr{H}^2(B_r, \mu)$ is the space of local L^2-holomorphic functions with radius of convergence at least r.

To conclude this section, we present a corollary which is an immediate consequence of Theorem 3.7.

Corollary 3.8 (i) *For $F \in \mathscr{H}^2(B, \mu)$,*

$$\int_{B_r} |F(z)|^2 \mu(dz) = \sum_{n=0}^{\infty} \int_{B_r} |J_{(n,0)} F(z)|^2 \mu(dz),$$

where $J_{(n,0)}$ is the orthogonal projection from $\mathscr{H}^2(B, \mu)$ onto $C_{(n,0)}$.

(ii) *Suppose that the norm $\|\cdot\|_B$ is completely rotation invariant with respect to a CONS $\{\varphi_k\} \subset H^{*(1,0)}$ and that $F \in C_{(n,0)}$ has an expression*

$$F = \sum_{k=0}^{\infty} a_k G_k,$$

where $\{a_k\} \in l^2$, while $\{G_k\}$ is a sequence of holomorphic monomials of degree n which are based on $\{\varphi_k\}$, normalized and mutually orthogonal in $\mathscr{H}^2(B, \mu)$. Then we have

$$\int_{B_r} |F(z)|^2 \mu(dz) = \sum_{k=0}^{\infty} |a_k|^2 \int_{B_r} |G_k(z)|^2 \mu(dz).$$

Remark 3.9 It is clearly possible to give a local version of Corollary 3.8 as well as of the theorem in the next section.

Remark 3.10 In [5], another type of localization of holomorphic Wiener functions is given in connection with SDEs with holomorphic coefficients.

4 Topological aspects of skeletons

In this section, we will consider the topological aspects of the skeletons of holomorphic Wiener functions.

Let $F \in \mathscr{H}^{1+}(B, \mu)$. As we mentioned in Definition 2.5, the skeleton of F is the mean of $F(z+h)$, which is equal to the local mean restricted to the centered ball B_r of arbitrary radius $r>0$ (Lemma 2.4). Here we will show that the fluctuation of $F(z+h)$ around the skeleton $F(h)$ in B_r decreases as the radius r tends to zero. Namely, we will show the following theorem.

Theorem 4.1 *Let $\|\cdot\|_B$ be completely rotation invariant and let B_r be the centered $\|\cdot\|_B$-ball with radius $r>0$. Then for each $F \in \mathscr{H}^{2+}(B, \mu)$,*

$$\lim_{r \to 0} \frac{1}{\mu(B_r)} \int_{B_r} |F(z+h)-F(h)|^2 \mu(dz) = 0, \quad h \in H. \tag{7}$$

In particular,

$$\lim_{\delta \to 0} \mu(|F(z)-F(h)|>\varepsilon \mid \|z-h\|_B<\delta) = 0, \quad \varepsilon>0. \tag{8}$$

Lemma 4.2 *Let $\|\cdot\|_B$ be completely rotation invariant and let B_r be the centered $\|\cdot\|_B$-ball with radius $r>0$. Then for each $F \in \mathscr{H}^2(B, \mu)$,*

$$\frac{1}{\mu(B_r)} \int_{B_r} |F(z)|^2 \mu(dz) \leqq \int_B |F(z)|^2 \mu(dz).$$

Proof. By Corollary 3.8(i), we have

$$\frac{1}{\mu(B_r)} \int_{B_r} |F(z)|^2 \mu(dz) = \sum_{n=0}^{\infty} \frac{1}{\mu(B_r)} \int_{B_r} |J_{(n,0)} F(z)|^2 \mu(dz),$$

and hence it is sufficient to show that, for each $n=0, 1, \ldots,$

$$\frac{1}{\mu(B_r)} \int_{B_r} |J_{(n,0)} F(z)|^2 \mu(dz) \leqq \int_B |J_{(n,0)} F(z)|^2 \mu(dz).$$

By Corollary 3.8(ii), we have only to show that

$$\frac{1}{\mu(B_r)} \int_{B_r} |G_k(z)|^2 \mu(dz) \leqq \int_B |G_k(z)|^2 \mu(dz) \tag{9}$$

for each holomorphic monomial G_k which appeared in Corollary 3.8(ii). Since G_k is a monomial, $|G_k(z)|^2$ increases as $\|z\|_B$ increases, so (9) is intuitively obvious. For a rigorous proof, see Appendix. $\square$

Before proving Theorem 4.1, it is important to notice that, if the norm is not rotation invariant, the assertion of Theorem 4.1 may be false. Indeed, by using the same method as in [11], we can construct a holomorphic Wiener function F and a measurable norm $\|\cdot\|_B$ without rotation invariance for which (8) does not hold. However we do not know whether "*complete* rotation invariance" is necessary or not. Indeed, if Lemma 4.2 holds for $\|\cdot\|_B$ which is not completely rotation invariant but only rotation invariant, then Theorem 4.1 would hold without complete rotation invariance of the norm.

Proof of Theorem 4.1 We will show (7) for $h=0$ only. For arbitrary $h\in H$, (7) and (8) are proved by using the Cameron–Martin density (cf. [11]).
Take an arbitrary $\varepsilon>0$. Since

$$\int_B \{F(z)-F(0)\}\mu(dz)=0,$$

there exists a continuous holomorphic polynomial G such that

$$\int_B G(z)\mu(dz)=0,$$

$$\int_B |F(z)-F(0)-G(z)|^2\mu(dz)<\tfrac{1}{4}\varepsilon.$$

Applying Lemma 4.2, we see that

$$\begin{aligned}
&\frac{1}{\mu(B_r)}\int_{B_r} |F(z)-F(0)|^2\mu(dz)\\
&\leqq\frac{2}{\mu(B_r)}\int_{B_r} |G(z)|^2\mu(dz)+\frac{2}{\mu(B_r)}\int_{B_r} |F(z)-F(0)-G(z)|^2\mu(dz)\\
&\leqq\frac{2}{\mu(B_r)}\int_{B_r} |G(z)|^2\mu(dz)+2\int_B |F(z)-F(0)-G(z)|^2\mu(dz)\\
&\leqq\frac{2}{\mu(B_r)}\int_{B_r} |G(z)|^2\mu(dz)+\tfrac{1}{2}\varepsilon.
\end{aligned}$$

Since G is continuous and

$$G(0)=\int_B G(z)\mu(dz)=0,$$

there exists an $r_0>0$ such that for $0<r<r_0$,

$$\frac{2}{\mu(B_r)}\int_{B_r} |G(z)|^2\mu(dz)<\tfrac{1}{2}\varepsilon,$$

which implies

$$\frac{1}{\mu(B_r)}\int_{B_r} |F(z)-F(0)|^2\mu(dz)<\varepsilon. \qquad \square$$

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Appendix: Proof of Lemma 4.2

Here we will present a rigorous proof for (9) to complete the proof of Lemma 4.2.

Let $\|\cdot\|_B$ be completely rotation invariant, and let $\{\varphi_k\}$ be the corresponding complete orthonormal system of $H^{*(1,0)}$, as in Definition 3.2.

Lemma A1 *Let φ be one of the φ_k's. Then*

$$\|(az_1)\oplus z_2\|_B, \quad z_1 \in H_\varphi, \quad z_2 \in B_\varphi^\perp,$$

is non-decreasing as $a>0$ increases.

Proof. It is sufficient to show that

$$\|(az_1)\oplus z_2\|_B \geqq \|z_1 \oplus z_2\|_B =: r, \quad a>1.$$

Assume that this is not true for some $a>1$. Then $(az_1)\oplus z_2$ must be an interior point of a ball B_r. Note that $0\oplus z_2 \in B_r$, because $(-z_1)\oplus z_2 \in B_r$ by the complete rotation invariance of $\|\cdot\|_B$ and $0\oplus z_2 = (1/2)\{z_1\oplus z_2 + (-z_1)\oplus z_2\}$. Since $z_1\oplus z_2$ can be expressed as a convex combination of $0\oplus z_2 \in B_r$ and $(az_1)\oplus z_2$, which is an interior point of B_r, $z_1\oplus z_2$ would be an interior point of B_r. This is a contradiction. □

For the moment, assume B to be of finite dimension; $B\cong \mathbf{C}^N$, and suppose that G has an expression

$$G(z_1,\dots,z_K) = \prod_{k=1}^K z_k^{m_k}, \quad (z_1,\dots,z_K)\in \mathbf{C}^K,$$

where $K\leqq N$. Letting $\mu(dz_1\dots dz_N)$ be the standard Gaussian measure on $\mathbf{C}^N$, we may ask whether

$$\frac{1}{\mu(B_r)}\int_{B_r} |G(z_1,\dots,z_K)|^2 \mu(dz_1\dots dz_N) \leqq \int_{\mathbf{C}^N} |G(z_1,\dots,z_K)|^2 \mu(dz_1\dots dz_N). \tag{10}$$

Here B_r denotes a centered ball of radius $r>0$ with respect to a completely rotation invariant norm.

In the following lemma, we will state an assertion under the identification $\mathbf{C}^N \cong \mathbf{R}^{2N}$, but using the same symbols.

Lemma A2 *Let $F:\mathbf{R}_+^{2N} := [0,\infty)^{2N} \to \mathbf{R}$ be non-decreasing in the following sense:*

$$F(x_1,\dots,x_{2N}) \leqq F(y_1,\dots,y_{2N}), \quad 0\leqq x_k \leqq y_k, \quad k=1,\dots,2N.$$

Then putting $B_r^+ := B_r \cap \mathbf{R}_+^{2N}$, $r>0$, we have

$$\frac{1}{\mu(B_r^+)}\int_{B_r^+} F(x_1,\dots,x_{2N})\mu(dx_1\dots dx_{2N}) \leqq \int_{\mathbf{R}_+^{2N}} F(x_1,\dots,x_{2N})\mu(dx_1\dots dx_{2N}).$$

Proof. Define a function $\rho_r(x)$, $x \in \mathbf{R}_+^{2N}$, $0 < r \leqq \infty$, by

$$\rho_r(x) := \frac{1}{\mu(B_r^+)} \mathbf{1}_{B_r^+}(x) g(x), \quad 0 < r < \infty,$$

$$\rho_\infty(x) := g(x),$$

where

$$g(x) := (2\pi)^{-N} \exp\left(- \sum_{k=1}^{2N} x_k^2/2 \right), \quad x = (x_1, \dots, x_{2N}).$$

Let $0 < r$, and define $x \vee y$ and $x \wedge y$, $x, y \in \mathbf{R}_+^{2N}$, by

$$x \vee y := (x_1 \vee y_1, \dots, x_{2N} \vee y_{2N}),$$

$$x \wedge y := (x_1 \wedge y_1, \dots, x_{2N} \wedge y_{2N}).$$

Thus the x_k and y_k are the components of x and y, respectively. Lemma A1 implies that the indicator function $\mathbf{1}_{B_r^+}$ has the property

$$\mathbf{1}_{B_r^+}(x \wedge y) \geqq \mathbf{1}_{B_r^+}(y).$$

On the other hand, we readily see that

$$g(x \vee y) g(x \wedge y) = g(x) g(y).$$

Hence it is easy to see that

$$\rho_\infty(x \vee y) \rho_r(x \wedge y) \geqq \rho_\infty(x) \rho_r(y).$$

Therefore it follows from a version of the FKG-inequality due to Preston [6, Theorem 3] that

$$\int_{\mathbf{R}_+^{2N}} F(x) \rho_r(x) \mu(dx) \leqq \int_{\mathbf{R}_+^{2N}} F(x) \rho_\infty(x) \mu(dx),$$

which completes the proof. □

By Lemma A2, it follows easily that (10) holds. Finally, letting the dimension N tend to infinity, we obtain (9). Thus the proof of Lemma 4.2 is complete.

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