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Probability Theory and Related Fields

Percolation of Poisson sticks on the plane

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Summary. We consider a percolation model on the plane which consists of 1-dimensional sticks placed at points of a Poisson process on $\mathbb{R}^2$; each stick having a random, but bounded length and a random direction. The critical probabilities are defined with respect to the occupied clusters and vacant clusters and they are shown to be equal. The equality is shown through a 'pivotal cell' argument, using a version of the Russo-Seymour-Welsh theorem which we obtain for this model.

1 Introduction

Consider a percolation model which consists of lines (called sticks hereafter) of random length and with random direction placed at points of a Poisson point process of intensity λ on a plane. The length and direction of sticks placed at different points are assumed to have an i.i.d. distribution. It can clearly be seen that, unless the sticks are placed in only one direction (i.e. the direction random variable has singleton support), the size of an 'occupied cluster' (i.e. a cluster of sticks forming a connected set) increases, in a stochastic sense, as λ increases. Also, if we look at the 'vacant cluster' (i.e. the cluster characterized by the absence of sticks), its size decreases in a stochastic sense as λ increases. This suggests a phase transition.

Domany and Kinzel [1] has obtained estimates of the critical density and scaling coefficients for this model through computer simulations. Hall [2] has shown that under suitable conditions, our model exhibits a sharp phase transition, i.e., there exists a λ_C finite positive, such that for $\lambda > \lambda_C$, the size of the occupied cluster is infinite with positive probability, and for $\lambda < \lambda_C$ the size of the occupied cluster is finite with probability 1. We investigate this model and define various notions of the critical density λ_C. We show that under suitable conditions, these various notions are identical, in the sense that the critical densities are all equal. Moreover, we also introduce various definitions of λ_C^* (the critical density corresponding to the phase transition point for the vacant cluster) and show that not only are all the λ_C^* equal, they equal λ_C.

The argument presented here is different from that of Menshikov [5] for percolation of d-dimensional spheres in a d-dimensional space. Menshikov cleverly exploited the fact that the spheres have non-zero d-dimensional volume to show the equality of the critical densities via an approximation with a discrete percolation model. Unfortunately, we cannot use that technique here. Moreover, Menshikov's method cannot handle the vacant clusters. Our argument relies on a Russo-Seymour-Welsh argument (see Kesten [4]) and hence is restricted to 2-dimensional spaces only. Our line of argument is similar to that used to prove equality of the critical parameters of site/bond percolation in 2-dimensions. We develop, for our model, a modified version of the 'pivotal site/bond' used in site/bond percolation models (Russo [7], Kesten [4]) and use it to construct an integral inequality which establishes the equality of the critical densities.

The RSW theorem and Russo's formula has been applied in discrete percolation models to obtain various scaling laws and power estimates, it is expected that the analogue of these theorems for our model can also be used for similar results.

2 The model and statement of results

Consider a Poisson process $\xi_0, \xi_1, \xi_2, \ldots$ of intensity λ on $\mathbb{R}^2$. Centered at each point ξ_i is a line (stick) $L(\xi_i)$ of a random length $2\rho_i$ and a random direction θ_i. We assume that $\rho_0, \rho_1, \rho_2, \ldots$ and $\theta_0, \theta_1, \theta_2, \ldots$ are i.i.d. sequences of random variables and are independent of each other with each θ_i having support in $[0, \pi)$. Let ρ and θ be independent random variables having the same distribution as that of ρ_i and θ_i respectively.

We say that two points x and y in a region $A \subset \mathbb{R}^2$ have an *occupied connection in* A (denoted by $x \overset{o}{\rightsquigarrow} y$ in A) if there exist Poisson points $\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_n}$ in A for some $0 \leqq i_1, i_2, \ldots, i_n$ and associated sticks $L(\xi_{i_1}), L(\xi_{i_2}), \ldots, L(\xi_{i_n})$ such that there is a continuous curve joining x and y which lies in $\bigcup_j L(\xi_{i_j})$. Such a continuous curve is called an *occupied path* joining x and y.

Without loss of generality, we assume that $\xi_0 = \mathbf{0}$, *the origin*, $\theta_0 = 0$, *i.e., the stick* $L(\xi_0)$ *of length* $2\rho_0$ *is centred at the origin and lies on the x-axis.* We define the *occupied cluster* as

$$W(\mathbf{0}) = \{x \in \mathbb{R}^2 : \mathbf{0} \overset{o}{\rightsquigarrow} x\}. \tag{2.1}$$

For any set A of $\mathbb{R}^2$ let $|A| = \sup\{d(x, y): x, y \in A\}$, where $d(.,.)$ is the Euclidean distance.

The *critical densities* corresponding to occupied clusters are

$$\lambda_H := \inf\{\lambda : P_\lambda\{|W(\mathbf{0})| = \infty\} > 0\}, \tag{2.2}$$

$$\lambda_T := \inf\{\lambda : E_\lambda(|W(\mathbf{0})|) = \infty\}. \tag{2.3}$$

Clearly, if θ has a degenerate distribution, then no two sticks intersect with probability 1 and thus there will not be any percolation via sticks, i.e., $\lambda_H = \lambda_T = \infty$.

If $\rho = 1$ and θ non-degenerate, Hall [2] has shown that $\lambda_H < \infty$. Moreover, $\lambda_H \geqq \lambda_H$(continuum), where λ_H(continuum) is the analogous critical density obtained by having disks of radius 1 centred at ξ_i (λ_D of Roy [6]) instead of sticks. Again, Hall [2] has shown that λ_H(continuum)>0. Thus, we have,

$$0<\lambda_H<\infty. \tag{2.4}$$

We shall assume the following:

$$0<\rho\leqq R \text{ for some constant } R>0, \tag{2.5}$$

$$\rho \text{ has a uniform distribution in } [0,\pi). \tag{2.6}$$

Our theorem will also hold when (2.6) is replaced by

$$P\{\theta(\mathrm{mod}\,\pi)\colon |\theta-\alpha|\leqq\beta\}\geqq C(\alpha)>0 \tag{2.7}$$

for any α, $\beta\in[0,\pi)$ with some $C(\alpha)$ independent of β.

In addition to the above critical densities, we have critical densities defined via the vacant clusters as in (2.9) and (2.10) below.

Two points x and y in a region $A\subset\mathbb{R}^2$ are said to have a *vacant connection* in A (denoted by $x \overset{v}{\rightsquigarrow} y$ in A) if there exists a continuous curve γ with x and y as its two end points and such that $\gamma\cap L(\xi_i)=\emptyset$ for all $i\geqq 0$. Such a continuous curve is called a *vacant path* joining x and y. We note here that the endpoints x and y of γ need not be in γ. Thus either of x or y or both may be in $\bigcup_i L(\xi_i)$.

The *vacant cluster* is defined as

$$W^*(\mathbf{0})=\{x\in\mathbb{R}^2\colon \mathbf{0}\overset{v}{\rightsquigarrow} x\}. \tag{2.8}$$

The critical densities corresponding to the vacant clusters are

$$\lambda_H^*:=\sup\{\lambda\colon P_\lambda\{|W^*(\mathbf{0})|=\infty\}>0\}, \tag{2.9}$$

$$\lambda_T^*:=\sup\{\lambda\colon E_\lambda(|W^*(\mathbf{0})|=\infty)\}. \tag{2.10}$$

In addition, two other critical densities are defined through crossing probabilities.

A $L-R$ *occupied* (respectively, *vacant*) crossing of a rectangle $[0,l_1]\times[0,l_2]$ is an occupied (respectively, vacant) path connecting the left edge $\{0\}\times[0,l_2]$ to the right edge $\{l_1\}\times[0,l_2]$ and which lies completely inside the rectangle. Similarly, we define the $T-B$ *occupied/vacant crossing* of a rectangle as an occupied/vacant path connecting the top and the bottom edges of the rectangle and which lies in the rectangle.

The crossing probabilities are defined by

$$\sigma((l_1,l_2),1,\lambda):=P_\lambda\{\exists \text{ an occupied } L-R \text{ crossing of } [0,l_1]\times[0,l_2]\},$$

$$\sigma((l_1,l_2),2,\lambda):=P_\lambda\{\exists \text{ an occupied } T-B \text{ crossing of } [0,l_1]\times[0,l_2]\}.$$

Similarly, $\sigma^*((l_1,l_2),1,\lambda)$ and $\sigma^*((l_1,l_2),2,\lambda)$ are defined by replacing occupied with vacant in the above definitions.

The critical densities are

(2.11) $$\lambda_S := \inf\{\lambda\colon \limsup_n \sigma((n, 3n), 1, \lambda) > 0\}$$

(2.12) $$\lambda_S^* := \sup\{\lambda\colon \limsup_n \sigma^*((n, 3n), 1, \lambda) > 0\}.$$

We prove the following

Theorem 2.1 *Under the assumptions* (2.5) *and* (2.6) $\lambda_H = \lambda_T = \lambda_S = \lambda_H^* = \lambda_T^* = \lambda_S^*$.

Remark. As stated earlier the theorem holds when the assumption (2.6) is replaced by the assumption (2.7).

The pivotal argument and the RSW argument we employ are developed in the next two sections. In Sect. 5 we prove the theorem.

3 The FKG inequality and Russo's formula

We consider the space $\mathscr{S} = \{-1, 1\}^{\mathbb{R}^2 \times \mathbb{R}_+ \times [0,\pi)}$ where $\mathbb{R}_+ = (0, \infty)$ and let $\mathscr{F}$ denote the Borel σ-field on $\mathscr{S}$. We equip $\mathscr{F}$ with the probability measure arising from our model. In other words, for any open set $A \subset \mathbb{R}^2 \times \mathbb{R}_+ \times [0, \pi)$, the number of points (z, r, s) in $A (z \in \mathbb{R}^2, r \in \mathbb{R}_+, s \in [0, \pi))$ with $\omega(z, r, s) = 1$ has a Poisson distribution with mean $l_\lambda \times \mu(A)$, where l_λ is the Lebesgue measure which assigns mass λ to the unit cube and μ is the probability measure on $\mathbb{R}_+ \times [0, \pi)$ corresponding to the length and direction of the stick, and the number of points (z, r, s) in $A_1, A_2, \ldots, A_k$ with $\omega(z, r, s) = 1$ are independent whenever $A_1, A_2, \ldots, A_k$ is a collection of disjoint sets in $\mathbb{R}^2 \times \mathbb{R}_+ \times [0, \pi)$. Pictorially, $\omega(z, r, s) = 1$ corresponds to a stick centred at z of length $2r$ and at an angle s w.r.t. the horizontal axis.

Let ω and ω' be two configurations in this space $\mathscr{S}$. We say that $\omega \preceq \omega'$ if $\omega'(z, r, s) = 1$ whenever $\omega(z, r, s) = 1$, for any $z \in \mathbb{R}^2$, $r \in \mathbb{R}_+$, $s \in [0, \pi)$. A function $f\colon \mathscr{S} \to \mathbb{R}$ is said to be *increasing* (respectively, *decreasing*) if for every $\omega \preceq \omega'$, $f(\omega) \leqq f(\omega')$ (respectively, $f(\omega) \geqq f(\omega')$). An event $A \in \mathscr{F}$ is increasing (decreasing) if 1_A is increasing (decreasing).

FKG inequality

If A and B are both increasing events or both decreasing events then $P(A \cap B) \geqq P(A) P(B)$.

Proof. Let $\{a_n\}$ be a sequence decreasing to zero and such that a_n is an integer multiple of a_{n+1} for each $n \geqq 1$. Consider the lattice $\mathbf{L} = (a_n \mathbf{Z})^2 \times (a_n \mathbf{Z}) \times [0, \pi)$. For any $k = (k_1, k_2) \in \mathbf{Z}^2$ and $l, m \in \mathbf{Z}_+$ let $C(k, l, m)$ denote the cell $\{(z, r, s)\colon (k_i - 1) a_n < z_i \leqq k_i a_n, (l-1) a_n < r \leqq l a_n, (m-1) a_n < s \leqq (m-1) a_n\}$ for $i = 1, 2$. Clearly, $C(k, l, m)$'s are disjoint for disjoint (k, l, m) and $\bigcup_{k,l,m} C(k, l, m) = \mathbb{R}^2 \times \mathbb{R}_+ \times [0, \pi)$.

Given a cell C in $\mathbf{L}_n$, let $N_n(C)(\omega) = \#\{(z, r, s) \in C\colon \omega(z, r, s) = 1\}$. Let $\mathscr{F}_n$ be the σ-field generated by $\{N_n(C), C \text{ a cell in } \mathbf{L}_n\}$. Since the vertex set of the lattice $\mathbf{L}_n$ is contained in the vertex set of the lattice $\mathbf{L}_{n+1}$, $\{E(1_A | \mathscr{F}_n)\}_{n \geqq 1}$ is a martingale for any $A \in \mathscr{F}$. So, by the martingale convergence theorem, $E(1_A | \mathscr{F}_n) \to 1_A$ w.p.1 as $n \to \infty$.

Now let $C = C(k, l, m)$ and suppose A is an increasing event in $\mathscr{F}$. We then have, for $\omega \preceq \omega'$, $N_n(C)(\omega) \leqq N_n(C)(\omega')$. Also, given $N_n(C)(\omega) = j$, the conditional distribution of the j points in C has the probability measure $v \times \mu_{l,m}$, where v is the uniform distribution on the cell $((k_1-1)\,a_n, k_1\,a_n] \times ((k_2-1)\,a_n, k_2\,a_n]$ and $\mu_{l,m}$ is the conditional distribution of μ given that $(l-1)\,a_n < r \leqq l\,a_n$ and $(m-1)\,a_n < s \leqq m\,a_n$. Thus, $E(1_A | \mathscr{F}_n)$ is a.s. increasing.

For two increasing events or two decreasing events A and B on a lattice with a partial order we have $E(E(1_A | \mathscr{F}_n)\, E(1_B | \mathscr{F}_n)) \geqq E(E(1_A | \mathscr{F}_n))\, E(E(1_B | \mathscr{F}_n)) \geqq E(1_A)\, E(1_B)$ (see Kemperman [3]). Thus, the dominated convergence theorem yields $E(1_A\, 1_B) \geqq E(1_A)\, E(1_B)$. □

Now we introduce the notion of a 'pivotal cell' and prove a version of the Russo's formula.

Let $\mathbf{L}$ be a lattice on $\mathbb{R}^2$. Given a configuration ω of the Poisson model and a cell C of $\mathbf{L}$ let ω_C denote the configuration which agrees with ω outside C in the model and for which there is no Poisson point situated inside C. Given an event A and a configuration ω, a cell C in $\mathbf{L}$ is said to be *pivotal* for (ω, A) if $\omega \in A$ and $\omega_C \notin A$. This definition of pivotal is different from Russo's definition (see Kesten [4]) in that, for a cell to be pivotal for an event, in this definition, the event must occur.

Suppose $\lambda_1 \leqq \lambda \leqq \lambda_2$, where λ_1 and λ_2 are fixed positive reals. Let a be such that $a^2 < 1/\lambda_2$ and R is an integer multiple of a. Consider the lattice $\mathbf{L} = (a\,\mathbf{Z}) \times (a\,\mathbf{Z})$. Let $C_1, C_2, \ldots, C_N$ be an enumeration of the cells of the lattice $\mathbf{L}$ which lie in the rectangle $[-2R, l_1 + 2R] \times [-2R, l_2 + 2R]$ for some l_1, $l_2 > 0$ and integer multiples of a.

Consider a Poisson model which consists of the union of various Poisson processes on disjoint regions each with the same i.i.d. distributions of the attached stick random variable. In particular, for every $1 \leqq i \leqq N$, let $\lambda(i) \in [\lambda_1, \lambda_2]$ be the intensity of the Poisson process on the cell C_i and let λ be the intensity of the process outside the rectangle $[-2R, l_1 + 2R] \times [-2R, l_2 + 2R]$. Let E be an increasing event which depends on the configuration in the rectangle $[0, l_1] \times [0, l_2]$.

Russo's formula

$$\lambda(i) \frac{d}{d\,\lambda(i)} P_\lambda(E)$$
$$= (E(\#\text{ of Poisson points in } C_i \,|\, C_i \text{ is pivotal for } E) - |C_i|)\; P\{C_i \text{ is pivotal for } E\}.$$

Proof. Fix a cell C_i in the lattice $\mathbf{L}$. Define $A_k = \{\exists$ exactly k Poisson points in $C_i\}$. Clearly,

$$\frac{d}{d\,\lambda(i)} P_\lambda(E) = \frac{d}{d\,\lambda(i)} \Big(\sum_{0 \leqq k < \infty} P_\lambda(E \,|\, A_k \text{ occurs})\, P_\lambda(A_k \text{ occurs}). \tag{3.1}$$

Given the Poisson points and their associated sticks inside the cell C_i, the outcome of the event E depends on the Poisson points situated outside C_i. Also, given that there are k Poisson points in the cell C_i, the conditional distribution of the position of the points inside C_i is uniform on C_i and hence is independent of $\lambda(i)$. Thus, $P_\lambda(E \mid A_k \text{ occurs})$ does not depend on $\lambda(i)$. The summation in (3.1) being absolutely convergent, we have,

$$\begin{aligned}
&\frac{d}{d\lambda(i)}\Big(\sum_{0\leqq k<\infty} P_\lambda(E\mid A_k \text{ occurs})\, P_\lambda\{A_k \text{ occurs}\}\Big)\\
&\quad= \sum_{0\leqq k<\infty} P_\lambda(E\mid A_k \text{ occurs})\,\frac{d}{d\lambda(i)}\, P_\lambda\{A_k \text{ occurs}\}\\
&\quad= \sum_{1\leqq k<\infty} P_\lambda(E\mid A_k \text{ occurs})\left(|C_i|\left[\frac{1}{k!}\,(-\lambda(i)\,C_i)^k \exp(-\lambda(i)\,C_i)\right.\right.\\
&\qquad\left.\left.+\frac{1}{(k-1)!}\,(\lambda(i)\,C_i)^{k-1}\exp(-\lambda(i)\,C_i)\right]\right)\\
&\qquad-|C_i|\exp(-\lambda(i)\,C_i)\,P_\lambda(E\mid \text{there is no Poisson point in } C_i)\\
&\quad=|C_i|\sum_{0\leqq k<\infty}\frac{1}{k!}\,(\lambda(i)\,C_i)^k\exp(-\lambda(i)\,C_i)\\
&\qquad\times\{P_\lambda(E\mid A_{k+1}\text{ occurs})-P_\lambda(E\mid A_k \text{ occurs})\}.
\end{aligned}$$

Let $J=J(\omega)=J(\omega;\, E,\, C_i)$ be defined as follows:

$$J(\omega)=\begin{cases}1 & \text{if } C_i \text{ is pivotal for } (\omega, E)\\ 0 & \text{if } C_i \text{ is not pivotal for } (\omega, E).\end{cases}$$

Then from the previous calculations we obtain,

$$\begin{aligned}
\frac{d}{d\lambda(i)}\, P_\lambda(E)=|C_i|&\sum_{0\leqq k<\infty}\frac{1}{k!}\,(\lambda(i)\,C_i)^k\exp(-\lambda(i)\,C_i)\\
&\times\{E_\lambda(1_E\, J\mid A_{k+1}\text{ occurs})-E_\lambda(1_E\, J\mid A_k\text{ occurs})\\
&+E_\lambda(1_E(1-J)\mid A_{k+1}\text{ occurs})-E_\lambda(1_E(1-J)\mid A_k\text{ occurs})\}.
\end{aligned}$$

Now $1_E(1-J)=1$ only if E occurs and C_i is not pivotal for (ω, E) and E is increasing so, $E_\lambda(1_E(1-J)\mid A_{k+1}\text{ occurs})=E_\lambda(1_E(1-J)\mid A_k\text{ occurs})$. From this and the previous calculations, we have after some computation,

$$\begin{aligned}
&\frac{d}{d\lambda(i)}\, P_\lambda(E)=P_\lambda\{C_i \text{ is pivotal for } E\}\times\\
&\qquad\left(\frac{1}{\lambda(i)}\, E_\lambda(\#\text{ of Poisson points in } C_i \mid C_i \text{ is pivotal for } E)-|C_i|\right). \quad \square
\end{aligned}$$

If E occurs and C_i is pivotal for E then there must be at least one Poisson point in C_i. Thus, we have the following:

Corollary 3.1 $\displaystyle \frac{d}{d\lambda(i)}\, P_\lambda(E)\geqq\left(\frac{1}{\lambda(i)}-|C_i|\right) P_\lambda\{C_i \textit{ is pivotal for } E\}.$

4 Some preliminary results

We first state the RSW theorem and a lemma which gives probabilistic bounds on the size of a cluster when the crossing probabilities are small.

RSW Lemma. *Let $\delta_1, \delta_2 > 0$ be such that $\sigma^*((l_1, l_2), 1, \lambda) \geqq \delta_1$ and $\sigma^*((l_3, l_2), 2, \lambda) \geqq \delta_2$ for some $l_1, l_2 \geqq 4R$ and $2R < l_3 < 3l_1/2$. For any integer $k \geqq 1$, we have $\sigma^*((kl_1, l_2), 1, \lambda) \geqq K_k(\lambda) f_k(\delta_1, \delta_2)$, where $K_k(\lambda) > 0$ is a constant independent of δ_1 and δ_2 and $f_k(\delta_1, \delta_2)$ is independent of λ.*

Lemma 4.1 *There exists a constant $\kappa > 0$ such that*
(i) *if for some $N > R$, $\sigma((N, 3N), 1, \lambda) \leqq \kappa$, then $P_\lambda\{|W(\mathbf{0})| > a\} \leqq C_1 \exp(-C_2 a)$ for all $a > 0$ and for some positive constants C_1, C_2 which depend on λ only,*
(ii) *if for some $N > R$, $\sigma^*((N, 3N), 1, \lambda) \leqq \kappa$, then $P_\lambda\{|W^*(\mathbf{0})| > a\} \leqq C_3 \exp(-C_4 a)$ for all $a > 0$ and for some positive constants C_3, C_4 which depend on λ only.*

The proof of this follows, after minor adjustments, from the analogous theorem for continuum percolation (Theorem 2.3, Roy [6]), while the proof of Lemma 4.1 is essentially the same as that of Theorem 5.1 of Kesten [4]. As such we omit these proofs.

We now show that

$$\lambda_S \leqq \lambda_T \leqq \lambda_H^* \leqq \lambda_T^* = \lambda_S^* = \lambda_H. \tag{4.1}$$

First, Lemma 4.1 yields

$$\lambda_S \leqq \lambda_T, \tag{4.2}$$

$$\lambda_T^* \leqq \lambda_S^*. \tag{4.3}$$

((4.2) has also been shown by a different method in Zuev and Sidorenko [8])

$$\lambda_H^* \leqq \lambda_T^* \tag{4.4}$$

follows trivially from the definitions of λ_H^* and λ_T^*. While

$$\lambda_S^* \leqq \lambda_H \tag{4.5}$$

is obtained by an RSW argument. Indeed, for $\lambda < \lambda_S^*$, we can construct infinitely many vacant circuits surrounding the origin $\mathbf{0}$ w.p.1, thereby showing that $\lambda \leqq \lambda_H$. Finally to complete the proof of (4.1) we need to show

$$\lambda_T \leqq \lambda_H^*, \tag{4.6}$$

$$\lambda_H \leqq \lambda_T^*. \tag{4.7}$$

To prove (4.7) we observe that for $\lambda > \lambda_T^*$, $E_\lambda\{|W^*(\mathbf{0})| < \infty\}$. Thus, if $S(i) = \{(x, y): |x| \leqq 2R, |y - i4R| \leqq 2R\}$ and $W^*(S(i)) = \bigcup_{x \in S(i)} W^*(x)$, then, by the FKG lemma,

$$\sum_{k \geqq 1} 3^k P_\lambda\{|W^*(S(0))| > 3^k\} < \infty. \tag{4.8}$$

But,

$$\begin{aligned} &P_\lambda\{\exists \text{ a vacant } L\text{–}R \text{ crossing of } [0, 3^k] \times [0, 3^{k+1}]\} \\ &\leqq P_\lambda\Big(\bigcup_{0 \leqq i \leqq 3^{k+1}/4R} \{|W^*(S(i))| \geqq 3^k\}\Big) \\ &\leqq (3^{k+1}/4R)\, P_\lambda\{|W^*(S(0))| \geqq 3^k\}. \end{aligned}$$

Hence, for $\lambda > \lambda_T^*$, by (4.8) and the Borel-Cantelli lemma,

$$(4.9) \qquad P_\lambda\{\exists \text{ a vacant } L-R \text{ crossing of } [0, 3^k] \times [0, 3^{k+1}] \text{ i.o.}\} = 0.$$

By the Jordan curve theorem, either there is a vacant $L-R$ crossing or an occupied $T-B$ crossing of $[0, 3^k] \times [0, 3^{k+1}]$. Thus,

$$(4.10) \quad P_\lambda\{\exists \text{ an occupied } T-B \text{ crossing of } [0, 3^k] \times [0, 3^{k+1}] \text{ for all large } k\} = 1.$$

By translation invariance, (4.8) is equivalent to

$$(4.11) \quad P_\lambda\{\exists \text{ an occupied } L-R \text{ crossing of } [0, 3^{k+1}] \times [0, 3^k] \text{ for all large } k\} = 1.$$

Now a $L-R$ crossing of $[0, 3^{k+1}] \times [0, 3^k]$ must intersect a $T-B$ crossing of $[0, 3^{k+1}] \times [0, 3^{k+2}]$. Continuing in this fashion for all k, we have a criss-cross of $L-R$ and $T-B$ crossings which extend all the way to infinity. Thus

$$(4.12) \qquad P_\lambda\{\exists \text{ an occupied infinite region in the first quadrant}\} = 1.$$

Dividing up the quadrant into countably many cells and using translation invariance, we have

$$(4.13) \qquad P_\lambda\{|W(S(0))| = \infty\} > 0.$$

Another translation invariance argument now yields

$$(4.14) \qquad P_\lambda\{|W(\mathbf{0})\} = \infty\} > 0.$$

Thus, if $\lambda > \lambda_T^*$, then $\lambda \geqq \lambda_H$, i.e., $\lambda_H \leqq \lambda_T^*$ as required.

In the argument to show (4.7) if we change occupied to vacant and vice versa then we see that $\lambda < \lambda_T$ implies $\lambda \leqq \lambda_H^*$, thus proving (4.6).

(4.2)–(4.7) yield (4.1).

5 Expected number of pivotal cells

In this section we show that, for a lattice $\mathbf{L}$ of size η, and for fixed λ_0, λ_1 with $\lambda_0 < \lambda_1$.

Lemma 5.1 *If there exist $\delta > 0$ and a sequence $\{l_n\}_{n \geqq 1}$ with $l_n \uparrow \infty$ as $n \to \infty$, such that, for all $n \geqq 1$ and for all $\lambda \in [\lambda_0, \lambda_1]$,*

$$(5.1) \qquad \sigma((l_n, 3\,l_n), 1, \lambda) > \delta \quad \text{and} \quad \sigma^*((l_n, 3\,l_n), 2, \lambda) > \delta$$

then

$$(5.2) \qquad \inf_{\lambda \in [\lambda_0, \lambda_1]} E_\lambda\{\#\text{ of pivotal cells of } \mathbf{L} \text{ for the event } E_n\} \to \infty$$

as $n \to \infty$, where $E_n = \{\exists$ a $L-R$ occupied crossing of $[0, l_n] \times [0, 3\,l_n]\}$.

Let $\mathbf{L}_m$ be a lattice of size a_m on $\mathbb{R}^2$. A cell C in this lattice is called *vacant* if $C \cap L(\xi_i) = \emptyset$ for all $i \geqq 1$ and it is called *occupied* if $C \cap L(\xi_i) \neq \emptyset$ for some $i \geqq 0$. A *(vacant/occupied)* $\mathbf{L}_m$ *path* is a collection $C_1, \ldots, C_k$ of (vacant/occupied) cells of $\mathbf{L}_m$ such that consecutively numbered cells have an edge in common. A *$L-R$ (vacant/occupied)* $\mathbf{L}_m$ crossing of a rectangle is a $\mathbf{L}_m$ path (vacant/occupied) which lies in the rectangle and the two end-cells of which lie, respectively, on the left and right edges of the rectangle.

Proof of Lemma 5.1. The proof of the theorem is broken into three steps.

Step 1. Construction of a pivotal cell.

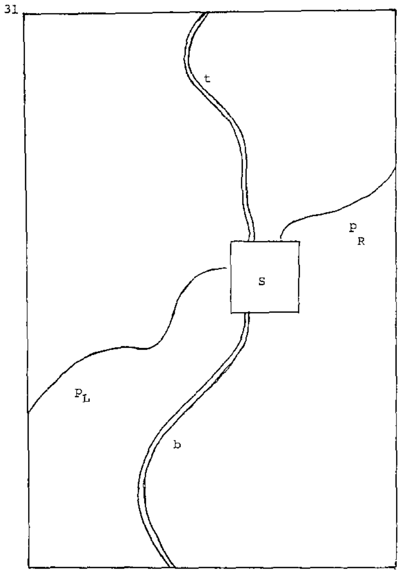

Fig. 1. The event $A(S)$

Fix ε and η such that

(5.3) $\quad 0<16\varepsilon<R,\ 0<10\eta<\varepsilon$ and $R/32$ is an integer multiple of η.

Since we approximate by lattices of size a_m and eventually let a_m decrease to 0, we take the sequence $\{a_m\}$ such that for all m,

(5.4) $\quad 2a_m,\ 3l_n$ and η are all integer multiples of $2a_{m+1}$.

We note here that the monotonicity properties of the crossing probabilities allow us to change the rectangles $[0, l_n]\times[0, 3l_n]$'s to slightly different rectangles whose sides satisfy the divisibility property (5.4) without violating (5.1). As our argument only needs that a sequence of growing rectangles whose larger side is $\leqq 3$ times the smaller side, the assumption (5.4) can be made without losing any generality.

For convenience in notation (unless we need to be specific) we denote l_n by l and E_n by E, D is the rectangle $[0, l_n]\times[0, 3l_n]$ and $\mathbf{L}$ is the lattice of size a_m. The pivotal cells will be constructed in the lattice of size η. This lattice will be called the η-lattice and a cell in this lattice will be an η-cell.

Let S be a square on the η-lattice of size d, where $26R\leqq d\leqq 30R$, and which lies inside D. Let $A(S)=A_m(S)$ be the set of configurations ω such that

(i) there is no occupied $L-R$ crossing of D consisting of sticks formed by the Poisson process outside S,

(ii) there are two occupied paths p_L and p_R in S, consisting of sticks formed by the Poisson process outside S, with one end-point of each p_L and p_R lying on the left edge and right edge of D, respectively, and the other end-point of each p_L and p_R at a distance of atmost $R/2$ from S,

(iii) there is a path b outside $\bigcup_{\xi\in S} L(\xi)$ which lies in D and connects the bottom edge of D to some side of S,

(iv) there is a $\mathbf{L}_m$ path t from the top edge of D to some side in S, which consists of cells C such that $C\cap L(\xi)=\emptyset$ for all $\xi\notin S$ (see Fig. 1).

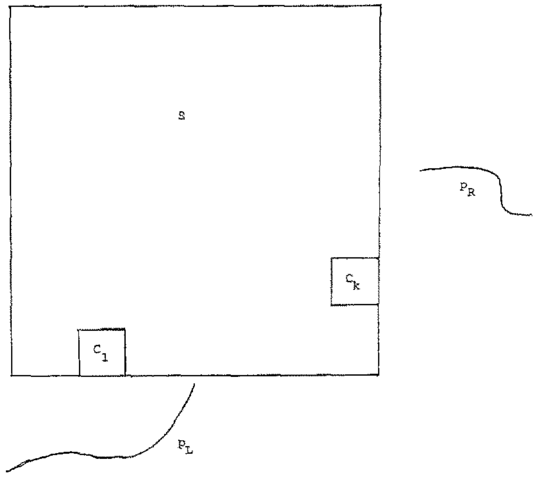

Fig. 2. The cells C_1 and C_k in the 'best case' for (5.5) to occur

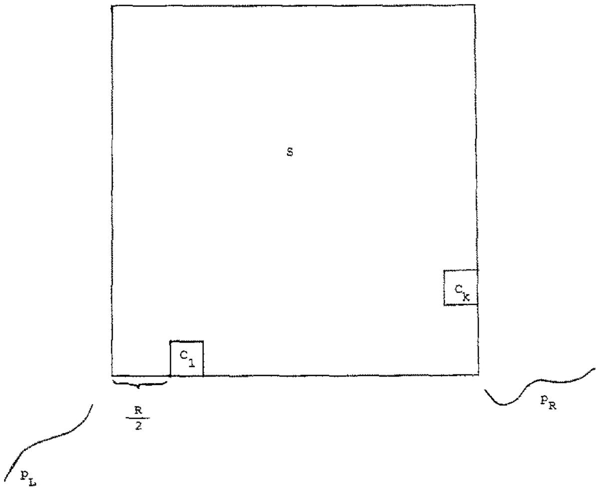

Fig. 3. The cells C_1 and C_k in the 'worst case' for (5.5) to occur

To construct a pivotal cell in S we observe that there exists $\alpha > 0$ such that for any configuration $\omega \in A(S)$

(5.5) there are cells $C_1, \ldots, C_k$ (not necessarily distinct) and indices $\beta_1, \ldots, \beta_k \in [0, \pi)$ (both depending on ω) such that if $\{L_1, \ldots, L_k\}$ is a collection of sticks centred in $C_1, \ldots, C_k$ respectively each of length $\geqq R - \varepsilon$ and L_i having a direction $\theta_i \pmod{\pi}$ for any $\theta_i \in [\beta_i - \alpha, \beta_i + \alpha]$, then (a) $L_1 \cap p_L \neq \emptyset$, $L_k \cap p_R \neq \emptyset$, (b) consecutive L_i's have non-empty intersection and (c) no two L_i's intersect unless they are consecutively numbered.

Indeed, if $A(S)$ occurs then (5.5) occurs; we note that having obtained C_1 and C_k, finding the correct $C_2, \ldots, C_{k-1}$ is easy. Also, for p_L, p_R as in Fig. 2 obtaining the subtending angle α' is no problem. In case p_L or p_R is as in Fig. 3, then also we can find a cell C adjacent to a side of S and at a distance $R/2$ from the corner of S which subtends an angle α''(say). Note, here we use the fact that the path p_L or p_R goes beyond a distance $3R/4$ from the square. In case both p_l and p_R are at a distance $\leqq 3R/4$ from the left edge of the square S, then of course, C_1 can be found easily. Similarly for the right edge. Moreover, Figs. 2, 3, are the 'best' and 'worst' respectively, for (5.5) to occur. Thus the minimum of these two subtending angles would be the value of α we would need for (5.5). A rigorous proof, based on this idea, can be written.

Clearly each of the cells $C_1, \ldots, C_k$ above is pivotal for E. Thus,

$$\begin{aligned} &P_\lambda(\exists \text{ a pivotal } \eta\text{-cell in } S \text{ for the event } E \mid A(S) \text{ occurs}) \\ &\geqq P_\lambda((5.5) \text{ occurs for } S) \\ &\geqq C(\lambda_0, \lambda_1, \eta, R\,\varepsilon), \end{aligned}$$

where the constant $C(\lambda_0, \lambda_1, \eta, R\,\varepsilon) > 0$.

Step 2. Location of the pivotal point.

Given a $L-R$ **L** crossing r of D, let r_i be the lowest $L-R$ **L** crossing of $[0, l] \times [0, 3l+20R]$ $(=\tilde{D}$ say) which is at a distance of at least $4iR+6a_m$ above r.

Let $A\,(=A_m)$ denote the (random) lowest vacant $L-R$ **L** crossing of the rectangle D and let A_i be as above.

Let $Q\,(=Q_m)$ be the event

$Q = \{A$ exists and the following hold

(i) $\exists$ a vacant **L** path t which lies above $\mathbf{L}_2$ and connects the top edge of $\tilde{D}$ to A_2,

(ii) for any point (x, y) in **L** and a $R/16$ neighborhood $N(x, y)$ of (x, y) there exists a vacant path b which lies below A and connects the bottom edge of S to $N(x, y)\}$.

Let (t_1, t_2) be some point on $t \cap A_2$, where t is as in the event Q. By definition of A_2, there is a point $(\tilde{t}_1, \tilde{t}_2)$ on A which is at most at a distance of $8R+16a_m$ away from (t_1, t_2). Since $16a_m \leqq R/2$, so $(9\tilde{t}_1, \tilde{t}_2)$ is in the interior of the four squares $[t_1-9R, t_1+9R] \times [t_2-18R, t_2]$, $[t_1-9R, t_1+9R] \times [t_2, t_2+18R]$, $[t_1-18R, t_1] \times [t_2-9R, t_2+9R]$ and $[t_1, t_1+18R] \times [t_2-9R, t_2+9R]$. W.l.o.g. assume that $(\tilde{t}_1, \tilde{t}_2) \in (t_1-9R, t_1+9R) \times (t_2-18R, t_2)$. Let $\tilde{S}$ be a square with sides of length $24R$ on the η-lattice and such that the square $[t_1-9R, t_1+9R] \times [t_2-18R, t_2]$ is contained in $\tilde{S}$ and its sides are at a distance of at least $2R$ from the sides of $\tilde{S}$.

By our construction, there is a vacant **L** path t' (contained in t) such that t' lies above A_2 and connects the top edge of $\tilde{D}$ to $\tilde{S}$. Also, since $(\tilde{t}_1, \tilde{t}_2)$ and $N(\tilde{t}_1, \tilde{t}_2)$ are both in the interior of $\tilde{S}$, there is a vacant path b' (contained in b) such that b' lies below A and connects the bottom edge of D to $\tilde{S}$.

Now let S be a square with sides on the η-lattice of length $26R$ and such that $\tilde{S} \subseteq S$. The existence of vacant paths t' and b' guarantee that there are no $L-R$ crossing of D lying outside S.

Fix $\omega \in Q$. For a choice of S as above (i), (iii) and (iv) of the definition of $A(S)$ holds. Also, let $d_L(S)$ $(d_R(S))$ be the minimum distance from the square

S to a path from the left (right, respectively) edge of D which lies in D. If $d_L(S)$, $d_R(S) \leqq R/2$ then (ii) in the definition of $A(S)$ holds. Otherwise, let S_1 be a square centred at the center of S, containing S, with sides on the η-lattice and such that either $R/2-\eta \leqq d_L(S) \leqq R/2$ or $R/2-\eta \leqq d_R(S) \leqq R/2$. Assume w.l.o.g. $R/2-\eta \leqq d_L(S) \leqq R/2$. If $d_R(S) \leqq R/2$ then (ii) holds, otherwise, considering a larger square S_2 (whose sides are at a distance of at least $R/2\sqrt{2}$ from each of the sides of S_1) we have both $d_L(S) \leqq R/2$ and $d_R(S) \leqq R/2$. Thus $\omega \in A(S_2)$. Hence if $\omega \in Q$ then $\omega \in A(S)$ for some square S on the η-lattice with sides $\leqq 30R$.

Step 3. Expected number of pivotal cells.

From Step 1 and Step 2 we have

$$E_\lambda\{\#\text{ of pivotal cells for the event } E_n\}$$
$$\geqq \frac{1}{K} \sum_S P_\lambda(\exists \text{ a pivotal cell in } S \text{ for the event } E_n \mid A(S) \text{ occurs})\, P_\lambda(A(S) \text{ occurs})$$
$$\geqq \frac{1}{K}\, C(\lambda_0, \lambda_1, \eta, R, \varepsilon) \sum_S P_\lambda(A(S) \text{ occurs}),$$

where, $K=(30R)^2/\eta^2 \geqq \#$ of squares S on the η-lattice which contain a fixed cell and $\sum_S$ is over all squares S in D which lie on the η-lattice and have sides of length $\geqq 26R$ and $\leqq 30R$. Thus, to prove the lemma, it suffices to show,

$$\inf_{\lambda \in [\lambda_0, \lambda_1]} \sum_S P_\lambda(A_m(S) \text{ occurs}) \to \infty \quad \text{as} \quad n \to \infty.$$

This can be obtained by constructing 'concentric squares' in D as in Chap. 7 of [4]. We provide a sketch of the argument here. Let Γ be the 'left most' vacant $\mathbf{L}_m$ path from the top edge of S to Λ_2. We obtain vacant paths from Γ to Λ_2 each of which lies in different annuli of these concentric squares. Each such vacant path provides a square S containing pivotal cells. Moreover, using the RSW lemma, we can provide a positive lower bound for the probability of the existence of these vacant paths. The number of such vacant paths will be of the order $O(n)$ for the event E_n as $n \to \infty$. Thus, we will have, $P_\lambda(A_m(S) \text{ occurs}) \to \infty$ as $n \to \infty$. □

6 The equality of the critical densities

In view of (4.1), to prove the equality of the critical densities defined in Sect. 2 we need to show

$$\lambda_S = \lambda_S^*. \tag{6.1}$$

We will prove this by contradiction.

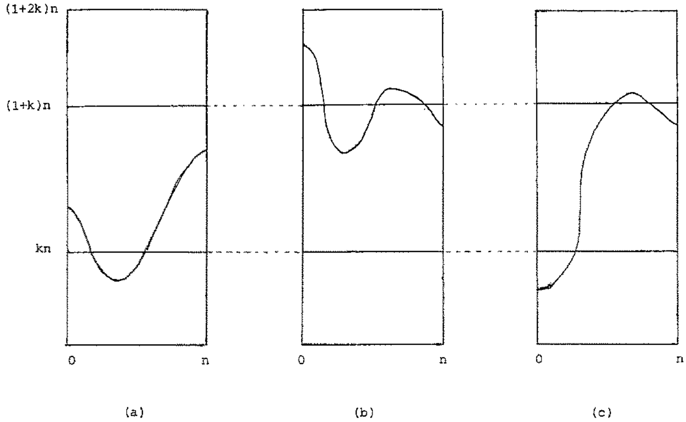

Fig. 4. A $L-R$ vacant crossing of $[0, n] \times [0, (1+2k)n]$ implies (a) a $L-R$ vacant crossing of $[0, n] \times [0, (1+k)n]$ or (b) a $L-R$ vacant crossing of $[0, n] \times [kn, (1+2k)n]$ or (c) a $T-B$ vacant crossing of $[0, n] \times [kn, (1+k)n]$

Lemma 6.1 *Suppose there exists λ_0 and λ_1 such that $\lambda_S < \lambda_0 < \lambda_1 < \lambda_S^*$, then there exists $\delta > 0$ and a sequence $\{l_n\}_{n \geqq 1}$ with $l_n \uparrow \infty$ as $n \to \infty$, such that, for all $n \geqq 1$ and for any $\lambda \in [\lambda_0, \lambda_1]$,*

(6.2) $$\sigma((l_n, 3\,l_n), 1, \lambda) > \delta,$$

(6.3) $$\sigma^*((l_n, 3\,l_n), 2, \lambda) > \delta.$$

Proof. First we note that,

(6.4) *if for some n, $k > 0$ and $\eta > 0$, $\sigma^*((n, (1+2k)n), 1, \lambda) > \eta$, then for any $t > 0$, and for some $f(t, k, \eta) > 0$, $\sigma^*((n, (1+2t)n), 1, \lambda) > f(t, k\eta)$.*

Indeed, from Fig. 4, we see that a $L-R$ vacant crossing of $[0, n] \times [0, (1+2k)n]$ entails either a $L-R$ vacant crossing of $[0, n] \times [0, (1+k)n]$ or a $L-R$ vacant crossing of $[0, n] \times [kn, (1+2k)n]$ or a $T-B$ vacant crossing of $[0, n] \times [kn, (1+k)n]$. By translation invariance and the FKG inequality we have, for each of these cases, $\sigma^*((n, (1+k)n), 1, \lambda) > 1-(1-\eta)^{1/3}$.

(6.4) is obtained by the repeated use of this technique.

From (6.4), we have that for λ_1, as in Lemma 6.1, there exists $\eta > 0$ and a sequence $\{l_n\}_{n \geqq 1}$ with $l_n \uparrow \infty$ as $n \uparrow \infty$ and $5\,l_{2n-1} > 4\,l_{2n}$ for all $n \geqq 1$, such that, $\sigma^*((l_{2n-1}, l_{2n}), 1, \lambda_1) > \eta$ and $\sigma^*((5\,l_{2n-1/4}, l_{2n}), 2, \lambda_1) > \eta$. This yields on an application of the RSW lemma, that there exists $\delta_1 > 0$ and a sequence $\{l_n\}_{m \geqq 1}$, $l_n \uparrow \infty$ as $n \uparrow \infty$, such that,

(6.5) $$\sigma^*((l_n, 3\,l_n), 2, \lambda_1) > \delta_1 \quad \text{for all } n.$$

Now if

(6.6) $$\limsup_n \sigma((l_n, 3\,l_n), 1, \lambda_1) = 0,$$

then, by Lemma 4.1, $E_\lambda(|W(\mathbf{0})|)<\infty$, which yields $\lambda_1 \leqq \lambda_T$. But, by (4.1), $\lambda_S=\lambda_T$, so (6.6) cannot be true. In other words, there exists $\delta_2>0$ and a subsequence $\{l'_n\}$ of $\{l_n\}$ such that

$$\sigma((l'_n, 3\,l'_n), 1, \lambda_1)>\delta_2 \quad \text{for all } n, \tag{6.7}$$

and (6.5) holds for δ_2 instead of δ_1.

If, for some subsequence $\{l''_n\}$ of $\{l'_n\}$, there exists a constant $\delta_3>0$ such that $\sigma((l''_n, 3\,l''_n), 1, \lambda_0)>\delta_3$ and $\sigma^*((l''_n, 3\,l''_n), 2, \lambda_0)>\delta_3$, then the monotonicity properties of the crossing probabilities in λ imply the lemma for $\delta=\min(\delta_1, \delta_2, \delta_3)$. Otherwise, for any subsequence $\{l''_n\}$ of $\{l'_n\}$, $\sigma^*((l''_n, 3\,l''_n), 2, \lambda_0)>\sigma^*((l''_n, 3\,l''_n), 2, \lambda_1)>\delta_1$, and $\limsup_n \sigma((l''_n, 3\,l''_n), 1, \lambda_0)=0$. Another application of Lemma 4.1 yields $\lambda_0 \leqq \lambda_T$, thereby providing a contradiction. □

To complete the proof of (6.1), suppose λ_0, λ_1, $\{l_n\}$ and δ are as in Lemma 6.1. We show that for any $\lambda \in (\lambda_0, \lambda_1)$,

$$\sigma((l_n, 3\,l_n), 1, \lambda) \to 0 \quad \text{as} \quad n \to \infty. \tag{6.8}$$

(6.8) clearly contradicts (6.2) and would thus prove (6.1).

To show (6.8) we use the pivotal formula. Let $\mathbf{L}$ be a lattice of size η, where $0<\eta^2<\lambda_1^{-2}$ (0; $\lambda_H=\lambda_S<\infty$, so this is possible). Let $E_n=\{\exists$ a $L-R$ occupied crossing of $[0, l_n]\times[0, 3\,l_n]\}$ and $N(n)=\#$ of pivotal cells in $\mathbf{L}$ for (ω, E_n). Then, by Corollary 3.1, for $0 \leqq t \leqq 1$ and $\lambda(t)=t\,\lambda_1+(1-t)\,\lambda_0$,

$$\frac{d}{dt} P_{\lambda(t)}(E_n) \geqq (\lambda_1-\lambda_0)\left(\frac{1}{\lambda_1}-\eta^2\right) E_{\lambda(t)}(N(n)). \tag{6.9}$$

By our choice of λ and $\{l_n\}$ satisfying Lemma 6.1, (6.9) yields

$$(P_{\lambda(t)}(E_n))^{-1} \frac{d}{dt} P_{\lambda(t)}(E_n) \geqq (\lambda_1-\lambda_0)\left(\frac{1}{\lambda_1}-\eta^2\right) E_{\lambda(t)}(N(n)\,|\,E_n).$$

This yields, on integrating,

$$P_{\lambda_0}(E_n) \leqq P_{\lambda_1}(E_n) \exp\left(-\alpha \int_0^1 E_{\lambda(t)}(N(n)\,|\,E_n)\right), \tag{6.10}$$

where $\alpha=(\lambda_1-\lambda_0)\left(\frac{1}{\lambda_1}-\eta^2\right)>0$.

(6.2) and (6.3) imply by Lemma 5.1 that $\inf_{\lambda \in [\lambda_0, \lambda_1]} E_\lambda(N(n)) \to \infty$ as $n \to \infty$. Also, $P_\lambda(E_n)=\sigma((l_n, 3\,l_n), 1, \lambda)>\delta$ for all $\lambda \in [\lambda_0, \lambda_1]$. Thus,

$$\inf_{\lambda \in [\lambda_0, \lambda_1]} E_\lambda(N(n)\,|\,E_n) \to \infty \quad \text{as} \quad n \to \infty. \tag{6.11}$$

(6.11) implies that the term on the right of (6.10) goes to 0 as $n \to \infty$. This shows that (6.8) holds, thereby proving Theorem 2.1.

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