Probab. Theory Relat. Fields 102, 567 – 573 (1995)

Probability Theory and Related Fields

A lim inf result for the increments of the Wiener process under the L_2-norm

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Received: 31 August 1994/In revised form: 14 March 1995

Summary. We prove an almost sure lower limit law for the square integral of the large increments of the Wiener process, extending results obtained by Li (1992).

Mathematics Subject Classification: 60F15, 60J65

1 Introduction

Let $(W(t),\ t \geqq 0)$ be a standard Wiener process. For $a < T$ define the increments

$$Y_T(t) = W(t+a) - W(t), \quad t \leqq T - a\,.$$

These increments have been extensively studied, in particular the quantities

$$M(T) = \sup_{t \leqq T-a} |Y(t)|\,,$$

see, e.g., [3] and the references therein.

We may consider $M(T)$ as a norm of the process Y_T. Looking at things that way, one may be tempted to try other norms, and one that immediately springs to mind is the square norm. This will be the subject of the present paper. In fact, we will give a lower limit law for the integral

$$\int_0^{T-a(T)} (W(t+a(T)) - W(t))^2\, dt\,,$$

where $a(T)$ will be some function tending to infinity.

★ Work supported by an NSERC Foreign Researcher Award at Carleton University

This question has been studied by Li [4]. His methods rest on an interesting comparison theorem, and depends on an eigenvalue computation which unfortunately breaks down when $a(T) < T/2$. His theorem states as follows:

Theorem A *If* $\theta \geqq \beta - \alpha > 0$ *and* $\alpha \geqq 0$ *then*

$$\liminf \frac{\log\log T}{T^2} \int_{\alpha T}^{\beta T} (W(t+\theta T) - W(T))^2 dt = \frac{(\beta-\alpha)^2}{4} \quad a.s.$$

We use a different approach based on a very useful inequality by Anderson [1] which will close the gap left by Theorem A. Our theorem states as follows:

Theorem 1 *Suppose that* $0 < \rho < 1$ *and that* $a(T)$ *is a nondecreasing function with* $a(T)/T \to \rho$. *Then*

$$\liminf \frac{\log\log T}{T^2} \int_0^{T-a(T)} (W(t+a(T)) - W(t))^2 dt = \gamma(\rho) \quad a.s.,$$

where

$$\gamma(\rho) = \frac{\rho^2}{8}\left(\theta \cot\frac{\pi}{4(m+1)} + (1-\theta)\cot\frac{\pi}{4m} - 1\right)^2$$

with

$$m = \left\lfloor \frac{1}{\rho} \right\rfloor, \quad \theta = \frac{1}{\rho} - m\,.$$

The same methods used in the proof of Theorem 1 can be used to prove the following theorem which we state without proof:

Theorem 2 *If* $a(T)$ *is nondecreasing,* $T/a(T)$ *is nondecreasing, and*

$$\frac{T}{a(T)\log\log T} \downarrow 0$$

then

$$\liminf \frac{\log\log T}{T^2} \int_0^{T-a(T)} (W(t+a(T)) - W(t))^2 dt = \frac{2}{\pi^2} \quad a.s.,$$

2 Proof of Theorem 1

The proof of the theorem rests on a good estimate for the probability

$$\mathbf{P}\left(\int_0^{T-a(T)} (W(t+a(T)) - W(t))^2 dt < x\right).$$

This will be provided by the following.

Lemma 1 *If m is a nonnegative integer an* $0 \leqq \theta \leqq 1$ *then*

$$\lim_{x\to 0} x \log\left(\mathbf{P}\left(\int_0^{m+\theta} (W(t+1)) - W(t))^2 dt < x\right)\right)$$

$$= \frac{1}{8}\left(\theta \cot \frac{\pi}{4(m+1)} + (1-\theta)\cot\frac{\pi}{4m} - 1\right)^2 .$$

Once this lemma is proven, Theorem 1 follows by a rescaling argument along with an application of the Borel–Cantelli lemma that seems to be well understood, so we will forgo that part of the proof.

It remains to prove Lemma 1. To this end, consider

$$\int_0^{m+\theta} (W(t+1) - W(t))^2 dt = \sum_{i=0}^{m}\int_0^{\theta} (W(t+i+1) - W(t+i))^2 dt$$

$$+ \sum_{i=0}^{m-1}\int_0^{1-\theta} (W(i+\theta+t+1) - W(i+\theta+t))^2 dt .$$

Now, for $0 \leqq t \leqq \theta$,

$$W(i+t) = W(i)\frac{\theta - t}{\theta} + W(i+\theta)\frac{t}{\theta} + \sqrt{\theta} B_i\left(\frac{t}{\theta}\right) = Y_i(t) + \sqrt{\theta} B_i\left(\frac{t}{\theta}\right) ,$$

and for $0 \leqq t \leqq 1-\theta$,

$$W(i+\theta+t) = W(i+\theta)\frac{1-\theta-t}{1-\theta} + W(i+1)\frac{t}{1-\theta} + \sqrt{1-\theta}\tilde{B}_i\left(\frac{t}{1-\theta}\right)$$

$$= \tilde{Y}_i(t) + \sqrt{1-\theta}\tilde{B}_i\left(\frac{t}{1-\theta}\right) ,$$

where B_i and $\tilde{B}_i$ are independent sequences of independent Brownian bridges that are independent of Y_i and $\tilde{Y}_i$.

This implies that

$$\int_0^{m+\theta} (W(t+1) - W(t))^2 dt$$

$$= \sum_{i=0}^{m}\int_0^{\theta}\left(Y_{i+1}(t) - Y_i(t) + \sqrt{\theta}\left(B_{i+1}\left(\frac{t}{\theta}\right) - B_i\left(\frac{t}{\theta}\right)\right)\right)^2 dt$$

$$+ \sum_{i=0}^{m-1}\int_0^{1-\theta}\left(\tilde{Y}_{i+1}(t) - \tilde{Y}_i(t) + \sqrt{1-\theta}\left(\tilde{B}_{i+1}\left(\frac{t}{1-\theta}\right) - \tilde{B}_i\left(\frac{t}{1-\theta}\right)\right)\right)^2 dt .$$

Now, Theorem 2 from Anderson [1] implies that

$$\mathbf{P}\left(\int_0^{m+\theta} (W(t+1)-W(t))^2 dt < x\right)$$

$$\leqq \mathbf{P}\left(\theta \sum_{i=0}^{m}\int_0^{\theta}\left(B_{i+1}\left(\frac{t}{\theta}\right)-B_i\left(\frac{t}{\theta}\right)\right)^2 dt + (1-\theta)\right.$$

$$\left.\times \sum_{i=0}^{m-1}\int_0^{1-\theta}\left(\tilde{B}_{i+1}\left(\frac{t}{1-\theta}\right)-\tilde{B}_i\left(\frac{t}{1-\theta}\right)\right)^2 dt < x\right).$$

On the other hand, if we let

$$I_1 = \theta \sum_{i=0}^{m}\int_0^{\theta}\left(B_{i+1}\left(\frac{t}{\theta}\right)-B_i\left(\frac{t}{\theta}\right)\right)^2 dt + (1-\theta)$$

$$\times \sum_{i=0}^{m-1}\int_0^{1-\theta}\left(\tilde{B}_{i+1}\left(\frac{t}{1-\theta}\right)-\tilde{B}_i\left(\frac{t}{1-\theta}\right)\right)^2 dt$$

and

$$I_2 = \sum_{i=0}^{m}\int_0^{\theta}(Y_{i+1}(t)-Y_i(t))^2 dt + \sum_{i=0}^{m-1}\int_0^{1-\theta}(\tilde{Y}_{i+1}(t)-\tilde{Y}_i(t))^2 dt\,,$$

then

$$\int_0^{m+\theta}(W(t+1)-W(t))^2 dt \leqq (\sqrt{I_1}+\sqrt{I_2})^2\,,$$

so

$$\mathbf{P}\left(\int_0^{m+\theta}(W(t+1)-W(t))^2 dt < x\right) \geqq \mathbf{P}(I_1 < x(1-\varepsilon)^2)\mathbf{P}(I_2 < x\varepsilon^2)\,.$$

The last probability can be estimated by

$$\mathbf{P}(I_2 < x\varepsilon^2)$$

$$\geqq \mathbf{P}\left(|W(i+\theta)-W(i)| < \frac{\varepsilon\sqrt{x}}{\sqrt{2(m+1)}},\right.$$

$$\left.|W(i+\theta)-W(i+1)| < \frac{\varepsilon\sqrt{x}}{\sqrt{2(m+1)}},\ 0 \leqq i \leqq m\right)$$

$$\geqq (Cx\varepsilon^2)^{m+1}\,,$$

with some constant C. This probability is negligible as compared to the main term which comes from I_1. We have

$$I_1 = \theta^2 \sum_{i=0}^{m}\int_0^1 (B_{i+1}(t)-B_i(t))^2 dt + (1-\theta)^2 \sum_{i=0}^{m-1}\int_0^1(\tilde{B}_{i+1}(t)-\tilde{B}_i(t))^2 dt\,.$$

Let now

$$Z_i(t) = B_{i+1}(t) - B_i(t) .$$

The processes $Z_i(t)$ have correlation structure

$$\mathbf{E}(Z_i(t)Z_j(s)) = (2\delta_{ij} - \delta_{|i-j|,1})(s \wedge t - st) ,$$

where δ is Kronecker's delta. Consider the $(m+1) \times (m+1)$ matrix A with entries

$$A_{ij} = 2\delta_{ij} - \delta_{|i-j|,1} .$$

This matrix has eigenvalues

$$\lambda_j = \left(2 \sin \frac{\pi j}{m+2}\right)^2 \quad (j = 1, \ldots, m+1)$$

with corresponding eigenvectors

$$x_j = (x_{j1}, \ldots, x_{j,m+1}) ,$$

where

$$x_{jl} = \frac{1}{\sqrt{m+2}} \sin \frac{\pi j l}{m+2} .$$

If we let

$$V_l(t) = \sum_{i=1}^{m+1} x_{il} Z_{i-1}(t) ,$$

then

$$\sum_{i=1}^{m+1} V_l^2(t) = \sum_{i=0}^{m} Z_i^2(t) ,$$

and

$$\mathbf{E}(V_l(t)V_j(s)) = \delta_{lj}\lambda_l(s \wedge t - st) .$$

This means that the processes

$$U_l(t) = \lambda_l^{-1/2} V_l(t)$$

are independent Brownian bridges. Carrying out the same analysis for $\tilde{B}_i$, we get (replacing m by $m-1$) eigenvalues $\tilde{\lambda}_l$ and independent Brownian bridges $\tilde{U}_l$. With these,

$$I_2 = \theta^2 \sum_{l=1}^{m+1} \lambda_l \int_0^1 U_l^2(t)dt + (1-\theta)^2 \sum_{l=1}^{m} \tilde{\lambda}_l \int_0^1 \tilde{U}_l^2(t)dt .$$

Now, by Anderson and Darling [2]:

Lemma 2

$$\mathbf{P}\left(\int_0^1 B^2(t)dt \leqq x\right) = \frac{4}{\sqrt{4\pi}} \exp\left(-\frac{1}{8x}\right)(1 + o(1))$$

as $x \to 0$.

In addition, we need the following simple fact:

Lemma 3 *If X and Y are independent nonnegative random variables with*

$$\lim_{x\to 0} x \log \mathbf{P}(X < x) = -A$$

and

$$\lim_{x\to 0} x \log \mathbf{P}(Y < x) = -B\,,$$

then

$$\lim_{x\to 0} x \log \mathbf{P}(X + Y < x) = -(\sqrt{A} + \sqrt{B})^2\,.$$

For the proof, first observe that for $\varepsilon > 0$ and x small enough:

$$\mathbf{P}(X < x) \geqq \exp\left(-\frac{A}{x}(1+\varepsilon)\right)$$

and

$$\mathbf{P}(Y < x) \geqq \exp\left(-\frac{B}{x}(1+\varepsilon)\right)\,.$$

Thus

$$\mathbf{P}(X + Y < x) \geqq \mathbf{P}\left(X < \frac{x\sqrt{A}}{\sqrt{A}+\sqrt{B}}\right)\mathbf{P}\left(Y < \frac{x\sqrt{B}}{\sqrt{A}+\sqrt{B}}\right)$$

$$\geqq \exp\left(-\frac{(\sqrt{A}+\sqrt{B})^2}{x}(1+\varepsilon)\right)\,.$$

On the other hand, again for x small enough,

$$\mathbf{P}(X < x) \leqq \exp\left(-\frac{A}{x}(1-\varepsilon)\right)$$

and

$$\mathbf{P}(Y < x) \leqq \exp\left(-\frac{B}{x}(1-\varepsilon)\right)\,.$$

Thus, if we choose $M > 1/\varepsilon$,

$$\mathbf{P}(X + Y < x) \leqq \sum_{k=0}^{M-1} \mathbf{P}\left(X < \frac{(k+1)x}{M}\right)\mathbf{P}\left(Y < \frac{(M-k)x}{M}\right)$$

$$\leqq M \exp\left(-\frac{(\sqrt{A}+\sqrt{B})^2}{x}(1-\varepsilon)\right)^2\,.$$

These inequalities, together with the fact that ε can be arbitrarily chosen, prove the lemma.

By repeated application of the last two lemmas, we have

$$x \log \mathbf{P}(I_1 < x) \to \frac{1}{8}\left(\theta \sum_{i=1}^{m+1} \sqrt{\lambda_i} + (1-\theta) \sum_{i=1}^{m} \sqrt{\tilde{\lambda}_i}\right)^2$$

$$= -\frac{1}{8}\left(\theta \cot \frac{\pi}{4(m+2)} + (1-\theta) \cot \frac{\pi}{4(m+1)} - 1\right)^2 .$$

This finally proves Lemma 1.

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