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Probability Theory and Related Fields

Stopping distributions for right processes

Neil Falkner[1,*] **and P.J. Fitzsimmons**[2,**]

[1] Department of Mathematics, The Ohio State University, Columbus, OH 43210-1174, USA
[2] Department of Mathematics, University of California, San Diego La Jolla, CA 92093-0112, USA

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Summary. Let X be a transient right process for which semipolar sets are polar. We characterize the measures which can arise as the distribution of X_T with T a non-randomized stopping time.

1. Introduction

Given a Markov process X with initial distribution μ and given a second measure ν, it is natural to ask when there exists a stopping time T such that the distribution of X_T is ν. One may or may not allow the stopping time to be randomized. (A *non-randomized* stopping time is a stopping time of the natural filtration of (X_t), whereas a *randomized* stopping time is a stopping time of the natural filtration of $((X_t, \Gamma))$ where Γ is a random variable uniformly distributed in $[0, 1]$ and independent of (X_t). In each case, the filtration is assumed to have been completed in the usual way.) In this paper, we shall be interested in conditions under which T may be taken to be non-randomized.

Let us consider some examples to illustrate the issues involved. First suppose X is Brownian motion in $\mathbb{R}^d$ starting from 0. Consider the probability measure ν on $\mathbb{R}^d$ that has half of its mass at 0 and half of its mass uniformly distributed on $A = \{x \in \mathbb{R}^d : \|x\| = 1\}$. Then there is an obvious randomized stopping time R such that the distribution of X_R is ν: Namely, $R = T(\Gamma)$ where

$$T(\gamma) = \begin{cases} 0 & \text{if } 0 \leqq \gamma < \frac{1}{2}, \\ D_A & \text{if } \frac{1}{2} \leqq \gamma \leqq 1, \end{cases}$$

and where $D_A = \inf\{t \geqq 0 : X_t \in A\}$.

If $d = 1$, then there is also an obvious non-randomized stopping time T such that the distribution of X_T is ν: We may take T to be the first time X visits $A \cup \{0\}$

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after visiting $\{-\frac{1}{2}, \frac{1}{2}\}$. However, if $d \geqq 2$, then there is no non-randomized stopping time T such that the distribution of X_T is ν: If T were such a time, then since $\{0\}$ is polar when $d \geqq 2$, we would have $P(T=0) = P(X_T = 0) = \frac{1}{2}$ whereas, by the Blumenthal 0–1 law, $P(T=0) = 0$ or 1. Now consider the probability measure ν on $\mathbb{R}^d$ that has mass p_n uniformly distributed on $A_n = \{x \in \mathbb{R}^d : \|x\| = 2^{-n}\}$ where $p_n > 0$ and $\sum p_n = 1$. Once again, there is an obvious randomized stopping time R such that the distribution of X_R is ν. However, it is less obvious that there is a non-randomized stopping time T such that the distribution of X_T is ν. See [**Fa80**, Example 2.10] for an elementary construction of a suitable T. It turns out that when X is Brownian motion in $\mathbb{R}^d$ starting from 0, if ν is a probability measure on $\mathbb{R}^d$ which is the distribution of X_R for some randomized stopping time R and if $\nu(\{0\}) = 0$, then ν is the distribution of X_T for some non-randomized stopping time T. It is instructive to compare this with what happens when X is uniform motion to the right in $\mathbb{R}$ starting from 0. In this case, given a probability measure ν on $\mathbb{R}$, if ν lives on $[0, \infty)$; then ν is the distribution of X_R for some randomized stopping time R, but if ν is the distribution of X_T for some non-randomized stopping time T, then ν is the unit point mass at x for some $x \geqq 0$. Note that Brownian motion satisfies Hunt's hypothesis H that semipolar sets are polar, whereas uniform motion to the right does not. For uniform motion to the right, the semipolar sets are precisely the countable subsets of $\mathbb{R}$, but only the empty set is polar.

Skorokhod [**Sk60, Sk65**] was the first to consider the stopping distribution problem. He showed that when X is Brownian motion in $\mathbb{R}$ starting from 0, if ν is a probability measure on $\mathbb{R}$ with mean 0 and finite variance, then ν is the distribution of X_R for some randomized stopping time R with $E(R) < \infty$ (and conversely). (As J.L. Doob observed in a postcard to P.A. Meyer, the stopping distribution problem for Brownian motion in $\mathbb{R}$ is trivial unless some condition is imposed on the size of the stopping time: Any probability measure ν on $\mathbb{R}$ is expressible as the distribution of $f(X_1)$ for a suitable Borel function $f: \mathbb{R} \to \mathbb{R}$ and hence as the distribution of X_T where $T = \inf\{t \geqq 1 : X_t = f(X_1)\}$. This sort of complication does not arise in the transient case.) Rost [**R70, R71, R73**] showed how to use potential theory to extend Skorokhod's result to a general strong Markov process satisfying mild regularity conditions; say a right process. We shall now give the precise statement of one of Rost's results.

Let X be a right process, as defined in [**Sh88**]. In particular, the state space E of X is assumed to be homeomorphic to a universally measurable subspace of a compact metric space. As usual, we write (P_t) for the transition function of X and U for the potential kernel of X. Thus $P_t(x, dy) = P^x(X_t \in dy)$, $U(x, dy) = \int_0^\infty P_t(x, dy)\,dt$, and if μ is a measure on E, then its potential μU is the measure on E given by $\mu U(dy) = \int_E \mu(dx) U(x, dy)$. If T is a random time, then μP_T denotes the measure on E which is the distribution of X_T under P^μ: $\mu P_T(dx) = P^\mu(X_T \in dx)$. Note that in forming μP_T, we discard any mass that X_T puts at the cemetery point Δ.

(1.1) Theorem. (*Rost* [**R70**]) *Let μ be a measure on E such that μU is σ-finite. Let ν be another measure on E. If $\mu U \geqq \nu U$, then there is a randomized stopping time R such that $\mu P_R = \nu$ (and conversely).*

The proof of the converse part is a simple application of the strong Markov property: Given such a time T, if f is any Borel function $\geqq 0$ on E, then $\mu U(f) = P^\mu(\int_0^\infty f(X_t)\,dt) \geqq P^\mu(\int_T^\infty f(X_t)\,dt) = P^\nu(\int_0^\infty f(X_t)\,dt) = \nu U(f)$. We should mention that we have stated Rost's theorem in slightly greater generality than he did, but no significant change in his proof is needed to establish (1.1). We remark that the assumption that μU is σ-finite is a type of transience hypothesis: It implies that there is a Borel function $f > 0$ on E such that $\mu U(f) < \infty$. Then μ and ν both live on $F = \{Uf < \infty\}$, F is absorbing, and the restriction of X to F is transient in the usual sense; see for instance **[G80]**. Let us also remark that recently the second-named author **[Fi91]** has shown that in Theorem (1.1), the time R may be taken to be of the form $R = T(\Gamma)$ where for $0 \leqq \gamma \leqq 1$, $T(\gamma) = \inf\{t \geqq 0; X_t \in B(\gamma)\}$ and where $B(\gamma)$, $0 \leqq \gamma \leqq 1$, is a decreasing family of finely closed sets. Moreover an R of this form is P^μ-essentially unique and there is a simple explicit description of a suitable family $B(\gamma)$, $0 \leqq \gamma \leqq 1$.

The stopping distribution problem with non-randomized stopping times has been studied by various authors, including Dubins **[Du68]**, Root **[Ro69]**, Baxter and Chacon **[BC74]**, Rost **[R76]**, Falkner **[Fa80, Fa81, Fa83]**, and Paul Chacon **[Ch85]**. In **[Ro69]**, **[R76]**, and **[Ch85]**, it was shown that under various special hypotheses on X, T may be taken to be of the form $T = \inf\{t \geqq 0: (t, X_t) \in B\}$ where B is a suitable subset of $[0, \infty) \times E$. **[Ro69]** and **[R76]** considered the case where B is a *barrier* (i.e., where $(t, x) \in B$ and $t < s$ imply $(s, x) \in B$) and **[Ch85]** considered the case where B is a *reverse barrier* (i.e., where $(t, x) \in B$ and $0 \leqq s < t$ imply $(s, x) \in B$). In this paper, we do not try to construct a T of any special form. Our aim is simply to prove existence of T under hypotheses more general than had previously been considered.

Once again, let X be a right process as defined in **[Sh88]**. Let $\mathscr{E}^e$ denote the σ-algebra on E generated by $\{f: f$ is q-excessive for some $q \geqq 0\}$. Then $\mathscr{E} \subseteq \mathscr{E}^e \subseteq \mathscr{E}^*$ where $\mathscr{E} = \text{Borel}(E)$ and $\mathscr{E}^*$ is the universal completion of $\mathscr{E}$. For a general right process, the excessive functions need not be nearly Borel and it is appropriate to replace the σ-algebra of nearly Borel sets by $\mathscr{E}^e$. We now state our main result.

(1.2) Theorem. *Let m be an excessive measure for X and assume that each $\mathscr{E}^e$-measurable semipolar set is m-polar. Let μ be a measure on E such that μU is σ-finite and $\mu U \ll m$. Let ν be another measure on E. If*

(1.3) $$\mu U \geqq \nu U$$

and

(1.4) *there exists a set $C \in \mathscr{E}^*$ such that for each m-polar set $Z \in \mathscr{E}^e$*

we have $\nu(Z) = \mu(Z \cap C)$,

then there is a non-randomized stopping time T such that $\mu P_T = \nu$ (and conversely).

For the proof of the converse part, we have already seen the necessity of (1.3). To see the necessity of (1.4), suppose we have such a time T. Since T is non-randomized, we have $P^x(T = 0) = 0$ or 1 for each $x \in E$, by the Blumenthal 0–1 law. Let $C = \{x \in E: P^x(T = 0) = 1\}$. Then $C \in \mathscr{E}^*$. If $Z \in \mathscr{E}^e$ is m-polar, then Z is μU-polar (since $\mu U \ll m$) so Z is μ-polar, so $\nu(Z) = \mu P_T(Z) = P^\mu(X_T \in Z) = P^\mu(X_T \in Z, T = 0) = \mu(Z \cap C)$.

If semipolars are μ-polar, one can take $m = \mu U$ in Theorem (1.2) and since m-polars and μ-polars are then the same, this results in an apparent weakening of the hypotheses of the theorem. However, in many situations there is a natural choice of the measure m; viz. Lebesgue measure for a Lévy process or Wiener measure for the infinite-dimensional Ornstein–Uhlenbeck process. These cases are discussed further below.

Theorem (1.2) generalizes the main result of [**Fa83**] in two ways. First, X is not required to have a reference measure. Second, X is not required to have a standard dual process. Our proof of Theorem (1.2) does use duality but, at least when X is a Borel right process, there always exists a "left process" $\hat{X}$ which is in weak duality with X with respect to a given excessive measure m. $\hat{X}$ is constructed from X by time-reversal.

The hypothesis in Theorem (1.2) that $\mu U \ll m$ is satisfied if and only if $\mu(G) = 0$ for each $\mathscr{E}^e$-measurable, finely open, m-polar set G. (The forward implication is trivial and the reverse implication is easy; see for instance [**Fi90**, p. 257, bottom].)

If X is a Lévy process, then its σ-ideal of sets of potential zero is translation invariant so if it has a reference measure, then Lebesgue measure is a reference measure; see for instance [**V70**, p. 19]. However, there do exist nontrivial examples of Lévy processes, even symmetric ones, which do not have reference measures; see [**Ha79**, p. 340]. If X is a symmetric Lévy process and m is Lebesgue measure, then semipolar sets are m-polar even if m is not a reference measure; see for instance [**FG88**, (2.9)]. More generally, if m is Lebesgue measure and X is Lévy process satisfying the "sector condition", then semipolar sets are m-polar even if m is not a reference measure; see [**Si77**] or [**Fi89**].

Another example of a process to which Theorem (1.2) applies is the infinite-dimensional Ornstein–Uhlenbeck process. For this process, $E = \{x \in C[0,1]: x(0) = 0\}$ and m is Wiener measure on E. The process (X_t) under P^x is equal in law to $(e^{-t/2}(x + B(e^t - 1)))$ where B is the Brownian motion in E associated with the Gaussian measure m. B starts from 0, has continuous paths and stationary independent increments, and the distribution of B_t is the image of m under the map $x \mapsto \sqrt{t}\, x$. The process X is symmetric with respect to m. (See [**M82**] for details concerning the preceding assertions in this paragraph.) Because of this symmetry, semipolar sets are m-polar; see for instance [**FG88**, (2.9)] again. Also, m is invariant and hence excessive for X. But, as we shall now explain, X does not have a reference measure. We are grateful to René Carmona and Bruce Driver for helpful discussions concerning this point. We begin by recalling the basic facts about the reproducing kernel Hilbert space H of m. E is a separable Banach space with norm $\|x\|_E = \sup\{|x(t)|: 0 \leqq t \leqq 1\}$ and m is a Gaussian measure on E. No proper closed linear subspace of E carries m. For some $\alpha > 0$, we have $\int_E \exp(\alpha\|x\|^2)m(dx) < \infty$. (This holds for any Gaussian measure: see [**Fe70**].) The dual space $E^* \subseteq L^2(m)$. Let H^* be the closure of E^* in $L^2(m)$ and define $J: H^* \to E$ by $J(y^*) = \int_E xy^*(x)m(dx)$. The integral exists in the sense of Bochner, and J is one-to-one. By definition, $H = J(H^*)$, with the norm $\|J(y^*)\|_H = \|y^*\|_{H^*}$. H^* may be viewed as the dual of H via the pairing

$$_{H^*}\langle x^*, J(y^*)\rangle_H = \int_E x^*(x)y^*(x)m(dx)\,.$$

In case $x^* \in E^*$, we have

$$_{H^*}\langle x^*, J(y^*)\rangle_H = x^*(J(y^*))\,.$$

Let $\beta_t(x) = x(t)$ for $x \in E$ and $0 \leqq t \leqq 1$, so that under m, $(\beta_t)_{0 \leqq t \leqq 1}$ is Brownian motion in $\mathbb{R}$ starting from 0. Suppose $x^* \in E^*$. Then x^* is represented by a real-valued Borel measure μ on $]0, 1]$. Integrating by parts, we get

$$x^* = \int_0^1 f(t)\,\mathrm{d}\beta_t \quad \text{(stochastic integral)}$$

where $f(t) = \mu]t, 1]$ for $0 \leqq t \leqq 1$. It is well-known that for $g \in L^2[0, 1]$, $\| g \|_{L^2[0,1]} = \| \int_0^1 g(t)\,\mathrm{d}\beta_t \|_{L^2(m)}$. It follows that $g \mapsto \int_0^1 g(t)\,\mathrm{d}\beta_t$ is a linear isometry of $L^2[0, 1]$ onto H^*. Suppose $g \in L^2[0, 1]$. Let $y^* = \int_0^1 g(t)\,\mathrm{d}\beta_t$ and let $\xi(s) = \int_0^s g(t)\,\mathrm{d}t$ for $0 \leqq s \leqq 1$. Note that $\xi \in E$. It is easy to check that $x^*(J(y^*)) = \int_0^1 \xi(s)\mu(\mathrm{d}s) = x^*(\xi)$. Thus $J(y^*) = \xi$ and $H = \{x \in E$: x is absolutely continuous and $x' \in L^2[0, 1]\}$. Now we proceed to the proof that X does not have a reference measure. Note that $m(H) = 0$ since Brownian paths have infinite variation. If $x \in E \backslash H$, then there is a Borel measurable linear subspace $L \subseteq E$ such that $x \notin L$, $H \subseteq L$, and $m(L) = 1$: See for instance [**Ca80**, Proposition 1]. Next, if F is a finite dimensional linear subspace of E such that $F \cap H = \{0\}$, then there is a Borel measurable linear subspace $L \subseteq E$ such that $F \cap L = \{0\}$, $H \subseteq L$, and $m(L) = 1$. This follows by induction on the dimension of F. For suppose $x \in F$ and $x \neq 0$. Then $x \in E \backslash H$ so there is a Borel measurable linear subspace $L_0 \subseteq E$ such that $x \notin L_0$, $H \subseteq L_0$, and $m(L_0) = 1$. Let $F_1 = F \cap L_0$. Then $\dim F_1 < \dim F$, so by the inductive hypothesis, there is a Borel measurable linear subspace $L_1 \subseteq E$ such that $F_1 \cap L_1 = \{0\}$, $H \subseteq L_1$, and $m(L_1) = 1$. Then $L = L_0 \cap L_1$ works for F. It follows that the same conclusion holds if the dimension of F is countably infinite. Hence $\dim(E/H)$ is uncountable because $m(H) = 0$. If L is a Borel measurable linear subspace of E such that $m(L) = 1$, then for each $x \in E$, the potential $U(x, \cdot)$ lives on $]0, \infty[\cdot x + L$. (Recall that (X_t) under P^x is equal in law to $(\mathrm{e}^{-t/2}(x + B(e^t - 1)))$.) If in addition x, $y \in E$ are linearly independent and $(\operatorname{span}\{x, y\}) \cap L = \{0\}$, then $U(x, \cdot)$ and $U(y, \cdot)$ are mutually singular because $(]0, \infty[\cdot x + L) \cap (]0, \infty[\cdot y + L) = \emptyset$. Hence the family of measures $U(x, \cdot)$, $x \in E$, includes uncountably many pairwise mutually singular measures. Thus X cannot have a reference measure. This argument appears to use the full axiom of choice but it can be rephrased to use only the principle of dependent choice. The key to this rephrasing is that if there were a reference measure, then we would have $\vee\{U(x, \cdot)$: $x \in E\} = \vee\{U(x, \cdot)$: $x \in A\}$ for some countable set $A \subseteq E$.

2. Tools

In this section we collect some of the definitions and results which are used in our proof of Theorem (1.2). For the time being, we continue to assume that X is an arbitrary right process, with state space E. (However, we shall soon specialize to the case of a Borel right process.) Let μ be an s-finite measure on E. (A measure is said to be a *s-finite* when it is the sum of countably many finite measures.) Let $A \in \mathscr{E}^e$. Recall that A is *μ-polar* when $P^\mu(X_t \in A$ for some $t > 0) = 0$. It is easy to see that A is μ-polar if and only if A is μU-polar. If ξ and m are s-finite measures on E with $\xi \ll m$ and if A is m-polar, then A is ξ-polar. Thus if $\mu U \ll m$ and A is m-polar, then A is μ-polar. The following result is a refinement of Proposition 6.5 in [**Hu57**].

(2.1) Lemma. *Let ν be a measure on E such that νU is σ-finite. Suppose $Z \in \mathscr{E}^e$ is ν-polar and $\nu(Z) = 0$. Then there exist an excessive function f and a decreasing sequence (G_n) of $\mathscr{E}^e$-measurable finely open sets such that $f = +\infty$ on $\bigcap_n G_n \supseteq Z$, $\nu(f) < \infty$, and $\nu(f) + \sum_n \nu(P_{G_n} f) < \infty$.*

For a proof of (2.1), see [**Fa83**, Lemma 1]. Once again, let μ be an s-finite measure on E and let $A \in \mathscr{E}^e$. Recall that A is *thin* when $P^x(T_A > 0) = 1$ for each $x \in E$, where $T_A = \inf\{t > 0;\ X_t \in A\}$. A is *semipolar* when A is the union of countably many thin $\mathscr{E}^e$-measurable sets. Hunt showed that if A is semipolar, then $P^\mu(X_t \in A$ for uncountably many $t) = 0$. When A satisfies the latter condition, we shall say A is *μ-semipolar*. Dellacherie showed that if A is μ-semipolar, then $A = B \cup C$ where $B \in \mathscr{E}^e$ is semipolar and $C \in \mathscr{E}^e$ is μ-polar; see [**De69a, De88**]. He also showed that if A is not μ-semipolar, then A carries a nonzero finite measure ρ which does not charge μ-semipolar sets; see [**De69b, De88**].

We now specialize to the case where X is a Borel right process. That is, we suppose X is a right process whose state space E is homeomorphic to a Borel subset of a compact metric space and for each Borel set $A \subseteq E$ and each $t > 0$, $x \mapsto P_t(x, A)$ is Borel measurable on E. In this case, each q-excessive function is nearly Borel, so each $\mathscr{E}^e$-measurable set is nearly Borel. Hence for $A \in \mathscr{E}^e$, if A is μ-polar (respectively, μ-semipolar), then $A \subseteq B$ for some Borel set $B \subseteq E$ such that B is μ-polar (respectively, μ-semipolar). In this context, for an arbitrary set $A \subseteq E$, we define A to be *μ-polar* (respectively, *μ-semipolar*) when $A \subseteq B$ for some Borel set $B \subseteq E$ such that B is μ-polar (respectively, μ-semipolar).

Now fix a measure m on E which is excessive for X. (Recall that then, in particular, m is σ-finite.) Let $(W, (Y_t)_{t\in\mathbb{R}}, Q_m)$ be the associated Kuznetsov process, which exists because X is a Borel right process. There is quite a lot of notation and theory associated with the Kuznetsov process and we shall recall only some of it. For the rest, we refer the reader to [**Fi90**] and to the papers cited there. W is the set of paths $w: \mathbb{R} \to E \cup \{\Delta\}$ that are E-valued and right-continuous on some open interval $]\alpha(w), \beta(w)[\subseteq \mathbb{R}$, taking the value Δ outside this interval. (Δ is a point not belonging to E and is called the "cemetery point." The "dead path" $[\Delta]$: $t \mapsto \Delta$ satisfies $\alpha([\Delta]) = +\infty$ and $\beta([\Delta]) = -\infty$.) $Y_t(w) = w(t)$ for $t \in \mathbb{R}$ and $w \in W$. Under Q_m, (Y_t) is stationary and after time α, (Y_t) is a strong Markov process with transition function (P_t). We may and do assume that the sample space for X is

$$\Omega = \{w \in W: \alpha(w) = 0 \text{ and } Y_{\alpha+}(w) \text{ exists in } E\} \cup \{[\Delta]\}$$

and that $X_t = Y_{t+}|\Omega$ for $t \geqq 0$. Let $l = {}^{\hat{p}}(1_{]\alpha,\beta[})$, the copredictable projection of $1_{]\alpha,\beta[}$. Intuitively, $l_t(w)$ is the conditional probability

$$Q_m(Y_t \in E \mid Y_s = w(s) \text{ for each } s > t)\,,$$

for $t \in \mathbb{R}$ and $w \in W$. One can choose a version of l with the following properties:

(i) l is optional as well as copredictable;

(ii) $0 \leqq l \leqq 1$;

(iii) $l_t \circ \sigma_s = l_{t+s}$ for $s, t \in \mathbb{R}$, where σ_s is the shift operator on W defined by $(\sigma_s w)(t) = w(s+t)$;

(iv) $]\!]\alpha, \beta[\![\subseteq \Lambda \subseteq [\![\alpha, \beta[\![$, and $Y_{t+}(w)$ exists in E for $(t, w) \in \Lambda$, where $\Lambda = \{(t, w) \in \mathbb{R} \times W: l_t(w) > 0\}$.

Now for $t \in \mathbb{R}$, let

$$\bar{Y}_t = \begin{cases} Y_{t+} & \text{if } t = \alpha \text{ and } l_\alpha > 0\,, \\ Y_t & \text{otherwise}\,. \end{cases}$$

The process $\bar{Y}$ is both optional and copredictable. (In contrast, Y is just optional.) For $A \in \mathscr{E}$, it is clear that A is m-polar if and only if $Q_m\,(Y_t \in A$ for some $t \in \mathbb{R}) = 0$, and A is m-semipolar if and only if $Q_m\,(Y_t \in A$ for uncountably many $t \in \mathbb{R}) = 0$. It can be shown that if A is m-polar, then $Q_m\,(\bar{Y}_t \in A$ for some $t \in \mathbb{R}) = 0$; the converse is clear. (In contrast, if A is m-polar, it does not follow that $Q_m\,(Y_{t+} \in A$ for some $t \in \mathbb{R}) = 0$. See **[FG90]** for a discussion of this point.)

There exists a Borel moderate Markov process $\hat{X} = (\hat{X}_t, \hat{P}^x)$ which is in weak duality with X with respect to m. The sample space of $\hat{X}$ is

$$\hat{\Omega} = \{w \in W : \beta(w) = 0\} \cup \{[\Delta]\}\ .$$

For $0 < t < \infty$, we have

$$\hat{X}_t = \bar{Y}_{-t}|\hat{\Omega}\ .$$

(Note that $\hat{X}_0$ is undefined.) For each x, the probability measure $\hat{P}^x$ is defined (at least initially) on the σ-algebra $\hat{\mathscr{F}}^0 = \sigma(\hat{X}_t : 0 < t < \infty)$. Of course $\hat{X}$ and the objects derived from it depend on m, but this dependence is often suppressed in what follows.

Let $\tau = \inf\{t > 0\colon X_t \neq X_0\}$ and for each x, let $\hat{\tau}^x = \inf\{t > 0\colon \hat{X}_t \neq x\}$. Let H and $\hat{H}$ be the sets of holding and coholding points respectively: $H = \{x \in E\colon P^x(\tau > 0) = 1\}$ and $\hat{H} = \{x \in E\colon \hat{P}^x(\hat{\tau}^x > 0) > 0\}$. (Note that $\hat{X}$ need not satisfy the Blumenthal 0–1 law.) It is easy to see that H and $\hat{H}$ are Borel sets. In the context of standard processes in strong duality, a point is holding if and only if it is coholding. In the present more general context, the family $(\hat{P}^x)_{x \in E}$ is determined only up to an m-polar set so the following result is the best we can hope for.

(2.2) Lemma. *The symmetric difference of H and $\hat{H}$ is m-polar.*

Proof. First let us show that $\hat{H} \backslash H$ is m-polar. We proceed by contradiction. Suppose $\hat{H} \backslash H$ is not m-polar. Then by the section theorem, there is a copredictable time T such that $Q_m(\bar{Y}_T \in \hat{H} \backslash H) > 0$. By the definition of $\hat{H}$,

$$Q_m(\hat{P}^{\bar{Y}_T}(\hat{\tau}^{\bar{Y}_T} > 0),\ \bar{Y}_T \in \hat{H} \backslash H) > 0\ .$$

(The ambiguity in this expression may be correctly resolved by recalling that $\bar{Y}_T$ is defined on W and for each x, $\hat{\tau}^x$ is defined on $\hat{\Omega}$.) By the moderate Markov property (in reverse time),

$$Q_m(\hat{\tau}^{\bar{Y}_T}(\hat{\theta}_T) > 0,\ \bar{Y}_T \in \hat{H} \backslash H) > 0$$

where $(\hat{\theta}_t w)(s) = w(t + s)$ for $s < 0$, Δ for $s \geqq 0$. That is,

$$Q_m(\bar{Y}_T \in \hat{H} \backslash H \text{ and there exists } \varepsilon > 0 \text{ such that } \bar{Y}_{T-t} = \bar{Y}_T \text{ for } 0 \leqq t \leqq \varepsilon) > 0\ .$$

Thus $\hat{H} \backslash H$ is not even m-semipolar. Hence by the second of the results of Dellacherie mentioned above, there is a finite measure ρ on E which does not charge m-semipolar sets but does charge $\hat{H} \backslash H$. In particular, ρ does not charge m-polar sets so by **[Fi87**, (5.22)], there is a Q_m-essentially unique optional copredictable homogeneous random measure κ carried by Λ, such that

$$(2.3) \qquad \int_{\mathbb{R}} \int_E f(t, x)\rho(\mathrm{d}x)\,\mathrm{d}t = \int_W \int_{\mathbb{R}} f(t, \bar{Y}_t(w))\kappa(w, \mathrm{d}t)Q_m(\mathrm{d}w)$$

for each non-negative $\mathscr{R} \otimes \mathscr{E}$-measurable function f on $\mathbb{R} \times E$. (Here $\mathscr{R}$ = Borel ($\mathbb{R}$).) The right hand side of (2.3) is usually abbreviated as $Q_m(\int_{\mathbb{R}} f(t, \bar{Y}_t)\kappa(dt))$. Since ρ does not charge m-semipolar sets, κ is diffuse; i.e., $\kappa(w, \cdot)$ is diffuse for Q_m-a.e. $w \in W$. Let $\varphi(x) = 1_{H^c}(x)$, $\psi(x, w) = 1_{\{\hat{\tau}^x(w) > 0\}}$, and let $Z_t(w) = \varphi(\bar{Y}_t(w))\psi(\hat{Y}_t(w), \hat{\theta}_t w)$. We claim that Q_m-a.s., $Z_t = 0$ for all but countably many t. First note that for Q_m-a.e. w, we have

$$\{t \in \Lambda(w) : \bar{Y}_t(w) \in H^c\} \subseteq \{t \in \Lambda(w) : \tau(\theta_t w) = 0\} ,$$

where $(\theta_t w)(s) = w(t + s)$ for $s > 0$, Δ for $s \leqq 0$. (For if $t \in \Lambda(\omega)$ and $\tau(\theta_t \omega) > 0$ then $\bar{Y}_{\cdot}(\omega)$ is constant on an interval of the form $[t, t + \varepsilon[$, and in this case there is a rational r in $[t, t + \varepsilon[$ such that $\tau(\theta_r \omega) > 0$ and $\bar{Y}_r(\omega) = \bar{Y}_t(\omega)$. Consequently

$$\begin{aligned} &Q_m(\tau \circ \theta_t > 0 \text{ and } \bar{Y}_t \in H^c \text{ for some } t \in \Lambda) \\ &\leqq Q_m(\tau \circ \theta_r > 0 \text{ and } \bar{Y}_r \in H^c \text{ for some rational } r \in \Lambda) \\ &\leqq \sum_r Q_m(\tau \circ \theta_r > 0, \bar{Y}_r \in H^c, r \in \Lambda) = 0 , \end{aligned}$$

where the final equality follows from the simple Markov property.) Now for each w, there exist at most countably many t's where both $\tau(\theta_t w) = 0$ and $\hat{\tau}^{\bar{Y}_t(w)}(\hat{\theta}_t w) > 0$. (Each such t is the right-hand endpoint of an interval of constancy of w.) This proves the claim. It follows that

$$Q_m\left(\int_{\mathbb{R}} Z_t \kappa(dt)\right) = 0$$

because κ is diffuse. Hence

$$Q_m\left(\int_{\mathbb{R}} {}^{\hat{p}}Z_t \kappa(dt)\right) = 0$$

since κ is copredictable. But it follows from the moderate Markov property (in reverse time) that for Q_m-a.e. w, for each t,

$${}^{\hat{p}}Z_t(w) = \varphi(\bar{Y}_t(w))\hat{P}^{\bar{Y}_t(w)}(\hat{\tau}^{\bar{Y}_t(w)} > 0)$$

so $\bar{Y}_t(w) \in \hat{H} \backslash H$ implies ${}^{\hat{p}}Z_t(w) > 0$. Hence

$$\rho(\hat{H} \backslash H) = Q_m\left(\int_0^1 1_{\hat{H} \backslash H}(\bar{Y}_t)\kappa(dt)\right) = 0 .$$

But this contradicts the fact that ρ charges $\hat{H} \backslash H$. Thus $\hat{H} \backslash H$ must be m-polar after all.

The proof that $H \backslash \hat{H}$ is m-polar is just the dual of the above argument. Since this fact is not needed for the proof of Theorem (1.2), we omit the details. □

An $\mathscr{E}^*$-measurable function $f \geqq 0$ on E is said to be *coexcessive* when it is excessive for $\hat{X}$; i.e., when $\hat{P}_t f \leqq f$ for $t > 0$ and $\hat{P}_t f \to f$ pointwise on E as $t \downarrow 0$, where $\hat{P}_t(y, dx) = \hat{P}^y(\hat{X}_t \in dx)$. A statement about points $x \in E$ is said to hold *m-quasi-everywhere* (abbreviated m-q.e.) when the set of points for which it does not hold is m-polar.

(2.4) Proposition. (*a*) *If f is coexcessive, then there exists a Borel coexcessive function g such that $f = g$ m-q.e. In particular, if f is coexcessive, then $f \circ \bar{Y}$ is optional and copredictable.*

(b) *If ξ is an excessive measure and $\xi \ll m$, then there exists a coexcessive version* $\hat{u}$ of $d\xi/dm$; if $\hat{u}'$ *is another coexcessive version of $d\xi/dm$, then $\hat{u} = \hat{u}'$ m-q.e.*

Proof. See **[Fi90**, (2.6)]. □

(2.5) Proposition. *Suppose ξ is an excessive measure and $\xi \ll m$. Then there is a version $\bar{u}$ of $d\xi/dm$ such that $\bar{u}$ is Borel measurable and Q_m-a.s., $\bar{u} \circ \bar{Y}$ is right-continuous on Λ; if $\bar{u}'$ is another such version of $d\xi/dm$, then $\bar{u} = \bar{u}'$ m-q.e. Moreover, there is a Borel m-polar set N, whose complement is absorbing for both X and $\hat{X}$, such that $\bar{u}|_{E\setminus N}$ is finite and finely continuous on $E\setminus N$. Finally, if $\hat{u}$ is a coexcessive version of $d\xi/dm$, then $\hat{u} \leqq \bar{u}$ m-q.e. and $\{\hat{u} \neq \bar{u}\}$ is m-semipolar.*

Proof. See **[Fi90**, (2.7)–(2.10), (3.7)] and **[Fi87**, (4.15)]. □

The function $\bar{u}$ in (2.5) is called a *fine version* of $d\xi/dm$. (The case where X is uniform motion to the right on $\mathbb{R}$ and m is Lebesgue measure is trivial but illustrative: $d\xi/dm$ may be taken to be increasing and $\hat{u}$ is its left-continuous version while $\bar{u}$ is its right-continuous version.)

As we have already remarked, if μ is a σ-finite measure on E, then $\mu U \ll m$ if and only if μ charges no $\mathscr{E}^e$-measurable finely open m-polar set. When expressed in terms of fine densities, the domination principle assumes the following form.

(2.6) Theorem. *Let λ be a measure on E such that λ charges no m-polar set and λU is σ-finite. Let ξ be an excessive measure such that $\xi \ll m$. Let $\bar{u}$ and $\bar{v}$ denote fine versions of $d(\lambda U)/dm$ and $d\xi/dm$ respectively. If $\bar{u} \leqq \bar{v}$ λ-a.e., then $\lambda U \leqq \xi$.*

For a proof of Theorem (2.6), see **[Fi90**, (2.13)]. (It is worth remarking that sharper versions of (2.5) and (2.6) exist: See **[FG90]**, and for an application see **[Fi91]**.) The "classical" version of the domination principle was expressed in terms of the coexcessive densities $\hat{u}$ and $\hat{v}$ and required the stronger hypothesis that λ charges no m-semipolar set. In our application of (2.6), this hypothesis would be satisfied. For us, the importance of (2.6) is that it is expressed in terms of densities which are (almost) finely continuous. This enables us to avoid considering the cofine topology, which is fortunate since in the present setting, the cofine topology need not exist; see **[SW73**, §3].

Suppose λ is a measure on E such that λ charges no m-polar set and λU is σ-finite. Let κ be a version of the corresponding optional copredictable homogeneous random measure carried by Λ, as in (2.3) (with ρ replaced by λ). We may and shall assume in addition that κ is *perfect*; see **[Fi87**, (5.23) and (5.27)]. Then

$$\hat{u}(x) = \hat{P}^x(\kappa]-\infty, 0[)$$

defines a coexcessive version $\hat{u}$ of $d(\lambda U)/dm$; see **[Fi90**, (3.7)]. (It is worth remarking that this fact depends heavily on the perfectness of κ; note that each $\hat{P}^x$ lives on $\hat{\Omega}$ which is a subset of W having Q_m-measure zero.) The next result is a version of the switching identity.

(2.7) Theorem. *Let λ and κ be as above. Let A be a Borel subset of E, let*

$$T_A = \inf\{t > 0: X_t \in A\}$$

and

$$\hat{T}_A = \inf\{t > 0: \hat{X}_t \in A\}$$

be the hitting and cohitting times of A, and let $P_A(x, \mathrm{d}y) = P^x(X(T_A) \in \mathrm{d}y)$ be the hitting kernel for A. Then

$$\hat{u}_A(x) = \hat{P}^x(\kappa] - \infty, -\hat{T}_A[)$$

defines a coexcessive version $\hat{u}_A$ of $\mathrm{d}(\lambda P_A U)/\mathrm{d}m$.

Proof. Let $\tau_A = \inf\{t > \alpha: Y_t \in A\}$. Note that $T_A = \tau_A | \Omega$. Since λU is excessive, we may consider the corresponding Kuznetsov measure $Q_{\lambda U}$ on W. For each Borel measurable function $f \geqq 0$ on E, we have

$$\lambda P_A U(f) = Q_{\lambda U}(f(Y_0), \tau_A < 0) \,. \tag{2.8}$$

This was shown in [**FM86**, (5.5)]. For the reader's convenience, we shall give a proof here. By [**GG87**, p. 36], the Kuznetsov measure corresponding to a potential has a particularly simple form, namely

$$Q_{\lambda U}(F) = \int_{-\infty}^{\infty} P^\lambda(F \circ \sigma_{-t})\,\mathrm{d}t = \int_{-\infty}^{\infty} P^\lambda(F \circ \sigma_t)\,\mathrm{d}t \tag{2.9}$$

for each $\mathscr{G}^*$-measurable function $F \geqq 0$ on W (where $\mathscr{G}^*$ is the universal completion of $\mathscr{G}^0 = \sigma\{Y_t: t \in \mathbb{R}\}$). Now $\tau_A \circ \sigma_t = \tau_A - t$, and P^λ lives on Ω so

$$\begin{aligned} Q_{\lambda U}(f(Y_0), \tau_A < 0) &= \int_{-\infty}^{\infty} P^\lambda(f(Y_t), t > \tau_A)\,\mathrm{d}t \\ &= P^\lambda\left(\int_{-\infty}^{\infty} f(Y_t) 1_{\{t > \tau_A\}}\,\mathrm{d}t\right) = P^\lambda\left(\int_{T_A}^{\infty} f(X_t)\,\mathrm{d}t\right) \\ &= \lambda P_A U(f)\,, \end{aligned}$$

which completes the proof of (2.8). Next, for each $t \in \mathbb{R}$, we introduce the operation b_t of birthing at time t: For $w \in W$,

$$(b_t w)(s) = \begin{cases} w(s) & \text{if } s > t\,, \\ \Delta & \text{if } s \leqq t\,. \end{cases}$$

Note that $b_t = \sigma_{-t} \circ \theta_t$. We claim that

$$Q_{\lambda U}(F) = Q_m\left(\int_{\mathbb{R}} F \circ b_t \kappa(\mathrm{d}t)\right) \tag{2.10}$$

for each $\mathscr{G}^*$-measurable function $F \geqq 0$ on W. By a routine completion argument it suffices to consider $\mathscr{G}^0$-measurable F. Let $Z_t = F \circ b_t$. Since $Z_t = F \circ \sigma_{-t} \circ \theta_t$, it follows from the strong Markov property that the optional projection of Z is ${}^oZ_t = P^{\bar{Y}_t}(F \circ \sigma_{-t})$. Hence

$$\begin{aligned} Q_m\left(\int_{\mathbb{R}} F \circ b_t \kappa(\mathrm{d}t) \right) &= Q_m\left(\int_{\mathbb{R}} P^{\bar{Y}_t}(F \circ \sigma_{-t}) \kappa(\mathrm{d}t) \right) \\ &= \int_{\mathbb{R}} \int_E P^x(F \circ \sigma_{-t}) \lambda(\mathrm{d}x)\, \mathrm{d}t \\ &= \int_{\mathbb{R}} P^\lambda(F \circ \sigma_{-t})\, \mathrm{d}t \\ &= Q_{\lambda U}(F) \end{aligned}$$

where the first step follows from the optionality of κ, the second from (2.3), and the last from (2.9). This completes the proof of (2.10). Combining (2.8) and (2.10), we obtain

$$\lambda P_A U(f) = Q_m\left(\int_{]-\infty, 0[} f(Y_0) 1_{\{\tau_A \circ b_t < 0\}} \kappa(\mathrm{d}t) \right). \tag{2.11}$$

Recall that $\hat{X}_t = \bar{Y}_{-t} | \hat{\Omega}$. Let $\tilde{X}_t = Y_{-t} | \hat{\Omega}$ and let $\tilde{T}_A = \inf\{t > 0\colon \tilde{X}_t \in A\}$. For each $t < 0$ and each $w \in W$ with $Y_0(w) \in E$, we have $\hat{\theta}_0 w \in \hat{\Omega}$ and we also have

$$\tau_A(b_t w) < 0$$

iff $\quad Y_s(w) \in A$ for some $s \in]t, 0[$ iff $\tilde{X}_s(\hat{\theta}_0 w) \in A$ for some $s \in]0, -t[$

iff $\quad \tilde{T}_A(\hat{\theta}_0 w) < -t$ iff $t < -\tilde{T}_A(\hat{\theta}_0 w)$.

Hence it follows from (2.11) that

$$\lambda P_A U(f) = Q_m(f(Y_0)\kappa] -\infty, -\tilde{T}_A \circ \hat{\theta}_0[) .$$

Since κ is perfect, we have $\kappa(w,]-\infty, t[) = \kappa(\hat{\theta}_0 w,]-\infty, t[)$ for each $t \leqq 0$ and each $w \in W$; see [**Fi87**, (5.23), (iv)]. Hence

$$\lambda P_A U(f) = Q_m(f(Y_0)\kappa(\hat{\theta}_0,]-\infty, -\tilde{T}_A \circ \hat{\theta}_0[) .$$

Applying the simple Markov property (in reverse time), we obtain

$$\lambda P_A U(f) = Q_m(f(Y_0)\hat{P}^{Y_0}(\kappa] -\infty, -\tilde{T}_A[)) .$$

Now observe that

$$\tilde{T}_A = \begin{cases} \hat{T}_A & \text{on} \quad \{\hat{T}_A \neq \hat{\zeta}\}, \\ \infty & \text{on} \quad \{\hat{T}_A = \hat{\zeta}\}, \end{cases}$$

where $\hat{\zeta} = -\alpha | \hat{\Omega}$ is the lifetime of $\hat{X}$. But since κ is perfect, we have $\kappa(w,]-\infty, \alpha(w)[) = 0$ for each $w \in W$; see [**Fi87**, (5.23), (iii)]. Hence $\kappa(w,]-\infty, -\tilde{T}_A(w)[) = \kappa(w,]-\infty, -\hat{T}_A(w)[)$ for each $w \in W$. Thus

$$\begin{aligned} \lambda P_A U(f) &= Q_m(f(Y_0)\hat{P}^{Y_0}(\kappa] -\infty, -\hat{T}_A[)) \\ &= Q_m(f(Y_0)\hat{u}_A(Y_0)) \\ &= \int_E f(x)\hat{u}_A(x) m(\mathrm{d}x) . \end{aligned}$$

Hence $\hat{u}_A$ is version of $d(\lambda P_A U)/dm$. Since κ is perfect, we have $\kappa(\sigma_s w, B) = \kappa(w, B + s)$ for each $w \in W$, each $s \in \mathbb{R}$, and each Borel set $B \subseteq \mathbb{R}$; see [**Fi87**, (5.23), (ii)]. Using this and the simple Markov property of $\hat{X}$, plus the fact that $\hat{\theta}_t = \hat{\theta}_0 \circ \sigma_{-t}$, we readily obtain

$$\hat{P}_t \hat{u}_A(x) = \hat{P}^x(\kappa] - \infty, -(t + \hat{T}_A \circ \hat{\theta}_t)[) .$$

Now $t + \hat{T}_A \circ \hat{\theta}_t = \inf\{s > t : \hat{X}_s \in A\} \downarrow \hat{T}_A$ as $t \downarrow 0$. Hence $\hat{u}_A$ is coexcessive. (Of course, we should point out that $\hat{u}_A$ is $\mathscr{E}^*$-measurable because κ is $\mathscr{G}^*$-measurable. The $\mathscr{G}^*$-measurability of κ is part of the definition of perfectness of κ; see [**Fi87**, (5.23), (i)].) This completes the proof of the theorem. □

For earlier versions of the switching identity, see [**B45**], [**Hu58**, eqn. 18.3], [**BG68**, VI(1.16)], [**A73**, 4.5 and 4.6], and [**GS84**, (11.6)]. We remark that (2.7) was implicit in the proof of [**Fi87**, (6.27)].

3. Proof of the main Theorem

In this section, we shall give the proof of Theorem (1.2). Assume for now that X is a Borel right process. Then all the results of Sect. 2 may be applied.

Step 1. Consider the case where μ charges no m-polar set. Let $\mathscr{T}$ be the set of all non-randomized stopping times T such that $\mu P_T U \geqq \nu U$. Then $\mathscr{T} \neq \emptyset$ since $0 \in \mathscr{T}$. Now the pointwise limit of any increasing sequence in $\mathscr{T}$ belongs to $\mathscr{T}$. (For if (T_n) is such a sequence and T is its limit, then for each Borel function $f \geqq 0$ on E with $\mu U(f) < \infty$, we have $\mu P_{T_n} U(f) = P^\mu(\int_{T_n}^\infty f(X_t)\,dt) \downarrow P^\mu(\int_T^\infty f(X_t)\,dt) = \mu P_T U(f)$, by dominated convergence so, as μU is σ-finite, $\mu P_T U \geqq \nu U$, whence $T \in \mathscr{T}$.) From this and the fact that P^μ is σ-finite it follows that $\mathscr{T}$ has P^μ-essentially maximal elements; let T be one of them. (Incidentally, the full axiom of choice is not needed for this. The principle of dependent choice is enough.) Let $\lambda = \mu P_T$. We claim that $\lambda = \nu$. For suppose not. Then we shall show that there is a non-randomized stopping time S such that $\lambda P_S U \geqq \nu U$ and $P^\lambda(S > 0) > 0$. Assume that this has been shown. We may suppose that $S([\Delta]) = 0$. Let $T' = T + S \circ \theta_T$. Then $T' \in \mathscr{T}$ because $\mu P_{T'} U = \mu P_T P_S U = \lambda P_S U \geqq \nu U$. But $T \leqq T'$ and $P^\mu(T < T') = P^\mu(S \circ \theta_T > 0, T < \infty) = P^\mu(P^{X(T)}(S > 0), T < \infty) = P^\lambda(S > 0) > 0$, which contradicts the maximality of T.

Now let us show how to construct S under the assumption that $\lambda \neq \nu$. Observe that λ charges no m-polar set. (For let A be a Borel m-polar set. Since $\mu U \ll m$, A is μU-polar. Equivalently, A is μ-polar. Hence $\lambda(A) = P^\mu(X_T \in A) = P^\mu(X_T \in A, T = 0) \leqq \mu(A) = 0$.) Also, since $\lambda U \geqq \nu U$, there is a randomized stopping time R such that $\lambda P_R = \nu$, by (1.1). Let τ, $\hat{\tau}^x$, H, and $\hat{H}$ be as above (2.2). Let

$$\lambda_1 = 1_{E \setminus H} \lambda, \quad \lambda_2 = 1_H \lambda$$

$$\nu_1 = \lambda_1 P_R, \quad \nu_2 = \lambda_2 P_R .$$

Then for $i = 1, 2$, λ_i charges no m-polar set and $\lambda_i U \geqq \nu_i U$. Moreover, either $\lambda_1 \neq \nu_1$ or $\lambda_2 \neq \nu_2$. Thus we may reduce to the case where either $\lambda(H) = 0$ or λ lives on H.

Case 1. Suppose $\lambda(H)=0$. Then $\lambda(\hat{H})=\lambda(\hat{H}\cap H)+\lambda(\hat{H}\backslash H)=0$ since, by (2.2), $\hat{H}\backslash H$ is m-polar. Let κ be as above (2.7) so that

$$\hat{u}(x)=\hat{P}^x(\kappa]-\infty,0[)$$

defines a coexcessive version $\hat{u}$ of $d(\lambda U)/dm$. By (2.4)(b), there also exists a coexcessive version $\hat{v}$ of $d(\nu U)/dm$. Let $V=\{\hat{u}>\hat{v}\}$. We claim that

$$\lambda(V)>0\,. \tag{3.1}$$

For suppose not. Then $\hat{v}\geqq\hat{u}$ λ-a.e. Let $\bar{u}$ and $\bar{v}$ be fine versions of $d(\lambda U)/dm$ and $d(\nu U)/dm$ respectively; see (2.5). Then $\{\hat{u}\neq\bar{u}\}$ and $\{\hat{v}\neq\bar{v}\}$ are m-semipolar. But we are assuming that m-semipolar sets are m-polar. Since λ charges no m-polar set, $\hat{u}=\bar{u}$ λ-a.e. and $\hat{v}=\bar{v}$ λ-a.e. Hence $\bar{v}\geqq\bar{u}$ λ-a.e. so $\nu U\geqq\lambda U$ by (2.6). But then $\lambda U=\nu U$ so $\lambda=\nu$, which is a contradiction. This proves (3.1).

We shall now obtain the desired stopping time S as the time at which X leaves a suitable Borel finely open set L. Let $\mathscr{B}=\{B_n\colon n=0,1,2,\ldots\}$ be a countable base for the topology of E. For each n, let $A_n=B_n^c$, let $\lambda_n=\lambda P_{A_n}$, let $\hat{T}_n=\hat{T}_{A_n}$, let $\hat{u}_n=\hat{P}^{\cdot}(\kappa]-\infty,\,-\hat{T}_n[)$ which, by (2.7), is a coexcessive version of $d(\lambda_n U)/dm$, and let $\bar{u}_n$ be a fine version of $d(\lambda_n U)/dm$. It follows readily from (2.5) that there is a Borel m-polar set N such that $E\backslash N$ is absorbing for X and such that the restrictions to $E\backslash N$ of $\bar{u},\bar{v},\bar{u}_n$ $(n=0,1,2,\ldots)$ are finely continuous on $E\backslash N$. Note that $E\backslash N$ is finely open. Let $\mathscr{U}$ be the topology on E generated by $\mathscr{B}$, $E\backslash N$, $\bar{u}$, $\bar{v}$, and $\{\bar{u}_n\colon n=0,1,2,\ldots\}$. Since $\mathscr{U}$ is second countable, any measure on E has a $\mathscr{U}$-support. (For the purposes of the present argument, this serves as a substitute for the fine support. The latter need not exist since X was not assumed to have a reference measure.) Let F be the $\mathscr{U}$-support of λ and let

$$G=F\cap(E\backslash\hat{H})\cap(E\backslash N)\cap\{\hat{u}=\bar{u}\}\cap\{\hat{u}_n=\bar{u}_n\text{ for each }n\}\,.$$

Then λ lives on G. (λ charges no m-polar set. Hence $\lambda(N)=0$. $\{\hat{u}_n\neq\bar{u}_n\}$ is m-semipolar and so, by hypothesis, m-polar. Hence $\hat{u}_n=\bar{u}_n$ λ-a.e.) Since $\lambda(V)>0$, $G\cap V\neq\emptyset$. Fix $z\in G\cap V$. For some sequence $(n(k))$ of natural numbers, we have $B_{n(k)}\downarrow\{z\}$. Let $\hat{\tau}_k=\hat{T}_{n(k)}$. Then $\hat{\tau}_k\downarrow\hat{\tau}^z$. But $\hat{\tau}^z=0$ $\hat{P}^z$-a.s. because $z\notin\hat{H}$. Hence $\hat{u}_{n(k)}(z)=\hat{P}^z(\kappa]-\infty,\,-\hat{\tau}_k[)\uparrow\hat{P}^z(\kappa]-\infty,0[)=\hat{u}(z)$. But $\hat{u}_n(z)=\bar{u}_n(z)$ and $\hat{u}(z)=\bar{u}(z)$ because $z\in G$. Hence $\bar{u}_{n(k)}(z)\uparrow\bar{u}(z)$. Now $\bar{u}(z)>\bar{v}(z)$ because $z\in V\cap G$. Thus there is an n (equal to $n(k)$ for some k) such that $z\in B_n$ and $\bar{u}_n(z)>\bar{v}(z)$. Let

$$L=B_n\cap\{\bar{u}_n>\bar{v}\}\cap(E\backslash N)\,.$$

Then L is $\mathscr{U}$-open and $L\cap F\neq\emptyset$ (since $z\in L\cap F$) so $\lambda(L)>0$. Let $D=L^c$, $S=T_D$, and $\hat{S}=\hat{T}_D$. We claim that

$$\lambda P_S U\geqq\nu U\,. \tag{3.2}$$

By (2.7), $\lambda P_S U=\hat{w}m$ where $\hat{w}=\hat{P}^{\cdot}(\kappa]-\infty,\,-\hat{S}[)$, so $\hat{w}\geqq\hat{u}_n$. Thus $\hat{w}>\hat{v}$ m-q.e. on L so (3.2) holds on subsets of L. Next, for each Borel function $f\geqq0$ on E such that $f=0$ on L, we have $\lambda P_S U(f)=\lambda(P_D Uf)=\lambda(Uf)\geqq\nu U(f)$. Thus (3.2) holds on subsets of D too. This completes the proof of (3.2). Finally, L is finely open so $P^\lambda(S>0)\geqq\lambda(L)>0$.

Case 2. Suppose λ lives on H. Recall that there is a randomized stopping time R such that $\lambda P_R=\nu$. For each x, let $\nu_x=\varepsilon_x P_R$ where ε_x is the unit point mass at x.

Then $\nu = \int_H \lambda(dx)\nu_x$ and $\varepsilon_x U \geqq \nu_x U$. Since λU is σ-finite, we have $\lambda U(f) < \infty$ for some Borel function $f > 0$ on E. Then $Uf < \infty$ λ-a.e., so $f(x)U(x, \{x\}) < \infty$ for λ-.a.e. x. Let

$$H_1 = \{x \in H: \varepsilon_x \neq \nu_x \text{ and } \varepsilon_x U(\{x\}) < \infty\} .$$

Then $\lambda(H_1) \neq 0$ because $\lambda \neq \nu$. If $x \in H_1$, then $P^x(R > 0) > 0$ so

$$(3.3) \qquad \infty > \varepsilon_x U(\{x\}) = P^x\left(\int_0^\infty 1_{\{x\}}(X_s)\,ds \right) > P^x\left(\int_R^\infty 1_{\{x\}}(X_s)\,ds \right) = \nu_x U(\{x\})$$

where in the third step we have used the fact that x is a holding point. For $x \in H_1$, let

$$t_x = \sup\left\{ t \geqq 0 \colon P^x\left(\int_t^\infty 1_{\{x\}}(X_s)\,ds \right) \geqq \nu_x U(\{x\}) \right\}$$

and for $x \in H_1^c$, let $t_x = 0$. For $x \in H_1$, we have $t_x > 0$ by (3.3) and we also have

$$P^x\left(\int_{t_x}^\infty 1_{\{x\}}(X_s)\,ds \right) = \nu_x U(\{x\}) .$$

Let $S = \tau \wedge t_{X_0}$. Then $P^\lambda(S > 0) = \lambda(H_1) > 0$. For each x,

$$\varepsilon_x P_S U(\{x\}) \geqq \nu_x U(\{x\})$$

and if A is a Borel subset of $E \backslash \{x\}$, then

$$\varepsilon_x P_S U(A) \geqq \varepsilon_x P_\tau U(A) \geqq \varepsilon_x P_A U(A) = \varepsilon_x U(A) \geqq \nu_x U(A) .$$

Thus $\varepsilon_x P_S U \geqq \nu_x U$ for each x. Hence $\lambda P_S U \geqq \nu U$. This completes Step 1.

Step 2. Consider the case where ν charges no m-polar set. We can choose a μ-essentially largest Borel m-polar set Z. If S is a random time and $S > 0$ P^μ-a.s. on $\{X_0 \in Z\}$, then μP_S charges no m-polar set. (For consider any Borel m-polar set N. Then N is μ-polar, so $P^\mu(X_S \in N) = P^\mu(X_S \in N, S = 0) \leqq P^\mu(X_0 \in N, X_0 \notin Z) = \mu(N \backslash Z) = 0$ where the last step follows from the maximality of Z.) We shall construct a non-randomized stopping time S such that $\mu P_S U \geqq \nu U$ and $S > 0$ P^μ-a.s. on $\{X_0 \in Z\}$. Then by Step 1 (with μ replaced by μP_S), there is a non-randomized stopping time T' such that $\mu P_S P_{T'} = \nu$. But then $\mu P_T = \nu$ where $T = S + T' \circ \theta_S$.

Now let us construct S. Since Z is m-polar, it is ν-polar. Also, by hypothesis, Z is ν-null. Hence, by (2.1), there is an excessive function f and a decreasing sequence (G_n) of finely open $\mathscr{E}^e$-measurable sets such that, letting $Z^* = \bigcap_n G_n$, we have $Z^* \supseteq Z$, $f = +\infty$ on Z^*, and $\nu(f) + \sum_n \nu(P_{G_n} f) < \infty$. Let $H_0 = E \backslash G_0$ and for $n \geqq 1$, let $H_n = G_{n-1} \backslash G_n$. Then $E = Z^* \cup (\bigcup_n H_n)$. (We remark that this is a disjoint union.) Let $\mu_n = \mu P_{H_n}$ and let $\nu_n = \nu P_{H_n}$. Then $\mu_n U \geqq \nu_n U$. (For consider any Borel function $h \geqq 0$ on E. Let $g = P_{H_n} Uh$. Then $\mu_n U(h) = \mu(g)$ and $\nu_n U(h) = \nu(g)$. But $\mu(g) \geqq \nu(g)$ because g is excessive, $\mu U \geqq \nu U$, and μU is σ-finite; see for instance [**Fa83**, p. 45].) Now $T_{H_n} \geqq T_{E \backslash G_n} > 0$ P^μ-a.s. on $\{X_0 \in Z\}$ so μ_n charges no m-polar set. Hence by Step 1, there is a non-randomized stopping time R_n such that $\mu_n P_{R_n} = \nu_n$. Let $S_n = T_{H_n} + R_n \circ \theta_{T_{H_n}}$. Then $\mu P_{S_n} = \nu_n$. Let $S = \inf_n S_n$. First, let us show that

$$\mu P_S U \geqq \nu U \; . \tag{3.4}$$

Consider a Borel function $h \geqq 0$ on E. If $h = 0$ outside H_n, then $P_{H_n} Uh = Uh$ so $\mu P_S U(h) \geqq \mu P_{S_n} U(h) = \nu_n U(h) = \nu(P_{H_n} Uh) = \nu U(h)$. Next, $f < \infty$ ν-a.e. and $f = \infty$ on Z^* so Z^* is ν-polar. Hence, if $h = 0$ outside Z^*, then $\mu P_S U(h) \geqq 0 = \nu U(h)$. Since $Z^* \cup \bigcup_n H_n = E$, (3.4) holds. Now let us show that $S > 0$ P^μ-a.s. on $\{X_0 \in Z\}$. Let $S'_n = S_0 \wedge \cdots \wedge S_n$. Then $S'_n \downarrow S$ so $f(X_{S'_n}) \to f(X_S)$ P^μ-a.s. (Since f is excessive, $f \circ X$ is a.s. right continuous.) Hence

$$\begin{aligned} P^\mu(f(X_S)) &\leqq \liminf_{n\to\infty} P^\mu(f(X_{S'_n})) \\ &\leqq \liminf_{n\to\infty} \sum_{k=0}^{n} P^\mu(f(X_{S_k}); S'_n = S_k) \\ &\leqq \sum_{k=0}^{\infty} P^\mu(f(X_{S_k})) = \sum_{k=0}^{\infty} \mu P_{S_k}(f) \\ &= \sum_{k=0}^{\infty} \nu_k(f) = \sum_{k=0}^{\infty} \nu P_{H_k}(f) \\ &\leqq \nu(f) + \sum_{l=0}^{\infty} \nu(P_{G_l} f) < \infty \; . \end{aligned}$$

But $f = +\infty$ on Z. Hence $P^\mu(X_S \in Z) = 0$ so $P^\mu(S = 0, X_0 \in Z) = 0$ as desired. This completes Step 2.

Step 3. In the general case, there is a trivial reduction to the setting of Step 2. For we can choose a ν-essentially largest Borel m-polar set M. Clearly the set C in (1.4) may be modified to be Borel measurable. Next, $\nu(M \cap C^c) = \mu(M \cap C^c \cap C) = 0$ so we may and do assume that $M \subseteq C$. Then for each Borel m-polar set Z, $\nu(Z) = \nu(Z \cap M) = \mu(Z \cap M \cap C) = \mu(Z \cap M)$. Thus μ and ν agree on subsets of M and ν charges no m-polar subset of $E \backslash M$ so we may finish by applying Step 2 to $1_{E \backslash M} \mu$ and $1_{E \backslash M} \nu$ in place of μ and ν.

Theorem (1.2) has now been completely proved in the case where X is a Borel right process. The reduction of the general case to this special case is essentially trivial but depends on a plethora of facts about the Ray–Knight compactification. Accordingly, for the reader's convenience, we give the details. Suppose X is a general right process. Let $(U^q)_{0<q<\infty}$ be the resolvent for X. Let $(\bar{\bar{E}}, \bar{\bar{U}}^q)$ be a Ray–Knight completion of (E, U^q); see [**Sh88**, p. 93]. ($\bar{\bar{E}}$ is compact and metrizable and $(\bar{\bar{U}}^q)_{0<q<\infty}$ is a Ray resolvent on $\bar{\bar{E}}$.) Let $(\bar{\bar{P}}_t)$ be the corresponding Ray semigroup on $\bar{\bar{E}}$ and let D be its set of non-branch points. Evidently the cemetery point Δ is a trap for $(\bar{\bar{U}}^q)$. In what follows Δ will serve as cemetery for $(\bar{\bar{U}}^q)$, so the convention that all functions and measures vanish at Δ remains in force. In particular, if we set $\bar{E} = D \backslash \{\Delta\}$ then $\bar{E}$ is a Borel subset of $\bar{\bar{E}}$ and for each $x \in \bar{E}$, $\bar{\bar{P}}_t(x, \cdot)$ lives on $\bar{E}$; see [**Sh88**, (9.11)]. Let $(\bar{P}_t)$ be the restriction of $(\bar{\bar{P}}_t)$ to $\bar{E}$. Let

$\bar{\mathscr{E}} = \text{Borel}(\bar{E})$ and let $\bar{\mathscr{E}}^*$ be the universal completion of $\bar{\mathscr{E}}$. We have $E \subseteq \bar{E}$, $E \in \bar{\mathscr{E}}^*$, the trace of $\bar{\mathscr{E}}^*$ on E is $\mathscr{E}^*$, and for each $x \in E$,

$$(3.5) \qquad \bar{P}_t(x, A) = P_t(x, A \cap E)$$

for each $A \in \bar{\mathscr{E}}^*$ and each $t \geq 0$; see [**Sh88**, §17, especially (17.10), (17.11), and (17.16)]. (We remark that the trace of $\bar{\mathscr{E}}$ on E is $\mathscr{E}^r$, the Ray σ-algebra on E. We have $\mathscr{E} \subseteq \mathscr{E}^r$. This inclusion may be strict.) It follows from (3.5) that a function on E is q-excessive for (P_t) if and only if it is the restriction to E of a function on $\bar{E}$ which is q-excessive for $(\bar{P}_t)$; see [**Sh88**, proof of (12.29)]. Hence

$$(3.6) \qquad \textit{the trace of } \bar{\mathscr{E}}^e \textit{ on } E \textit{ is } \mathscr{E}^e ,$$

where $\bar{\mathscr{E}}^e$ is the σ-algebra on $\bar{E}$ generated by $\{f : f \text{ is } q\text{-excessive for } (\bar{P}_t), \text{ for some } q \geq 0\}$. Now $(\bar{P}_t)$ is a Borel right semigroup on $\bar{E}$ and may be realized by a Borel right process $\bar{X}$ on a canonical path space $\bar{\Omega}$:

> $\bar{\Omega}$ is the set of paths $\bar{\omega} : [0, \infty[\to \bar{E} \cup \{\Delta\}$ which are $\bar{E}$-valued and right-continuous into $\bar{E}$ on some interval $[0, \bar{\zeta}(\bar{\omega})[$, taking the value Δ outside this interval; $\bar{X}_t(\bar{\omega}) = \bar{\omega}(t)$ for $\bar{\omega} \in \bar{\Omega}$ and $t \in [0, \infty[$;

see [**Sh88**, (9.13)]. Define Φ on Ω by $\Phi(\omega) = (X_t(\omega))_{0 \leq t < \infty}$. We may and do assume that $\Phi : \Omega \to \bar{\Omega}$; see [**Sh88**, (18.1)]. Clearly

$$(3.7) \qquad \bar{X}_t \circ \Phi = X_t \textit{ for each } t .$$

Let $\mathscr{F}^{\cdot} = \sigma(X_t^{-1}(A) : A \in \mathscr{E}^*, \ 0 \leq t < \infty)$ and let $\bar{\mathscr{F}}^{\cdot} = \sigma(\bar{X}_t^{-1}(A) : \ A \in \bar{\mathscr{E}}^*, \ 0 \leq t < \infty)$. It follows from (3.7) that

$$(3.8) \qquad \Phi \textit{ is } \mathscr{F}^{\cdot}/\bar{\mathscr{F}}^{\cdot}\textit{-measurable.}$$

For each s-finite measure λ on E, define $\bar{\lambda}$ on $\bar{E}$ by $\bar{\lambda}(A) = \lambda(A \cap E)$ for $A \in \bar{\mathscr{E}}^*$. It follows readily from (3.5) and (3.7) that

$$(3.9) \qquad \bar{P}^{\bar{\lambda}}(\bar{F}) = P^{\lambda}(\Phi^{-1}(\bar{F})) \textit{ for each } \bar{F} \in \bar{\mathscr{F}}^{\cdot} .$$

Let $\mathscr{F}^{\lambda}$ be the P^{λ}-completion of $\mathscr{F}^{\cdot}$ and let $\bar{\mathscr{F}}^{\bar{\lambda}}$ be the $\bar{P}^{\bar{\lambda}}$-completion of $\bar{\mathscr{F}}^{\cdot}$. It follows from (3.8) and (3.9) that

$$(3.10) \quad \Phi \text{ is } \mathscr{F}^{\lambda}/\bar{\mathscr{F}}^{\bar{\lambda}}\textit{-measurable and } \bar{P}^{\bar{\lambda}}(\bar{F}) = P^{\lambda}(\Phi^{-1}(\bar{F})) \textit{ for each } \bar{F} \in \bar{\mathscr{F}}^{\bar{\lambda}} .$$

Consider $\bar{A} \in \bar{\mathscr{E}}^e$. Let $A = \bar{A} \cap E$. Then $A \in \mathscr{E}^e$ by (3.6). Let $\bar{F} = \{\bar{X}_t \in \bar{A}$ for uncountably many $t\}$ and let $F = \{X_t \in A$ for uncountably many $t\}$. Then $\bar{F} \in \bar{\mathscr{F}}^{\bar{\lambda}}$ and $F = \Phi^{-1}(\bar{F})$ so by (3.10), $\bar{P}^{\bar{\lambda}}(\bar{F}) = P^{\lambda}(F)$. It follows that

$$(3.11) \quad \bar{A} \textit{ is } \bar{\lambda}\textit{-semipolar for } \bar{X} \textit{ if and only if } \bar{A} \cap E \text{ is } \lambda\textit{-semipolar for } X .$$

By a similar argument,

$$(3.12) \qquad \bar{A} \textit{ is } \bar{\lambda}\textit{-polar for } \bar{X} \textit{ if and only if } \bar{A} \cap E \textit{ is } \lambda\textit{-polar for } X .$$

Let m, μ, ν and C be as in Theorem (1.2). It follows from (3.11) and (3.12) that each $\bar{\mathscr{E}}^e$-measurable semipolar set for $\bar{X}$ is $\bar{m}$-polar and that for each $\bar{m}$-polar set $Z \in \bar{\mathscr{E}}^e$, we have $\bar{\nu}(Z) = \nu(Z \cap E) = \mu(Z \cap E \cap C) = \bar{\mu}(Z \cap C)$. From (3.5), it follows that for each s-finite measure λ on E, we have $\bar{\lambda}\bar{U} = \overline{\lambda U}$. Hence $\bar{\mu}\bar{U} \geq \bar{\nu}\bar{U}$. Thus the hypotheses of Theorem (1.2) hold with $\bar{X}$, m, μ, and ν replaced by $\bar{X}$, $\bar{m}$, $\bar{\mu}$, and $\bar{\nu}$. Since $\bar{X}$ is a Borel right process, there is a non-randomized stopping time $\bar{T}$ for $\bar{X}$ such that $\bar{\mu}\bar{P}_{\bar{T}} = \bar{\nu}$. For $0 \leqq t < \infty$, let $\mathscr{F}^{\cdot}_t = \sigma(X_s^{-1}(A): A \in \mathscr{E}^*, 0 \leqq s \leqq t)$ and let $\bar{\mathscr{F}}^{\cdot}_t = \sigma(\bar{X}_s^{-1}(A): A \in \bar{\mathscr{E}}^*, 0 \leqq s \leqq t)$. It follows from (3.7) that

$$\text{(3.13)} \qquad \Phi \textit{ is } \mathscr{F}^{\cdot}_t/\bar{\mathscr{F}}^{\cdot}_t\textit{-measurable for each } t\,.$$

Let $\mathscr{N}^{\mu} = \{F \in \mathscr{F}^{\mu}: P^{\mu}(F) = 0\}$, let $\bar{\mathscr{N}}^{\bar{\mu}} = \{\bar{F} \in \bar{\mathscr{F}}^{\bar{\mu}}: \bar{P}^{\bar{\mu}}(\bar{F}) = 0\}$, and for $0 \leqq t < \infty$, let $\mathscr{F}^{\mu}_t = \mathscr{F}^{\cdot}_t \vee \mathscr{N}^{\mu}$ and $\bar{\mathscr{F}}^{\bar{\mu}}_t = \bar{\mathscr{F}}^{\cdot}_t \vee \bar{\mathscr{N}}^{\bar{\mu}}$. It follows from (3.10) and (3.13) that

$$\text{(3.14)} \qquad \Phi \textit{ is } \mathscr{F}^{\mu}_t/\bar{\mathscr{F}}^{\bar{\mu}}_t\textit{-measurable for each } t\,.$$

Now $\bar{T}$ is an $(\bar{\mathscr{F}}^{\bar{\mu}}_t)$-stopping time. Let $T = \bar{T} \circ \Phi$. It follows from (3.14) that T is an $(\mathscr{F}^{\mu}_t)$-stopping time. Consider $A \in \mathscr{E}^*$. Then $A \in \bar{\mathscr{E}}^*$, $\{\bar{X}_{\bar{T}} \in A\} \in \bar{\mathscr{F}}^{\bar{\mu}}$, and $\Phi^{-1}(\{\bar{X}_{\bar{T}} \in A\}) = \{X_T \in A\}$, so $\mu P_T(A) = P^{\mu}(X_T \in A) = \bar{P}^{\bar{\mu}}(\bar{X}_{\bar{T}} \in A) = \bar{\mu}\bar{P}_{\bar{T}}(A) = \bar{\nu}(A) = \nu(A)$.

Thus $\mu P_T = \nu$. This completes the proof of Theorem (1.2) for a general right process.

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