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Max-infinitely divisible and max-stable sample continuous processes

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Summary. Conditions for a process ξ on a compact metric space S to be simultaneously max-infinitely divisible and sample continuous are obtained. Although they fall short of a complete characterization of such processes, these conditions yield complete descriptions of the sample continuous non-degenerate max-stable processes on S and of the infinitely divisible non-void random compact subsets of a Banach space under the operation of convex hull of unions.

0 Introduction

Let S be a topological space, ξ be a stochastic process on S and $\mathscr{F}(S)$ be a function space over S. ξ is max-infinitely divisible (max-i.d.) in $\mathscr{F}(S)$ if ξ has a version with all of its sample paths in $\mathscr{F}(S)$ and if for each $n \in \mathbb{N}$ there exist independent identically distributed processes ξ_{ni} with sample paths in $\mathscr{F}(S)$ such that

$$\mathscr{L}(\xi)=\mathscr{L}\left(\bigvee_{i=1}^{n} \xi_{ni}\right), \qquad n \in \mathbb{N}$$

as probability laws in $\mathscr{F}(S)$, where $\left(\bigvee_{i=1}^{n} \xi_{ni}\right)(t,\omega) \equiv \bigvee_{i=1}^{n} \xi_{ni}(t,\omega)$, i.e. the maxima are taken pointwise. If $\xi_{ni}=a_n^{-1}(\xi_i-b_n)$ where ξ_i are i.i.d. with law $\mathscr{L}(\xi)$, $a_n>0$ and b_n are functions in $\mathscr{F}(S)$, then ξ is max-stable in $\mathscr{F}(S)$. These processes arise as limits of pointwise maxima of i.i.d. random processes. (For an early example see Brown and Resnick [5] which considers the pointwise maxima of independent Brownian motions). If S is a finite set, these are the max-i.d.

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and max-stable random vectors on $\mathbb{R}^d$ which were studied by Balkema and Resnick [3], de Haan and Resnick [10] and Pickands [19]. The classical max-stable distributions in $\mathbb{R}$ go back to Fisher and Tippett [6] and Gnedenko [8]. For S countable and $\mathscr{F}(S)=\ell^\infty(S)$, the spectral representation of max-i.d. laws is obtained in Vatan [22], and that of max-stable laws in deHaan [9] and Vatan [22]. c_0 and $\ell^\infty(\mathbb{R})$ are also considered in Vatan [21] with the latter considered in deHaan [9] as well. Norberg [18] appears to be the first author to consider sample path properties when S is uncountable; he obtains the representation of max-i.d. laws in $U(S)$, the space of non-negative, not necessarily finite upper semicontinuous functions on a locally compact space S. We borrow from Norberg's result and proof to obtain fairly complete results (Theorems 2.1 and 2.4) on the representation of max-i.d. laws in $C(S)$, the space of continuous functions on a compact metric space. Our results give a complete characterization of only a subclass of such laws, those with a continuous vertex, but they contain a procedure to produce a large variety of max-i.d. sample continuous processes. Several examples and counterexamples in Sect. 2 give an idea of the scope of these results.

The results on max-i.d. laws in $C(S)$ are then applied to obtain in Theorem 3.7 the spectral representation of all the non-degenerate max-stable laws in $C(S)$ (non-degenerate in the sense that no one dimensional projections are degenerate). In another application (Theorem 4.1), the representation of a.s. non-void infinitely divisible random compact subsets of a Banach space with respect to the operation of convex hull of unions is also obtained (see also Norberg [18]). Stable random sets in a restricted sense are also characterized in the finite dimensional case. It is somewhat surprising that incomplete results for the general case do imply complete characterizations in these three important particular settings.

There is a recognized duality between results for max-i.d. or max-stability on the one hand and those for infinite divisibility or stability for sums on the other. As shown by Norberg in great generality, upper semi-continuity appears to be a very natural sample path property for max-i.d. and max-stable processes. This is analogous to the fact that a natural space for the paths of stable (for sums) Lévy processes is $D[0, 1]$. In this latter case, deep studies of sample continuity have unearthed a wealth of interesting stable processes with sample continuous paths, such as certain random Fourier series (Marcus and Pisier [14]), even though no complete characterization of sample continuity for stable processes has been found. As shown in this article, the situation (and of course the methods of study) in the max-stable case is completely different and less complex, but it is possible to provide a complete characterization of sample continuity for max-stable processes.

From a practical point of view, sample continuity of the max-stable limiting process is a pertinent question when considering max-domains of attraction and the appropriate space in which weak convergence is taking place. A typical motivating example in some of the initial studies of max-stability is the determination of bank heights for a river by recording the levels along a stretch of the river on a number of different occasions. Since the level of the river is often a random continuous function, weak convergence results will provide the most information if the maxima can be normalized to converge weakly in the space of continuous functions, which of course requires the weak limit to be sample continuous.

1 Preliminaries and notation

Our starting point is Norberg's representation of max-i.d. processes with upper semicontinuous non-negative sample paths ([18], Theorem 5.1). His result is based on Matheron's representation of random closed sets which are infinitely divisible with respect to unions ([15], Prop. 3.2.1) which in turn rests upon Choquet's Theorem characterizing alternating capacities of infinite order (e.g. [15], Theorem 2.2.1). (Matheron's representation can also be deduced from more general results on semigroups appearing in [4]).

In what follows, S is a compact metric space and (U, v) is the space of non-negative not necessarily finite upper semicontinuous real functions on S, endowed with the vague topology (cf. [17, 18, 23]). Given a function $h \in U$, $h \not\equiv -\infty$, $U^h = \{f \in U : f \not\equiv h, f \geqq h\}$. $C(S)$ or simply C denotes the space of continuous functions on S, with the sup norm, and for $h \in U$, $h \not\equiv -\infty$, we let $C^h = \{f \in C : f \not\equiv h, f \geqq h\}$. A measure on S will always mean a Borel measure. For functions f on S and sets $A \subset S$, we let $f(A) = \sup_{s \in A} f(s)$. (This notation closely follows that of [18].) Throughout the paper, maxima of functions is understood to mean pointwise maxima.

1.1. Definition. A stochastic process ξ on a compact metric space S is max-i.d. in $C(S)$ (max-i.d. in (U, v)) if it is sample continuous (has a version with sample paths in U) and for each $n \in \mathbb{N}$ there exist i.i.d. sample continuous processes ξ_{ni} (i.i.d. (U, v)-valued processes ξ_{ni}) such that

$$\mathscr{L}(\xi) = \mathscr{L}\left(\bigvee_{i=1}^{n} \xi_{ni}\right). \tag{1.0}$$

It will be understood, throughout the paper, that if a process ξ is sample continuous, then reference is always to a version of it with continuous sample paths.

We state Norberg's Theorem here for the reader's convenience:

1.2. Theorem (Norberg [18]). *ξ is* max-i.d. *in (U, v) and ξ is not carried by $\{\infty\}$ if and only if there is a (unique) pair (h, v) satisfying*

(i) $h: S \to [0, \infty]$, $h \not\equiv \infty$, *is an upper semicontinuous function and*
(ii) v *is a locally finite (finite on compact sets) measure on* U^h

such that

$$h(K) = \sup[x : P\{\xi(K) \geqq x\} = 1], \qquad K \subset S \text{ closed}, \tag{1.1}$$

$$v\left(\bigcup_{i=1}^{n} \{f : f(K_i) \geqq x_i\}\right) = -\log P\left(\bigcap_{i=1}^{n} \{\xi(K_i) < x_i\}\right), \qquad n \in \mathbb{N}, \quad K_i \subset S \text{ closed}, \quad x_i > h(K_i), \tag{1.2}$$

and

$$\mathscr{L}(\xi) = \mathscr{L}\left(h \vee \left(\bigvee_{i=1}^{\infty} \eta_i\right)\right) \tag{1.3}$$

where $\{\eta_i\}$ *is the collection of atoms of the Poisson point process with mean measure* ν.

h is called the *vertex* and ν the *max-Lévy measure* of the max-i.d. process ξ. We refer to Resnick [20] and Karr [13] for the definition and properties of Poisson point processes.

1.3. Remark. Obviously in Theorem 1.2, U can be replaced by the upper semicontinuous functions with values in $[c, \infty]$, $c \in \mathbb{R}$, and then h takes values in $[c, \infty]$. We are also interested in the possibility of taking $c = -\infty$. Norberg's proof shows that if ξ is upper semicontinuous and takes its values in $(-\infty, \infty)$, then there still exist h and ν. In this case, h is upper semicontinuous and takes values in $[-\infty, \infty)$, whereas ν is defined on the space of functions from S to $[-\infty, \infty]$ whose hypographs ($=\text{hypo} f = \{(s, x)\colon -\infty < x \leqq f(s)\}$) are closed sets that contain the hypograph of h. These functions are upper semicontinuous but may take the values $+\infty$ and $-\infty$ (although $\nu\{f(S) = \infty\} = -\log P\{\xi(S) < \infty\} = 0$). But given a function h and a locally finite measure ν with these properties then the corresponding max-i.d. process ξ may take the value $-\infty$. (The way to prove these observations is, as in Norberg's Theorem, to pass to hypographs and then apply Matheron's Theorem; the existence of h upper semicontinuous requires an easy extension of Matheron's Theorem ([16], Theor. 2.4).)

An important sub-class of max-i.d. processes are the max-stable ones that will be considered in Sect. 3.

1.4. Definition. A stochastic process ξ on a compact metric space S, such that $\xi(t)$ is non-degenerate for all $t \in \mathbb{R}$, is *max-stable* in $C(S)$ if ξ is sample continuous and for each n there exist continuous functions $a_n(t) > 0$, $b_n(t)$, $t \in S$, such that if ξ_i are i.i.d. copies of ξ,

$$\mathscr{L}(\xi) = \mathscr{L}\left(a_n^{-1}\left(\bigvee_{i=1}^{n} \xi_i - b_n\right)\right). \tag{1.4}$$

It is well-known ([24, 25]) that every nondegenerate max-stable distribution function on $\mathbb{R}$ is of the type of one and only one distribution in the parametric family

$$F_\gamma(x) = \exp\{-(1+\gamma x)^{-1/\gamma}\}, \qquad \gamma x \geqq -1, \qquad \gamma \in \mathbb{R} \tag{1.5}$$

where it is understood that if $\gamma > 0$ (type I) then $F_\gamma(x) = 0$ for $x \leqq -1/\gamma$, that if $\gamma < 0$ (type II) then $F_\gamma(x) = 1$ for $x \geqq -1/\gamma$, and that if $\gamma = 0$ (type III) then $(1+\gamma x)^{-1/\gamma}$ is an abuse of language for e^{-x}, $-\infty < x < \infty$. This is a continuous parameterization: if $\gamma_k \to \gamma$ then $F_{\gamma_k}(x) \to F_\gamma(x)$, $x \in \mathbb{R}$, and in fact, since F_γ is continuous, $\|F_{\gamma_k} - F_\gamma\|_\infty \to 0$. Since the maxima in (1.4) are taken pointwise, if ξ is max-stable in $C(S)$ (or, in general, if ξ is a max-stable process with non-degenerate marginals), then for each $t \in S$ the real random variable $\xi(t)$ is max-stable and non-degenerate. Therefore, there exist functions $a(t) > 0$, $b(t) \in \mathbb{R}$, $t \in S$, such that for all $x \in \mathbb{R}$

$$P((\xi(t) - b(t))/a(t) \leqq x) = F_{\gamma(t)}(x). \tag{1.6}$$

The relationship between the functions $a_n(t)$ and $b_n(t)$ of Definition (1.4) and the functions $a(t)$, $b(t)$ and $\gamma(t)$ of (1.6) for a given $\xi(t)$ is easily seen to be:

(1.7) $$a_n(t)=n^{\gamma(t)} \quad \text{(for any value of } \gamma(t)),$$

(1.8) $$b_n(t)=(n^{\gamma(t)}-1)(a(t)/\gamma(t)-b(t)) \quad \text{for } \gamma(t)\neq 0$$

and

(1.9) $$b_n(t)=a(t)\log n \quad \text{for } \gamma(t)=0.$$

The continuity of $a_n(t)$ and $b_n(t)$, together with the sample continuity of ξ, imply the continuity of $\gamma(t)$, $a(t)$ and $b(t)$ (Lemma 3.5). This continuity is essential in order to reduce the general case to the case of simple max-stable processes (i.e. those whose marginals have distribution $\Phi_{1,1}(x)\equiv e^{-x^{-1}}I(x>0)$, $x\in\mathbb{R}$). The theory to be developed in Sect. 2 is suitable for handling the simple max-stable processes. Then, via the reduction, a complete characterization of all max-stable processes in $C(S)$ is deduced in Theorem 3.7.

In Sect. 4, the theory from the previous two sections will be applied to obtain representations of random compact sets which are i.d. or finite-dimensional random compact sets which are stable for the operation of convex hulls of unions (see [18] for related results on max-i.d. random sets). The standard facts on random sets needed for this will be recalled at the beginning of Sect. 4.

2 Max-i.d. sample continuous processes

Our first result gives a fairly general (although not a completely general) way of constructing max-i.d. sample continuous processes.

2.1. Theorem. *Let h be an upper semicontinuous function on S with $-\infty\leqq h(s)<\infty$ for all $s\in S$ and let $C^h\equiv\{f\in C(S)\colon f\neq h, f\geqq h\}$. Let ν be an infinite, locally finite measure on C^h such that*

(2.1) $$\nu\{f\in C^h\colon f(I)\geqq x\}<\infty \quad \text{if } x>h(I)$$

for all closed balls $I\subset S$. Let $\{\eta_i\}$ be the points of a Poisson point process η_ν on $C(S)$ with mean measure ν. Then the process

(2.2) $$\xi=\bigvee_{i=1}^{\infty}\eta_i\left(=h\vee\left(\bigvee_{i=1}^{\infty}\eta_i\right)\text{a.s.}\right)$$

is a max-i.d. *sample continuous process and the relations* (1.1) *and* (1.2) *hold for ξ, h and ν. If h is continuous, the result holds also for finite ν.*

Proof. For simplicity, we write C for $C(S)$ and η for η_ν. Since ν is infinite, $P\{\eta_\nu(C)=0\}=e^{-\nu(C)}=0$. Therefore, $h\vee\left(\bigvee_{i=1}^{\infty}\eta_i\right)=\bigvee_{i=1}^{\infty}\eta_i$ as $\eta_i\geqq h$ for all i. Since each η_i is in C, the process $\xi = \bigvee_{i=1}^{\infty} \eta_i = \lim_{n\to\infty} \bigvee_{i=1}^{n} \eta_i$ is lower semicontinuous and has $\inf_{s\in S} \alpha(s) > -\infty$. Next we show that the process ξ is a.s. bounded. Let $A_I^x = \{f \in C^h\colon f(I) \geqq x\}$ with $x > h(I)$. Then by (2.1),

$$P\{\eta(A_I^x) < \infty\} = P\{N_{\nu(A_I^x)} < \infty\} = 1$$

where N_λ is a Poisson random variable with mean λ. Since $h(s) < \infty$ for all s and h is upper semicontinuous, every $s \in S$ has a neighborhood where h is bounded from above so that by compactness $h(S) < \infty$. Taking $I = S$ in (2.3) shows that for each $x > h(S)$ the number of $\eta_i(S)$ larger than x is a.s. finite, and therefore $P\{\xi(S) = \infty\} = 0$, i.e. ξ is a.s. bounded. So, since $|\xi|(S) < \infty$ and ξ has a.s. lower semicontinuous paths, in order to show that ξ is sample continuous it is enough to prove that ξ has upper semicontinuous sample paths with probability one.

Write η_ω for $\eta(\omega)$. Let Ω_1 be the set of ω's for which $\eta_\omega(A_I^x) < \infty$ for all $x > h(I)$ rational and all closed balls I with rational radii and centers in a countable dense set of points of S, and for which $\eta_i(\omega) \geqq h$ for all i. Then $P(\Omega_1) = 1$ by (2.3). Take now $\omega \in \Omega_1$ and $s \in S$. Assume $\xi(\omega)(s) < x$ for some $x < \infty$ (which implies, in particular, $h(s) < x$). Take $I_n \downarrow \{s\}$, I_n closed balls in the countable collection just described, with $s \in (I_n)^\circ$. Then $A_{I_n}^x \downarrow A_{\{s\}}^x$. Moreover, h being upper semicontinuous, $h(s) < x$ implies $h(I_n) < x$ from some n on; hence $\eta_\omega(A_{I_n}^x) < \infty$ by assumption. So, we can apply downward continuity of η_ω and obtain $\eta_\omega(A_{I_n}^x) \downarrow \eta_\omega(A_{\{s\}}^x)$. But $\xi(\omega)(s) < x$ implies $\eta_\omega(A_{\{s\}}^x) = 0$, and since η is integer valued, $\eta_\omega(A_{I_n}^x) = 0$ from some $n(\omega)$ on. Equivalently, $\sup_{u \in I_n} \eta_i(\omega)(u) < x$ for $n > n(\omega)$ and $i = 1, \ldots$. Hence $\overline{\lim}_{u\to s} \xi(\omega)(u) \leqq \overline{\lim}_{n\to\infty} \sup_{u\in I_n} \bigvee_{i=1}^{\infty} \eta_i(\omega)(u) \leqq x$. This shows that for all $\omega \in \Omega_1$, $s \in S$, $\overline{\lim}_{u\to s} \xi(\omega)(u) \leqq \xi(\omega)(s)$, i.e. that $\xi(\omega)$ is upper semicontinuous. We have thus proved that ξ is sample continuous.

Note that, since h is upper semicontinuous, a simple compactness argument shows that if (2.1) holds, then it holds for all $K \subset S$ closed. Now, $\xi(K) > x$ if and only if $\eta_i(K) > x$ for some i, i.e. $\{\xi(K) > x\} = \{\eta\{f(K) > x\} \neq 0\}$. Therefore, $P\{\xi(K) > x\} = 1$ implies that $P(\eta\{f(K) > x\} = 0) = 0$ which in turn implies that $\nu\{f(K) > x\} = \infty$ (since $\nu(A) = p < \infty \Rightarrow P(\eta(A) = 0\} = e^{-p} \neq 0$), hence by (2.1), that $x \leqq h(K)$. Hence

$$h(K) \geqq \sup\{x\colon P\{\xi(K) \geqq x\} = 1\} \qquad (= \sup\{x\colon P\{\xi(K) > x\} = 1\}).$$

The reverse inequality is trivial from $h(K) \leqq \xi(K)$ a.s. Hence, (1.1) is proved.

To prove (1.2) note that the above observation shows that if $x_i > h(K_i)$, $i = 1, \ldots n$, then

$$\begin{aligned} (2.4) \qquad P\{\xi(K_i) \leqq x_i, \quad i = 1, \ldots, n\} &= P\{\eta\{f(K_i) > x_i\} = 0, \quad i = 1, \ldots, n\} \\ &= P\left(\eta\left(\bigcup_{i=1}^{n} \{f(K_i) > x_i\}\right) = 0\right) \\ &= \exp\left\{-\nu\left(\bigcup_{i=1}^{n} \{f(K_i) > x_i\}\right)\right\}. \end{aligned}$$

Since (2.4) is equivalent to (1.3), this completes the proof of the theorem for ν infinite. The only obstruction in the finite case is that $\xi = h \vee \left(\bigvee_{i=1}^{\infty} \eta_i\right)$ takes the value h with probability $e^{-\nu(C)} \neq 0$. Hence, in this case, we need h to be continuous.

It remains to verify that ξ is max-i.d. This is obvious because for all n, the pairs $(h, \nu/n)$ satisfy the hypotheses of the theorem, hence there is a sample continuous process ξ_n which is related to ν/n and h by (1.1)–(1.3). Consequently, if ξ_{ni}, $i=1, \ldots, n$, are i.i.d. copies of ξ_n, then for any $t_1, \ldots, t_m \in [0,1]$, $x_1, \ldots, x_m \in \mathbb{R}$ with $h(t_j) > x_j$, (1.2) immediately yields $P\left(\bigcap_{r=1}^{m} \{\xi(t_r) < x_r\}\right)$ $= P\left(\bigcap_{r=1}^{m} \left\{\bigvee_{i=1}^{n} \xi_{ni}(t_r) < x_r\right\}\right)$, i.e. $\mathscr{L}(\xi) = \mathscr{L}\left(\bigvee_{i=1}^{n} \xi_{ni}\right)$. □

By Dini's lemma, the convergence of the partial maxima $\bigvee_{i=1}^{n} \eta_i$ to ξ in Theorem 2.1 is uniform for almost every ω.

2.2 Remark. Theorem 2.1 applies to give the sample continuity of the limiting process arising from the normalized maxima of Brownian motions considered in Brown and Resnick [5].

We only have a converse to the above theorem if h is continuous. (Note that, by Theorem 1.2, if ξ is upper semicontinuous – hence, *a fortiori*, if ξ is continuous – then the function h defined by (1.2) is also upper semicontinuous.) Before proving the partial converse, we note that condition (2.1) in Theorem 2.1 cannot be relaxed.

2.3 Example. Let

$$f_i(s) = \begin{cases} is & 0 \leqq s \leqq \frac{1}{i} \\ 1 & \frac{1}{i} \leqq s \leqq 1. \end{cases}$$

$$\nu\{f_i\} = \frac{1}{i}, \qquad i=1, \ldots, \qquad h = f_1.$$

Then ν lives in C^h and h is the largest function for which this is true. We have $\eta = \sum_{i=1}^{\infty} N_{1/i} \delta_{f_i}$ with $N_{1/i}$ independent Poisson random variables with mean $1/i$. So, $\eta\{f_i\} = N_{1/i}$, $P\{\eta\{f_i\} \neq 0\} = 1 - e^{-1/i}$, and the sets $\{\eta\{f_i\} \neq 0\}$ are independent. Hence, since $\Sigma(1 - e^{-1/i}) = \infty$, we have $P\{\eta\{f_i\} \neq 0 \text{ i.o.}\} = 1$, i.e. the atoms $\{\eta_i\}$ of the point process with mean measure ν contain infinitely many different f_i with probability 1. This shows that $\xi = \bigvee_{i=1}^{\infty} \eta_i = I_{(0,1]}$ a.s., i.e. ξ is not continuous (in fact it is not even upper semicontinuous). □

2.4 Theorem. *Let ξ be a* max-*i.d. sample continuous process on S such that the function defined by*

$$h(s) = \sup\{x\colon P\{\xi(s) \geqq x\} = 1\}, \qquad s \in S$$

is continuous. Then there is a locally-finite Borel measure ν *on* C^h *such that, if* $\{\eta_i\}$ *are the atoms of a Poisson point process with mean measure* ν, *then*

$$\xi \overset{\mathscr{D}}{=} h \vee \left(\bigvee_{i=1}^{\infty} \eta_i \right),$$

and the relations (1.1) *and* (1.2) (*hence also* (2.4) *and* (2.1)) *hold for* ξ, h *and* ν.

Proof. Since h is continuous, $h(s) > -\infty$ for all s. In what follows, by subtracting h from ξ if necessary, we may assume $h \equiv 0$. By Norberg's Theorem 5.1 in [18] (Theorem 1.2 above with $h=0$), there is a locally finite (hence σ-finite) measure ν on $\mathscr{U}^h$ (endowed with the vague topology) such that (1.2) and (1.3) hold. If $\nu \equiv 0$, then $\xi \equiv 0$ in which case the theorem is obvious. Hence, it may also be assumed that $\nu \not\equiv 0$. The proof consists of showing that $\nu^*(C^c)=0$ (actually, $\nu_*(C^c)=0$ would be enough) since then the result follows by Theorem A, Sect. 17 in Halmos [11] (note that the trace of the Borel σ-algebra of (U, v) on C contains the balls).

Consider

$$\nu|_{\{\|f\|_\infty > \varepsilon\}} \equiv \nu^\varepsilon \quad \text{and} \quad \nu|_{\{\|f\|_\infty \leqq \varepsilon\}} \equiv \nu_\varepsilon$$

and let W and V be independent max-i.d. processes with trajectories in U associated to the pairs $(0, \nu^\varepsilon)$ and $(0, \nu_\varepsilon)$ respectively. Then $Y = W \vee V$ has law $Q = \mathscr{L}(\xi)$. Furthermore,

$$\|V(\omega)\|_\infty \leqq \varepsilon \quad a.s. \tag{2.5}$$

Note that $|\nu^\varepsilon| \equiv \nu^\varepsilon(U) < \infty$. This follows because $\nu\{\|f\|_\infty > \varepsilon\} = -\log P\{\|\xi\|_\infty \leqq \varepsilon\} < \infty$ since $h \equiv 0$ satisfies (1.1). So, we can take W to be the process

$$W \equiv \bigvee_{j=1}^{N_{|\nu^\varepsilon|}} Z_j$$

where $Z = Z_1, Z_2, \ldots$ are i.i.d. with $\mathscr{L}(Z) = \dfrac{\nu^\varepsilon}{|\nu^\varepsilon|}$, independent of V, and $N_{|\nu^\varepsilon|}$ is Poisson with parameter $|\nu^\varepsilon|$ independent of both $\{Z_i\}$ and V. (For verification, simply compute the characteristic functions.) In particular,

$$\{W = Z\} \supseteq \{N_{|\nu^\varepsilon|} = 1\} \quad \text{and} \tag{2.6}$$
$$P(N_{|\nu^\varepsilon|} = 1) = \nu\{\|f\|_\infty > \varepsilon\}\, e^{-\nu\{\|f\|_\infty > \varepsilon\}} > 0$$

for all $\varepsilon > 0$ small enough (if $\nu \not\equiv 0$ then there is $\varepsilon_0 > 0$ such that $\nu^\varepsilon \not\equiv 0$ for all $0 < \varepsilon < \varepsilon_0$).

Since $\mathscr{L}(W \vee V) = \mathscr{L}(\xi)$ and ξ is sample continuous, it follows that

$$P^*(W \vee V \in C^c) = 0. \tag{2.7}$$

(If we call Q the law of ξ in C, and i the inclusion of $(C, \|\cdot\|_\infty)$ into $(U\, v)$, which is continuous, then $\mathscr{L}(W \vee V) = Q \circ i^{-1}$ gives mass 1 to a σ-compact subset D of (W, v) contained in $i(C)$ because Q is tight in C. Therefore, $P_*\{W \vee V \in C\}$

$\geq P\{W \vee V \in D\} = Q(i^{-1}(D)) = P(Z \in D) = 1$.) Assuming, without loss of generality, that $N_{|\nu^\varepsilon|}$, V and Z_i, $i \in \mathbb{N}$, are defined as coordinates in a product probability space, we have by (2.6) and (2.7)

$$0 = P^*(W \vee V \in C^c) \geq P^*(Z \vee V \in C^c, N_{|\nu^\varepsilon|} = 1) = P^*(Z \vee V \in C^c)\, P(N_{|\nu^\varepsilon|} = 1).$$

(This requires the obvious fact that $(\mu \times \gamma)^*(A \times B) = \mu^*(A)\, \gamma^*(B)$.) Therefore

$$P^*(Z \vee V \in C^c) = 0. \tag{2.8}$$

Let

$$A_m^r = \Big\{f \geq 0\colon\ f \in U,\ \|f\|_\infty > r, f(t) \text{ is continuous if } f(t) > \frac{1}{m},$$
$$\overline{\lim_{t_n \to t}}\, f(t_n) \leq \frac{1}{m} \quad \text{for all } t_n \to t \quad \text{if} \quad f(t) = 0\Big\}.$$

Let m_0 be such that $\nu^{1/m} \neq 0$ for all $m \geq m_0$. For some $m > m_0$, let $\varepsilon = 1/m$, and let Z and V be as above for this ε. Then, since $\|V\|_\infty \leq \varepsilon$, in order that $V(\omega) \vee Z(\omega)$ be a continuous function it is necessary that $Z \in A_m^r$ whenever $\|Z\|_\infty > r$, $r \geq \varepsilon$. Thus, by (2.5), (2.8) and the perfectness of coordinates in a product probability space (e.g. Andersen [1], Prop. 3.1) we have, for all $r \geq \varepsilon$

$$0 = P^*\{Z \vee V \in C^c\} \geq P^*\{Z \in \{\|f\|_\infty > r\} \setminus A_m^r\} = \left(\frac{\nu^\varepsilon}{|\nu^\varepsilon|}\right)^* (\{\|f\|_\infty > r\} \setminus A_m^r).$$

Since for $r \geq \varepsilon$, $\nu_\varepsilon\{\|f\|_\infty > r\} = 0$ ($\{\|f\|_\infty > \varepsilon\}$ is a Borel set of (U, v)), we obtain

$$\nu^*(\{\|f\|_\infty > r\} \setminus A_m^r) = 0, \qquad r \geq \varepsilon.$$

Let now $C_r = C \cap \{f \geq 0, \|f\|_\infty > r\}$. Since $C_r = \bigcap_{m > 1/r} A_m^r$ (note $A_m^r \downarrow$ as $m \uparrow$, for fixed r), we obtain

$$\nu^*(C^c \cap \{\|f\|_\infty > r\}) = \nu^*(\{\|f\|_\infty > r\} \setminus C_r) \leq \sum_{m \geq 1/r} \nu^*(\{\|f\|_\infty > r\} \setminus A_m^r) = 0.$$

Since ε is only required to be $0 < \varepsilon = 1/m < 1/m_0$ in this argument, it follows that the identity $\nu^*(C^c \cap \{\|f\|_\infty > r\}) = 0$ holds for all $r > 0$. Then

$$\nu^*(C^c) \leq \sum_{r^{-1} = m_0 + 1}^{\infty} \nu^*(C^c \cap \{\|f\|_\infty > r\}) = 0. \quad \square$$

Theorem 2.1 and 2.3 completely characterize the max-i.d. laws in $C(S)$ *which have a continuous vertex*. It is possible for the vertex h to be upper semicontinuous but not continuous. When this happens ξ may be sample-continuous even if ν is not concentrated on the continuous functions as the following example indicates.

2.5. Example. Take for $n \in \mathbb{N}$,

$$f_n(s) = \begin{cases} 0 & 0 \leq s \leq 1 - 1/n \\ (n-1)\, s - n + 2 - 1/n & 1 - 1/n \leq s < 1 \\ 1 & s = 1, \end{cases}$$

and let $\nu\{f_n\}=\frac{1}{n}$, $n\in\mathbb{N}$. The vertex is

$$h(s)=I_{\{1\}}(s),$$

which is upper semi-continuous but not continuous.

Let $\{\eta_i\}$ be the atoms of a Poisson point process with mean measure ν. Then

$$\xi=\bigvee_{i=1}^{\infty}\eta_i$$

is max-i.d. (at least in (U, v)) by Norberg's Theorem.

Only finitely many of the f_n's are non-zero on each interval $[0, t]$, $t<1$. Therefore, $\xi(s)$ is the maximum of at most finitely many non-zero continuous functions and hence is continuous at $s\neq 1$. Furthermore, ξ is continuous at 1 because almost always occur infinitely many f_n's among the η_i's (since ν has infinite mass), and along any subsequence $(n_k)\subset(n)$

$$\lim_{s\to 1}\bigvee_{k=1}^{\infty} f_{n_k}(s)=1=\bigvee_{k=1}^{\infty} f_{n_k}(1).$$

Thus, ξ is sample continuous max-i.d. even though ν is concentrated on C^c. □

Although the max-Lévy measure ν of a max-i.d. sample continuous process need not live on C, it cannot be concentrated too far away from C. In fact, if $h=-\infty$, then ν is supported by the continuous functions on S with values in $[-\infty, \infty)$.

2.6. Corollary. *Let ξ be a* max-i.d. *sample continuous process on S, and let (h, ν) be its associated pair. Let $g\geqq h$ be a continuous function. Then*

$$\nu^*\{f\in U^h\colon f\vee g\in C^c\}=0.$$

In particular, $\nu^*\{f\geqq g\colon f\in C^c\}=0$. *If* $h=-\infty$ *then*

$$\nu^*\{f\in U^h\colon f\vee M\in C^c \text{ for some } M>-\infty\}=0. \tag{2.9}$$

Proof. For any Borel set A of U^g, let

$$\nu_g(A)=\nu\{f\in U^h\colon f\vee g\in A\}.$$

Then (g, ν_g) is the pair associated to the max-i.d. sample continuous process $\xi\vee g$, which therefore satisfies the hypotheses of Theorem 2.4. Then

$$\nu_g^*(C^c)=0$$

by Theorem 2.4. Hence, $\nu^*\{f\colon f\vee g\in C^c\}=0$. □

It is possible for $h\equiv-\infty$ and for ν to satisfy (2.9) in Corollary 2.6, but for ξ to fail to be sample continuous:

2.7. *Example.* Let $v(f_n)=\frac{1}{n}$ for $n=2, 3, \ldots$ where f_n are constructed so that for $j>1$

$$f_j(t) \quad \text{is continuous on} \quad (2^{-j}, 2^{-j+1})$$

with $f_j(3\cdot 2^{-j-1})=0$

$$\lim_{t\uparrow 2^{-j+1}} f_j(t)=-\infty$$

$$\lim_{t\downarrow 2^{-j}} f_j(t)=-\infty$$

$$f_j(t)=-\infty \quad \text{on} \quad (2^{-j}, 2^{-j+1})^c.$$

Then ξ is discontinuous at 0. □

3 Max-stable sample continuous processes

Proposition 3.2 below provides the representation theorem for max-stable processes of a special type. This is one of the two main steps in the proof of the general result, Theorem 3.7.

3.1. Definition. A stochastic process $\xi(t)$, $t\in S$, is *simple* max-*stable* in $C(S)$ if it is max-stable in $C(S)$ and if for every $t\in S$, $\xi(t)$ has a $\Phi_{1,1}$ distribution, that is

$$P\{\xi(t)\leqq x\}=e^{-x^{-1}}I(x>0), \qquad t\in S, \qquad x\in\mathbb{R}. \tag{3.1}$$

(Note that $\Phi_{1,1}(x)=F_1(x-1)$.)

3.2. Proposition. *For a stochastic process ξ on S the following are equivalent:*

(i) *ξ is simple* max-*stable in $C(S)$.*

(ii) *There is a finite Borel measure σ on $C_1^+=\{f\in C(S)\colon \|f\|_\infty=1, f\geqq 0\}$ such that for all $t\in S$, $\int_{C_1^+} f(t)\,d\sigma(f)=1$, and if v denotes the measure $dv=d\sigma\times dr/r^2$ on $C_1^+\times\mathbb{R}_+=C(S)^+=\{f\in C(S)\colon f\geqq 0, f\not\equiv 0\}$, then*

$$\mathscr{L}(\xi)=\mathscr{L}\left(\bigvee_{i=1}^{\infty}\eta_i\right) \tag{3.2}$$

where $\{\eta_i\}$ are the points (functions) of the Poisson process with mean measure v.

(iii) *There is a finite Borel measure σ on C_1^+ with $\int_{C_1^+} f(t)\,d\sigma(f)=1$ for all $t\in S$ such that for $K_1, \ldots, K_n$ compact subsets of S, $x_1, \ldots, x_n>0$, $n\in\mathbb{N}$,*

$$-\log P\left(\bigcap_{i=1}^{n}\{\xi(K_i)<x_i\}\right)=\int_{C_1^+}\max_{i\leqq n}\frac{g(K_i)}{x_i}\,d\sigma(g). \tag{3.3}$$

(iv) *ξ is sample continuous and there is a finite Borel measure σ on C_1^+ with*

$$\int_{C_1^+} f(t)\,d\sigma(f)=1 \quad \text{for all } t\in S \quad \text{such that for all } f\in C(S), \quad f>0,$$

(3.4)
$$-\log P\{\xi<f\}=\int_{C_1^+} \|g/f\|_\infty\,d\sigma(g).$$

Proof. (i)$\Rightarrow$(ii): If ξ is simple max-stable in $C(S)$ then in particular ξ is max-i.d. in $C(S)$. Moreover,

$$h(t)=\sup\{x\colon P\{\xi(t)\ge x\}=1\}=\sup\{x\colon e^{-x^{-1}}I(x>0)=0\}=0, \qquad t\in S,$$

i.e. the vertex of ξ is $h\equiv 0$ which is continuous. Hence, by Theorem 2.4 there is ν on $C(S)^+$ such that

(3.5)
$$\mathscr{L}(\xi)=\mathscr{L}\left(h\vee\left(\bigvee_{i=1}^{\infty}\eta_i\right)\right)$$

where $\{\eta_i\}$ are the points of the Poisson point process with mean measure ν. Now, as is customarily done ([10]), we obtain the form of ν from the fact that if ξ_i are i.i.d. with $\mathscr{L}(\xi_i)=\mathscr{L}(\xi)$, then

(3.6)
$$\mathscr{L}\left(n^{-1}\bigvee_{i=1}^{n}\xi_i\right)=\mathscr{L}(\xi)$$

(which is obvious from (3.1)). Let $\bar{K}=\{K_1,\dots,K_n\}$, $\bar{x}=\{x_1,\dots,x_n\}$, $n\in\mathbb{N}$, where K_i are compact subsets of S and x_i are positive numbers. For ease of notation, define

$$A_{\bar{K},\bar{x}}=\{f\in C(S)^+\colon f(K_i)\le x_i,\ i=1,\dots,n\}.$$

Then

(3.7)
$$\begin{aligned}
\nu(A^c_{\bar{K},\bar{x}}) &= -\log P\{\xi\in A_{\bar{K},\bar{x}}\} \qquad \text{by (2.4)}\\
&= -\log P\left\{\bigvee_{i=1}^{n}\xi_i\in n\,A_{\bar{K},\bar{x}}\right\} \qquad \text{by (3.6)}\\
&= -n\log P\{\xi\in n\,A_{\bar{K},\bar{x}}\}\\
&= -n\log P\{\xi\in A_{\bar{K},n\bar{x}}\}\\
&= n\,\nu(nA^c_{\bar{K},\bar{x}}) \qquad \text{by (2.4).}
\end{aligned}$$

Since the sets $A^c_{\bar{K},\bar{x}}$ determine Borel measures on $C(S)^+$, it follows that

$$\nu(A)=n\,\nu(nA), \qquad n\in\mathbb{N}, \qquad A\in\mathscr{B}(C(S)^+).$$

Hence (by standard arguments),

(3.8)
$$\nu(A)=s\,\nu(sA), \qquad s\in\mathbb{R}_+, \qquad A\in\mathscr{B}(C(S)^+).$$

It follows from this, as in e.g. [10], that if a measure σ is defined on C_1^+ by

$$\sigma(A)=\nu\{g\colon g/\|g\|_\infty\in A,\ \|g\|_\infty\ge 1\}, \qquad A\in\mathscr{B}(C_1^+),$$

then, with the "polar coordinates" identification

$$C(S)^+ \leftrightarrow C_1^+ \times \mathbb{R}_+$$
$$f \leftrightarrow (f/\|f\|_\infty, \|f\|_\infty),$$

we have

$$dv = d\sigma \times dr/r^2. \tag{3.9}$$

In particular, $v(C(S)^+ = \infty$ so that (3.5) becomes (3.2). Finally, for all $t \in S$,

$$\begin{aligned}(3.10)\quad 1 &= -\log P\{\xi(t) < 1\} \quad \text{by (3.1)}\\ &= v\{g:\ g(t) \geqq 1\} \quad \text{by (2.4)}\\ &= \int_{C_1^+ \times \mathbb{R}_+} I((f, r):\ \|f\|_\infty = 1, r \geqq 1/f(t))\, d\sigma(f) \times dr/r^2 \quad \text{by (3.9)}\\ &= \int_{C_1^+} \left(\int_{1/f(t)}^{\infty} \frac{dr}{r^2}\right) d\sigma(f) = \int_{C_1^+} f(t)\, d\sigma(f),\end{aligned}$$

thereby completing the proof of (ii).

(ii)⇒(i): $h=0$ and v as in (ii) are respectively the vertex and the max-Lévy measure of a max-i.d. sample continuous process ξ on S by Theorem 2.1. By undoing the steps in the previous proof it is obvious that (3.9) implies (by (3.8) and (3.7))

$$P\{\xi \in A_{\bar{K}, \bar{x}}\} = P\left\{\bigvee_{i=1}^{n} \xi_i/n \in A_{\bar{K}, \bar{x}}\right\}$$

for all $\bar{K}$, $\bar{x}$, where ξ_i are i.i.d. with law $\mathscr{L}(\xi)$. Hence

$$\mathscr{L}(\xi) = \mathscr{L}\left(n^{-1} \bigvee_{i=1}^{n} \xi_i\right),$$

i.e. ξ is max-stable in $C(S)$. Finally, a computation completely analogous to that in (3.10), together with (2.4), gives

$$x^{-1} = x^{-1} \int_{C_1^+} f(t)\, d\sigma(f) = v\{g: g(t) \geqq x\} = -\log P\{\xi(t) < x\}$$

for all $t \in S$ and $x > 0$, i.e. (3.1). Thus, ξ is simple max-stable in $C(S)$.

(iii)⇔(ii): This is just a computation similar to (3.10): if v is defined by (3.9) then

$$\begin{aligned}(3.11)\quad v(A^c_{\bar{K}, \bar{x}}) &= \int_{C_1^+ \times \mathbb{R}_+} I((r, f): \|f\|_\infty = 1, r \geqq \min_{i \leqq n} x_i/f(K_i))\, d\sigma(f) \times dr/r^2\\ &= \int_{C_1^+} \left(\int_{\min_{i \leqq n} x_i/f(K_i)}^{\infty} dr/r^2\right) d\sigma(f)\\ &= \int_{C_1^+} \max_{i \leqq n} \frac{f(K_i)}{x_i}\, d\sigma(f).\end{aligned}$$

So, (3.3) simply expresses the fact that the max-Lévy measure of ξ is $d\nu = d\sigma \times dr/r^2$ (note that (3.3) does determine the finite dimensional distributions of ξ).

(iv)$\Leftrightarrow$(iii): Since (3.4) is nothing but

$$-\log P\{\xi < f\} = \nu\{g \in C(S)^+ :\ g \not< f\}, \tag{3.4'}$$

as a computation not unlike (3.11) shows, it follows that (iii) implies (iv) (use for instance (4.9) and (4.10) below to obtain (3.4′) from (3.3) via (3.11)). (Note that the $<$ in (3.3), (3.4) and (3.4′) can be replaced by $\leqq$). If σ is as in (iv), i.e. as in (ii), then $d\nu = d\sigma \times dr/r^2$ is the max-Lévy measure of a simple max-stable sample continuous process $\bar{\xi}$ as shown above. Then (iii) holds for $\bar{\xi}$, hence so does (3.4′) by the argument just mentioned. Hence $P\{\bar{\xi} < f\} = P\{\xi < f\}$ for all $f > 0$, $f \in C(S)$, and therefore $\mathscr{L}(\bar{\xi}) = \mathscr{L}(\xi)$ since these quantities do determine the finite dimensional distributions of sample continuous processes. □

Next we show that simple max-stable processes have almost all their sample paths strictly positive. Since we will need to prove similar properties for other max-stable processes, we isolate the argument that gives them. It uses random closed sets. We refer to Matheron [15] for generalities on random closed sets and their laws.

3.3. Lemma. (i) *Let A be a random closed subset of a σ-compact space T satisfying*

$$\mathscr{L}(A) = \mathscr{L}\left(\bigcap_{i=1}^{n} A_i\right), \quad n \in \mathbb{N} \tag{3.12}$$

for A_i i.i.d. with $\mathscr{L}(A_i) = \mathscr{L}(A)$, and

$$P\{t \in A\} < 1 \quad \textit{for all } t \in T. \tag{3.13}$$

Then

$$P\{A = \phi\} = 1. \tag{3.14}$$

(ii) *If, with the same notation, A satisfies* (3.13) *and*

$$\mathscr{L}(A) = \mathscr{L}\left(\bigcup_{i=1}^{n} A_i\right), \quad n \in \mathbb{N} \tag{3.15}$$

then A also satisfies (3.14).

Proof. It is enough to assume T is compact. (i): For K compact, let $T_A(K) \equiv P\{A \cap K \neq \phi\}$ be the hitting functional of A. Since

$$\left\{\left(\bigcap_{i=1}^{n} A_i\right) \cap K \neq \phi\right\} \subset \bigcap_{i=1}^{n} \{A_i \cap K \neq \phi\},$$

it follows that, by (3.12), $T_A(K) \leqq (T_A(K))^n$ for all $n \in \mathbb{N}$. This implies that $T_A(K)$ is either 0 or 1. Hence, by (3.13), $T_A(\{t\}) = 0$ for all $t \in T$. But by monotone continuity of T_A ($K_n \downarrow K$ with K_n and K compact, implies $T_A(K_n) \downarrow T_A(K)$) and compactness of T, for each $t \in T$ there is an open neighborhood G_t such that $T_A(G_t)=0$. A finite number of the G_i, $i=1, \ldots, r$, cover T so that

$$P\{A \neq \phi\} = P\{A \cap T \neq \phi\} = P\left\{A \cap \left(\bigcup_{i=1}^{r} G_i\right) \neq \phi\right\} \leqq \sum_{i=1}^{r} P\{A \cap G_i \neq \phi\} = 0.$$

Thus, (3.14) is proved.

For (ii), let $Q_A(K) = 1 - T_A(K) = P\{A \cap K = \phi\}$. If (3.15) holds, then from

$$\left\{\left(\bigcup_{i=1}^{n} A_i\right) \cap K = \phi\right\} \subset \bigcap_{i=1}^{n} \{A_i \cap K = \phi\}$$

we obtain $Q_A(K) \leqq (Q_A(K))^n$ for all $n \in \mathbb{N}$ and $K \subset T$ compact. Hence T_A is either 0 or 1 and (3.14) follows as in case (i). □

3.4. Corollary. *Let ξ be a sample continuous simple* max-*stable process on S. Then*

(3.16) $$P\{\xi > 0\} = 1.$$

Proof. Take $A = \{t\colon \xi(t) = 0\}$. Since by definition $\mathscr{L}(\xi) = \mathscr{L}\left(\frac{1}{n} \bigvee_{i=1}^{n} \xi_i\right)$, ξ_i i.i.d. with law $\mathscr{L}(\xi)$, *and* $\xi(t)$, $\xi_i(t) \geqq 0$, it follows that the random sets $\left\{t\colon \frac{1}{n} \bigvee_{i=1}^{n} \xi_i(t) = 0\right\} = \bigcap_{i=1}^{n} \{t\colon \xi_i(t) = 0\}$ have the same law as $\{t\colon \xi(t) = 0\}$, that is, (3.12) holds. Moreover $P\{t \in A\} = P\{\xi(t) = 0\} = 0$ by definition, so that (3.13) is satisfied. Hence, Lemma 3.3 yields $P\{\{t\colon \xi(t) = 0\} = \phi\} = 1$, i.e. (3.16). □

3.5. Lemma. *Let ξ be max-stable in $C(S)$, and let $a(t)$, $b(t)$ and $\gamma(t)$ be the real functions* $(a(t) > 0)$ *defined in* (1.6.) *Then*

(i) *the functions a, b and γ are continuous*;

(ii) $P(1 + \gamma(t)(\xi(t) - b(t))/a(t) \neq 0$ *for all* $t) = 1$.

Proof. The continuity of $\gamma(t)$ follows from (1.7) and the assumed continuity of $a_n(t) > 0$ in Definition 1.4. Then, if $t_k \to t$, it follows that

(3.17) $$\|F_{\gamma(t_k)} - F_{\gamma(t)}\|_\infty \to 0$$

by an observation made after (1.5) in Sect. 1. Since $\xi(t_k) \to \xi(t)$ a.s. and the distribution function of $\xi(t)$, $F_{\gamma(t)}(a(t)\,x + b(t))$, is continuous, it also follows that

(3.18) $$\sup_{x \in \mathbb{R}} |F_{\gamma(t_k)}(a(t_k)\,x + b(t_k)) - F_{\gamma(t)}(a(t)\,x + b(t))| \to 0.$$

So, (3.17) and (3.18) imply

(3.19) $$F_{\gamma(t)}(a(t_k)\,x + b(t_k)) \to F_{\gamma(t)}(a(t)\,x + b(t)), \qquad x \in \mathbb{R}.$$

Obviously (see (1.5)), $F_{\gamma(t)}$ is one-to-one on the set $\{u\colon F_{\gamma(t)}(u)\in(0,1)\}$, which includes at least a half-line. This and (3.19) show that there is a half-line L such that

$$(3.20)\qquad a(t_k)\,x+b(t_k)\to a(t)\,x+b(t) \quad \text{for all } x\in L.$$

Consequently, the continuity of the functions $a(t)$ and $b(t)$ is established, thereby verifying (i).

We prove (ii) by showing that

$$(3.21)\qquad P(1+\gamma(t)(\xi(t)-b(t))/a(t)=0 \textit{ for some } t\in A_i)=0 \quad \text{for } i=1,2,3,$$

where

$$(3.22)\qquad A_1=\{t\colon \gamma(t)>0\},\quad A_2=\{t\colon \gamma(t)<0\} \quad\text{and}\quad A_3=\{t\colon \gamma(t)=0\}.$$

(3.21) is obvious for $i=3$. Let

$$\zeta(t)=[1+\gamma(t)(\xi(t)-b(t))/a(t)]^{1/\gamma(t)},\qquad t\in A_1\cup A_2.$$

Then a simple computation shows that $\zeta(t)$ is simple max-stable. For $i=1$, (3.21) becomes

$$P(\zeta(t)\neq 0,\, t\in A_1)=1,$$

which follows from Corollary 3.4 (since Lemma 3.3 holds for σ-compact spaces, so does Corollary 3.4). The relevant case is $i=2$. Let

$$\rho(t)=-1-\gamma(t)(\xi(t)-b(t))/a(t),\qquad t\in A_2.$$

Define $A=\{t\in A_2\colon \rho(t)=0\}$. Then by (i), A is a random closed set of A_2 (which is σ-compact). Since for $t\in A_2, (\xi(t)-b(t))/a(t)<-\dfrac{1}{\gamma}$ a.s. (since this process has distribution F_γ), it follows that

$$P(t\in A)=P(\rho(t)=0)=0 \quad\text{and}\quad \rho(t)<0 \quad \text{a.s.}$$

In particular, (3.13) holds for A. Moreover, ρ is max-stable and

$$\mathscr{L}\left(\left\{n^{-\gamma(t)}\bigvee_{i=1}^{n}\rho_i(t)\colon\ t\in A_2\right\}\right)=\mathscr{L}(\{\rho(t)\colon\ t\in A_2\})$$

by (1.7) and (1.8) (or, better, ρ is max-stable because ξ is, and for each $t\in A_2$, $x\in\mathbb{R}$, $P(\rho(t)\leqq x)=e^{-(-x)^{-1/\gamma}}I(x<0)+I(x>0)$). So, since $\rho\leqq 0$, $\{t\in A_2\colon n^{-\gamma(t)}\bigvee_{i=1}^{n}\rho_i(t)=0\}=\bigcup_{i=1}^{n}\{t\in A_2\colon \rho_i(t)=0\}$ and therefore, A also satisfies (3.15). Lemma 3.3 gives $P(A=\emptyset)=1$, that is $P(A\neq\emptyset)=P(\rho(t)=0 \text{ for some } t\in A_2)=0$, i.e. (3.21) holds for $i=2$. □

3.6. Corollary. *Let ξ be* max-*stable in $C(S)$ and let $a(t)>0$, $b(t)$ and $\gamma(t)$ be the functions defined in* (1.6). *Let $\zeta(t)$, $t\in S$, be the process defined by*

$$\zeta(t)=[1+\gamma(t)(\xi(t)+b(t))/a(t)]^{1/\gamma(t)}, \qquad t\in S \tag{3.23}$$

where, by abuse of notation, the formula means

$$\zeta(t)=e^{(\xi(t)-b(t))/a(t)} \qquad \text{if } \gamma(t)=0. \tag{3.23'}$$

Then $\zeta(t)$, $t\in S$, is simple max-*stable in $C(S)$.*

Proof. ζ is well-defined and sample continuous by Lemma 3.5. Simple computation from (1.5) and (1.6) establishes that $\mathscr{L}(\zeta(t))=\Phi_{1,1}$ for all t, and it is equally easy to prove that the finite-dimensional distributions of $n^{-1}\bigvee_{i=1}^{n}\zeta_i(t)$, $t\in S$, equal the corresponding ones of $\zeta(t)$, $t\in S$, if ζ_i are i.i.d. with law $\mathscr{L}(\zeta)$. That is, ζ is max-stable. Therefore, ζ is simple max-stable in $C(S)$. □

Suppose now we are given ζ simple max-stable in $C(S)$ and continuous functions $a(t)>0$, $b(t)$, and $\gamma(t)$. Then a new process $\xi(t)$ may be defined by inverting equation (3.23), namely

$$\xi(t)=a(t)\frac{\zeta^{\gamma(t)}-1}{\gamma(t)}+b(t), \qquad t\in S \tag{3.24}$$

where, with the usual abuse of notation,

$$\xi(t)=a(t)\ln\zeta(t)+b(t) \qquad \text{if } \gamma(t)=0. \tag{3.24'}$$

Then, $\xi(t)$ is well-defined by Corollary 3.4, sample continuous, max-stable in $C(S)$, and $P((\xi(t)-b(t))/a(t)\leqq x)=F_{\gamma(t)}(x)$.

Collecting this last observation, Corollary 3.6 and Proposition 3.2 yields the following characterization of processes which are max-stable in $C(S)$.

3.7. Theorem. *A process ξ on a compact metric space S is (strictly non-degenerate)* max-*stable in $C(S)$ if and only if there exist*

(a) *continuous functions $a(t)>0$, $b(t)$, and $\gamma(t)$ and*

(b) *a finite measure σ on $C_1^+(S)$ satisfying $\int_{C_1^+} f(t)\,d\sigma(f)=1$ for all $t\in S$,*

such that, if $\zeta_\sigma=\bigvee_{i=1}^{\infty}\eta_i$ where $\{\eta_i\}$ are the points of a Poisson point process on $C(S)$ with means measure $d\nu=d\sigma\times\frac{dr}{r^2}$, then the law of ξ coincides with the law of the process defined by the equations (3.24) *and* (3.24′) *with ζ_σ in place of ζ. If this is the case, then the norming functions $a_n(t)$ and the location functions $b_n(t)$ of Definition* 1.4 *are given by the equations* (1.7)–(1.9), *and the variable $(\xi(t)-b(t))/a(t)$ has the $F_{\gamma(t)}$ distribution for all $t\in S$.*

Although we do not have a complete spectral representation of max-i.d. laws in $C(S)$ the situation is not unlike the situation of infinitely divisible laws from sums for which there is not a complete characterization of Lévy measures in $C(S)$. In contrast, Theorem 3.7 gives a complete description of the representation for max-stable laws in $C(S)$ while the results for stable laws in $C(S)$ are less complete (e.g. [2]).

4 An application to random sets

We obtain, as an application of the results in Sect. 2, a representation for non-void random compact convex subsets of a separable Banach space which are infinitely divisible with respect to the operation "convex hull of unions". Norberg [18] obtains a representation of max-i.d. random compact sets of $\mathbb{R}^d$ which are allowed to be $\emptyset$ and also the whole space. Norberg's result is obtained as a consequence of his representation of max-i.d. random variables taking values in continuous semilattices (this general result contains also the representation for upper semicontinuous functions quoted in Theorem 1.2, and its proof, considerably more involved, is based on an extension of Choquet's theorem; it does not seem to contain, however, the results from Sect. 2). We also obtain a characterization of a somewhat restricted notion of stable sets for convex hulls of unions.

4.1. Definition. A random compact convex subset Ξ of a separable Banach space B *is i.d. for convex hulls of unions* if for each n

$$\mathscr{L}(\Xi)=\mathscr{L}\left(\overline{\mathrm{co}}\bigcup_{i=1}^{n}\Xi_{ni}\right)$$

where Ξ_{ni} are independent identically distributed random compact convex sets.

For random compact convex sets and their support functions see e.g. Hormander [12] and Giné and Hahn [7]. We recall a few facts. Given a compact convex set $K\subset B$, its support function $k\colon B^*\to\mathbb{R}$, where B^* is the (topological) dual of B, is

$$k(y^*)=\sup_{x\in K} y^*(x), \qquad y^*\in B^*.$$

Let B_1^* be the unit ball of B^* and w^* denote the weak*-topology. Let $S\subset C(B_1^*, w^*)$ be the set of w^*-continuous, subadditive and positively homogeneous functions on B_1^*. Let $\mathscr{K}$ be the collection of all nonempty compact convex subsets of B. Then the map

$$\mathscr{K}\xrightarrow{\ s\ }S$$

that assigns to each set its support function is $1-1$ and onto, is an isometry between the Hausdorff distance (d) and the sup-norm ($\|\cdot\|_\infty$), and takes convex hulls of unions into max's, i.e.

$$\overline{\mathrm{co}}\left(\bigcup_{i=1}^{n} K_i\right)\leftrightarrow\bigvee_{i=1}^{n} k_i, \qquad n\leqq\infty$$

(where $n=\infty$ is only allowed if the left side is in $\mathscr{K}$). Since (B_1^*, w^*) is compact metric, this isometric isomorphism allows the representation of max-i.d. compact convex sets to be reduced to the terms of Theorems 2.1 and 2.4. This gives a characterization because, as we will see below, if $\xi \in S$ is max-i.d. then the vertex of its support, h, is automatically continuous. Up to a few not necessarily trivial details, this is the essence of the proof below.

On notation: given compact sets or random sets K, Ξ, we denote their respective support functions or processes by k, ξ. A measure ν on $\mathscr{K}$ means a Borel measure on $(\mathscr{K}, d)$, which becomes a Borel measure $\nu \circ s^{-1}$ on $(S, \|\cdot\|_\infty)$ by the isometry s, and we continue to denote $\nu \circ s^{-1}$ by ν.

4.2. Theorem. *Let H be a non-void compact convex subset of B and let ν be a locally finite Borel measure on*

$$\mathscr{K}^H = \{K \in \mathscr{K}:\ K \supseteq H, K \neq H\}$$

such that

$$\nu\{K \in \mathscr{K}^H:\ K \nsubseteq D\} < \infty, \qquad D \in \mathscr{K}^H. \tag{4.1}$$

Let $\{K_i\}$ be the points of a Poisson point process on $\mathscr{K}$ with mean measure ν. Then

$$\Xi \equiv \overline{\mathrm{co}}\left(H \cup \left(\bigcup_{i=1}^{\infty} K_i\right)\right) \tag{4.2}$$

is an infinitely divisible random compact convex set for convex hulls of unions. Moreover

$$H = \overline{\mathrm{co}}\left(\bigcup [K:\ K \in \mathscr{K}, P\{\Xi \supseteq K\} = 1]\right) \tag{4.3}$$

and

$$\nu\{K \in \mathscr{K}^H:\ K \nsubseteq D\} = -\log P\{\Xi \subseteq D\}, \qquad D \in \mathscr{K}^H. \tag{4.4}$$

Conversely, if Ξ is infinitely divisible for convex hulls of unions, then equations (4.3) *and* (4.4) *define respectively a non-void compact convex set H and a locally finite measure ν on $\mathscr{K}^H$ satisfying* (4.1) *such that* (4.2) *holds in distribution.*

Proof. The idea is to pass to support functions and processes, and apply Theorem 2.1 for the direct part and Theorem 2.4 for the converse. Assume (4.1). In order to apply Theorem 2.1, it must first be shown that (4.1) implies (2.1), or equivalently,

$$\nu\{k \in S^h: k \nleq g\} < \infty, \qquad g \in S^h, \tag{4.5}$$

where $S^h = \{g \in S:\ g \geq h,\ g \neq h\}$, implies

$$\nu\{k \in S^h: k(I) > x\} < \infty \tag{4.6}$$

for all $x < h(I)$ and all w^*-compact subsets I of B_1^*. This is obviously true for $I = \{0\}$ ($h(0) = 0$ and $g \in S \Rightarrow g(0) = 0$). If $I \neq \{0\}$, let $v \in B$ be such that $\sup_{y^* \in I} y^*(v) = x$, let $k_v \in S$ be defined by $k_v(y^*) = y^*(v)$, $y^* \in B_1^*$, and let $g = h \vee k_v$. Then $g \in S^h$ and $g(I) = x$. Hence $\{k \in S^h:\ k(I) > x\} \subseteq \{k \in S^h:\ k \nleq g\}$ from which it follows that (4.5) implies (4.6).

Since $H \neq \phi$, $h \in S \subset C(B_1^*, w^*)$. So, Theorem 2.1 shows that if $\{\eta_i\}$ are the points of a Poisson point process with mean measure ν supported by S^h, then $\xi = h \vee \left(\bigvee_{i=1}^{\infty} \eta_i\right)$ is a sample continuous max-i.d. process. Since subadditivity and positive homogeneity are preserved by the max operation, it follows that the paths of ξ are in S. Passing to sets gives (4.2).

Next we prove that if

$$\tilde{h} = \sup[f \in S : P\{\xi \geqq f\} = 1]$$

then $h = \tilde{h}$, i.e. (4.3) holds. By Theorem 2.1, h satisfies (1.1). Therefore we have

$$(4.7) \qquad S \ni f \leqq \tilde{h} \Rightarrow P\{\xi \geqq f\} = 1 \Rightarrow P\{\xi(y^*) \geqq f(y^*)\} = 1 \; \forall y^* \subset B_1^* \Rightarrow h \geqq f$$

and

$$(4.8) \qquad S \ni f \leqq h \Rightarrow P\{\xi(y^*) \geqq f(y^*)\} = 1 \; \forall y^* \in B_1^* \Rightarrow P\{\xi \geqq f\} = 1$$
$$\text{(by continuity of } \xi \text{ and } f \text{ and by separability of } B_1^*) \Rightarrow f \leqq \tilde{h}.$$

Taking $f = h$ in (4.8) gives $h \leqq \tilde{h}$, whereas (4.7) implies $h \geqq \sup[f \in S\colon f \leqq \tilde{h}] = \tilde{h}$. Hence, $h = \tilde{h}$ and (4.3) is proved.

By Theorem 2.1, ν satisfies (1.2). But (1.2) implies (4.4) because if $\{y_n^*\}$ is a countable w^*-dense set of B_1^*, then

$$(4.9) \qquad \bigcap_{n=1}^{\infty} \{g \in C(B_1^*)\colon \; g(y_n^*) \leqq f(y_n^*)\} = \{g \in C(B_1^*)\colon \; g \leqq f\}$$

and

$$(4.10) \qquad \bigcup_{n=1}^{\infty} \{g \in C(B_1^*)\colon \; g(y_n^*) > f(y_n^*)\} = \{g \in C(B_1^*)\colon \; g \nleqq f\}.$$

Then

$$\begin{aligned} \nu\{g \in S^h\colon \; g \nleqq f\} &= \nu\left(\bigcup_{n=1}^{\infty} \{g \in S^h\colon \; g(y_n^*) > f(y_n^*)\}\right) \\ &= \lim_{m \to \infty} \nu\left(\bigcup_{n=1}^{m} \{g \in S^h\colon \; g(y_n^*) > f(y_n^*)\}\right) \\ &= \lim_{m \to \infty} \left[-\log P\left(\bigcap_{n=1}^{m} \{\xi(y_n^*) \leqq f(y_n^*)\}\right)\right] \\ &= -\log P\{\xi \leqq f\}. \end{aligned}$$

The direct part of the Theorem is thus proved.

For the converse, let ξ be a subadditive and positively homogeneous max-i.d. sample continuous process on (B_1^*, w^*). Then, by Remark 1.3, there is an upper semicontinuous function $h\colon B_1^* \to [-\infty, \infty]$ and a measure ν on not necessarily finite functions on B_1^* with closed graphs such that (1.1) and (1.2) in Theorem 1.2 hold. Since $\xi(y^*) < \infty$, it follows that $h(y^*) < \infty$ for all $y^* \in B_1^*$. Also, $h(0) = 0$

because $\xi(0)=0$ a.s. Suppose $h(y^*)=-\infty$ for some $y^*\in B_1^*$, $y^*\neq 0$, and let $\lambda\in\mathbb{R}$ be such that $P\{\xi(-y^*)\geqq\lambda\}<1$ (i.e. $\lambda>h(-y^*)$); such a λ exists since $h(-y^*)<\infty$. Then, by (1.2), for all $M>0$,

$$\begin{aligned}\infty &> v\{f\geqq h:\ f\neq h, f(y^*)\geqq -M\}+v\{f\geqq h:\ f\neq h, f(-y^*)\geqq\lambda\}\\ &\geqq v\{f\geqq h:\ f\neq h,\ f(y^*)\geqq -M \text{ or } f(-y^*)\geqq\lambda\}\\ &= -\log P\{\xi(y^*)<-M, \xi(-y^*)<\lambda\}.\end{aligned}$$

Hence,

$$P\{\xi(y^*)<-M, \xi(-y^*)<\lambda\}\neq 0.$$

Therefore, for some ω in the set of probability 1 where $\xi(\omega)$ is subadditive, $\xi(\omega, y^*)\geqq h(y^*)$, $\xi(\omega, -y^*)\geqq h(-y^*)$ and $\xi(\omega, 0)=0=h(0)$, we have

$$0=h(0)\leqq\xi(\omega, y^*+(-y^*))\leqq\xi(\omega, y^*)+\xi(\omega, -y^*)<-M+\lambda,$$

which implies $\lambda>M$ for all M, i.e. $\lambda=\infty$. In other words, $\lambda>h(-y^*)\Rightarrow\lambda=+\infty$, that is $h(-y^*)=+\infty$, a contradiction. Thus we have proved that h takes values in $(-\infty,\infty)$. An argument entirely similar to the previous one shows that h is subadditive (given $y^*, z^*\in B_1^*$, (1.2) shows, as above, that for all $\varepsilon>0$

$$P\{\xi(y^*)<h(y^*)+\varepsilon, \xi(z^*)<h(z^*)+\varepsilon\}\neq 0, \quad \text{hence} \quad h(y^*+z^*)\leqq h(y^*)+h(z^*)+2\varepsilon).$$

But h is also upper semicontinuous (Remark 1.3). Therefore h is continuous at 0: on the one hand, by upper semicontinuity, $\overline{\lim}_{y^*\to 0} h(y^*)\leqq h(0)=0$, while on the other, by subadditivity, $0=h(0)\leqq h(y^*)+h(-y^*)$ and these two conditions obviously imply $\lim_{y^*\to 0} h(y^*)=0=h(0)$. But by subadditivity, h is then continuous for all $y^*\in B_1^*$ since $|h(y^*)-h(z^*)|\leqq h(y^*-z^*)\vee h(z^*-y^*)\to 0$ as $y^*\to z^*$. Since h is continuous, the measure v is supported by $C^h\equiv C(B_1^*, \omega^*)^h$ by Theorem 2.1, and v satisfies (1.2), hence (2.1).

Suppose $f\in C^h$, $f\in\operatorname{supp} v$ and f is not positively homogeneous, i.e. $v\{g:\ \|f-g\|_\infty>\varepsilon,\ g\in C^h\}>0$ for all $\varepsilon>0$, and for some $\alpha>0$ and $s\in B_1^*, f(\alpha s)\neq\alpha f(s)$. Since h is positively homogeneous and $f\geqq h$, either $f(s)>h(s)$ or $f(\alpha s)>h(\alpha s)$ or both. Let us assume, without loss of generality, that $f(s)>h(s)$ and $h(\alpha s)\leqq f(\alpha s)<\alpha f(s)$. Let $\delta\equiv\alpha f(s)-f(\alpha s)$. Then $f(s)\geqq h(s)+\delta/\alpha$. Let $\delta' =\frac{\delta}{3\alpha}\wedge\frac{\delta}{3}$, and let

$$\mathcal{N}=\{g\in C^h:\ \|f-g\|_\infty<\delta'\}$$

$$\mathcal{M}=\{g\in C^h:\ g(s)\geqq f(s)-\delta' \text{ or } g(\alpha s)\geqq f(\alpha s)+\delta'\}.$$

Then

$$0<v(\mathcal{N})<\infty \tag{4.11}$$

since $v\{g\in C^h:\ g(s)>f(s)-\delta'\}<\infty$ by (2.1) and

$$v(\mathcal{M})<\infty \tag{4.12}$$

by (2.1). Let ξ_1, ξ_2, ξ_3 be independent sample continuous max-i.d. processes respectively associated to $(h, v|_{\mathcal{N}})$, $(h, v|_{\mathcal{M}\setminus\mathcal{N}})$, and $(h, v|_{\mathcal{M}^c})$. Then $\mathcal{L}(\xi)$

$=\mathscr{L}(\xi_1 \vee \xi_2 \vee \xi_3)$. We note that if $g \in \mathcal{N}$ then $\alpha g(s) - g(\alpha s) \geqq \alpha f(s) - \alpha\delta' - f(\alpha s) - \delta' = \delta - \delta' - \alpha\delta' \geqq \delta/3$. Likewise, if $g \in \mathcal{N}$ and $g' \in \mathcal{M}^c$, then $\alpha(g \vee g')(s) - (g \vee g')(\alpha s) \geqq \delta/3$. Hence, if $g_i \in \mathcal{N}$ then $\bigvee g_i$ is not positively homogeneous, and the same is true for $(\bigvee g_i) \vee (\bigvee g_i')$ if $g_i \in \mathcal{N}$ and $g_i' \in \mathcal{M}^c$. Therefore, if $\xi_1 \neq h$ and $\xi_2 = \mathrm{h}$ then $\xi_1 \vee \xi_2 \vee \xi_3 = (\bigvee g_i) \vee (\bigvee g_i')$, $g_i \in \mathcal{N}$, $g_i' \in \mathcal{M}^c$, and we have

$$P\{\xi \text{ is not positively homogeneous}\} \geqq P\{\xi_1 \neq h, \xi_2 = h\} = (1 - e^{-\nu(\mathcal{N})})\, e^{-\nu(\mathcal{M} \setminus \mathcal{N})} > 0$$

by (4.11) and (4.12), a contradiction. Hence, $f \notin \operatorname{supp} \nu$, i.e. ν contains only positively homogeneous functions.

A similar proof shows that ν is supported by subadditive functions. Suppose that $f \in \operatorname{supp} \nu$ and is not subadditive. Then there are $s, t \in B_1^*$ such that $f(s+t) > f(s) + f(t)$, and $\nu\{g \in C^h\colon \|f - g\|_\infty < \varepsilon\} > 0$ for all $\varepsilon > 0$. Take $\delta \equiv f(s+t) - f(s) - f(t)$, $\delta' = \delta/4$, $\mathcal{N} = \{g \in C^h\colon \|f - g\|_\infty < \delta'\}$ and $\mathcal{M} = \{g \in C^h\colon g(s+t) \geqq f(s+t) - \delta'$ or $g(s) \geqq f(s) + \delta'$ or $g(t) \geqq f(t) + \delta'\}$. Then, $0 < \nu(\mathcal{N}) < \infty$, $\nu(\mathcal{M}) < \infty$, $g \in \mathcal{N} \Rightarrow g(s+t) > g(s) + g(t) + \frac{\delta}{4}$ and

$$g \in \mathcal{N}, g' \in \mathcal{M}^c \Rightarrow (g \vee g')(s+t) > (g \vee g')(s) + (g \vee g')(t) + \frac{\delta}{4}.$$

Thus, if ξ_1, ξ_2, ξ_3 are as above for these $\mathcal{N}$ and $\mathcal{M}$, then

$$P\{\xi \text{ is not subadditive}\} \geqq P\{\xi_1 \neq h, \xi_2 = h\} > 0,$$

a contradiction. Hence the support of ν consists entirely of subadditive functions. Then ν is concentrated on S^h, hence passing back to sets, on $\mathscr{K}^h$.

Finally, observe that (4.9) and (4.10) together with (1.2) imply

$$\nu\{f \in S^h\colon f \leqq g\} = -\log P\{\xi \leqq g\}, \qquad g \in S^h$$

as in the proof of the direct part. That is, ν satisfies (4.4). Now (4.1) is a consequence of (4.3) and (4.4) □

A consequence of this Theorem is that if Ξ is a random compact convex set of B which is infinitely divisible for convex hulls of unions and such that $\Xi \neq \emptyset$ a.s., then the compact set H defined by Eq. (4.3) is non-void, i.e. Ξ contains a.s. a fixed non-void compact convex set.

We now consider a restricted notion of stable random set for convex hulls of unions when $B = \mathbb{R}^d$.

4.3. Definition. The compact convex random set Ξ in $B = \mathbb{R}^d$ is stable for convex hulls of unions if $\xi(y^*)$ is a non-degenerate random variable for all $y^* \in B_1^* \setminus \{0\}$ and if there exist compact convex sets K_n and positive real numbers a_n, $n \in \mathbb{N}$, such that

$$\mathscr{L}(\Xi) = \mathscr{L}\left(\frac{\overline{\mathrm{co}} \bigcup_{i=1}^{n} \Xi_i + K_n}{a_n}\right), \qquad n \in \mathbb{N} \tag{4.13}$$

where $\{\Xi_i\}_{i=1}^\infty$ are i.i.d. copies of Ξ.

This definition is less general than it looks at first glance.

4.4. Proposition. *A random compact convex set Ξ in $B=\mathbb{R}^d$ is stable for convex hulls of unions if and only if there exist $\gamma>0$ and a compact convex set H such that*

$$\mathscr{L}(\Xi)=\mathscr{L}\left(n^{-\gamma}\left(\overline{\mathrm{co}}\bigcup_{i=1}^{n}\Xi_i+(n^{\gamma}-1)H\right)\right),\qquad n\in\mathbb{N}, \tag{4.14}$$

where $\{\Xi_i\}_{i=1}^{\infty}$ are i.i.d. copies of Ξ. Moreover, there exists a compact convex random set Θ containing zero as an interior point almost surely such that

$$\mathscr{L}(\Xi)=\mathscr{L}(\Theta+H). \tag{4.15}$$

Proof. If Ξ satisfies (4.14) it is certainly stable since a fortiori Ξ satisfies (4.13). Suppose now that Ξ is stable. By passing to support processes we see that $\xi\in S$, the support process of Ξ, is max-stable with $b_n(y^*)=-k_n(y^*)$, $y^*\in B_1^*$, and $a_n(y^*)=a_n$. Although ξ is degenerate at $0\in B_1^*$, most of Proposition 3.5 can be applied. In particular $a(y^*)>0$, $b(y^*)$ and $\gamma(y^*)$ are continuous on $B_1^*\backslash\{0\}$ and $B_1^*\backslash\{0\}=A_1\cup A_2\cup A_3$, where, as in the proof of Lemma 3.5, A_1, A_2 and A_3 are respectively the subsets of $B_1^*\backslash\{0\}$ where γ is positive, negative, and zero. Since $a_n=n^{-\gamma(y^*)}$ it follows that γ is constant. Hence, $B_1^*\backslash\{0\}=A_i$ for some $i=1, 2, 3$.

We next establish that $B_1^*\backslash\{0\}=A_1$. Suppose $B_1^*\backslash\{0\}=A_3$. Then, by (1.9), $-k_n(y^*)=b_n(y^*)=a(y^*)\log n>0$ which is impossible because, by subadditivity, either $k_n(y^*)\geqq 0$ or $k_n(-y^*)\geqq 0$. Suppose $B_1^*\backslash\{0\}=A_2$. Then, by (1.8), $-k_n(y^*)=b_n(y^*)=(n^{\gamma}-1)(a(y^*)/\gamma-b(y^*))$ so that, since $n^{\gamma}-1<0$, $a/\gamma-b\in S$. Therefore, $\xi+(a/\gamma-b)\in S$ a.s.. But, by the definition of A_2, we also have $P\{(\xi(y^*)-b(y^*))/a(y^*)\leqq -1/\gamma\}=1$ for all $y^*\in B_1^+\backslash\{0\}$. (See the comments following (1.5).) So, $P\{\xi(y^*)+(a(y^*)/\gamma-b(y^*))\leqq 0, \xi(-y^*)+(a(-y^*)/\gamma-b(-y^*))\leqq 0\}=1$ which contradicts $\xi+(a/\gamma-b)\in S$ a.s. as in the previous case. Therefore $B_1^*\backslash\{0\}=A_1$. In this case, by (1.8), $k_n=-b_n=(n^{\gamma}-1)(b-a/\gamma)$ so that $h\equiv b-a/\gamma\in S$, and, by (1.7), $a_n=n^{\gamma}$ with $\gamma>0$. So, (4.14) follows from (4.13) by letting H be the random set with support function h. In particular, $H(h)$ is the vertex of the support of Ξ (ξ).

Since $\xi-h$ is obviously positively homogeneous (both ξ and h are), and $\xi-h>0$ in $B_1^*\backslash\{0\}$ a.s. (by Lemma 3.3; see Corollary 3.4, that applies to σ-compact sets), in order to prove (4.15) it is enough to show that $\xi-h$ is also subadditive (since then it will be the support process of a random set Θ that contains 0 as an interior point). By sample continuity of ξ it suffices to prove a.s. subadditivity for a countable dense set of points in B_1^*, hence for two.

For $n\in\mathbb{N}$, define random processes

$$X_n=n^{-\gamma}\bigvee_{i=m_{n-1}+1}^{m_n}\xi_i-n^{-\gamma}h$$

where ξ_i are i.i.d. copies of ξ and $m_n=n(n+1)/2$. Then the processes X_n are independent and their common distribution is $\mathscr{L}(\xi-h)$ by (4.14). Given x^*, $y^*\in B_1^*\backslash\{0\}$, let Y_n be the random variable

$$Y_n=X_n(x^*+y^*)-X_n(x^*)-X_n(y^*),\qquad n\in\mathbb{N},$$

and let

$$Y=(\xi-h)(x^*+y^*)-(\xi-h)(x^*)-(\xi-h)(y^*).$$

For ω in the set of probability one where all the processes ξ_i are subadditive, we have by the subadditivity of $\bigvee \xi_i$,

$$Y_n(\omega) \geqq -\frac{1}{n^\gamma}(h(x^*-y^*)-h(x^*)-h(y^*)) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, $\liminf_{n\to\infty} Y_n(\omega) > -\frac{1}{r}$ for all $r>0$, i.e.

$$P\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{\omega : Y_n(\omega) > -\frac{1}{r}\right\}\right)=1.$$

Thus, by the Borel-Cantelli Lemma, $\sum P\left\{Y_n \leqq -\frac{1}{r}\right\} < \infty$ for all r, i.e. $P\left\{Y \leqq -\frac{1}{r}\right\}=0$ and $P\{Y<0\}=0$. This shows that $\xi - h$ is a.s. subadditive. □

In order to give the spectral representation of Ξ, define

(4.16) $$S^{(h)}=\{f \in S \colon f \geqq h, f \neq h, f-h \in S\}$$

for any given $h \in S$, and

(4.17) $$T_{\gamma,a,h}(f)=\left(\frac{f-h}{a}\right)^{\gamma^{-1}}, \quad f \in S^{(h)}$$

for any $\gamma>0$, any continuous and positively homogeneous function a on $B_1^* \setminus \{0\}$, and any $h \in S$. Actually $T_{\gamma,a,h}(S^{(h)})$ is a cone of $C(B_1^* \setminus \{0\})$ consisting of functions which are constant on rays λy^*, $\lambda>0$. The second assertion is obvious; as for the first, for $\lambda>0, f \in S^{(h)}$,

$$\lambda T_{\gamma,a,h}(f)=\left(\frac{\lambda^\gamma(f-h)}{a}\right)^{\gamma^{-1}}=T_{\gamma,a,h}(\lambda^\gamma(f-h)+h),$$

$$f \in S^{(h)} \Rightarrow \lambda^\gamma(f-h) \in S$$

and

$$f-h \geqq 0 \quad \text{and} \quad f-h \neq 0 \Rightarrow \lambda^\gamma(f-h)+h \in S^{(h)}.$$

The transformation $T_{\gamma,a,h}$ is introduced so that $T_{\gamma,a,h}(\xi)$ is $\Phi_{1,1}$ for the max-stable support process ξ of interest.

4.5. Theorem. *Ξ is a compact convex random set stable for convex hulls of unions if and only if there exist a positive number γ, a continuous positively homogeneous function a on $B_1^* \setminus \{0\}$, a compact convex set H, and a finite measure σ on*

$$T_{\gamma,a,h}(S^{(h)})_1=\{g \in T_{\gamma,a,h}(S^{(h)}) \colon \|g\|_\infty=1\},$$

satisfying

$$\int f(y^*)\, d\sigma(f)=1 \qquad \text{for any } y^* \neq 0,$$

such that for all compact convex sets K with $H \subset K^{\circ}$

$$(4.18) \qquad -\log P\{\Xi \subseteq K\} = \int_{T_{\gamma,a,h}(S^{(h)})_1} \left\| g \Big/ \left(\frac{k-h}{a}\right)^{\frac{1}{\gamma}} \right\|_{\infty} d\sigma(g).$$

If $H \not\subset K^{\circ}$, then $P\{\Xi \subseteq K\} = 0$.

Proof. Let Ξ be stable for convex hulls of unions. Then Ξ is infinitely divisible and ξ is max-i.d. with support in $S^{(h)}$, where h is the support function of H in (4.14). The argument at the end of the proof of Theorem 4.2 applied to $\xi - h$ shows that the support of the max-Lévy measure ν of ξ consists of continuous functions f such that $f-h$ is non-negative and subadditive. Since the support of ξ is also contained in S^h, it follows that

$$(4.19) \qquad \operatorname{supp} \nu \subseteq S^{(h)}.$$

Before drawing conclusions from (4.19), let us deduce properties for the function a. The number $\gamma > 0$ and the function a on $B^* \setminus \{0\}$ are defined by the requirement that the random variable $((\xi(y^*) - b(y^*))/a(y^*))^{\gamma^{-1}}$ be $\Phi_{1,1}$, for every $y^* \in B_1^* \setminus \{0\}$, which is possible because, as proved in Proposition 4.4., $A_1 = B_1^* \setminus \{0\}$. From

$$P\{(\xi(\lambda y^*) - b(\lambda y^*))^{\gamma^{-1}} \leq x\} = e^{-a(\lambda y^*)^{\gamma^{-1}}/x}, \qquad \lambda > 0, \qquad x > 0,$$

and

$$\begin{aligned} P\{(\xi(\lambda y^*) - b(\lambda y^*))^{\gamma^{-1}} \leqq x\} &= P\{(\xi(y^*) - b(y^*))^{\gamma^{-1}} \leqq x/\lambda^{\gamma^{-1}}\} \\ &= \exp\{-\lambda^{\gamma^{-1}} a(y^*)^{\gamma^{-1}}/x\}, \qquad \lambda > 0, \qquad x > 0, \end{aligned}$$

it follows that a is positively homogeneous. Also, a is continuous on $A_1 = B_1^* \setminus \{0\}$ by Proposition 3.5.

Suppose that η_i are the points of a Poisson point process with mean measure ν. Then

$$\xi = h \vee (\bigvee \eta_i) \quad \text{and} \quad T_{\gamma,a,h}(h \vee (\bigvee \eta_i)) = T_{\gamma,a,h}(h) \vee (\bigvee T_{\gamma,a,h}(\eta_i))$$

where we define $T_{\gamma,a,h}(h) = 0$. Therefore, the max-Lévy measure of $T_{\gamma,a,h}(\xi) = \left(\frac{\xi - h}{a}\right)^{\gamma^{-1}}$ is $\mu = \nu \circ T_{\gamma,a,h}^{-1}$, a locally finite measure supported by the cone $T_{\gamma,a,h}(S^{(h)})$, by (4.19). On the other hand, since $\left(\frac{\xi - h}{a}\right)^{\gamma^{-1}}$ is $\Phi_{1,1}$ on $B_1^* \setminus \{0\}$, the computations (3.7) and (3.10) from Proposition 3.2 apply to μ and yield

$$(4.20) \qquad d\mu = d\sigma \times dr/r^2$$

and, $T_{\gamma,a,h}(S^{(h)})$ being a cone, σ is supported by $T_{\gamma,a,h}(S^{(h)})_1$. Moreover, σ is finite and $\int f(y^*)\, d\sigma(f) = 1$ for all $y^* \neq 0$.

Conversely, if σ verifies these properties, the the max-i.d. process $\bigvee \eta_i$ on $B_1^* \setminus \{0\}$, where η_i are the points of a Poisson process with mean measure μ defined by (4.20), has almost all of its trajectories in $T_{\gamma,a,h}(S^{(h)})$ because

(i) it is sample continuous by the proof of Theorem 2.1 (the compactness of the space in Theorem 2.1 is only used to obtain that the vertex of the support is finite, which is trivial in this case because the vertex is 0) and

(ii) if $\eta_i \in T_{\gamma,a,h}(S^{(h)})$ and $\bigvee_{i=1}^{\infty} \eta_i \in C(B_1^* \setminus \{0\})$ then $\bigvee_{i=1}^{\infty} \eta_i \in T_{\gamma,a,h}(S^{(h)})$. (More specifically, if $\eta_i = \left(\frac{f_i - h}{a}\right)^{\gamma^{-1}}$, with $f_i \in S^{(h)}$, then $\bigvee_{i=1}^{\infty} \eta_i = \left(\frac{\vee f_i - h}{a}\right)^{\gamma^{-1}}$ which is continuous. Consequently, $\bigvee_{i=1}^{\infty} f_i$ is continuous on $B_1^* \setminus \{0\}$ because h and a are, but then $\bigvee_{i=1}^{\infty} f_i$ is bounded outside an open neighborhood of 0 and positive homogeneity gives continuity as 0. Subadditivity of $\bigvee_{i=1}^{\infty} f_i$ is obvious and therefore $\bigvee_{i=1}^{\infty} f_i \in S^{(h)}$, which implies that $\bigvee_{i=1}^{\infty} \eta_i \in T_{\gamma,a,h}(S^{(h)})$).

Now $\bigvee_{i=1}^{\infty} \eta_i$ is a simple max-stable process by the structure of μ (see the proof of Prop. 3.2, (ii)$\Rightarrow$(i)). Then $\xi = T_{\gamma,a,h}^{-1}\left(\bigvee_{i=1}^{\infty} \eta_i\right)$ is the support process of a stable random set Ξ with the characteristics γ^{-1}, a, h and σ. □

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