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Probability Theory and Related Fields

Optimal partitioning of a measurable space into countably many sets

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Summary. A notion of an optimal partition of a measurable space into countably many sets according to given nonatomic probability measures is defined. It is shown that the set of optimal partitions is nonempty. Bounds for the optimal value are given and the set of optimal partitions is characterized. Finally, an example related to statistical decision theory is presented.

1. Introduction

Suppose we are given countably many nonatomic probability measures $\{\mu_i\}_{i=1}^{\infty}$ on the same measurable space $(\mathscr{X}, \mathscr{B})$. Let $\mathscr{P}$ denote the set of all measurable partitions $P=\{A_i\}_{i=1}^{\infty}$ of the space $(\mathscr{X}, \mathscr{B})$. Here by a partition is meant a sequence of countably many disjoint subsets from $\mathscr{B}$ the union of which is contained in $\mathscr{X}$. Let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of positive numbers with $\sum_{i=1}^{\infty} \alpha_i = 1$.

Definition. A partition $P^*=\{A_i^*\}_{i=1}^{\infty} \in \mathscr{P}$ is said to be α-optimal if it maximizes the number $\inf_{i \in N}[\alpha_i^{-1} \mu_i(A_i)]$ with $P=\{A_i\}_{i=1}^{\infty} \in \mathscr{P}$ ranging over $\mathscr{P}$, i.e.

$$\inf_{i \in N}[\alpha_i^{-1} \mu_i(A_i^*)] = \sup\{\inf_{i \in N}[\alpha_i^{-1} \mu_i(A_i)] : P=\{A_i\}_{i=1}^{\infty} \in \mathscr{P}\},$$

where $N=\{1, 2, \dots\}$.

The purpose of this paper is to prove the existence of an α-optimal partition and to find its form and upper and lower bounds for $\inf_{i \in N}[\alpha_i^{-1} \mu_i(A_i^*)]$. These results generalize those obtained in the finite case, where one considers optimal partitions of a measurable space into finitely many sets according to the measures $\{\mu_i\}_{i=1}^{n}$ and numbers $\{\alpha_i\}_{i=1}^{n}$ (see Legut and Wilczynski [9]). The problem of α-optimal partitioning a measurable space $(\mathscr{X}, \mathscr{B})$ can be interpreted as a problem of fair division of an object $\mathscr{X}$ among countably many participants (cf. [7]). In this problem each μ_i represents the individual evaluation of sets from $\mathscr{B}$.

The problem of fair division with a finite number of participants has been considered since 1946 (cf. [2, 5–13]).

We assume throughout this paper that $\sum_{i=1}^{\infty} \alpha_i$ is finite to ensure that the above problem has nontrivial solutions for every sequence $\{\mu_i\}_{i=1}^{\infty}$ of nonatomic probability measures on $(\mathscr{X}, \mathscr{B})$. For example, if $\mu_1 = \mu_2 = \ldots = \mu_n = \ldots$ and if $\sum_{i=1}^{\infty} \alpha_i = \infty$ then

$$\sup\{\inf_{i \in N}[\alpha_i^{-1}\mu_i(A_i)]: P = \{A_i\}_{i=1}^{\infty} \in \mathscr{P}\} = 0$$

and thus any partition is α-optimal.

The paper is organized as follows: In Sect. 2 we prove existence of an α-optimal partition $P^* = \{A_i^*\}_{i=1}^{\infty} \in \mathscr{P}$ and obtain lower and upper bounds for the number

$$t^* = \sup\{\inf_{i \in N}[\alpha_i^{-1}\mu_i(A_i)]: P = \{A_i\}_{i=1}^{\infty} \in \mathscr{P}\}$$

In Sect. 3 we characterize the α-optimal partitions and in Sect. 4 we give an example of application of our main result to statistical decision theory.

2. The existence and bounds

To prove that there exists a partition $P^* = \{A_i^*\}_{i=1}^{\infty}$ maximizing the number $\inf_{i \in N}[\alpha_i^{-1}\mu_i(A_i)]$ we need the following result due to Eisele [4].

Theorem 1. *If $\{\mu_i\}_{i=1}^{\infty}$ is a sequence of nonatomic finite measures, then the range X of the mapping $\bar{\mu}: \mathscr{P} \to R^N$ defined by*

$$\bar{\mu}(P) = (\mu_1(A_1), \mu_2(A_2), \ldots) \in R^N, P = \{A_i\}_{i=1}^{\infty} \in \mathscr{P}$$

is convex and weakly compact in $\sigma(R^N, (R^N)^*)$ topology.

Making use of Theorem 1, Legut [7] proved:

Theorem 2. *There exists a partition $P^0 = \{A_i^0\}_{i=1}^{\infty} \in \mathscr{P}$ of $\mathscr{X}$ satisfying $\mu_i(A_i^0) = \alpha_i$ for all $i \in N$.*

It is clear that

$$t^* = \sup\{t \in R: (t\alpha_1, t\alpha_2, \ldots) \in X\} \tag{1}$$

and thus Theorem 2 implies that $t^* \geqq 1$.

Let us denote by X^n the range of the mapping $\mu^n: \mathscr{P} \to R^n$ defined by

$$\mu^n(P) = (\mu_1(A_1), \mu_2(A_2), \ldots, \mu_n(A_n)) \in R^n, \quad P = \{A_i\}_{i=1}^{\infty} \in \mathscr{P}.$$

It is known that X^n is convex and compact in R^n (cf. [3]). Further, let

$$t_n^* = \sup\{\inf_{i \leqq n}[\alpha_i^{-1}\mu_i(A_i)]: P = \{A_i\}_{i=1}^{\infty} \in \mathscr{P}\},$$

or equivalently

$$t_n^* = \sup\{t \in R: (t\alpha_1, t\alpha_2, \ldots, t\alpha_n) \in X^n\}.$$

Theorem 3. *There exists an α-optimal partition $P^* = \{A_i^*\}_{i=1}^{\infty}$ satisfying $\mu_i(A_i^*) = t^*\alpha_i$ for each $i \in N$. Moreover, $t^* = \lim_{n\to\infty} t_n^*$.*

Proof. For each $n \in N$ the compactness of X^n yields $t_n^*(\alpha_1, \alpha_2, \ldots, \alpha_n) = (t_n^*\alpha_1, t_n^*\alpha_2, \ldots, t_n^*\alpha_n) \in X^n$, implying that $t_n^*(\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots) \in X$, $n \geqq 1$. Moreover, $\{t_n^*\}_{n=1}^{\infty}$ is a nonincreasing sequence of numbers bounded from below by t^*. Denote $t^0 = \lim_{n\to\infty} t_n^*$. Obviously, $t_0 \geqq t^*$ and $t_n^*(\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)$ converges weakly to $t_0\alpha = t_0(\alpha_1, \alpha_2, \ldots)$. From the weak compactness of X we have $t^0\alpha \in X$ and hence $t^0 \leqq t^*$. But this implies that $t^0 = t^*$ and that $t^*\alpha \in X$, which by (1) completes the proof.

Now we find estimates on the number t^*.

Theorem 4. *If at least two of the measures $\{\mu_i\}_{i=1}^{\infty}$ are different, then $1 < t^* \leqq M$, where $M := \sup\left\{\sum_{i=1}^{\infty} \mu_i(A_i): P = \{A_i\}_{i=1}^{\infty} \in \mathscr{P}\right\}$.*

Proof. First we show the inequality $t^* \leqq M$. There is nothing to prove if $M = \infty$. Then assume that $M < \infty$ and suppose that $t^* > M$. The definition of t^* yields $\alpha_i^{-1}\mu_i(A_i) > M$ for all $i \in N$, where $P^* = \{A_i^*\}_{i=1}^{\infty}$ is an α-optimal partition. Hence we have $\sum_{i=1}^{\infty} \mu_i(A_i^*) > M$, which contradicts the definition of the number M.

Now we prove the inequality $t^* > 1$. By our assumption there exists an $m \in N$ such that $M_m > 1$ (cf. [5]), where

$$M_m := \sum_{i=1}^{m} \mu_i(A_i^0) = \sup\left\{\sum_{i=1}^{m} \mu_i(A_i): P = \{A_i\}_{i=1}^{\infty} \in \mathscr{P}\right\}.$$

Denote $r_i = \mu_i(A_i^0)$, $i := 1, 2, \ldots, m$. It follows from the convexity and the weak compactness of the set $X \subseteq R^N$ that for each sequence $(\beta_i)_{i=0}^{\infty}$ with $\sum_{i=0}^{\infty} \beta_i = 1$ and for each $u^i \in X$, $i \in N \cup \{0\}$, the sequence $\sum_{i=0}^{\infty} \beta_i u^i$ belongs to X. Let

$$V := \left\{u \in R^N: u = \beta_0 r + \sum_{i=1}^{\infty} \beta_i e^i, \beta_i \geqq 0, i \in N \cup \{0\}, \sum_{i=0}^{\infty} \beta_i = 1\right\},$$

where $r=(r_1, r_2, \ldots, r_m, 0, \ldots)\in X$ and $e^i=(0, \ldots, 0, 1, 0, \ldots)\in X$ (1 is placed at the i-th coordinate, $i\in N$). Clearly $V\subseteq X$. Now it is sufficient to compute the real number

$$s^*:=\max\{s\in R: s(\alpha_1, \alpha_2, \ldots)\in V\}.$$

The set on the right hand side is not empty, because $s(\alpha_1, \alpha_2, \ldots)\in V$ iff the following system of equations for β_i, $i\in N\cup\{0\}$ has a solution, and for $s=1$ this is the case:

$$\beta_0 r_i+\beta_i=\alpha_i s \quad \text{for } i\leqq m,$$

$$\beta_i=\alpha_i s \quad \text{for } i>m,$$

$$\sum_{i=0}^{\infty} \beta_i=1.$$

We see further by solving these equations, that

$$s^*=\min\{r_i[r_i-\alpha_i(M_m-1)]^{-1}: [r_i-\alpha_i(M_m-1)]>0, i\leqq m\}$$

and finally we get $1<s^*\leqq t^*$, which completes the proof.

3. The main result

In this section we will prove our main theorem. For this we need the following three lemmas.

Lemma 1. *Let*

$$S^n:=\left\{s=(s_1, s_2, \ldots, s_n)\in R^n: s_i>0 \text{ for all } i\leqq n \text{ and } \sum_{i=1}^{n} s_i=1\right\}.$$

Then the following equalities hold

$$t_n^*=\max_{a\in X^n}\min_{p\in\bar{S}^n}\sum_{i=1}^{n} p_i\alpha_i^{-1}a_i=\min_{p\in\bar{S}^n}\max_{a\in X^n}\sum_{i=1}^{n} p_i\alpha_i^{-1}a_i,$$

where $\bar{S}^n$ stands for the closure of S^n in R^n.

The proof of this lemma can be obtained by using the minimax theorem of Sion (cf. [1]). For more details see Legut and Wilczynski [9].

Lemma 2. *Let*

$$S=\left\{s=(s_1, s_2, \ldots)\in\ell^1: s_i\geqq 0 \text{ for all } i\in N \text{ and } \sum_{i=1}^{\infty} s_i=1\right\}$$

and let

$$S_0=\{s\in S: s_i\leqq 2t^*\alpha_i \text{ for all } i\in N\}.$$

Then

(2) $$t^*=\sup_{a\in X}\inf_{p\in S}\sum_{i=1}^{\infty} p_i\alpha_i^{-1}a_i=\inf_{p\in S}\sup_{a\in X}\sum_{i=1}^{\infty} p_i\alpha_i^{-1}a_i$$

and

(3) $$t^*=\sup_{a\in X}\inf_{p\in S_0}\sum_{i=1}^{\infty} p_i\alpha_i^{-1}a_i=\inf_{p\in S_0}\sup_{a\in X}\sum_{i=1}^{\infty} p_i\alpha_i^{-1}a_i$$

Proof. Since

$$\sup\{\inf_{i\in N}[\alpha_i^{-1}\mu_i(A_i)]: P=\{A_i\}_{i=1}^{\infty}\in\mathscr{P}\}$$

$$=\sup_{a\in X}\inf_{i\in N}[\alpha_i^{-1}a_i]=\sup_{a\in X}\inf_{p\in S}\sum_{i=1}^{\infty} p_i\alpha_i^{-1}a_i$$

the first equality in (2) follows from Theorem 3. Moreover, by Lemma 1 and Theorem 3, we have

$$\inf_{p\in S}\sup_{a\in X}\sum_{i=1}^{\infty} p_i\alpha_i^{-1}a_i\leqq\inf_{p\in S^n}\sup_{a\in X}\sum_{i=1}^{n} p_i\alpha_i^{-1}a_i$$

$$=\inf_{p\in S^n}\sup_{a\in X^n}\sum_{i=1}^{n} p_i\alpha_i^{-1}a_i=t_n^*\to t^*=\sup_{a\in X}\inf_{p\in S}\sum_{i=1}^{\infty} p_i\alpha_i^{-1}a_i,$$

which yields (2) (sup inf$\leqq$inf sup always holds). We now start proving (3). By (2), for each $0<\varepsilon<t^*$ there exists $p(\varepsilon)\in S$ such that

$$t^*+\varepsilon\geqq\sup_{a\in X}\sum_{i=1}^{\infty} p_i(\varepsilon)\alpha_i^{-1}a_i,$$

hence

$$t^*+\varepsilon\geqq\sup_{i\in N} p_i(\varepsilon)\alpha_i^{-1}.$$

Therefore $p(\varepsilon)\in S_0$. By (2) we have

$$\inf_{p\in S_0}\sup_{a\in X}\sum_{i=1}^{\infty} p_i\alpha_i^{-1}a_i=\inf_{p\in S}\sup_{a\in X}\sum_{i=1}^{\infty} p_i\alpha_i^{-1}a_i$$

$$=\sup_{a\in X}\inf_{p\in S}\sum_{i=1}^{\infty} p_i\alpha_i^{-1}a_i\leqq\sup_{a\in X}\inf_{p\in S_0}\sum_{i=1}^{\infty} p_i\alpha_i^{-1}a_i\leqq\inf_{p\in S_0}\sup_{a\in X}\sum_{i=1}^{\infty} p_i\alpha_i^{-1}a_i,$$

which completes the proof of Lemma 2.

Lemma 3. *There exists* $p^0=(p_1^0, p_2^0, \ldots)\in S_0$ *such that*

$$t^*=\sup_{a\in X}\sum_{i=1}^{\infty} p_i^0\alpha_i^{-1}a_i.$$

Proof. It follows from (3) and from the compactness of S_0 that there exist $p^0, p^1, \ldots$ from S_0 such that $p^n \to p^0$ in ℓ^1 and

$$t^* = \lim_{n\to\infty} \sup_{a\in X} \sum_{i=1}^{\infty} p_i^n \alpha_i^{-1} a_i. \tag{4}$$

It is clear that

$$t^* = \inf_{p\in S_0} \sup_{a\in X} \sum_{i=1}^{\infty} p_i \alpha_i^{-1} a_i \leqq \sup_{a\in X} \sum_{i=1}^{\infty} p_i^0 \alpha_i^{-1} a_i.$$

To prove the converse inequality let us suppose that

$$\sup_{a\in X} \sum_{i=1}^{\infty} p_i^0 \alpha_i^{-1} a_i = t^* + 3\varepsilon$$

for some $\varepsilon > 0$. Then there exist $b \in X$ and an integer number $k = k(\varepsilon)$ such that

$$\sum_{i=1}^{\infty} p_i^0 \alpha_i^{-1} b_i \geqq t + 2\varepsilon \quad \text{and} \quad \sum_{i=1}^{k} p_i^0 \alpha_i^{-1} b_i \geqq t^* + \varepsilon. \tag{5}$$

Let $L := \max_{i\leqq k} \alpha_i^{-1} b_i$ and let

$$X_L = \{a \in X : \alpha_i^{-1} a_i \leqq L \text{ for each } i \in N\}.$$

Then

$$\left| \sup_{a\in X_L} \sum_{i=1}^{\infty} p_i^n \alpha_i^{-1} a_i - \sup_{a\in X_L} \sum_{i=1}^{\infty} p_i^0 \alpha_i^{-1} a_i \right|$$
$$\leqq \sup_{a\in X_L} \left| \sum_{i=1}^{\infty} (p_i^n - p_i^0) \alpha_i^{-1} a_i \right| \leqq \mathrm{L} \sum_{i=1}^{\infty} |p_i^n - p_i^0|$$

which implies ($L < \infty$) that

$$\lim_{n\to\infty} \sup_{a\in X_L} \sum_{i=1}^{\infty} p_i^n \alpha_i^{-1} a_i = \sup_{a\in X_L} \sum_{i=1}^{\infty} p_i^0 \alpha_i^{-1} a_i. \tag{6}$$

Hence, by (5), (6) and (4)

$$t^* + \varepsilon \leqq \sum_{i=1}^{k} p_i^0 \alpha_i^{-1} b_i \leqq \sup_{a\in X_L} \sum_{i=1}^{\infty} p_i^0 \alpha_i^{-1} a_i$$
$$= \lim_{n\to\infty} \sup_{a\in X_L} \sum_{i=1}^{\infty} p_i^n \alpha_i^{-1} a_i \leqq \lim_{n\to\infty} \sup_{a\in X} \sum_{i=1}^{\infty} p_i^n \alpha_i^{-1} a_i = t^*.$$

But this is a contradiction, which completes the proof of Lemma 3.

As a consequence of the last two lemmas we obtain the following

Corollary. *Let $P^* = \{A_i^*\}_{i=1}^{\infty} \in \mathscr{P}$ be an α-optimal partition. Then*

$$(7) \qquad \inf_{p \in S_0} \sum_{i=1}^{\infty} p_i \alpha_i^{-1} \mu_i(A_i^*) = \sum_{i=1}^{\infty} p_i^0 \alpha_i^{-1} \mu_i(A_i^*) = \sup_{P \in \mathscr{P}} \sum_{i=1}^{\infty} p_i^0 \alpha_i^{-1} \mu_i(A_i),$$

where p^0 is as in Lemma 3.

Proof. Since $\mu_i(A_i^*) = t^* \alpha_i$ for each $i \geqq 1$ by Lemma 2 and Lemma 3 we have

$$t^* = \sup_{P \in \mathscr{P}} \inf_{p \in S_0} \sum_{i=1}^{\infty} p_i \alpha_i^{-1} \mu_i(A_i) = \inf_{p \in S_0} \sum_{i=1}^{\infty} p_i \alpha_i^{-1} \mu_i(A_i^*)$$
$$= \sum_{i=1}^{\infty} p_i^0 \alpha_i^{-1} \mu_i(A_i^*) \leqq \sup_{P \in \mathscr{P}} \sum_{i=1}^{\infty} p_i^0 \alpha_i^{-1} \mu_i(A_i) = t^*,$$

and hence (7) holds.

Before coming to the main theorem we will introduce the following notation. For each $i \in N$, let $f_i = d\mu_i/dv$ denote the Radon-Nikodym derivative, where $v = \sum_{i=1}^{\infty} 2^{-i} \mu_i$. Moreover, let $\{B_i^*\}_{i=1}^{\infty}$ and $\{C_i^*\}_{i=1}^{\infty}$ be two sequences of measurable sets from $(\mathscr{X}, \mathscr{B})$ defined by

$$B_i^* = \bigcap_{\substack{j=1 \\ j \neq i}}^{\infty} \{x \in \mathscr{X} : p_i^0 \alpha_i^{-1} f_i(x) > p_j^0 \alpha_j^{-1} f_j(x)\}, \qquad i \in N,$$

$$C_i^* = \bigcap_{\substack{j=1 \\ j \neq i}}^{\infty} \{x \in \mathscr{X} : p_i^0 \alpha_i^{-1} f_i(x) \geqq p_j^0 \alpha_j^{-1} f_j(x)\}, \qquad i \in N.$$

Now we may state the main result of the paper.

Theorem 5. *Let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of positive numbers with $\sum_{i=1}^{\infty} \alpha_i = 1$. Then there exist a point $p^0 \in S_0$ and a corresponding α-optimal partition $P^* = \{A_i^*\}_{i=1}^{\infty} \in \mathscr{P}$ satisfying*

(i) $B_i^* \subset A_i^* \subset C_i^*$,

(ii) $\mu_1(A_1^*)/\alpha_1 = \mu_2(A_2^*)/\alpha_2 = \ldots = t^*$.

Moreover, any partition $P^* = \{A_i^*\}_{i=1}^{\infty} \in \mathscr{P}$ which satisfies (i) *and* (ii) *is α-optimal.*

Proof. The existence of the α-optimal partition $P^* = \{A_i^*\}_{i=1}^{\infty} \in \mathscr{P}$ satisfying (ii) follows from Theorem 3. By Corollary, this partition maximizes the number $\sum_{i=1}^{\infty} p_i^0 \alpha_i^{-1} \mu_i(A_i)$ with $P = \{A_i\}_{i=1}^{\infty}$ ranging over $\mathscr{P}$, i.e. $\sum_{i=1}^{\infty} p_i^0 \alpha_i^{-1} \mu_i(A_i^*) = \sup_{P \in \mathscr{P}} \sum_{i=1}^{\infty} p_i^0 \alpha_i^{-1} \mu_i(A_i)$, and hence (i) must hold. Since all measures μ_i are non-atomic the rest of the proof is straightforward.

4. Statistical interpretation

Suppose that Y is an $\mathscr{X}$ valued random variable having an unknown distribution μ which belongs to the set $\{\mu_1, \mu_2, \ldots\}$. Given a single observation $Y=y$ it is to be decided which is the true distribution of Y. A decision rule is a measurable partition $P=\{A_i\}_{i=1}^{\infty}$ of $\mathscr{X}$ with the understanding that if Y falls in A_i then $\mu=\mu_i$ is guessed. If that guess is correct and the true value of μ is μ_j, $j \geqq 1$, then the gain equals α_j^{-1}, otherwise it is equal to zero. It is easy to see that the α-optimal partition is minimax in the following problem: Find a partition which maximizes the minimal expected payoff R, where

$$R=\inf_{i \in N}[\alpha_i^{-1} \mu_i(A_i)].$$

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