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Probability Theory and Related Fields

First passage percolation: the stationary case

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Summary. If the passage time of the edges of the $\mathbb{Z}^d$ lattice are stationary, ergodic and have finite moment of order $p>d$, then a.s. the set of vertices that can be reached within time t, has an asymptotic shape as $t\to\infty$.

1. Introduction

In Section 2, there is a precise description of the model. For now, consider $\mathbb{Z}^2$ as the vertices of the square lattice. To each edge of the square lattice assign, independently and according to the same probability law, a nonnegative number called the passage time of the edge. If r is a path in the square lattice, the passage time of r is the sum of the passage time of all the edges in r. The travel time between two vertices x, y is the infimum of the passage time over all the paths in the square lattice from x to y.

This model was introduced in [6] by Hammersley and Welsh. Early results can be found in [12]. Then Cox and Durrett showed that under some moment conditions, the set of vertices, one can reach within time t has a nonrandom asymptotic shape as t increases for almost all realizations of the model [4]. Also see [7], and [5] where there are some computer simulations.

An equivalent formulation of the result of Cox and Durrett is the following. Let $F(t)$ be the distribution function of the passage time of an edge and for $x\in\mathbb{R}^2$, let $|x|=|x_1|+|x_2|$.

Theorem. [4] *If the random variable with distribution function $1-(1-F(t))^4$ has finite second moment then there is a deterministic continuous and nonnegative function μ on $\{x\in\mathbb{R}^2; |x|=1\}$ such that*

$$\lim_{\substack{|x|\to\infty\\ x\in\mathbb{Z}^2}} \left(|x|^{-1}\, T(0,x)-\mu\left(\frac{x}{|x|}\right)\right)=0 \quad a.s.$$

In [7] the proof is given for $\mathbb{Z}^d$, $d\geqq 2$ under the assumption that $F(t)$ has finite second moment. Then in [7], Derriennic noticed that this result is true if the passage time of the edges are not necessarily independent but only stationary and bounded. The purpose of this paper is to show that if the passage time of the edges are stationary, ergodic and have a finite moment $>d$ (i.e. strictly greater than the dimension of the lattice) then the theorem holds. The proof is given in Section 3.

Actually, the natural integrability conditions which appear are those of the Lorentz spaces $L(d, 1)$ ([10], see also [13, V. 3] and [2]). If f is a real valued measurable function on a probability space $(\Omega, \mathscr{F}, P)$ then the Lorentz norm of f is

$$\|f\|_{d,1}=\int_0^1 f^*(s)\, s^{(1/d)-1}\, ds$$

where f^*: $[0,1]\to\mathbb{R}^+$, is the nonincreasing right continuous function which has the same distribution as $|f|$. Then $L(d,1)=\{f: \|f\|_{d,1}<\infty\}$. $\|f\|_{d,1}$ is a norm, $L(d,1)$ is a Banach space and there are constants c and c' such that

$$c'\|f\|_{d+\varepsilon}\geqq\|f\|_{d,1}\geqq c\|f\|_d \quad \text{where } \|f\|_p^p=\int_\Omega |f|^p\, dP.$$

Using a d-dimensional Rohlin tower ([11] for example), one can construct, for any $p<d$, a counterexample where the passage time of the edges have a moment of order p and are stationary.

We end this section with some notation. $e_i=(0, \ldots, 1, \ldots 0)$ where 1 is at the ith position. $\mathbb{R}^+=\{\lambda\in\mathbb{R}; \lambda\geqq 0\}$. For $x\in\mathbb{R}^d$, $|x|_\infty=\max(|x(i)|; 1\leqq i\leqq d)$, $|x| =|x(1)|+\ldots+|x(d)|$ and $\|x\|^2=|x(1)|^2+\ldots+|x(d)|^2$ and then $\|x\|\leqq|x| \leqq\sqrt{d}\,\|x\|$. m will be the counting measure on $\mathbb{Z}^d$ and $\mathbb{1}_A$, the indicator function of the set A. $[\lambda]$ is the greatest integer $\leqq\lambda$.

2. Description of the model and statement of the result

Consider an ergodic measure preserving $\mathbb{Z}^d$-action $(\tau_x : x\in\mathbb{Z}^d)$ of a probability space $(\Omega, \mathscr{F}, P)$, that is

(i) $\tau_x: \Omega\to\Omega$ is measurable for all $x\in\mathbb{Z}^d$
(ii) $P(\tau_x A)=P(A)$ for all $A\in\mathscr{F}$ and $x\in\mathbb{Z}^d$
(iii) $\tau_x\tau_y=\tau_{x+y}$ for all $x, y\in\mathbb{Z}^d$
(iv) if $f: \Omega\to\mathbb{R}$ is a measurable function such that $f(\tau_x\omega)=f(\omega)$
for some $x\in\mathbb{Z}^d$, $x\neq 0$, then $f=$constant a.e.

To assign a passage time to each edge of $\mathbb{Z}^d$, we choose d measurable and nonnegative functions $f_i: \Omega\to\mathbb{R}^+$, $1\leqq i\leqq d$. The passage time of the edge from x to $x+e_i$ (in the realization ω of the model) is $t(\omega; x, x+e_i)=f_i(\tau_x\omega)$. Two vertices x, y of $\mathbb{Z}^d$ are adjacent if $|y-x|=1$. For any n pairs of adjacent vertices (x_j, y_j), the distribution of $(t(\omega; x_1+z, y_1+z), \ldots, t(\omega; x_n+z, y_n+z))$ is independent of $z\in\mathbb{Z}^d$. An equivalent description of the model would be to say that we are given a sequence of random variables $\{t(\omega; x, x+e_i): 1\leqq i\leqq d, x\in\mathbb{Z}^2\}$ which is stationary and ergodic (see [9, p. 22ff] for example). The case considered in [4] and [7] is where $\{t(\omega; x, x+e_i): 1\leqq i\leqq d, x\in\mathbb{Z}^d\}$ is a sequence of independent identically distributed random variables.

We now proceed as in [7] to define the travel time between two vertices and the time constants. A path r in $\mathbb{Z}^d$ is a finite sequence of adjacent vertices $r=(x_0, \ldots, x_n)$. The passage time of r is $T(\omega; r)=\sum_{j=0}^{n-1} t(\omega; x_j, x_{j+1})$ and if x and y are two vertices of $\mathbb{Z}^d$, the travel time from x to y is $T(\omega; x, y)=\inf\{T(\omega; r)$: r is a path from x to $y\}$. Then for all $x, y\in\mathbb{Z}^d$, $T(\omega; 0, x+y)\leqq T(\omega; 0, x) + T(\omega; x, x+y)= T(\omega; 0, x)+ T(\tau_x \omega; 0, y)$, that is $T(\omega; 0, x)$ is subadditive. Subsequently, ω will not appear explicitly in most expressions.

As in the independent case, it follows from Kingman's subadditive theorem [8] that if the f_i, $1\leqq i\leqq d$ are in $L^1(P)$ then there is a continuous function $\mu: \mathbb{R}^d \to \mathbb{R}^+$ such that for all $x\in\mathbb{Z}^d$, $\lim_{n\to\infty} n^{-1} T(0, nx)=\mu(x)$ a.e. and in L^1, and the following properties hold:

(i) $\mu(0)=0$
(ii) $\mu(\lambda x)=|\lambda|\,\mu(x)$ for all $\lambda\in\mathbb{R}$ and $x\in\mathbb{R}^d$
(iii) $\mu(x+y)\leqq\mu(x)+\mu(y)$ for all $x, y\in\mathbb{R}^d$
(iv) $|\mu(y)-\mu(x)|\leqq\bar{\mu}\,|y-x|$ where $\bar{\mu}=\max(\max_{1\leqq i\leqq d}\mu(e_i), 1)$.

Theorem. *If $f_i\in L(d, 1)$ for all $1\leqq i\leqq d$, then* $\lim_{|x|\to\infty}\left(\frac{1}{|x|}T(0, x)-\mu\left(\frac{x}{|x|}\right)\right)=0$ *a.e.*

The proof of the theorem relies on the following maximal lemma which was needed in a different context in [1].

Maximal Lemma. *If $f_i\in L(d, 1)$ for all $1\leqq i\leqq d$ then $P\{\sup_{x\neq 0}|x|^{-1}T(0, x)>\lambda\} <K\lambda^{-d}\sup_{1\leqq i\leqq d}\|f_i\|_{d,1}^d$ for all $\lambda>0$ where K is constant that depends only on the dimension d.*

The proof of this lemma is essentially the same as in [1; Theorem 6]. However, a self-contained proof, including the transference principle, is given below for the two-dimensional case. This might facilitate a possible reference to [1].

Fix a point x in $\mathbb{Z}^2$, $x\neq 0$. The construction given below is for x in the region $x(2)\geqq x(1)$ and $x(2)>-x(1)$. The construction for x in the other regions of $\mathbb{Z}^2$ is simply obtained by a rotation.

Among all the paths from $(0, 0)$ to x, choose a set ξ_x of $|x|$ paths r_j, $1\leqq j\leqq|x|$ such that r_j goes from $(0, 0)$ to $(x(1)-[(|x|+1)/2]+j, 0)$, then upward to $(x(1)-[(|x|+1)/2]+j,\ x(2)-[|x|/2])$ and then to x in such a way that the edge between $(k, x(2)-\ell)$ and $(k+1,\ x(2)-\ell)$, $x(1)-\ell-1\leqq k\leqq x(1)+\ell-1$, $0\leqq\ell\leqq[|x|/2]-1$ belongs to at most $|x|/2(\ell+1)$ paths, the edge between $(k, x(2)-\ell)$ and $(k, x(2)-\ell+1)$, $x(1)-\ell\leqq k\leqq x(1)+\ell$, $1\leqq\ell\leqq[|x|/2]$ belongs to at most $|x|/2\ell$ paths and such that no other edge of the graph belongs to these paths.

Since $T(0, x)\leqq T(r_j)$ for all r_j in ξ_x, $T(0, x)\leqq|x|^{-1}\sum_{j=1}^{|x|} T(r_j)$ and $|x|^{-1}T(0, x) \leqq|x|^{-2}\sum_{j=1}^{|x|} T(r_j)$.

This is a sum of terms of the form $w(y) f_1(\tau_y \omega)$ and $v(y) f_2(\tau_y \omega)$ where $w(y)$ (resp. $v(y)$) is the number of paths in ξ_x which go through the edge from y to $y+(1,0)$ (resp. y to $y+(0,1)$). For all y in $\mathbb{Z}^2$, $w(y) \leqq |x|$ since the paths are without loops. For y in the region $x(1)-[(|x|+1)/2]+1 \leqq y(1) \leqq x(2)-[(|x|+1)/2]+|x|$, and $1 \leqq y(2) < x(2)-[|x|/2]$, $w(y)=0$ and $v(y)=1$. And for $y=(k, x(2)-\ell)$ for $x(1)-\ell-1 \leqq k \leqq x(1)+\ell$, $0 \leqq \ell \leqq [|x|/2]$, $w(y) \leqq |x|/2(\ell+1) \leqq 2|x||y-x|^{-1}$, $y \neq x$ and $v(y) \leqq |x|/2\ell \leqq 2|x||y-x|^{-1}$, $y \neq x$.

$$\begin{aligned} |x|^{-1} T(0,x) \leqq C((2|x|+1)^{-1} & \sum_{i=-|x|}^{|x|} f_1(\tau_{(i,0)}\omega) \\ & +(2|x|+1)^{-2} \sum_{i,j=-|x|}^{|x|} f_2(\tau_{(i,j)}\omega) \\ & +|x|^{-1}(f_1(\tau_x \omega) + \sum_{0<|y-x| \leqq |x|} |y-x|^{-1} f_1(\tau_y \omega)) \\ & +|x|^{-1}(f_2(\tau_x \omega) + \sum_{0<|y-x| \leqq |x|} |y-x|^{-1} f_2(\tau_y \omega))). \end{aligned}$$

We must prove a maximal inequality for each of these terms. The first two terms are classical averages. Since the maximal function for these two averages is in L^2, the weak maximal inequality obtained from Chebyshev's would be enough for our purpose. But we prefer to treat all these four terms the same way. The first step is to show a maximal inequality for the real-variable case.

Let $f\colon \mathbb{Z}^d \to \mathbb{R}^+$, then $\|f\|_{2,1} = \frac{1}{2} \int_0^\infty f^*(t)\, t^{-1/2}\, dt = \sum_{i=1}^\infty f^*(i-1)(\sqrt{i}-\sqrt{i-1})$ where $f^*(i)$ is the sequence of values of f rearranged in a decreasing order.

Lemma 1. *For $f\colon \mathbb{Z} \to \mathbb{R}^+$, the following maximal inequality holds:*

$$m\{z \in \mathbb{Z} : \sup_{n \geqq 0} (2n+1)^{-1} \sum_{i=-n}^{n} f(z+i) > \lambda\} \leqq 12 \lambda^{-2} \|f\|_{2,1}^2 .$$

Lemma 2. *For $f\colon \mathbb{Z}^2 \to \mathbb{R}^+$, the following maximal inequality holds:*

$$m\{z \in \mathbb{Z}^2 : \sup_{n \geqq 0} (2n+1)^{-2} \sum_{|y|_\infty \leqq n} f(z+y) > \lambda\} < 36 \lambda^{-2} \|f\|_{2,1}^2 .$$

Lemma 3. *For $f\colon \mathbb{Z}^2 \to \mathbb{R}^+$, the following maximal inequality holds:*

$$m\{z \in \mathbb{Z}^2 : \sup_{x \neq 0} |x|^{-1}(f(x+z) + \sum_{0<|y-x| \leqq |x|} |y-x|^{-1} f(y+z)) > \lambda\} < c \lambda^{-2} \|f\|_{2,1}^2 .$$

Proof of Lemma 2. Write $A_n f(z) = (2n+1)^{-2} \sum_{|y|_\infty \leqq n} f(z+y)$. It is sufficient to consider the case where f has finite support. Following the classical covering Lemma [13, p. 54], we can build a finite sequence (n_j, z_j) such that

$$\begin{aligned} & A_{nj} f(z_j) > \lambda \ \forall j \\ & I_j = [z(1)-n_j, z(1)+n_j] \times [z(2)-n_j, z(2)+n_j] \text{ are pairwise disjoint} \\ & m(I_j) \geqq \tfrac{1}{9} \mathrm{m}\{z \in \mathbb{Z}^2 : \sup_n A_n f(z) > \lambda\}. \end{aligned}$$

Put $L = \cup I_j$. Then

(1)
$$\lambda < \sum_j \frac{m(I_j)}{m(L)} A_{n_j} f(z_j)$$
$$= \frac{1}{\sqrt{m(L)}} \sum_j \frac{1}{\sqrt{m(L)}} \sum_{s \in I_j} f(s) \text{ since } m(I_j) = (2n_j + 1)^2$$
$$\leqq \frac{1}{\sqrt{m(L)}} \sum_{i=1}^{m(L)} i^{-1/2} f^*(i-1) \leqq \frac{2}{\sqrt{m(L)}} \|f\|_{2,1}$$

and
$$m\{z: \sup_{n \geqq 1} A_n f(z) > \lambda\} \leqq 9m(L) < 9(4\lambda^{-1} \|f\|_{2,1})^2.$$

Proof of Lemma 3. The proof is similar to the one of Lemma 2 except for the calculation needed to obtain inequality (1).

Write $A_x f(z) = |x|^{-1}(f(x+z) + \sum_{0<|y-x| \leqq |x|} |y-x|^{-1} f(y+z))$. Consider the case where f has finite support. By the covering lemma, we have a finite sequence (x_j, z_j) such that

$$A_{x_j} f(z_j) > \lambda \quad \text{for all } j$$
$$I_j = \{y \text{ in } \mathbb{Z}^2 : |y - z_j| \leqq |x_j|\} \text{ are pairwise disjoint}$$
$$m(I_j) \geqq 9^{-1} m\{z: \sup_{x \neq 0} A_x f(z) > \lambda\}.$$

Put $L = \cup I_j$. Then

$$\lambda < \sum_j \frac{m(I_j)}{m(L)} A_{x_j} f(z_j)$$
$$\leqq \frac{4}{\sqrt{m(L)}} \sum_j \frac{m(I_j)}{|x_j| \sqrt{m(L)}} (f(x_j + z_j) + \sum_{0<|y-x_j| \leqq |x_j|} |y - x_j|^{-1} f(y + z_j)).$$

Let $p(y)$ be the weights which appear in this average, that is $p(y) = m(I_j)|x_j|^{-1}(m(L))^{-1/2}|y-x_j|^{-1}$ if $0 < |y - x_j| \leqq |x_j|$, $p(x_j) = m(I_j)|x_j|^{-1} (m(L))^{-1/2}$ and $p(y) = 0$ for all other y in $\mathbb{Z}^2$. Then

$$m\{y \text{ in } L: p(y) \geqq 3i^{-1/2}\} \leqq \sum_j m\{y \text{ in } I_j : |y - x_j| \leqq \sqrt{i}\, m(I_j)/3|x_j| \sqrt{m(L)}\}$$
$$\leqq \sum_j \frac{im(I_j)}{(3|x_j|)^2} \frac{m(I_j)}{m(L)} \leqq i.$$

Therefore $\lambda < \frac{4}{\sqrt{m(L)}} \sum_{i=1}^{m(L)} 3i^{-1/2} f^*(i-1) \leqq \frac{24}{\sqrt{m(L)}} \|f\|_{2,1}$.

Using the transference method as it is explained in [3], we obtain a maximal inequality for each term of (1). The only modification needed because we are working with Lorentz spaces, is the use of Minkowski's inequality. This was first done in [2] for similar averages with a continuous parameter.

Lemma 4. *For $f\colon \Omega \to \mathbb{R}^+$, the following maximal inequality holds*

$$P\{\omega\colon \sup_{n \geqq 0} (2n+1)^{-2} \sum_{|y|_\infty \leqq n} f(\tau_y \omega) > \lambda\} \leqq c(\lambda^{-1} \|f\|_{2,1})^2.$$

Proof. For k and u, positive integers, let

$$E = \{\omega\colon \sup_{n \leqq k} (2n+1)^{-2} \sum_{|y|_\infty \leqq n} f(\tau_y \omega) > \lambda\}$$

and

$$\bar{E} = \{(z, \omega)\colon \sup_{n \leqq k} (2n+1)^{-2} \sum_{|y|_\infty \leqq n} f(\tau_{y+z} \omega) > \lambda \quad \text{and} \quad |z|_\infty \leqq u\}.$$

$(m \times P)(\bar{E}) = \int_\Omega m(\bar{E}_\omega)\, dP = \sum_z P(\bar{E}_z)$ where $\bar{E}_\omega$ and $\bar{E}_z$ are the ω- and z-section of $\bar{E}$ respectively.

Since τ_x is measure preserving, $P(\bar{E}_z) = P(E)$ for all $|z|_\infty \leqq u$

$$\begin{aligned} P(\bar{E}_\omega) &= m(z\colon \sup_{|n| \leqq k} (2n+1)^{-1} \sum_{|y|_\infty \leqq n} f(z_{y+z} \omega) > \lambda, |\tau|_\infty \leqq u\} \\ &\leqq c(\lambda^{-1} \|F_{u+k}(\cdot, \omega)\|_{2,1})^2 \end{aligned}$$

by Lemma 1, where $F_{u+k}(y, \omega) = f(\tau_y \omega)$ for $|y|_\infty \leqq u+k$ and 0 otherwise.

By an integration by parts, the Lorentz norm can be written as $\|F_{u+k}\|_{2,1} = \int_0^\infty (m\{F_{u+k}(\cdot, \omega) > \lambda\})^{1/2} d\lambda$. Minkowski's inequality yields

$$\begin{aligned} \int_\Omega m(\bar{E}_\omega)\, dP &\leqq c\lambda^{-2} \int_\Omega \left(\int_0^\infty (m\{F_{u+k}(\cdot, \omega) > \lambda\})^{1/2} d\lambda \right)^2 dP \\ &\leqq c\lambda^{-2} \left(\int_0^\infty \left(\int_\Omega m\{F_{u+k}(\cdot, \omega) > \lambda\}\, dP \right)^{1/2} d\lambda \right)^2 \\ &= c\lambda^{-2} \left(\int_0^\infty \left(\sum_{|x|_\infty \leqq u+k} P\{f(\tau_z \omega) > \lambda\} \right)^{1/2} d\lambda \right)^2 \\ &\leqq c\lambda^{-2} (2(u+k)+1)^2 \|f\|_{2,1}^2. \end{aligned}$$

Therefore $(2u+1)^2 P(E) \leqq c(2(u+k)+1)^2 \lambda^{-2} \|f\|_{2,1}^2$. Let $u \to \infty$ and then $k \to \infty$ to obtain $P(\sup_{n \neq 0} (2n+1)^{-2} \sum_{|z|_\infty \leqq n} f(\tau_z \omega) > \lambda\} \leqq c(\lambda^{-1} \|f\|_{2,1})^2$.

The same proof works for the other two types of averages.

3. Proof of the theorem

For each positive integer λ put $A_\lambda=\{\omega;\ \sup_{x\neq 0}|x|^{-1}T(0,x)<\lambda\}$. For each integer $M\geqq 1$, $V_M=\left\{\frac{x}{M};x\in\mathbb{Z}^d\right\}$, $V=\bigcup_{M\geqq 1} V_M$ and $B=\{z\in\mathbb{R}^d;z\in V \text{ and } |z|=1\}$. For each $z\in B$, there is a rotation $\varphi_z\in SO(d)$ such that $\varphi_z(e_d)=\frac{z}{\|z\|}$. For each positive integer k and $\rho>0$, let $C(k,\rho)$ be the conical volume:

$$C(k,\rho)=\left\{x\in\mathbb{R}^d: x(1)^2+\ldots+x(d-1)^2\leqq\left(\frac{x(d)}{k}\right)^2, 0\leqq x(d)\leqq\rho\right\}.$$

Then $\varphi_z(C(k,\rho)$ does not depend on the element φ_z of $SO(d)$ chosen.

For $h\in L^1(\Omega)$, we use $M_\rho(k,z)h$ to denote the average $M_\rho(k,z)h(\omega)=m(\varphi_z C(k,\rho))^{-1}\sum h(\tau_y\omega)$ where the sum is over all y in $\varphi_z C(k,\rho)\cap\mathbb{Z}^d$.

It is simple to check that the ergodic theorem holds a.e. in these cones; that is if $h\in L^1(P)$ then $M_\rho(k,z)h(\omega)\to\int h\,dP$ a.e. as $\rho\to\infty$.

Then we can find a set $\Omega'\subset\Omega$ of measure one such that for all positive integers λ, k and all $z\in B$, we have:

$$M_\rho(k,z)\mathbb{1}_{A_\lambda}(\omega)\to P(A_\lambda)\quad\text{as}\quad\rho\to\infty\qquad\text{for all }\omega\in\Omega'. \tag{2}$$

Therefore for all $\omega\in\Omega'$, there is an integer $N_1=N_1(\omega,k,z,\lambda)$ such that for all $\rho\geqq N_1$, $P(A_\lambda)-K_1\lambda^{-d}<M_\rho(k,z)\mathbb{1}_{A_\lambda}<P(A_\lambda)+K_1\lambda^{-d}$ where $K_1=K\sup_{1\leqq i\leqq d}\|f_i\|_{d,1}^d$.

Consider a fixed $\omega\in\Omega'$ and positive integers k, λ and $z\in B$. Assume that $2K_1\lambda^{-d}<1/4$. We claim that if $\rho>\max(N_1(\omega,k,z,\lambda),2k)$ and if $\rho'-\rho\geqq\max(2,K_6\rho\lambda^{-d})$ where K_6 is a constant that depends only on K_1 and d, then there is a $z'\in\mathbb{Z}^d\cap(\varphi_z C(k,\rho')\setminus\varphi_z C(k,\rho))$ such that $\tau_{z'}\omega\in A_\lambda$.

To establish the claim, put $\Delta'=\mathbb{Z}^d\cap(\varphi_z C(k,\rho')\setminus\varphi_z C(k,\rho))$, $\Delta=\mathbb{Z}^d\cap\varphi_z C(k,\rho)$, $D'=\{y\in\Delta':\tau_y\omega\in A_\lambda\}$ and $D=\{y\in\Delta:\tau_y\omega\in A_\lambda\}$.

Then

$$\begin{aligned}1-2K_1\lambda^{-d}&<P(A_\lambda)-K_1\lambda^{-d}\quad\text{by the maximal lemma}\\&<M_\rho(k,z)\mathbb{1}_{A_\lambda}\\&\leqq(m(D')+m(D))/(m(\Delta')+m(\Delta)).\end{aligned}$$

Thus $M(D')>m(\Delta')-2K_1\lambda^{-d}(m(\Delta)+m(\Delta'))$.

We would like to have $(1-2K_1\lambda^{-d})m(\Delta')>2K_1\lambda^{-d}m(\Delta)+1$ or still, $m(\Delta')>8K_1\lambda^{-d}m(\Delta)$. But $m(\Delta')>K_2\left(\frac{\rho}{k}-2\right)^{d-1}(\rho'-\rho-1)>K_3\left(\frac{\rho}{k}\right)^{d-1}(\rho'-\rho)$ and $m(\Delta)<K_4\frac{(\rho+3k)^d}{2k^{d-1}}<K_5 k\left(\frac{\rho}{k}\right)^d$. Therefore it would be enough to have $\rho'-\rho>\frac{8K_1K_5}{K_3}\cdot\frac{\rho}{\lambda^d}$.

We complete the proof by contradiction. Suppose that the theorem is false. Then there exists a measurable set F of positive measure such that, for all $\omega \in F$,

$$(3) \qquad \limsup_{|x| \to \infty} \left| |x|^{-1} T(0, x) - \mu\left(\frac{x}{|x|}\right) \right| > 0.$$

Then fix ω in $F \cap \Omega'$ and find a sequence (x_i) in $\mathbb{Z}^d$ and $\varepsilon, 0 < \varepsilon < 1$ such that $|x_i| \to \infty$, $\frac{x_i}{|x_i|} \to y$ as $i \to \infty$ and $\left| \frac{1}{|x_i|} T(0, x_i) - \mu\left(\frac{x_i}{|x_i|}\right) \right| > \varepsilon$ for all i.

To obtain a contradiction we make the following choices:

1. Choose the integer λ large enough so that $\lambda^{1-d} < \varepsilon/(32\sqrt{d} K_6)$ and $2K_1/\lambda^d < 1/4$.

2. Choose k large enough so that $\lambda k^{-1} < \varepsilon/(8\sqrt{d}\,\lambda)$.

3. Choose M large enough to be able to find z in V_M such that $|z| = 1$ and $|y - z| < \varepsilon/10\,\bar{\mu}\lambda$.

Let n_i be the integer such that $n_i M \leqq |x_i| < (n_i + 1) M$ $\left(\text{and hence, } 0 \leqq 1 - \frac{n_i M}{|x_i|} < \frac{M}{|x_i|}\right)$.

4. Choose N_0 large enough such that for all $i > N_0$, $M|x_i|^{-1} < \varepsilon/20\,\bar{\mu}\lambda$, $\left| y - \frac{x_i}{|x_i|} \right| < \varepsilon/10\,\bar{\mu}\lambda$, $|(n_i M)^{-1} T(0, n_i M z) - \mu(z)| < \varepsilon/10$ and $\|n_i M z\| > \max(2k, N_1(\omega, k, z, \lambda), \lambda^d/K_6)$.

Then for these choices and for all $i > N_0$ we have,

$$(4) \qquad \begin{aligned} \left| |x_i|^{-1} T(0, x_i) - \mu\left(\frac{x_i}{|x_i|}\right) \right| \leqq & \left| |x_i|^{-1} T(0, x_i) - |x_i|^{-1} T(0, n_i M z) \right| \\ & + \left| |x_i|^{-1} T(0, n_i M z) - (n_i M)^{-1} T(0, n_i M z) \right| \\ & + \left| (n_i M)^{-1} T(0, n_i M z) - \mu\left(\frac{x_i}{|x_i|}\right) \right|. \end{aligned}$$

It follows from the claim (with $\rho = \|n_i M z\|$ and $\rho' - \rho = 2K_6\,\rho\,\lambda^{-d}$) that there is a $z' \in \mathbb{Z}^d \cap (\varphi_z C(k, \rho') \setminus \varphi_z C(k, \rho))$ such that $\tau_{z'}\,\omega \in A_\lambda$. And therefore

$$(5) \qquad \begin{aligned} ||x_i|^{-1} T(0, x_i) - |x_i|^{-1} T(0, n_i M z)| & \leqq |x_i|^{-1} T(x_i, n_i M z) \\ & \leqq |x_i|^{-1} (T(x_i, z') + T(z', n_i M z)) \\ & \leqq \lambda |x_i|^{-1} (|x_i - z'| + |z' - n_i M z|) \\ & \leqq \lambda |x_i|^{-1} (|x_i - n_i M z| + 2|z' - n_i M z|). \end{aligned}$$

(6) But $\quad |x_i - n_i M z| \leqq |x_i| (|y - |x_i|^{-1} x_i| + |y - z| + (1 - |x_i|^{-1} n_i M))$,

and

$$\begin{aligned} |z' - n_i M z| & \leqq \sqrt{d}\, \|z' - n_i M z\| \\ & \leqq \sqrt{d} (\rho'/k + (\rho' - \rho)) \\ & \leqq \sqrt{d} (2K_6 \|n_i M z\|/k\lambda^d + \|n_i M z\|/k + 2K_6 \|n_i M z\|/\lambda^d) \\ & \leqq 4\sqrt{d} K_6\, n_i M \lambda^{-d} + \sqrt{d}\, n_i M k^{-1}. \end{aligned}$$

Taking (6) and (7) into account and properties 1. to 4., from (5) we obtain

$$||x_i|^{-1} T(0, x_i) - |x_i|^{-1} T(0, n_i M z)| \leqq \frac{\varepsilon}{2}.$$

The second term in (4) is bounded by $M|x_i|^{-1}((n_i M)^{-1} T(0, n_i M z)) < \varepsilon/10$ and the third one by $|(n_i M)^{-1} T(0, n_i M z) - \mu(z)| + \bar{\mu}|z-y| + \bar{\mu}|y - |x_i|^{-1} x_i| < 3\varepsilon/10$. This is a contradiction to (1).

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