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Isoperimetric constants and estimates of heat kernels of pre Sierpinski carpets

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Summary. The author calculated isoperimetric constants of the n-dimensional pre Sierpinski carpet $\mathscr{Y}_n$. As an application, he obtained the following estimate of the Neumann heat kernel $p_n(t, x, y)$ on $\mathscr{Y}_n$;

$$p_n(t, x, y) \leqq \text{const.}\, t^{-d(n)/2} \quad \text{for } 1 \leqq t < \infty, \quad x, y \in \mathscr{Y}_n,$$

where

$$d(n) = \log(3^n - 1)/\{\log(3^n - 1) - \log(3^{n-1} - 1)\}.$$

0. Introduction

The purpose of this paper is to calculate the isoperimetric constants of the n-dimensional pre Sierpinski carpet and, as an application, to present an estimate of the Neumann heat kernel on the n-dimensional pre Sierpinski carpet.

Let $C^\circ_{n,i}$ be the open set in $\mathbb{R}^n$ defined by

$$C^\circ_{n,i} = \sum_{\mathbb{j} \in \mathbb{Z}^n} \{2 \cdot 3^i \mathbb{j} + R^\circ_{n,i}\},$$

where $i \in \mathbb{Z}$ and $R^\circ_{n,i}$ is the open rectangle

$$R^\circ_{n,i} = \{x = (x_k) \in \mathbb{R}^n; \quad 2 \cdot 3^{i-1} < x_k < 4 \cdot 3^{i-1} \quad \text{for } k = 1, 2, \ldots\ldots n\}.$$

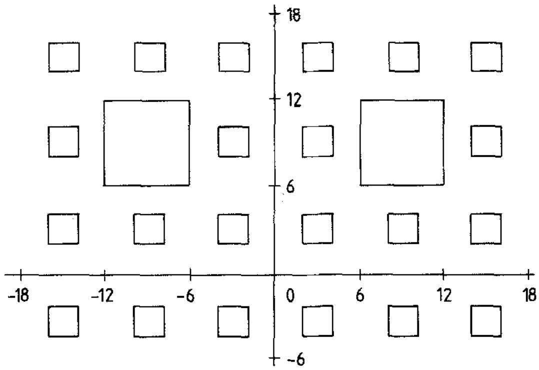
We set

$$\mathscr{S}_n = \mathbb{R}^n - \bigcup_{i \in \mathbb{Z}} C^\circ_{n,i}$$

and

$$\mathscr{Y}_n = \mathbb{R}^n - \bigcup_{i \in \mathbb{N}} C^\circ_{n,i}, \tag{0.1}$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$.



$\mathscr{Y}_2$

Fig. 1

The fractal $\mathscr{S}_2$ is called the Sierpinski carpet [11]. $\mathscr{S}_n$ was taken by Kusuoka as the generalization of $\mathscr{S}_2$ for $n \geqq 3$ (in private communication). $\mathscr{S}_n$ is called n-dimensional Sierpinski carpet, and $\mathscr{Y}_n$ the n-dimensional *pre* Sierpinski carpet, see Fig. 1. We refer to [1–3] for work on the Sierpinski carpet, and to [7, 10] for the physical background relating fractals.

We introduce now the notions of the isoperimetric constants. For this we denote by $\mathcal{O}_{n,b}$ the totality of bounded open sets in $\mathbb{R}^n$ with smooth boundaries. Let O be an open set in $\mathbb{R}^n$ with a sufficiently smooth boundary. Set

$$\mathscr{I}_d^+(O) = \inf \frac{\|O \cap (\partial q)\|_n^d}{|O \cap q|_n^{d-1}}. \tag{0.2}$$

Here the infimum is taken over $q \in \mathcal{O}_{n,b}$ with $1 \leqq |O \cap q|_n < \infty$; we denote by $|\cdot|_n$ (resp. $\|\cdot\|_n$) the n dimensional ($n-1$ dimensional) volume in $\mathbb{R}^n$ induced by Lebesgue measure, and by ∂q the boundary of q in $\mathbb{R}^n$. Similarly we set

$$\mathscr{I}_d^-(O) = \inf \frac{\|O \cap (\partial q)\|_n^d}{|O \cap q|_n^{d-1}}.$$

Here the infimum is taken over $q \in \mathcal{O}_{n,b}$ with $0 < |O \cap q|_n \leqq 1$. $\mathscr{I}_d^+(O)$ (resp. $\mathscr{I}_d^-(O)$) is called the *large* (*small*) *scale isoperimetric constant of* O with index d.

Now we state our results.

Theorem 1. *Let* $\mathscr{Y}_n^\circ$ *be the open kernel of* $\mathscr{Y}_n$, *and set*

$$d(n) = \log(3^n - 1)/\{\log(3^n - 1) - \log(3^{n-1} - 1)\}. \tag{0.4}$$

Then

$$d(n) = \sup\{d; \mathscr{I}_d^+(\mathscr{Y}_n^\circ) > 0\}. \tag{0.5}$$

Moreover,

$$\mathscr{I}_{d(n)}^+(\mathscr{Y}_n^\circ) > 0 \quad \text{and} \quad \mathscr{I}_n^-(\mathscr{Y}_n^\circ) > 0. \tag{0.6}$$

Remark. $d(2) = 3/2$, $d(3) = 2.764\ldots$. $d(n)/n \to 1$ as $n \to \infty$.

The significance of isoperimetric constants lies in the fact that they give bounds on the heat kernel for large time. We quote the following lemma from [9].

Lemma. *Let $p_O(t, x, y)$ be the Neumann heat kernel on O.* Suppose $\mathcal{I}_d^+(O)>0$ *and $\mathcal{I}_e^-(O)>0$ for some $d, e\geqq 1$. Then*

$$p_O(t,x,y)\leqq \text{const.}\, t^{-d/2} \quad \textit{for}\ 1\leqq t<\infty, \quad x,y\in O,$$

and

$$p_O(t,x,y)\leqq \text{const.}\, t^{-e/2} \quad \textit{for}\ 0<t<1, \quad x,y\in O.$$

This lemma follows from a combination of Federer-Fleming's theorem and Nash's theorem (and its extentions due to Carlen-Kusuoka-Stroock [5]).

Theorem 2. *Let $p_n(t, x, y)$ be the Neumann heat kernel on the n-dimensional pre Sierpinski carpet. Then*

$$p_n(t,x,y)\leqq \text{const.}\, t^{-d(n)/2} \quad \textit{for}\ 1\leqq t<\infty, \quad x,y\in \mathcal{Y}_n. \tag{0.8}$$

Kusuoka conjectured Theorem 1 and 2 in private communication. He also proved Theorem 1 and 2 for $n=2$ with a different method from ours. However, his method is not effective for $n\geqq 3$, because he used some special property of $n=2$.

Let $\tilde{d}(n)$ denote the order of the decay of the Neumann heat kernel of the *n-dimensional pre Sierpinski carpet*:

$$\tilde{d}(n)=-2\cdot\lim_{t\to\infty}((\log p_n(t,x,x))/\log t),$$

if the limit of the right hand side exists and is independent of x. By Theorem 2 we have

$$d(n)\leqq\tilde{d}(n).$$

Hence we obtain lower bounds on $\tilde{d}(n)$ (if it exists). It is also known ([8]) that

$$\tilde{d}(n)\leqq\log_3(3^n-1).$$

To prove the existence of $\tilde{d}(n)$ and to calculate the precise value of $\tilde{d}(n)$ are still open problems for $n\geqq 3$. Recently Barlow-Bass-Sherwood [3] proved the existence of $\tilde{d}(2)$.

One motivation of our work is to obtain lower bounds on the *spectral dimension* of the n-dimensional Sierpinski carpet, denoted by $d_S(n)$. The spectral dimension is defined in terms of the *density of states*, that is, the asymptotic frequency of the large eigenvalues of the *Laplacian* on a bounded region. In our case the construction of the Laplacian itself is a problem. One possible idea is to construct the *Brownian motion*, a nondegenerate diffusion process with sufficiently many invariant properties, in order to define the *Laplacian* as its generator. If we obtain $\tilde{d}(n)$ and show that

$$C_1\cdot t^{-\tilde{d}(n)/2}\leqq p_n(t,x,x)\leqq C_2\cdot t^{-\tilde{d}(n)/2} \quad \text{for all}\ x\in\mathcal{Y}_n, \quad 1\leqq t<\infty,$$

then we may construct the Brownian motion as a limit of $\{3^{-k}\cdot X_{t\cdot 3^{\rho k}}\}$ $(k\to\infty)$, where $\{X_t\}$ is the reflecting Brownian motion on $\mathscr{Y}_n$ and $\rho=(\log_3(3^n-1))/\tilde{d}(n))$. If this procedure is justified, the resulting Brownian motion has the transition probability density $p(t, x, y)$ with respect to μ (the limit of $\mu_k(dx)=(3^n/(3^n-1))^k\cdot 1_{\mathscr{Y}_n}(x/3^k)\,dx(k\to\infty)$ in the vague topology), such that

$$p(t, x, y)\leqq \text{const.}\, t^{-\tilde{d}(n)/2} \qquad \text{for all } 0<t<\infty.$$

Hence from Mercer's theorem we have $\tilde{d}(n)=d_S(n)$ (see [4], p. 618). In the case of the 2-dimensional Sierpinski carpet, Barlow-Bass-Sherwood [3] proved $d_S(2)=\tilde{d}(2)$.

In Barlow-Perkins [4], Goldstein [6] and Kusuoka [8], the spectral dimension of another fractal, the Sierpinski gasket was obtained to be $\log_5 9$. The large scale isoperimetric constant equals 0 for $d>1$, since the Sierpinski gasket is a *finitely ramified* fractal, that is, it can be disconnected by removing finitely many points. Hence for the Sierpinski gasket isoperimetric constants yield only a trivial estimate.

We prepare the following notation in order to explain the idea of the proof.

For $i\in\mathbb{Z}$ and $\mathbb{j}=(j_1, j_2, \ldots, j_n)\in\mathbb{Z}^n$, let $u_{n,i,\mathbb{j}}$ denote the open rectangle defined by

$$u_{n,i,\mathbb{j}}=\{x=(x_k)\in\mathbb{R}^n;\quad 3^i\cdot j_k<x_k<3^i\cdot(j_k+1)\}.$$

We set

$$\mathscr{U}_{n,i}=\{u_{n,i,\mathbb{j}};\ \mathbb{j}\in\mathbb{Z}^n\} \quad\text{and}\quad \mathscr{U}_n=\bigcup_{i\in\mathbb{Z}}\mathscr{U}_{n,i}. \tag{0.9}$$

Note that elements of $\mathscr{U}_n$ have the following property:

$$u\supset u' \quad\text{or}\quad u\subset u' \quad\text{if } u\cap u'\neq\phi, \quad (u, u'\in\mathscr{U}_n). \tag{0.10}$$

Let

$$\mathbb{U}_n=\{\mathbb{u}=\{u_i\};\ u_i\in\mathscr{U}_n,\ u_i\cap u_j\neq\phi \quad\text{if } i\neq j\}, \tag{0.11}$$

and set for $\mathbb{u}=\{u_i\}\in\mathbb{U}_n$

$$[\mathbb{u}]=\overline{\bigcup_i u_i}. \tag{0.12}$$

Here $\bar{A}$ stands for the closure of A in $\mathbb{R}^n$.

For $i\in\{1, 2, \ldots, n\}$ and $j\in\mathbb{Z}$, let $H_{n,i,j}$ denote the $n-1$ dimensional plane defined by

$$H_{n,i,j}=\bigcup_{m\in\mathbb{Z}}\{x=(x_k)\in\mathbb{R}^n;\ x_i=3^j\cdot(2m+1)\}.$$

We set

$$H_{n,i}=\bigcup_{j\in\mathbb{Z}}H_{n,i,j} \quad\text{and}\quad H_n=\bigcup_{1\leqq i\leqq n}H_{n,i}. \tag{0.13}$$

We denote by $F_i(u)$ the face of $u\in\mathscr{U}_n$ included by $H_{n,i}$;

(0.14) $$F_i(u)=(\partial u)\cap H_{n,i}.$$

We set

(0.15) $$F(u)=\bigcup_{i=1}^{n} F_i(u).$$

Note that

(0.16) $$\|F_i(u)\cap\mathscr{Y}_n^\circ\|_n=\frac{1}{n}\cdot\|F(u)\cap\mathscr{Y}_n^\circ\|_n \quad \text{for all } i=1,2,\dots,n.$$

We now explain the idea of our method. We first observe that

(0.18) $$\frac{\|F(u)\cap\mathscr{Y}_n^\circ\|_n^{d(n)}}{|u\cap\mathscr{Y}_n^\circ|_n^{d(n)-1}}=n^{d(n)} \quad \text{for } u\in\mathscr{U}_{n,j} \quad \text{with } j\geqq 0 \quad \text{and} \quad u\cap\mathscr{Y}_n^\circ\neq\phi.$$

So our strategy is to show for all $q\in\mathscr{O}_{n,b}$ and $x\in q\cap\mathscr{Y}_n\bigcap\{\bigcup_{i\in\mathbb{Z}}\bigcup_{u\in\mathscr{U}_{n,i}} u\}$ there exists $u\in\mathscr{U}_n$ satisfying $x\in u$ and

(0.19) $$\|(\partial q)\cap u\cap\mathscr{Y}_n^\circ\|_n\geqq\varepsilon_n\|F(u)\cap\mathscr{Y}_n^\circ\|_n.$$

Here ε_n is a positive constant depending only on the dimension n. From (0.19) we conclude that there exists $\mathfrak{u}\in\mathbb{U}_n$ such that

(0.20) $$q\cap\mathscr{Y}_n^\circ\subset[\mathfrak{u}],$$

and that all elements u of $\mathfrak{u}$ satisfy (0.19), and then we obtain Theorem 1.

(0.19) is the main ingredient in the proof of Theorem 1, and will be proved by induction *via* the dimension n. The condition that q is bounded is essential for (0.19). For example, if $q=\{x;|x|>1\}$, then there exists no $\varepsilon>0$ satisfying (0.19) for all $x\in q$. Indeed, the size of u containing x and $\partial q=\{x;|x|=1\}$ becomes bigger as x goes far away from the origin while $\|(\partial q)\cap u\cap\mathscr{Y}_n^\circ\|_n\leqq\|\partial q\|_n$.

In Section 1 we prepare some notation and definitions, and restate Theorem 1 in its general form. Section 2 presents a reduction of Theorem 1. Section 3 completes the proof of Theorem 1.

1. Notation and definitions

In this section we prepare some notation. We shall prove $\mathscr{I}_{d(n)}^{+}(O)>0$ for a class of open sets containing $\mathscr{Y}_n^\circ$.

Let $\theta=(\theta_i)$ $(i\in\mathbb{N}=\{1,2,3,\dots\})$ be a sequence of $\{0,1\}$, and let Θ be the totality of θ, that is

$$\Theta=\{\theta=(\theta_i);\theta_i=0 \text{ or } 1\,(i\in\mathbb{N})\}.$$

We set

(1.1) $$\mathbb{0}=(0,0,\dots) \quad \text{and} \quad \mathbb{1}=(1,1,\dots)\in\Theta.$$

Let $C_n(\theta)$ be the closed set in $\mathbb{R}^n$ defined by

$$C_n(\theta)=\bigcup_{\{i;\,\theta_i=1\}} C_{n,i}, \tag{1.2}$$

where

$$C_{n,i}=\sum_{\mathbb{j}\in\mathbb{Z}^n}\{2\cdot 3^i\mathbb{j}+R_{n,i}\},$$

and $R_{n,i}$ is the closed rectangle

$$R_{n,i}=\{x=(x_k)\in\mathbb{R}^n;\ 2\cdot 3^{i-1}\leqq x_k\leqq 4\cdot 3^{i-1} \text{ for all } k=1,2,\dots n\}.$$

Let $O_n(\theta)$ be the open set in $\mathbb{R}^n$ defined by

$$O_n(\theta)=\mathbb{R}^n-C_n(\theta). \tag{1.3}$$

Obviously we have for $\theta\in\Theta$

$$\phi=C_n(\mathbb{0})\subset C_n(\theta)\subset C_n(\mathbb{1}) \quad\text{and}\quad \mathbb{R}^n=O_n(\mathbb{0})\supset O_n(\theta)\supset O_n(\mathbb{1}). \tag{1.4}$$

Moreover $O_n(\mathbb{1})$ is the open kernel of n dimensional pre Sierpinski carpet;

$$\mathcal{Y}_n^{\circ}=O_n(\mathbb{1}). \tag{1.5}$$

We shall show in Section 2 and 3 the following theorem.

Theorem 1.1.

$$\mathcal{I}_{d(n)}^{+}(O_n(\theta))>0, \qquad \mathcal{I}_n^{-}(O_n(\theta))>0 \qquad \textit{for all}\ \ \theta\in\Theta. \tag{1.6}$$

Theorem 1 comes from Theorem 1.1. Indeed, $\sup\{d;\ \mathcal{I}_d^{+}(\mathcal{Y}_n^{\circ})>0\}\geqq d(n)$ is clear. Let $w_r=\{x=(x_i);\ -3^r<x_i<3^r \text{ for } i=1,2,\dots,n\}$. Then

$$\inf_{r\geqq 1}\frac{\|(\partial w_r)\cap\mathcal{Y}_n^{\circ}\|_n^d}{|w_r\cap\mathcal{Y}_n^{\circ}|_n^{d-1}}=0 \qquad \text{for all} \quad d>d(n).$$

This implies $\sup\{d;\ \mathcal{I}_d^{+}(\mathcal{Y}_n^{\circ})>0\}\leqq d(n)$. We, therefore, obtain (0.5). (0.6) is clear from (1.6).

We define measures associated with $\theta\in\Theta$. Let $|\cdot|_{n,\theta}$ (resp. $\|\cdot\|_{n,\theta}$) be the n dimensional volume ($n-1$ dimensional volume) defined by

$$|\cdot|_{n,\theta}=|\cdot\cap O_n(\theta)|_n, \qquad \|\cdot\|_{n,\theta}=\|\cdot\cap O_n(\theta)\|_n. \tag{1.7}$$

Here $|\cdot|_n$ (resp. $\|\cdot\|_n$) is the $n(n-1)$ dimensional volume induced by Lebesgue measure. Obviously we have

$$|\cdot|_{n,\mathbb{0}}=|\cdot|_n, \qquad \|\cdot\|_{n,\mathbb{0}}=\|\cdot\|_n,$$

$$\|F_i(u)\|_{n,\theta}=\frac{1}{n}\cdot\|F(u)\|_{n,\theta} \qquad \text{for all } \ i=1,2,\dots,n.$$

We observe, if $u, u'\in\mathcal{U}_{n,i}$ and $|u|_{n,\theta}, |u'|_{n,\theta}>0$, then

$$|u|_{n,\theta}=|u'|_{n,\theta}, \qquad \|F(u)\|_{n,\theta}=\|F(u')\|_{n,\theta}.$$

Now we define functions $\Lambda_{n,\theta}(i,j)\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{R}$ by

$$\Lambda_{n,\theta}(i,j)=\left(\frac{3^n-1}{3^n}\right)^{\lambda(i,j,\theta)}, \tag{1.8}$$

where $\lambda(i,j,\theta)$ is the function defined by

$$\lambda(i,j,\theta)=\#\{k\in\mathbb{N};\theta_k=1, i<k\leqq j\} \qquad \text{for } i\leqq j,$$
$$\lambda(i,j,\theta)=-\#\{k\in\mathbb{N};\theta_k=1, j<k\leqq i) \qquad \text{for } i\geqq j, \qquad (\theta=(\theta_k), k\in\mathbb{N}).$$

Note that $\Lambda_{n,\theta}(i,j)\geqq 1$ for $i\geqq j$ and $\Lambda_{n,\theta}(i,j)=1$ if both of i and j are smaller than one.

The following lemma is an immediate consequence of these definitions.

Lemma 1.2. *Let $\theta\in\Theta$ and $u\in\mathscr{U}_{n,i}$ with $|u|_{n,\theta}=0$. Then*

$$|u|_{n,\theta}=\Lambda_{n,\theta}(0,i)\,|u|_{n,\mathbb{0}}, \tag{1.9}$$

and

$$\|F(u)\|_{n,\theta}=\Lambda_{n-1,\theta}(0,i)\,\|F(u)\|_{n,\mathbb{0}}. \tag{1.10}$$

We finish this section with the following lemma, which will be used in Sections 2 and 3.

Lemma 1.3. *Let $\{a_i\}$ and $\{b_i\}$ be sequences of positive numbers such that $(\sum_i b_i)<\infty$. Let c be a constant with $c\geqq 1$. Then*

$$(\sum_i a_i)^c/(\sum_i b_i)\geqq \inf_i\left(\frac{(a_i)^c}{b_i}\right). \tag{1.13}$$

Proof. Since $c\geqq 1$, we obtain

$$(\sum_{i\leqq k} a_i)^c\geqq \sum_{i\leqq k} a_i^c=\sum_{i\leqq k}\left(\frac{a_i^c}{b_i}\right)\cdot b_i\geqq \inf_{i\leqq k}\left(\frac{a_i^c}{b_i}\right)\cdot(\sum_{i\leqq k} b_i)$$

for all k, which implies (1.13) immediately.

2. The reduction of Theorem 1

In this section we obtain the reduction of Theorem 1 and 1.1. We begin by introducing a notion of θ *proper* to an open set q.

Let $u\in\mathscr{U}_n$. u is said to be θ *proper* to an open set q, if

$$\|\overline{(q\cap u)}\cap F_i(u)\|_{n,\theta}\geqq \mu\,\|F_i(u)\|_{n,\theta} \qquad \text{for all } i=1,2,\dots,n. \tag{2.1}$$

Here $\overline{q\cap u}$ is the closure of $q\cap u$, and μ is a positive constant satisfying

$$1/2<\mu<1. \tag{2.2}$$

μ will be fixed throughout this paper. Note that, if $q \supset u$, then u is θ *proper* to q, and that, if u is θ *proper* to q, then u is θ *proper* to all open sets including q.

Recall that $\mathcal{O}_{n,b}$ denotes the totallity of the bounded open sets with smooth boundary, see (0.9) for the definition of $\mathcal{U}_{n,i}$.

Theorem 2.1. *Let $q \in \mathcal{O}_{n,b}$ and $u \in \mathcal{U}_{n,i}$. Suppose u includes $u^- \in \mathcal{U}_{n,i-1}$ satisfying*

$$(2.3) \qquad u^- \text{ is } \theta \text{ proper to } q \text{ and } |u^-|_{n,\theta} > 0.$$

Then, at least, one of the following holds:

$$(2.4) \qquad \|(\partial q) \cap u\|_{n,\theta} \geqq \varepsilon_n \|F(u)\|_{n,\theta},$$

or

$$(2.5) \qquad u \text{ is } \theta \text{ proper to } q \text{ and } |u|_{n,\theta} > 0.$$

Here ε_n is the constant defined by

$$(2.6) \qquad \varepsilon_2 = (1-\mu)/3, \qquad \varepsilon_n = n^{-1} \cdot 3^{-2} \cdot \min\{(1-\mu)/2, \mu/3^n\} \cdot \varepsilon_{n-1}.$$

We shall prove Theorem 2.1 in Section 3. We derive here Theorem 1 and 1.1 from Theorem 2.1.

Let $\mathfrak{u} \in \mathbb{U}_n$ (see (0.12) for the definition of $\mathbb{U}_n$). $\mathfrak{u}$ is said to be a θ *exhausion* of an open set q if $\mathfrak{u}$ satisfies the following conditions;

$$(2.7) \qquad q \cap \mathcal{O}(\theta) \subset [\mathfrak{u}], \quad \text{where } [\mathfrak{u}] \text{ is defined by (0.12)},$$

$$(2.8) \qquad \|(\partial q) \cap u\|_{n,\theta} \geqq \varepsilon_n \|F(u)\|_{n,\theta} \quad \text{for all } u \in \mathfrak{u},$$

(2.9) each $u \in \mathfrak{u}$ includes $u^- \in \mathcal{U}_n$ such that
(i) $u^- \in \mathcal{U}_{n,i-1}$, where i is the integer such that $u \in \mathcal{U}_{n,i}$,
(ii) u^- is θ *proper* to q and $|u^-|_{n,\theta} > 0$.

Proposition 2.2. (i) *Let $\theta \in \Theta$ and $q \in \mathcal{O}_{n,b}$. Suppose that Theorem* 2.1 *holds for n. Then there exists a θ exhaustion $\mathfrak{u}$ of q.*
(ii) *Let $q' \in \mathcal{O}_{n,b}$ with $q \subset q'$. Then there exists θ exhaustion $\mathfrak{u}$ of q and $\mathfrak{u}'$ of q' such that*

$$(2.10) \qquad [\mathfrak{u}] \subset [\mathfrak{u}'].$$

Proof. Let $q_0 = q \cap O_n(\theta)$, and set

$$(2.11) \qquad \mathcal{U}^-(x) = \{u \in \mathcal{U}_n; x \in u, u \text{ is } \theta \text{ proper to } q \text{ and } |u|_{n,\theta} > 0\}.$$

Then

$$(2.12) \qquad \mathcal{U}^-(x) \neq \phi \quad \text{for all } x \in q_{00} = q_0 \bigcap_{i \in \mathbb{Z}} \{ \bigcup_{u \in \mathcal{U}_{n,i}} u\}.$$

This is because, for all $x \in q_{00}$, there exists a unit u with $x \in u \subset q_0$ and u is θ *proper* to q and $|u|_{n,\theta} > 0$ if $u \subset q_0$. Let $u^-(x)$ denote the element of $\mathcal{U}^-(x)$

with $u^-(x)\supset u$ for all $u\in\mathscr{U}^-(x)$. Then $u^-(x)$ exists uniquely, since q is bounded. Furthermore, we denote by $u(x)$ the element of $\mathscr{U}_n$ satisfying

$$u^-(x)\subset u(x) \quad \text{and} \quad u(x)\in\mathscr{U}_{n,i(x)}, \tag{2.13}$$

where $i(x)$ is the integer such that $u^-(x)\in\mathscr{U}_{n,i(x)-1}$. From Theorem 2.1 $u(x)$ satisfies (2.8).

We denote the collections of $u(x)$ over $x\in q_{00}$ by $\mathscr{U}(q)$, and set

$$\mathfrak{u}=\{u\in\mathscr{U}(q);\text{there exists no } u'\in\mathscr{U}(q) \text{ with } u\subsetneqq u'\}. \tag{2.14}$$

Since q is bounded, $\mathfrak{u}$ is not empty and $q_0\subset[\mathfrak{u}]$. It is easy to see that $\mathfrak{u}$ is θ *exhaustion* of q. Hence we obtain (i).

Next we show (ii). Let

$$\mathscr{U}^-(x,q')=\{u\in\mathscr{U}_n;\, x\in u,\, u \textit{ is } \theta \textit{ proper to } q' \text{ and } |u|_{n,\theta}>0\}. \tag{2.15}$$

Then by the definition of θ *proper* we have

$$\mathscr{U}^-(x)\subset\mathscr{U}^-(x,q') \qquad \text{for all } x\in q_{00}. \tag{2.16}$$

(ii) follows from this immediately. Q.E.D.

Lemma 2.3. *Suppose Proposition 2.2. holds for n Then*

$$\mathscr{I}^+_{d(n)}(O_n(\theta))\geqq(\varepsilon_n)^{d(n)-1} \qquad \textit{for all } \theta\in\Theta. \tag{2.17}$$

and

$$\mathscr{I}^-_n(O_n(\theta))>0 \qquad \textit{for all } \theta\in\Theta. \tag{2.18}$$

Proof. We first prove (2.17). Let $q\in\mathscr{O}_{n,b}$ with $|q|_{n,\theta}\geqq 1$ and $\mathfrak{u}$ be a θ exhaustion of q. Set $D=D(n)=\dfrac{d(n)}{d(n)-1}=\dfrac{\log(3^n-1)}{\log(3^{n-1}-1)}$ and $\varepsilon=\varepsilon_n$. Then

$$\frac{(\|\partial q\|_{n,\theta})^D}{|q|_{n,\theta}}\geqq\frac{(\sum_{\mathfrak{u}}\|(\partial q)\cap u\|_{n,\theta})^D}{\sum_{\mathfrak{u}}|q\cap u|_{n,\theta}}\geqq\varepsilon^D\frac{(a'+a'')^D}{(b'+b'')}, \tag{2.19}$$

where

$$a'=\sum_{\mathfrak{u}'}\|F(u)\|_{n,\theta},\qquad b'=\sum_{\mathfrak{u}'}|u|_{n,\theta},\qquad a''=\sum_{\mathfrak{u}''}\|F(u)\|_{n,\theta},\qquad b''=\sum_{\mathfrak{u}''}|u|_{n,\theta},$$

and

$$\mathfrak{u}'=\{u;\,u\in\mathfrak{u},\,\|F(u)\|_{n,\theta}\geqq 1\},\qquad \mathfrak{u}''=\{u;\,u\in\mathfrak{u},\,0<\|F(u)\|_{n,\theta}<1\}.$$

Now we divide the case into two parts: $b'\geqq b''$ and $b'<b''$.

Suppose that $b'\geqq b''$. Then by Lemma 1.3 we have

$$\frac{(a'+a'')^D}{(b'+b'')}\geqq\frac{a'^D}{2b'}\geqq\frac{1}{2}\cdot\inf_{u\in\mathfrak{u}'}\frac{(\|F(u)\|_{n,\theta})^D}{|u|_{n,\theta}}\geqq\frac{1}{2}\cdot n^D>1. \tag{2.20}$$

We next suppose $b'<b''$. Then, by $b'+b''=|q|_{n,\theta}\geqq 1$, we have $b''\geqq 1/2$. Let $N=\dfrac{n}{n-1}$. Then we have from Lemma 1.3

$$(2.21)\qquad \frac{(a'')^N}{b''}\geqq \inf_{u\in\mathfrak{u}''}\frac{(\|F(u)\|_{n,\theta})^N}{|u|_{n,\theta}}=\inf_{u\in\mathfrak{u}''}\frac{(\|F(u)\|_{n,\mathbb{0}})^N}{|u|_{n,\mathbb{0}}}=n^N.$$

We used here $\mathcal{O}_n(\theta)\supset u$ for $u\in\mathfrak{u}''$ to pass from the second term to the third. Hence we obtain from $D>N$ and $a''\geqq 1$ that

$$(2.22)\qquad \frac{(a'')^D}{b''}\geqq\frac{(a'')^N}{b''}\geqq n^N.$$

Then we have by Lemma 1.3 that

$$(2.23)\qquad \frac{(a'+a'')^D}{(b'+b'')}\geqq\min\left\{\frac{(a')^D}{b'},\frac{(a'')^D}{b''}\right\}\geqq\min\{n^D,n^N\}>1.$$

From (2.19), (2.20) and (2.23) we conclude (2.17).

Second, we prove (2.18). Let $q\in\mathcal{O}_{n,b}$ with $0<|q|_{n,\theta}\leqq 1$ and $\mathfrak{u}$ be a θ exhaustion of q. Set $N=N(n)=n/(n-1)$ and $\varepsilon=\varepsilon_n$. Let a', b', ... be defined as before.

We divide the case into two parts: $a'>0$ and $a'=0$:

Suppose that $a'>0$. Then $a'\geqq 1$. This with $|q|_{n,\theta}\leqq 1$ yields

$$(2.24)\qquad \frac{(\|\partial q\|_{n,\theta})^N}{|q|_{n,\theta}}\geqq\Big(\sum_{\mathfrak{u}}\|(\partial q)\cap u\|_{n,\theta}\Big)^N\geqq\varepsilon^N(a'+a'')^N\geqq\varepsilon^N.$$

Next we suppose $a'=0$. Then $b'=0$. Hence we have

$$(2.25)\qquad \frac{(\|\partial q\|_{n,\theta})^N}{|q|_{n,\theta}}\geqq\varepsilon^N\frac{(a'+a'')^N}{(b'+b'')}=\varepsilon^N(a'')^N/b''\geqq\varepsilon^N n^N.$$

From (2.24) and (2.25) we conclude (2.18). Q.E.D.

We next present a simple observation for *exhaustion*, which will be used in the proof of Proposition 3.6.

Lemma 2.4. *Let $\theta\in\Theta$, $U\in\mathcal{U}_{n,i}$ with $|U|_{n,\theta}>0$ and $u\in\mathcal{U}_n$.*
(i) *Let $q\in\mathcal{O}_{n,b}$ with $q\subset U$ and suppose that u is θ proper to q. Then*

$$(2.27)\qquad u\subset U.$$

(ii) *Let $q,q^*\in\mathcal{O}_{n,b}$ with $q\cap q^*=\phi$ and $\overline{q\cup q^*}=\bar{U}$. Let $\mathfrak{u}$(resp. $\mathfrak{u}^*$) be a θ exhaustion of $q(q^*)$. Then*

$$(2.28)\qquad [\mathfrak{u}],[\mathfrak{u}^*]\subset\overline{U^+},$$

where $U^+\in\mathcal{U}_{n,i+1}$ such that $U\subset U^+$, and

$$(2.29)\qquad [\mathfrak{u}]\dot{\subset}\bar{U}\quad or\quad [\mathfrak{u}^*]\subset\bar{U}.$$

Proof. We first prove (i). Assume $u \supsetneqq U$ or $u \cap U = \phi$. Then, by $q \subset U$ and this assumption, we have

$$\|(\overline{q \cap u}) \cap F_i(u)\|_{n,\theta} \leqq \|(\overline{U \cap u}) \cap F_i(u)\|_{n,\theta} \leqq 2^{-1} \|F_i(u)\|_{n,\theta},$$

which contradicts to (2.1).

Second, we prove (ii). Let $u \in \mathfrak{u}$ with $u \in \mathscr{U}_{n,j}$. Then u contains a unit $u^- \in \mathscr{U}_{n,j-1}$, which is θ proper to q. From (i) we have $u^- \subset U$. Hence $u \subset U^+$, which implies $[\mathfrak{u}] \subset \overline{U^+}$. Similarly we have $[\mathfrak{u}^*] \subset \overline{U^+}$. We, therefore, conclude (2.28).

Now we proceed with the proof of (2.29). Suppose that (2.29) is false. Then from (i) and (2.28), $\mathfrak{u}$ and $\mathfrak{u}^*$ consist of the single element U^+, and U is θ *proper* to q and $|U|_{n,\theta} > 0$. Therefore,

$$\|(\overline{q \cap U}) \cap F_i(U)\|_{n,\theta} \geqq \mu \|F_i(U)\|_{n,\theta} \quad \text{for all } i = 1, 2, \ldots, n, \tag{2.31}$$

and

$$\|(\overline{q^* \cap U}) \cap F_i(U)\|_{n,\theta} \geqq \mu \|F_i(U)\|_{n,\theta} \quad \text{for all } i = 1, 2, \ldots, n. \tag{2.32}$$

The sum of the first terms of (2.31) and (2.32) equals $\|F_i(U)\|_{n,\theta}$, since $q \cap q^* = \phi$ and $\overline{q \cup q^*} = \bar{U}$. Hence

$$\|F_i(U)\|_{n,\theta} \geqq 2\mu \|F_i(U)\|_{n,\theta}.$$

This yields contradiction, because $1 < 2\mu$ by (2.2). Q.E.D.

3. Proof of Theorem 2.1

In this section, we shall complete the proof of Theorems 1, 1.1 and 2.1.

First of all we prepare a couple of notations. We set for $\mathfrak{u} \in \mathbb{U}_n$ and $\theta \in \Theta$

$$F[\mathfrak{u} \mid \theta] = \sum_{u \in \mathfrak{u}} \|F(u)\|_{n,\theta}, \tag{3.1}$$

and

$$[\mathfrak{u}] = \overline{\bigcup_{u \in \mathfrak{u}} u}. \tag{3.2}$$

We call $\hat{\mathfrak{u}} \in \mathbb{U}_n$ a minimal covering of $\mathfrak{u} \in \mathbb{U}_n$ if $\hat{\mathfrak{u}}$ satisfies the following conditions:

$$[\mathfrak{u}] \subset [\hat{\mathfrak{u}}], \tag{3.3}$$

$$F[\hat{\mathfrak{u}} \mid \mathbb{0}] \leqq F[\mathfrak{v} \mid \mathbb{0}] \quad \text{for all } \mathfrak{v} \in \mathbb{U}_n \quad \text{with} \quad [\mathfrak{u}] \subset [\mathfrak{v}]. \tag{3.4}$$

Proposition 3.1. *Let $\mathfrak{u} \in \mathbb{U}_n$ and suppose $[\mathfrak{u}]$ is a bounded set. Then there exists a minimal covering $\hat{\mathfrak{u}}$ of $\mathfrak{u}$.*

Proof. Since $[\mathfrak{u}]$ is a bounded set in $\mathbb{R}^n$, we can assume

(3.5) $$[\mathfrak{u}]\subset[-3^k,3^k]\times[-3^k,3^k]\times\ldots\times[-3^k,3^k] \quad \text{for some } k.$$

Let $I=\{(i,j);\, i,j\in\mathbb{Z}\cup\{\infty\},\, i\leqq 0,\, i\leqq j\}$, and let $\to$ denote the order on I defined by the following conditions:

(3.5) $$(i,j)\to(i',j')$$

if and only if

(3.6) $$i>i' \quad \text{or} \quad i=i' \quad j<j'.$$

For example,

$$(0,0)\to(0,1)\to(0,2)\to\ldots\to(0,\infty)\to(-1,-1)\to(-1,0)\to\ldots\to(-1,\infty)$$
$$\to(-2,-2)\to(-2,-1)\to(-2,0)\to\ldots\to(-2,\infty)\to(-3,-3)\to(-3,-2)\to\ldots.$$

Let $\{\mathfrak{u}(i,j)\}$, $(i,j)\in I$, be a sequence of elements of $\mathbb{U}_n$, defined by induction *via* (i,j):

(3.7) $$\mathfrak{u}(0,0)=\bigcup_{i\geqq 0}\mathfrak{u}_i, \quad \text{where} \quad \mathfrak{u}_i=\mathfrak{u}\cap\mathscr{U}_{n,i}.$$

(3.8) $$\mathfrak{u}(i,j)=\mathfrak{u}'(i,j)\cup\{\mathfrak{u}(i,j-1)-\{u\in\mathfrak{u}(i,j-1);\, u\subset[\mathfrak{u}'(i,j)]\}\},$$

where

$$\mathfrak{u}'(i,j)=\{u\in\mathscr{U}_{n,j};\, \|F(u)\|_{n,\mathbb{O}}\leqq F[\mathfrak{u}(i,j-1;u)\,|\,\mathbb{O}]\}$$

and

$$\mathfrak{u}(i,j-1;u)=\{v\in\mathfrak{u}(i,j-1);\, v\subset u\}.$$

(3.9) $$\lim_{j\to\infty}\mathfrak{u}(i,j)=\mathfrak{u}(i,\infty).$$

(3.10) $$\mathfrak{u}(i-1,i-1)=\mathfrak{u}(i,\infty)\cup\{\mathfrak{u}_{i-1}-\{u\in\mathfrak{u}_{i-1};\, u\subset[\mathfrak{u}(i,\infty)]\}\}.$$

Note that from (3.5) the limit (3.9) exists for all i. This is because $\mathfrak{u}(i,j)=\mathfrak{u}(i,k)$ for all $j\geqq k$. Now we define

(3.11) $$\hat{\mathfrak{u}}=\lim_{i\to-\infty}\mathfrak{u}(i,\infty).$$

It is not difficult to see that $\hat{\mathfrak{u}}$ is a minimal covering of $\mathfrak{u}$. Q.E.D.

Remark. The minimal covering is not always unique. For example, let $\mathfrak{u}=\{u_i\}_{i=1,2,3,\ldots}\in\mathbb{U}_2$ satisfying that

$$[\mathfrak{u}]\subset[0,1]\times[0,1], \quad u_{2i-1},u_{2i}\in\mathscr{U}_{2,-i} \quad \text{and} \quad u_i\cap u_j=\phi \quad \text{for } i\neq j.$$

Then both of $\mathfrak{u}$ and $[0,1]\times[0,1]$ are the minimal coverings of $\mathfrak{u}$.

Lemma 3.2. *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{U}_n$, $\theta \in \Theta$. *Suppose*

$$|v|_{n,\theta} > 0 \quad \textit{for all } v \in \mathbf{v}$$

and

$$F[\mathbf{u} \mid \mathbb{0}] \leqq F[\mathbf{v} \mid \mathbb{0}]. \tag{3.12}$$

Then

$$\sum_i \Lambda_{n-1,\theta}(i,m)\, F[\mathbf{u} \cap \mathscr{U}_{n,i} \mid \theta] \leqq \sum_i \Lambda_{n-1,\theta}(i,m)\, F[\mathbf{v} \cap \mathscr{U}_{n,i} \mid \theta] \tag{3.13}$$

for all $m \in \mathbb{Z}$, *where* $\Lambda_{n-1,\theta}$ *is the function defined by* (1.8).

Proof. Clearly we can assume $|u|_{n,\theta} > 0$ for all $u \in \mathbf{u}$. Note that from the definition of $\Lambda_{n-1,\theta}$, we have

$$\Lambda_{n-1,\theta}(0,i) \cdot \Lambda_{n-1,\theta}(i,m) = \Lambda_{n-1,\theta}(0,m) \quad \text{for all } i. \tag{3.14}$$

Then (3.13) follows from Lemma 1.2. Indeed,

$$\begin{aligned} F[\mathbf{u} \mid \mathbb{0}] &= \sum_i F[\mathbf{u} \cap \mathscr{U}_{n,i} \mid \mathbb{0}] \\ &= \{\Lambda_{n-1,\theta}(0,m)\}^{-1} \cdot \sum_i \Lambda_{n-1,\theta}(i,m)\, F[\mathbf{u} \cap \mathscr{U}_{n,i} \mid \theta]. \end{aligned} \tag{3.15}$$

Similarly, we have

$$F[\mathbf{v} \mid \mathbb{0}] = \{\Lambda_{n-1,\theta}(0,m)\}^{-1} \cdot \sum_i \Lambda_{n-1,\theta}(i,m)\, F[\mathbf{v} \cap \mathscr{U}_{n,i} \mid \theta]. \tag{3.16}$$

Combining (3.15) and (3.16) with (3.12) yields (3.13). Q.E.D.

Let $\mu\colon \Theta \to \{-\infty\} \cup \mathbb{N} \cup \{\infty\}$ denote the function defined by

$$\mu(\theta) = \sup\{i; \theta_i = 1\} \quad \text{if } \theta = (\theta_i) \neq \mathbb{0}, \qquad \mu(\theta) = -\infty \quad \text{if } \theta = \mathbb{0}. \tag{3.17}$$

Note that $\mu(\theta) = -\infty$ if and only if $\theta = \mathbb{0}$.

As a corollary of Lemma 3.2 we have the following:

Lemma 3.3. *Let* $\theta \in \Theta$, $m = \mu(\theta)$ *and* $\mathbf{u}, \mathbf{v} \in \mathbb{U}_n$. *Suppose that* $-\infty < m < \infty$,

$$|v|_{n,\theta} > 0 \quad \textit{for } v \in \mathbf{v}, \tag{3.18}$$

and that

$$[\mathbf{u}] \subset [\mathbf{v}]. \tag{3.19}$$

Then

$$\sum_i \Lambda_{n-1,\theta}(i,m)\, F[\hat{\mathbf{u}} \cap \mathscr{U}_{n,i} \mid \theta] \leqq F[\mathbf{v} \mid \theta], \tag{3.20}$$

where $\hat{\mathbf{u}}$ *is a minimal covering of* $\mathbf{u}$.

Proof. By (3.4) and (3.19) we have

$$F[\hat{\mathfrak{u}}|\Theta] \leqq F[\mathfrak{v}|\Theta]. \tag{3.21}$$

Then from Lemma 3.2 we obtain

$$\sum_i A_{n-1,\theta}(i,m)\,F[\hat{\mathfrak{u}}\cap\mathcal{U}_{n,i}|\theta] \leqq \sum_i A_{n-1,\theta}(i,m)\,F[\mathfrak{v}\cap\mathcal{U}_{n,i}|\theta] \tag{3.22}$$
$$\leqq \sum_i F[\mathfrak{v}\cap\mathcal{U}_{n,i}|\theta] = F[\mathfrak{v}|\theta].$$

Here we used $A_{n-1,\theta}(i,m)\leqq 1$ for all i. Q.E.D.

Let $U\in\mathcal{U}_{n,k}$ be fixed, and suppose

$$U = T\times A, \tag{3.23}$$

where

$$T=(\tau,\tau') \quad\text{and}\quad A\in\mathcal{U}_{n-1,k}.$$

Note that $\tau'-\tau=3^k$ and $\tau=i_0\,3^k$ for some $i_0\in\mathbb{Z}$. Without loss of generality we can assume that τ satisfies

$$\{\tau\}\times A = F_1[U], \tag{3.24}$$

where $F_1[U]$ is the first face of U defined by (0.14). This assumption implies that i_0 is odd.

Let

$$U^r = T^r\times A,\qquad T^r=(\tau+r\cdot 3^{k-1},\ \tau+(r+1)\cdot 3^{k-1})\qquad (r=0,1,2). \tag{3.25}$$

We set $T^r=(\rho,\rho')$. Without loss of generality, we can assume

$$\{\rho\}\times A\supset F_1[u] \quad\text{for all } u\in\mathcal{U}_{n,k-1} \text{ such that } u\subset U^r. \tag{3.26}$$

We set for $\theta\in\Theta$ and $t\in T$,

$$\theta(t)=(\theta_i(t)),\qquad \tilde{\theta}(t)=(\tilde{\theta}_i(t)), \tag{3.27}$$

where

$$\theta_i(t)=1\ (\{C_n(\theta^{(i)})\cap\Pi_t\cap U\}\neq\phi),\qquad \theta_i(t)=0\ (\{C_n(\theta^{(i)})\cap\Pi_t\cap U\}=\phi),$$

and

$$\tilde{\theta}_i(t)=\theta_i(t)\ (i\neq k),\qquad \tilde{\theta}_i(t)=0\ (i=k).$$

Here $C_n(\cdot)$ is the closed set defined by (1.2), and $\theta^{(i)}=(\theta^{(i)}_j)$ is the element of Θ defined by $\theta^{(i)}_j=0(j\neq i)$ and $\theta^{(i)}_j=\theta_i(j=i)$, and Π_t is the $n-1$ dimensional plane defined by

$$\Pi_t=\{x=(x_i)\in\mathbb{R}^n;\ x_1=t\}. \tag{3.28}$$

We see

$$\{t\}\times\{C_{n-1}(\theta(t))\cap A\} = C_n(\theta)\cap\Pi_t\cap U. \tag{3.29}$$

Now we consider the functions $\Phi^r_{\theta,U}$; $\mathbb{U}_{n-1} \to \mathbb{R}$ $(r=0, 1, 2)$ defined by

$$\Phi^r_{\theta,U}[\mathfrak{a}] = \int_{T^r} \sum_{i \geqq \tilde{m}(t)} F[\hat{\mathfrak{a}} \cap \mathscr{U}_{n-1,i} | \theta(t)]\, dt. \tag{3.31}$$

See (3.25) for the definition of T^r. Here

$$\tilde{m}(t) = \mu(\tilde{\theta}(t)),$$

where $\mu(\cdot)$ is defined by (3.17), $\theta(t)$ is (3.27) and $\hat{\mathfrak{a}}$ is the minimal covering of $\mathfrak{a}$ which is constructed by Proposition 3.1.

Similarly we define

$$m(t) = \mu(\theta(t)). \tag{3.32}$$

Note that

$$-\infty \leqq \tilde{m}(t) \leqq m(t) \leqq k, \qquad \#\{i;\, \theta_i(t)=1, \tilde{m}(t) < i \leqq m(t)\} \leqq 1, \tag{3.33}$$

and

$$A_{n-1,\theta(t)}(i, m(t)) \geqq \left(\frac{3^{n-1}-1}{3^{n-1}}\right) \quad \text{for all } i \geqq \tilde{m}(t). \tag{3.34}$$

See (1.8) for the definition of $A_{n-1,\theta(t)}(i, m(t))$.

Lemma 3.4. *Let $n \geqq 3$, and let $\mathfrak{a} \in \mathbb{U}_{n-1}$ such that $[\mathfrak{a}] \subset \bar{A}$. Then*

$$\Phi^r_{\theta,U}[\mathfrak{a}] \geqq (1/2) \cdot \sum_{a \in \mathfrak{a}} |a|_{n-1,\theta(\rho)}. \tag{3.35}$$

Here ρ is defined by (3.26).

Proof. Let $a \in \mathscr{U}_{n-1}$ and $i(a)$ be the integer such that $a \in \mathscr{U}_{n-1,i(a)}$. We prove this lemma in the case of $r=0$. Set $T(a) = \{t \in T^0;\ \tilde{m}(t) \leqq i(a)\}$, and suppose $|a|_{n-1,\theta(\rho)} > 0$.

We first see that

$$\int_{T(a)} \|F(a)\|_{n-1,\theta(t)}\, dt \geqq (1/2) \cdot |a|_{n-1,\theta(\rho)}. \tag{3.36}$$

Indeed, if $i(a)=k$, that is $a=A$, then $T(a)=T^0$. Hence we have

$$\begin{aligned} &\int_{T(a)} \|F(a)\|_{n-1,\theta(t)}\, dt \\ &\quad = (n-1) \cdot \|F_2(U) \cap \{x \in \mathbb{R}^n;\, x_1 \in T^0\}\|_{n,\theta} \\ &\quad \geqq (n-1) \cdot \left(\frac{3^{n-2}-1}{3^{n-1}-1}\right) \cdot |a|_{n-1,\theta(\rho)} \geqq (1/2) \cdot |a|_{n-1,\theta(\rho)}. \end{aligned}$$

Suppose $i(a)<k$, and set $I=(\rho'-3^{i(a)}, \rho')$. Here ρ' is the number such that $(\rho,\rho')=T^0$, which means $\rho'=\rho+3^{k-1}$. Then $I\subset T(a)$ and

$$\int_{T(a)} \|F(a)\|_{n-1,\theta(t)}\,dt \geqq \int_I \|F(a)\|_{n-1,\theta(t)}\,dt$$
$$=(n-1)\cdot\|F_2(U(a))\|_{n,\theta}=(n-1)\cdot|a|_{n-1,\theta(\rho)},$$

where $U(a)\in\mathscr{U}_n$ is defined by $U(a)=I\times a$.

It follows from (3.36) that

$$\Phi^0_{\theta,U}[\mathfrak{a}]=\int_{T^0}\sum_{i\geqq\hat{m}(t)}\sum_{a\in\hat{\mathfrak{a}}\cap\mathscr{U}_{n-1,i}}\|F(a)\|_{n-1,\theta(t)}\,dt \tag{3.37}$$
$$=\sum_{a\in\hat{\mathfrak{a}}}\int_{T(a)}\|F(a)\|_{n-1,\theta(t)}\,dt$$
$$\geqq(1/2)\cdot\sum_{a\in\hat{\mathfrak{a}}}|a|_{n-1,\theta(\rho)}\geqq(1/2)\cdot\sum_{a\in\mathfrak{a}}|a|_{n-1,\theta(\rho)}.$$

We can prove similarly the case of $r=1, 2$. Q.E.D.

We present some notation which will be used in the rest of this section. Let $\pi_{i,u}$ be the projection from a unit u to F_i(u), the face of u defined by (0.14). We set

$$b_i\equiv b_i(q,u)=\overline{(q\cap u)}\cap\pi_{i,u}((\partial q)\cap u), \tag{3.41}$$
$$a_i\equiv a_i(q,u)=F_i(u)\cap\overline{(q\cap u)}-b_i,$$

where $\bar{q}$ is the closure of q and ∂q is the boundary of q in $\mathbb{R}^n$.

$$\beta_i\equiv\beta_i(q,u,\theta)=\|b_i\|_{n,\theta}, \tag{3.42}$$
$$\alpha_i\equiv\alpha_i(q,u,\theta)=\|a_i\|_{n,\theta}.$$

Moreover,

$$\alpha_i^*\equiv\alpha_i^*(q,u,\theta)=\alpha_i(q^*,u,\theta), \tag{3.43}$$
$$\beta_i^*\equiv\beta_i^*(q,u,\theta)=\beta_i(q^*,u,\theta),$$

where $q^*=(\bar{q})^C$.

We often omit (q, u, θ) if no confusion occurs.

Lemma 3.5. *For all* $(q,u,\theta)\in(\mathcal{O}_{n,b},\mathscr{U}_n,\Theta)$ *and* $i\in\{1, 2, \ldots, n\}$, *we have*

(i) $\alpha_i+\beta_i+\alpha_i^*+\beta_i^*=\|F_i(u)\|_{n,\theta}$,
(ii) $\alpha_i+\beta_i=\|\overline{(q\cap u)}\cap F_i(u)\|_{n,\theta}$,
(iii) $\|(\partial q)\cap u\|_{n,\theta}\geqq\beta_i+\beta_i^*$.

Proof. Lemma 3.5 follows from the above definitions immediately. Q.E.D.

Proposition 3.6. *Let* $n\geqq 3$ *and suppose Theorem* 2.1 *holds for* $n-1$. *Let* $U\in\mathscr{U}_{n,k}$ *satisfy* (3.23), *and let* U^r $(r=0, 1, 2)$ *be the subset of* U *defined by* (3.25). *Set*

$$\alpha(r)=\sum\alpha_1(q,u,\theta) \quad\text{and}\quad \alpha^*(r)=\sum\alpha_1^*(q,u,\theta), \tag{3.46}$$

where summations are taken over $\{u;\, u\in\mathscr{U}_{n,k-1},\, u\subset U^r\}$. *Then*

$$\|(\partial q)\cap U\|_{n,\theta}\geqq\varepsilon_{n-1}\,3^{-2}\cdot\min\{\alpha(r),\alpha^*(r)\}, \tag{3.47}$$

where ε_{n-1} *is the positive constant in Theorem* 2.1.

Proof. Let

$$\tilde{a}(r)=\cup a_1(q,u)\quad\text{and}\quad\tilde{a}^*(r)=\cup a_1(q^*,u), \tag{3.48}$$

where unions are taken over $\{u;\, u\in\mathscr{U}_{n,k-1},\, u\subset U^r\}$ and $a_1(\cdot,\, u)$ is defined by (3.41). Since $\tilde{a}(r)$ and $\tilde{a}^*(r)$ are contained in the $n-1$ dimensional plane $\{x=(x_i);\, x_1=\rho\}$ (see (3.26) for ρ), we can rewrite

$$\tilde{a}(r)=\{\rho\}\times a(r),\qquad \tilde{a}^*(r)=\{\rho\}\times a^*(r),$$

where $a(r),\, a^*(r)\subset\mathbb{R}^{n-1}$. Set

$$a=\{a(r)\cap O_{n-1}(\theta(\rho))\}^\circ,\qquad a^*=\{a^*(r)\cap O_{n-1}(\theta(\rho))\}^\circ. \tag{3.49}$$

Here $O_{n-1}(\cdot)$ is defined by (1.3) and A° means open kernel of A in $\mathbb{R}^{n-1}$. Note that

$$|a|_{n-1,\theta(\rho)}=\|\tilde{a}(r)\|_{n,\theta}=\alpha(r),\qquad |a^*|_{n-1,\theta(\rho)}=\|\tilde{a}^*(r)\|_{n,\theta}=\alpha^*(r). \tag{3.50}$$

Let $\mathbb{a}(t)$ (resp. $\mathbb{a}^*(t)$) denote a $\theta(t)$ exhaustion of $a(a^*)$. Such exhaustions exist by Theorem 2.1 for $n-1$ and Proposition 2.2. We can choose $\mathbb{a}(t)$ and $\mathbb{a}^*(t)$ such that $\mathbb{a}(t)=\mathbb{a}(s)$ and $\mathbb{a}^*(t)=\mathbb{a}^*(s)$ if $\theta(t)=\theta(s)$. Then there exist $\mathbb{v}$ and $\mathbb{v}^*\in\mathbb{U}_{n-1}$ satisfying

$$a\subset[\mathbb{v}]\subset[\mathbb{a}(t)],\qquad a^*\subset[\mathbb{v}^*]\subset[\mathbb{a}^*(t)]\qquad\text{for all } t\in T^r, \tag{3.51}$$

and

$$|v|_{n-1,\theta(t)}>0\qquad\text{for all } v\in\mathbb{v}\cup\mathbb{v}^*\quad\text{and}\quad t\in T^r.$$

Indeed, we can construct $\mathbb{v}$ and $\mathbb{v}^*$ in the following way:

$$\mathbb{v}=\bigwedge_{t\in T^r}\mathbb{a}(t),\qquad \mathbb{v}^*=\bigwedge_{t\in T^r}\mathbb{a}^*(t),$$

where $\wedge$ is defined by

$$\mathbb{a}\wedge\mathbb{b}=\{u\in\mathbb{a}\cup\mathbb{b};\ \text{there exists no } v\in\mathbb{a}\cup\mathbb{b} \text{ such that } v\subsetneqq u\}.$$

Let

$$\tilde{q}(t)=\cup\pi_{1,u}(q\cap u\cap\Pi_t)\quad\text{and}\quad\tilde{q}^*(t)=\cup\pi_{1,u}(q^*\cap u\cap\Pi_t),$$

where unions are taken over $\{u;\, u\in\mathscr{U}_{n,k-1},\, u\subset U^r\}$ and $\pi_{1,u}$ is the projection from u to $F_1(u)$, and $\Pi_t=\{x=(x_i)\in\mathbb{R}^n;\, x_1=t\}$. Let $q(t)$ and $q^*(t)$ be open sets in $\mathbb{R}^{n-1}$ such that

$$\tilde{q}(t)=\{\rho\}\times q(t),\qquad \tilde{q}^*(t)=\{\rho\}\times q^*(t).$$

Then

$$a\subset q(t),\qquad a^*\subset q^*(t)\qquad\text{for all } t\in T^r. \tag{3.52}$$

From Theorem 2.1 for $n-1$ and Proposition 2.2 there exists a $\theta(t)$ exhaustion $\mathbb{v}(t)$ (resp. $\mathbb{v}^*(t)$) of $q(t)(q^*(t))$ such that

$$[\mathbb{a}(t)]\subset[\mathbb{v}(t)],\qquad [\mathbb{a}^*(t)]\subset[\mathbb{v}^*(t)]\qquad \text{for all } t\in T^r.$$

Hence we see from (3.51) that

$$[\mathbb{v}]\subset[\mathbb{v}(t)] \quad\text{and}\quad [\mathbb{v}^*]\subset[\mathbb{v}^*(t)]\qquad \text{for all } t\in T^r. \tag{3.53}$$

Let $\hat{\mathbb{v}}$ and $\hat{\mathbb{v}}^*$ be the minimal coverings of $\mathbb{v}$ and $\mathbb{v}^*$, respectively.
Set

$$A_{\theta(t)}(i)=A_{n-1,\theta(t)}(i,m(t)), \tag{3.54}$$

where $A_{n-1,\theta(t)}(i,j)$ is the function defined by (1.8) and $m(t)=\mu(\theta(t))$. See (3.17) for the definition of $\mu(\theta)$ and (3.27) for the definition of $\theta(t)$. From Lemma 3.3 and (3.53), we have

$$\sum_i A_{\theta(t)}(i)\,F[\hat{\mathbb{v}}\cap\mathscr{U}_{n-1,i}|\theta(t)]\leqq F[\mathbb{v}(t)|\theta(t)] \tag{3.55}$$

and

$$\sum_i A_{\theta(t)}(i)\,F[\hat{\mathbb{v}}^*\cap\mathscr{U}_{n-1,i}|\theta(t)]\leqq F[\mathbb{v}^*(t)|\theta(t)]. \tag{3.56}$$

Let $T_1=\{t\in T^r;\ [\mathbb{v}(t)]\subset A\}$ and $T_2=\{t\in T^r;\ [\mathbb{v}^*(t)]\subset A\}$. Then by Lemma 2.4 we have

$$T_1\cup T_2=T^r. \tag{3.57}$$

From (2.8) we have

$$\varepsilon_{n-1}\|F(v)\|_{n-1,\theta(t)}\leqq\|(\partial q(t))\cap v\|_{n-1,\theta(t)},$$

for $v\in\mathbb{v}(t)\cup\mathbb{v}^*(t)$ with $v\subset A$. This is beause for $v\subset A$,

$$\|(\partial q(t))\cap v\|_{n-1,\theta(t)}=\|(\partial q^*(t))\cap v\|_{n-1,\theta(t)}.$$

Hence taking the summation with respect to v over $\mathbb{v}(t)$ and $\mathbb{v}^*(t)$, respectively, yields

$$\varepsilon_{n-1}F[\mathbb{v}(t)|\theta(t)]\leqq\|(\partial q(t))\cap A\|_{n-1,\theta(t)}\qquad (t\in T_1) \tag{3.58}$$

and

$$\varepsilon_{n-1}F[\mathbb{v}^*(t)|\theta(t)]\leqq\|(\partial q(t))\cap A\|_{n-1,\theta(t)}\qquad (t\in T_2). \tag{3.59}$$

We obtain from (3.55) and (3.58) that

$$\varepsilon_{n-1}\sum_i A_{\theta(t)}(i)\,F[\hat{\mathbb{v}}\cap\mathscr{U}_{n-1,i}|\theta(t)]\leqq\|(\partial q(t))\cap A\|_{n-1,\theta(t)} \tag{3.60}$$

for $t \in T_1$, and from (3.56) and (3.59) that

$$\varepsilon_{n-1} \sum_i A_{\theta(t)}(i) F[\hat{\mathbb{V}}^* \cap \mathscr{U}_{n-1,i} | \theta(t)] \leqq \|(\partial q(t)) \cap A\|_{n-1,\theta(t)} \tag{3.61}$$

for $t \in T_2$.

Now we divide the case into two parts:

$$F[\hat{\mathbb{V}} | \mathbb{0}] \leqq F[\hat{\mathbb{V}}^* | \mathbb{0}] \tag{3.62}$$

and

$$F[\hat{\mathbb{V}} | \mathbb{0}] \geqq F[\hat{\mathbb{V}}^* | \mathbb{0}]. \tag{3.63}$$

First we suppose (3.62). Then by Lemma 3.2 with the notation in (3.54) we have

$$\sum_i A_{\theta(t)}(i) F[\hat{\mathbb{V}} \cap \mathscr{U}_{n-1,i} | \theta(t)] \leqq \sum_i A_{\theta(t)}(i) F[\hat{\mathbb{V}}^* \cap \mathscr{U}_{n-1,i} | \theta(t)].$$

Combining this with (3.61) and noting (3.60) and $T^r = T_1 \cup T_2$ yield

$$\varepsilon_{n-1} \sum_i A_{\theta(t)}(i) F[\hat{\mathbb{V}} \cap \mathscr{U}_{n-1,i} | \theta(t)] \leqq \|(\partial q(t)) \cap A\|_{n-1,\theta(t)}$$

for all $t \in T^r$. By (3.34) and (3.54) we have

$$A_{\theta(t)}(i) \geqq \left(\frac{3^{n-1}-1}{3^{n-1}}\right) \quad \text{for all } i \geqq \tilde{m}(t).$$

Hence we see

$$\varepsilon_{n-1} \cdot \left(\frac{3^{n-1}-1}{3^{n-1}}\right) \cdot \sum_{i \geqq \tilde{m}(t)} F[\hat{\mathbb{V}} \cap \mathscr{U}_{n-1,i} | \theta(t)] \leqq \|(\partial q(t)) \cap A\|_{n-1,\theta(t)} \tag{3.64}$$

for all $t \in T^r$. Integrating both sides of (3.64) over $t \in T^r$, we obtain

$$\varepsilon_{n-1} \cdot \left(\frac{3^{n-1}-1}{3^{n-1}}\right) \cdot \Phi^r_{\theta,U}[\mathbb{V}] \leqq \|(\partial q) \cap U^r\|_{n,\theta}.$$

See (3.31) for the definition of $\Phi^r_{\theta,U}$. Combining this with Lemma 3.4 and noting $\left(\frac{3^{n-1}-1}{3^{n-1}}\right) \cdot \left(\frac{1}{2}\right) \geqq 3^{-2}$ yield

$$\varepsilon_{n-1} \cdot 3^{-2} \cdot \sum_{v \in \mathbb{V}} |v|_{n-1,\theta(\rho)} \leqq \|(\partial q) \cap U^r\|_{n,\theta}.$$

We, therefore, obtain from (3.50) and (3.51) that

$$\begin{aligned} \varepsilon_{n-1} \cdot 3^{-2} \cdot \alpha(r) &\leqq \varepsilon_{n-1} \cdot 3^{-2} \cdot \sum_{v \in \mathbb{V}} |v|_{n-1,\theta(\rho)} \\ &\leqq \|(\partial q) \cap U^r\|_{n,\theta} \leqq \|(\partial q) \cup U\|_{n,\theta}. \end{aligned} \tag{3.65}$$

Second, we suppose (3.63). Then by the same argument as above we have

$$\varepsilon_{n-1}\cdot 3^{-2}\cdot\alpha^*(r)\leqq\|(\partial q)\cap U\|_{n,\theta}. \tag{3.66}$$

Combining (3.65) and (3.66) completes the proof. Q.E.D.

Proposition 3.7. *Let $n\geqq 3$, and suppose Theorem 2.1 holds for $n-1$. Then Theorem 2.1 holds for n with the positive constant ε_n defined by*

$$\varepsilon_n=n^{-1}3^{-2}\varepsilon\cdot\varepsilon_{n-1}, \tag{3.67}$$

where

$$\varepsilon=\min\{(1-\mu)/2,\ \mu/3^n\}. \tag{3.68}$$

Proof. Let $\theta\in\Theta$, $q\in\mathcal{O}_{n,b}$ and $U\in\mathcal{U}_{n,k}$. Suppose U contains $U^-\in\mathcal{U}_{n,k-1}$ satisfying (2.3). We shall show that U satisfies (2.4) or (2.5):

$$\|(\partial q)\cap U\|_{n,\theta}\geqq\varepsilon_n\|F_i(U)\|_{n,\theta}, \tag{2.4}$$

or

$$U \text{ is } \theta \text{ proper to } q \text{ and } |U|_{n,\theta}>0. \tag{2.5}$$

We divide the case into three parts:

(I) $\beta_i(q,U,\theta)+\beta_i^*(q,U,\theta)\geqq\varepsilon\|F_i(U)\|_{n,\theta}$ for some i.
(II) $\beta_i(q,U,\theta)+\beta_i^*(q,U,\theta)<\varepsilon\|F_i(U)\|_{n,\theta}$ for all i,
$\alpha_i^*(q,U,\theta)<\varepsilon\|F_i(U)\|_{n,\theta}$ for all i.
(III) $\beta_i(q,U,\theta)+\beta_i^*(q,U,\theta)<\varepsilon\|F_i(U)\|_{n,\theta}$ for all i,
$\alpha_i^*(q,U,\theta)\geqq\varepsilon\|F_i(U)\|_{n,\theta}$ for some i.

Here $i\in\{1,2,\ldots,n\}$.

First suppose (I): Then from (iii) of Lemma 3.5 we have

$$\|(\partial q)\cap U\|_{n,\theta}\geqq\beta_i(q,U,\theta)+\beta_i^*(q,U,\theta)\geqq\varepsilon\|F_i(U)\|_{n,\theta},$$

which implies (2.4).

Second we suppose (II): By Lemma 3.5 we have

$$\begin{aligned}&\|(\overline{q\cap U})\cap F_i(U)\|_{n,\theta}\\ &\quad=\|F_i(U)\|_{n,\theta}-\alpha_i^*(q,U,\theta)-\beta_i^*(q,U,\theta)\\ &\quad\geqq(1-2\varepsilon)\|F_i(U)\|_{n,\theta}\geqq\mu\|F_i(U)\|_{n,\theta}.\end{aligned}$$

We, therefore, conclude U is θ proper to q, which is (2.5).

Finally suppose (III): Without loss of generality we can assume $i=1$; $\alpha_1^*(q,U,\theta)\geqq\varepsilon\|F_1(U)\|_{n,\theta}$. Hence we have

$$\alpha^*(r)\geqq\alpha_1^*(q,U,\theta)\geqq\varepsilon\|F_1(U)\|_{n,\theta}\quad\text{for } r=1,2,3. \tag{3.72}$$

Here $\alpha^*(r)$ is defined by (3.46).

Now, there exists r such that $U^- \subset U^r$, where U^r is the subset of U defined by (3.25). Since $\alpha_1(q, U^-, \theta)+\beta_1(q, U^-, \theta)=\|(\overline{q\cap U^-})\cap F_1(U^-)\|_{n,\theta}$ and U^- is θ proper to q, we have

$$(3.73)\quad \alpha_1(q, U^-, \theta)+\beta_1(q, U^-, \theta)=\|(\overline{q\cap U^-})\cap F_1(U^-)\|_{n,\theta} \geqq \mu\cdot\|F_1(U^-)\|_{n,\theta}\geqq \mu\cdot 3^{-n+1}\cdot\|F_1(U)\|_{n,\theta}.$$

If $\|(\partial q)\cap U\|_{n,\theta}\geqq \varepsilon\|F_1(U)\|_{n,\theta}$, we obtain (2.4). Then we assume $\|(\partial q)\cap U\|_{n,\theta} < \varepsilon\|F_1(U)\|_{n,\theta}$, which implies

$$(3.74)\qquad \beta_1(q, U^-, \theta)+\beta_1^*(q, U^-, \theta)\leqq \|(\partial q)\cap U^-\|_{n,\theta}\leqq \varepsilon\|F_1(U)\|_{n,\theta}.$$

Combining (3.73) and (3.74) yields

$$\alpha_1(q, U^-, \theta)\geqq(\mu\cdot 3^{-n+1}-\varepsilon)\|F_1(U)\|_{n,\theta}\geqq \varepsilon\|F_1(U)\|_{n,\theta}.$$

Hence we obtain

$$(3.75)\qquad \alpha(r)\geqq\alpha_1(q, U^-, \theta)\geqq\varepsilon\|F_1(U)\|_{n,\theta}.$$

Combining (3.72) and (3.75) with Proposition 3.6 yields

$$\|(\partial q)\cap U\|_{n,\theta}\geqq\varepsilon_{n-1}\cdot 3^{-2}\cdot\min\{\alpha(r), \alpha^*(r)\} \geqq\varepsilon_{n-1}\cdot 3^{-2}\cdot\varepsilon\|F_1(U)\|_{n,\theta}=\varepsilon_n\|F(U)\|_{n,\theta},$$

which implies (2.4). Q.E.D.

Proof of Theorem 1, 1.1 and 2.1. As we see in Section 2, Theorem 2.1 implies Theorem 1 and 1.1. Hence from Proposion 3.7, what remainds is to show Theorem 2.1 for $n=2$. We use the notation α_i, β_i,... as before. Let $q\in\mathcal{O}_{2,b}$ and $U\in\mathcal{U}_{2,k}$. Suppose U includes $U^-\in\mathcal{U}_{2,k-1}$ satisfying (2.3).

We set

$$(3.76)\qquad \varepsilon_2=2\varepsilon/3, \quad \varepsilon=(1-\mu)/2.$$

We divide the case into three parts:

(I) $\beta_i(q, U, \theta)+\beta_i^*(q, U, \theta)\geqq\varepsilon\|F_i(U)\|_{2,\theta}$ for some i.
(II) $\beta_i(q, U, \theta)+\beta_i^*(q, U, \theta)<\varepsilon\|F_i(U)\|_{2,\theta}$ for all i, and $\alpha_i^*(q, U, \theta) <\varepsilon\|F_i(U)\|_{2,\theta}$ for all i.
(III) $\beta_i(q, U, \theta)+\beta_i^*(q, U, \theta)<\varepsilon\|F_i(U)\|_{2,\theta}$ for all i, and $\alpha_i^*(q, U, \theta) \geqq\varepsilon\|F_i(U)\|_{2,\theta}$ for some i.

Here $i=\{1, 2\}$.

First suppose (I): Then (2.4) follows from (iii) of Lemma 3.5.

Second suppose (II): Then (2.5) follows from (i) of Lemma 3.5.

Now we easily see

$$(3.77)\qquad \alpha_1(q, U, \theta)=0 \quad \text{if } \alpha_2^*(q, U, \theta)>0,$$

$$(3.78)\qquad \alpha_2(q, U, \theta)=0 \quad \text{if } \alpha_1^*(q, U, \theta)>0.$$

This is because $n=2$.

Finally we suppose (III): Without loss of generality we can assume

$$\alpha_2^*(q, U, \theta)\geqq\varepsilon\|F_2(U)\|_{2,\theta}>0.$$

Hence from (3.77)

$$\alpha_1(q, U, \theta) = 0. \tag{3.79}$$

This with the first hypothesis of (III) and Lemma 3.5 implies

$$\alpha_1^*(q, U, \theta) \geqq (1-\varepsilon) \cdot \|F_1(U)\|_{2,\theta} > 0.$$

Then from (3.78), we have

$$\alpha_2(q, U, \theta) = 0. \tag{3.80}$$

Now there exists ι such that

$$\begin{aligned}&\alpha_\iota(q, U, \theta) + \beta_\iota(q, U, \theta) + \beta_\iota^*(q, U, \theta)\\ &\qquad \geqq \alpha_\iota(q, U^-, \theta) + \beta_\iota(q, U^-, \theta) + \beta_\iota^*(q, U^-, \theta).\end{aligned}$$

This with (3.79) and (3.80) yields

$$\begin{aligned}\beta_\iota(q, U, \theta) &+ \beta_\iota^*(q, U, \theta)\\ &\geqq \alpha_\iota(q, U^-, \theta) + \beta_\iota(q, U^-, \theta) + \beta_\iota^*(q, U^-, \theta)\\ &\geqq \alpha_\iota(q, U^-, \theta) + \beta_\iota(q, U^-, \theta)\\ &\geqq \mu \|F_\iota(U^-)\|_{2,\theta} \geqq \mu \cdot 3^{-1} \|F_\iota(U)\|_{2,\theta} \geqq \varepsilon_2 \|F(U)\|_{2,\theta}.\end{aligned} \tag{3.81}$$

Here we used the assumption on U^- to pass from the third line to the fourth, and $\mu/3 \geqq 1/6 \geqq (1-\mu)/3$ by (2.2). This implies (2.4). Q.E.D.

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