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On dissipative stochastic equations in a Hilbert space

A.S. Holevo

Steklov Mathematical Institute, Vavilova 42, 117966 Moscow, Russia
(e-mail: holevo@class.mian.su)

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Summary. A general existence and uniqueness theorem for solutions of linear dissipative stochastic differential equation in a Hilbert space is proved. The dual equation is introduced and the duality relation is established. Proofs take inspirations from quantum stochastic calculus, however without using it. Solutions of both equations provide classical stochastic representation for a quantum dynamical semigroup, describing quantum Markovian evolution. The problem of the mean-square norm conservation, closely related to the unitality (non-explosion) of the quantum dynamical semigroup, is considered and a hyperdissipativity condition, ensuring such conservation, is discussed. Comments are given on the existence of solutions of a nonlinear stochastic differential equation, introduced and discussed recently in physical literature in connection with continuous quantum measurement processes.

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0 Introduction

Consider the linear stochastic equation

$$\psi(t) = \psi_0 + \int_0^t \sum_j L_j \psi(s)\, dW_j(s) - \int_0^t K\psi(s)\, ds \tag{0.1}$$

in a separable (complex) Hilbert space $\mathscr{H}$, where $W_j(t)$, $j = 1, 2, \ldots$, are independent standard Wiener processes, L_j, K are, in general, unbounded operators defined on a dense domain $\mathscr{D} \subset \mathscr{H}$ and $\psi_0 \in \mathscr{H}$. The equation is called *dissipative* if

$$\sum_j \|L_j\psi\|^2 - 2\,\mathrm{Re}(K\psi|\psi) \leqq c\|\psi\|^2, \quad \psi \in \mathscr{D}\,. \tag{0.2}$$

Results on the equation of this type can be found in the book of Rozovsky [19], where existence and uniqueness of the strong solution of Eq. (0.1) in a scale of Hilbert spaces is established under additional hypotheses (cf. Sect. 4). Many other works use conditions of the coercivity type, stronger than (0.2) (see [5] for a survey of various approaches and results).

In this paper we make an observation that under very mild conditions it is possible to prove the existence and uniqueness for solutions of dissipative equations in the weak topology sense (Sect. 1). The proof takes some inspirations from quantum stochastic calculus (although not using it!), namely, from Frigerio–Fagnola's existence proof and Mohari's uniqueness proof for quantum stochastic differential equations (see [18, Chap. VI] for a survey of these ideas).

In Sect. 2 we introduce the dual stochastic differential equation, establish existence and uniqueness of its solution, and the duality relation. These concepts are the classical pattern of the "dual cocycle", introduced and studied in the noncommutative situation by Journé [15].

The reminiscence of quantum stochastic calculus is not occasional. In fact, the conservative stochastic differential equations are closely related to some important concepts in the noncommutative probability, such as dynamical semigroup and nonlinear stochastic equation for the normalized posterior wave function (Sects. 3, 4). We call the linear equation (0.1) *conservative* if the left-hand side in (0.2) is identically zero,

$$\sum_j \|L_j\psi\|^2 - 2\,\mathrm{Re}(K\psi|\psi) = 0, \quad \psi \in \mathscr{D}\,. \tag{0.3}$$

If the operators L_j, K are bounded, then this condition implies mean-square norm conservation for the solution:

$$\mathbf{M}\|\psi(t)\|^2 = \|\psi_0\|^2\,, \tag{0.4}$$

however in general one will have only $\mathbf{M}\|\psi(t)\|^2 \leqq \|\psi_0\|^2$ with the possibility of strict inequality (see Sect. 3). Solutions of (0.1) provide classical stochastic representation for a quantum dynamical semigroup (relation (3.2)), and the property (0.4) is closely related to the unitality (non-explosion) of this dynamical semigroup. In Sect. 4 we derive (0.4) from (0.3) and a further condition of *hyperdissipativity* which means dissipativity in a Hilbert scale associated to the operator $K + K^*$. This gives quite a different view on the conditions for unitality of quantum dynamical semigroup, obtained by operator-theoretic methods in [4].

The nonlinear stochastic equation, satisfied by the normalized posterior wave function in a continuous quantum measurement process, was extensively discussed recently in physical literature (see [7, 2]). This equation is also interesting from a mathematical point of view. A general study of this type of equations with arbitrary driving martingales, and with *bounded* operator coefficients is presented in [1]. The case of unbounded operator coefficients is substantially more complicated, essentially with the possibility of violation of the property (0.4). In the paper [11] Gatarek and Gisin proved existence of solution for the nonlinear stochastic equation by using reduction to the linear equation (0.1) via Girsanov's transformation, in the case of one-dimensional Wiener process. They also commented that the multidimensional case can be treated in the same way. However the argument in fact substantially relies upon the property (0.4), which is automatic in their case. To extend this argument to the general case, one has to complement it by a condition, ensuring the mean-square norm conservation (Sect. 4).

1 Existence and uniqueness of the weak topology solution

The condition (0.2) implies closability of K since $K_c = K + (c/2)I$ is accretive:

$$\mathrm{Re}(\psi|K_c\psi) \geqq 0, \quad \psi \in \mathscr{D},$$

and the closure $\bar{K}_c$ of K_c is accretive (see [16]). Let $\mathscr{D}^*$ be a core for K^*. We assume that

(I.1) *L_j are closable with L_j^* defined on $\mathscr{D}^*$, and*

$$\sum_j \|L_j^*\psi\|^2 < \infty, \quad \psi \in \mathscr{D}^*.$$

We say that Eq. (0.1) has *weak topology* (*wt-*) *solution* on $[0, T]$, if there exists random process $\psi(t)$, $t \in [0, T]$, with values in $\mathscr{H}$, weakly continuous in probability, and such that

$$(\phi|\psi(t)) - (\phi|\psi_0) = \int_0^t \sum_j (L_j^*\phi|\psi(s))\, dW_j(s) - \int_0^t (K^*\phi|\psi(s))\, ds \tag{1.1}$$

for all $\phi \in \mathscr{D}^*$, $t \in [0, T]$.

Theorem 1 *Let the condition* (I.1) *be fulfilled. Then there exists a wt-solution of Eq.* (0.1), *satisfying*

$$\mathbf{M}\|\psi(t)\|^2 \leqq \|\psi_0\|^2 e^{ct}, \tag{1.2}$$

$$\mathbf{M}|(\phi|\psi(t) - \psi(s))|^2 \leqq c_{\phi,T}|t-s|\ \|\psi_0\|^2, \quad \phi \in \mathscr{D}^*. \tag{1.3}$$

If moreover

(I.2) *$\bar{K}_c$ maximal accretive (m-accretive), i.e. $-\bar{K}_c$ is generator of a contraction semigroup, then such solution is a.s. unique.*

Proof. Let $\tilde{K}_c$ be any m-accretive extension of K_c, then $R_n = (I + (1/n)\tilde{K}_c)^{-1}$; $n = 1, 2, \ldots$, are bounded operators, converging strongly to I as $n \to \infty$ [16]. Consider

$$L_j^n = L_j R_n, \quad K^n = R_n^* K R_n,$$

then these are bounded operators satisfying the dissipativity condition (0.2) with the same constant. Moreover

$$L_j^n\psi \to L_j\psi, \quad K^n\psi \to K\psi, \qquad \psi \in \mathscr{D} \tag{1.4}$$

(see [9]). The equation

$$d\psi^n(t) = \sum_{j=1}^n L_j^n\psi^n(t)\, dW_j(t) - K^n\psi^n(t)\, dt, \quad \psi^n(0) = \psi_0 \tag{1.5}$$

has unique strong solution which is Markov process in $\mathscr{H}$. The dissipativity condition implies

$$\mathbf{M}\|\psi^n(t)\|^2 \leqq \|\psi_0\|^2 e^{ct} . \tag{1.6}$$

This follows from the fact that by the Ito formula for square of norm, the process $\|\psi(t)\|^2 e^{ct}$ satisfies an exponential equation with nonpositive coefficient of dt, hence is a supermartingale (cf. [1]).

Consider the family

$$\psi^n(t); \quad n = 1, 2, \dots ;$$

of functions of $t \in [0, T]$ with values in the separable Hilbert space $\mathscr{H} \otimes \mathscr{L}_T^2(W)$, where $\mathscr{L}_T^2(W)$ is the space of complex random variables with finite second moment, generated by $W_j(t)$; $t \in [0, T]$. For any $\phi \in \mathscr{D}^*$ by (1.5), (1.6)

$$\mathbf{M}|(\phi|\psi^n(t) - \psi^n(s))|^2 \leqq c_{\phi,T}|t-s| \ \|\psi_0\|^2 . \tag{1.7}$$

By using (1.6), (1.7), the Arzela–Ascoli theorem, separability of the Hilbert space and diagonalization, one can find a subsequence $\{n_k\}$ such that for all $\phi \in \mathscr{H}$, $\xi \in \mathscr{L}_T^2(W)$ the sequences $\mathbf{M}\bar{\xi}(\phi|\psi^{n_k}(t))$ will converge uniformly in $t \in [0, T]$. By (1.6) and the weak compactness of the unit ball in $\mathscr{H} \otimes \mathscr{L}_T^2(W)$, there is $\psi(\,\cdot\,)$, such that

$$\mathbf{M}\bar{\xi}(\phi|\psi^{n_k}(t)) \rightrightarrows \mathbf{M}\bar{\xi}(\phi|\psi(t)), \quad \phi \in \mathscr{H}, \ \xi \in \mathscr{L}_T^2(W) . \tag{1.8}$$

Taking limits in (1.6), (1.7) we obtain (1.2), (1.3). In particular, $\psi(t)$ is weakly continuous in the mean-square sense, hence in probability. Let us show that we can pass to the limit in Eqs. (1.5), i.e. in

$$\mathbf{M}\bar{\xi}(\phi|\psi^{n_k}(t)) = \mathbf{M}\bar{\xi}(\phi|\psi_0) + \mathbf{M}\bar{\xi}\int_0^t \sum_j (L_j^{n_k *}\phi|\psi^{n_k}(s))\, dW_j(s)$$

$$+ \mathbf{M}\bar{\xi}\int_0^t (K^{n_k *}\phi|\psi^{n_k}(s))\, ds, \quad \phi \in \mathscr{D}^*, \ \xi \in \mathscr{L}_T^2(W) .$$

By (1.8) one can pass to the limit in the left side and in the second term on the right side, and it remains to show the weak convergence of the stochastic integral. Since by (1.6) and (1.2)

$$\mathbf{M}\left|\int_0^t \sum_j (L_j^{n_k *}\phi|\psi^{n_k}(s))\, dW_j(s)\right|^2 \leqq c\|\psi_0\|^2 \sum_j \|L_j^*\phi\|^2 , \tag{1.9}$$

it is sufficient to show this for the dense subset of random variables $\{I_p(t)\}$, defined as

$$I_0(t) = 1, \quad I_p(t) = \int_0^t I_{p-1}(s) \sum_j a_j(s)\, dW_j(s) ,$$

where $a_j(t)$ are arbitrary functions, satisfying $\Sigma_j |a_j(t)|^2 \leqq \text{const}$. We have

$$\mathbf{M}\overline{I_p(t)}\int_0^t \sum_j (L_j^{n_k *}\phi|\psi^{n_k}(s))\, dW_j(s) = \int_0^t \mathbf{M}\overline{I_{p-1}(s)}\left(\sum_j a_j(s) L_n^{n_k *}\phi|\psi^{n_k}(s)\right) ds . \tag{1.10}$$

By conditions (I.1), (I.2) the series $\phi(s) = \Sigma_j a_j(s) L_j^{n_k *} \phi$ is convergent with the norm bounded uniformly in s, k. From (1.8), (1.6), (1.4) and (I.1) one sees that the integrand in the right-hand side of (1.10) converges pointwise as $k \to \infty$. This proves convergence of the stochastic integrals as $k \to \infty$, and hence the existence of the wt-solution.

Let ψ_1, ψ_2 be two solutions, such that $\psi_1(0) = \psi_2(0)$, and satisfying (1.2). Then $\delta(t) = \psi_1(t) - \psi_2(t)$ satisfies

$$(\phi|\delta(t)) = \int_0^t \sum_j (L_j^* \phi|\delta(s))\, dW_j(s) - \int_0^t (K^* \phi|\delta(s))\, ds\,.$$

To prove uniqueness it is sufficient to show

$$m_p(t) \equiv \mathbf{M}\overline{I_p(t)}\delta(t) = 0; \quad p = 0, 1, \ldots\,, \tag{1.11}$$

for arbitrary $a_j(s)$, defining $I_p(t)$. By the Ito product formula

$$\begin{aligned}(\phi|\overline{I_p(t)}\delta(t)) &= \int_0^t \sum_j ((L_j^* I_p(s) + a_j(s) I_{p-1}(s))\phi|\delta(s))\, dW_j(s) \\ &\quad + \int_0^t \left(\left(\sum_j a_j(s) L_j^* I_{p-1}(s) - K^* I_p(s)\right)\phi\,\middle|\,\delta(s)\right) ds\,.\end{aligned} \tag{1.12}$$

By (1.2) one has $\mathbf{M}\|\overline{I_p(t)}\delta(t)\| \leqq$ const, hence taking into account (I.1), the stochastic integral is a martingale. Taking expectation gives

$$(\phi|m_p(t)) = \int_0^t \left(\sum_j a_j(s) L_j^* \phi\,\middle|\, m_{p-1}(s)\right) ds - \int_0^t (K^* \phi|m_p(s))\, ds, \quad \phi \in \mathscr{D}^*\,,$$

for $p \geqq 1$ and

$$(\phi|m_0(t)) = -\int_0^t (K^* \phi|m_0(s))\, ds, \quad \phi \in \mathscr{D}^*\,.$$

By (I.2) the operator K_c^* is m-accretive, so that $(\lambda I + K^*)\mathscr{D}^*$ is dense in $\mathscr{H}$ for $\lambda > c/2$. Using this fact and taking Laplace transform as in [18], one gets $m_p(t) = 0$. Then by induction one arrives at (1.11).

The Markov property follows from the fact that $\psi^{n_k}(t)$ are Markov and from the uniqueness of the solution. □

Remark. 1. The proof of existence can be extended to the case where $L_j = L_j(t, \omega)$ (respectively $K = K(t, \omega)$) are regular left continuous in t in the strong operator topology on the domain $\mathscr{D}$, which is assumed independent of t, ω (respectively in the strong resolvent sense), the dissipativity condition (0.2) holds with a constant independent of t, ω, and (I.1) is replaced with

$$\sum_j \|L_j^*(t, \omega)\psi\|^2 \leqq c_\psi, \quad \psi \in \mathscr{D}^*\,,$$

where c_ψ and $\mathscr{D}^*$ are independent of t, ω.

2 The dual equation

According to the Riesz theorem the one-to-one correspondence between vectors $\phi \in \mathscr{H}$ and linear bounded functionals ϕ^* on $\mathscr{H}$ is established by the formula $\phi^* : \psi \to (\phi|\psi)$, $\psi \in \mathscr{H}$. Consider the stochastic differential equation

$$d\phi^*(t) = \sum_j \phi^*(t) L_j \, dW_j(t) - \phi^*(t) K \, dt; \quad \phi^*(0) = \phi_0^* ,$$

which we call the *dual equation.* Precisely, we shall call random process $\phi(t)$; $t \in [0, T]$, with values in $\mathscr{H}$ *wt-solution of the dual equation* if $\phi(t)$ is weakly continuous in probability and satisfies

$$(\phi(t)|\psi) - (\phi_0|\psi) = \int_0^t \sum_j (\phi(s)|L_j\psi) \, dW_j(s) - \int_0^t (\phi(s)|K\psi) \, ds \tag{2.1}$$

for all $\psi \in \mathscr{D}$, $t \in [0, T]$. Note that the right-hand side is defined due to the dissipativity condition (0.2). Let us denote S_t the unitary operator of time reversal in $\mathscr{L}_T^2(W)$, uniquely determined by the relation

$$S_t W(s) = \begin{cases} W(s), & s \geqq t, \\ W(t) - W(t-s), & s < t . \end{cases}$$

Operator S_t is an involution, $S_t^2 = I$.

Theorem 2 *Under the conditions of Theorem* 1 *there exists a unique wt-solution of Eq.* (2.1), *satisfying*

$$\mathbf{M}|(\phi(t)|\psi_0)|^2 \leqq \|\phi_0\|^2 \|\psi_0\|^2 e^{ct} . \tag{2.2}$$

Moreover, this solution satisfies the duality relation

$$\mathbf{M}\bar{\xi}(\phi(t)|\psi_0) = \mathbf{M}\overline{S_t \xi}(\phi_0|\psi(t)) , \tag{2.3}$$

where $\xi \in \mathscr{L}_T^2(W)$, *and* $\psi(t)$ *is the solution of* (1.1).

Proof. Consider the approximating equation

$$d\phi^{n*}(t) = \sum_{j=1}^n \phi^{n*}(t) L_j^n \, dW_j(t) - \phi^{n*}(t) K^n \, dt; \quad \phi^{n*}(0) = \phi_0^* .$$

It has unique strong solution $\phi^n(t)$, such that $\mathbf{M}\|\phi^n(t)\|^2 < \infty$. This solution can be represented as multiplicative stochastic integral

$$\phi^{n*}(t) = \phi_0^* \lim_{N \to \infty} \overrightarrow{\prod_{k=1}^{N}} \left(I + \sum_{j=1}^n L_j^n \Delta W_j^{(k)} - K^n \Delta t \right) ,$$

where $\Delta t = t/N$, $W_j^{(k)} = W_j(k\Delta t) - W_j((k-1)\Delta t)$ and the product is time-ordered as indicated. This follows from a result of Emery [8] in finite-dimensional case and can be generalized to infinite dimensions by using the technique of [13]. The limit then should be understood as strong limit in probability.

In the same way

$$\psi^n(t) = \lim_{N\to\infty} \overleftarrow{\prod_{k=1}^{N}} \left(I + \sum_{j=1}^{n} L_j^n \Delta W_j^{(k)} - K^n \Delta t \right) \psi_0 ,$$

for the solution of (1.5). Since $S_t \Delta W_j^{(k)} = \Delta W_j^{(N-k+1)}$; $k = 1, \dots, N$, we have

$$\mathbf{M}\bar{\xi}(\phi^n(t)|\psi_0) = \mathbf{M}\overline{S_t\xi}(\phi_0|\psi^n(t)) . \tag{2.4}$$

Applying this twice with $\xi = (\phi^n(t)|\psi_0)$, we obtain

$$\mathbf{M}|(\phi^n(t)|\psi_0)|^2 = \mathbf{M}|(\phi_0|\psi^n(t))|^2 . \tag{2.5}$$

From (1.6) it follows then

$$\mathbf{M}|(\phi^n(t)|\psi_0)|^2 \leqq \|\phi_0\|^2 \|\psi_0\|^2 e^{ct} . \tag{2.6}$$

Also from (1.7) we obtain

$$\mathbf{M}|(\phi^n(t) - \phi^n(s)|\psi_0)|^2 = \mathbf{M}|(\phi_0|\psi^n(t) - \psi^n(s))|^2 \leqq c_{\phi_0,T}|t-s|\|\psi_0\|^2 . \tag{2.7}$$

From (2.4) and (1.8) it follows that $\mathbf{M}\bar{\xi}(\phi^n(t)|\psi_0)$ converge uniformly in t for $\psi_0 \in \mathscr{H}$, $\xi \in \mathscr{L}_T^2(W)$. Let $\phi(\cdot)$ be the limiting random function. Then (2.3) follows from (2.4) and (2.2) from (2.5).

Due to (2.7) $\phi(t)$ is weakly continuous in probability. The proof that $\phi(t)$ is wt-solution of the dual equation (2.1) proceeds like in Theorem 1, but making use of the duality relation (2.3). For example, the estimate (1.9) is replaced with

$$\begin{aligned} &\mathbf{M}\left|\int_0^t \sum_j (\phi^{n_k}(s)|L_j^{n_k}\psi_0)\, dW_j(s)\right|^2 \\ &= \int_0^t \sum_j \mathbf{M}|(\phi^{n_k}(s)|L_j^{n_k}\psi_0)|^2\, ds \\ &= \int_0^t \sum_j \mathbf{M}|(\phi_0|\psi_j^{n_k}(s)))|^2\, ds \leqq c\|\phi_0\|^2 \sum_j \|L_j^{n_k}\psi_0\|^2 \\ &\leqq c'\|\phi_0\|^2(\|\psi_0\|^2 + \|K\psi_0\|^2) , \end{aligned}$$

for $\psi_0 \in \mathscr{D}$, where $\psi_j^n(t)$ is the solution of (0.1), satisfying $\psi_j^n(0) = L_j^n\psi_0$, and (1.6), (0.2) were used.

Uniqueness is also proved as in Theorem 1, but using the estimate (2.2) instead of (1.6). For example, to prove that the stochastic integral in the analog of formula (1.12) is martingale, we observe that for a wt-solution $\phi(t)$

$$\begin{aligned} &\mathbf{M}\sum_j |(\phi(s)|(L_j I_p(s) + a_j(s) I_{p-1}(s))\psi_0)| \\ &\leqq \|\phi_0\|^2 \left(\mathbf{M}|I_p(s)|^2 \sum_j \|L_j\psi_0\|^2 + \mathbf{M}|I_{p-1}(s)|^2 \|\psi_0\|^2 \sum_j |a_j(s)|^2 \right) . \qquad \square \end{aligned}$$

Similarly to (2.5) we obtain from (2.3)

$$\mathbf{M}|(\phi(t)|\psi_0)|^2 = \mathbf{M}|(\phi_0|\psi(t))|^2 . \tag{2.8}$$

3 Quantum dynamical semigroups

In what follows we assume that the condition (0.2) holds with $c = 0$, i.e.

$$\sum_j \|L_j\psi\|^2 \leqq 2\,\mathrm{Re}(\psi|K\psi), \quad \psi \in \mathscr{D} . \tag{3.1}$$

The notion of dynamical semigroup is a noncommutative analog of that of (sub-)Markov semigroup: while the latter are semigroups of maps in functional spaces, the former are semigroups of maps in operator algebras, having certain properties of positivity and normalization. These semigroups satisfy differential equations, that are noncommutative generalization of the Kolmogorov equations (see Appendix).

Let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded operators in $\mathscr{H}$. Since it is dual Banach space of the space of trace-class operators, it is supplied with the *weak* topology*. On norm-bounded sets this topology coincides with the weak operator topology (see e.g. [3]). A map Φ in $\mathscr{B}(\mathscr{H})$ is called completely positive if

$$\sum_{i,j} (\psi_i|\Phi_t[X_i^* X_j]\psi_j) \geqq 0$$

for any finite sets $\{\psi_j\} \in \mathscr{H}$, $\{X_j\} \in \mathscr{B}(\mathscr{H})$. By a *dynamical semigroup* in $\mathscr{B}(\mathscr{H})$ (or *quantum dynamical semigroup*) we shall call a semigroup Φ_t; $t \geqq 0$, of weak* continuous completely positive maps in $\mathscr{B}(\mathscr{H})$, satisfying $\Phi_0 = \mathrm{Id}$ (the identity map of $\mathscr{B}(\mathscr{H})$), and $\Phi_t[I] \leqq I$ (the unit operator in $\mathscr{H}$). Moreover for any X the function $t \to \Phi_t[X]$ must be weak*-continuous. Φ_t is called *unital* if $\Phi_t[I] = I$.

Proposition *Under the assumptions* (I.1), (I.2) *the relation*

$$(\psi_0|\Phi_t[X]\psi_0) = \mathbf{M}(\psi(t)|X\,\psi(t)), \quad t \geqq 0 , \tag{3.2}$$

defines a dynamical semigroup Φ_t *in* $\mathscr{B}(\mathscr{H})$. *If* $\mathscr{D}$ *(corr.* $\mathscr{D}^*$*) is an invariant domain of* $e^{-\bar{K}t}$ *(corr. of* e^{-K^*t}*) for* $t \geqq 0$, *then this semigroup is the minimal solution of the forward and backward Markovian master equations* (A.6), (A.2).

Proof. For fixed t the relation (3.2) defines a bounded linear map $X \to \Phi_t[X]$ in $\mathscr{B}(\mathscr{H})$. In fact by (1.2) the right side is a bounded form in ψ_0, X. The semigroup property follows from the Markov property of solutions. From the definition (3.2) one can see that the maps $X \to \Phi_t[X]$ are weak* continuous and completely positive:

$$\sum_{i,j} (\psi_i|\Phi_t[X_i^* X_j]\psi_j) = \mathbf{M}\left\|\sum_j X_j\psi_j(t)\right\|^2 \geqq 0 ,$$

where $\psi_j(0) = \psi_j$.

Moreover, the function $t \to \Phi_t[X]$ is weak* continuous for all $X \in \mathscr{B}(\mathscr{H})$. To see this it is sufficient to establish continuity of functions $t \to (\phi_0|\Phi_t[X]\psi_0)$ at $t = 0$. From (1.2) we have

$$\mathbf{M}\|\psi(t) - \psi_0\|^2 \leqq 2[\|\psi_0\|^2 - \operatorname{Re}\mathbf{M}(\psi_0|\psi(t))] , \tag{3.3}$$

which tends to zero as $t \to 0$ by (1.3). From the definition (3.2) the required continuity then follows.

From (1.2) with $c = 0$

$$\Phi_t[I] \leqq I . \tag{3.4}$$

We shall use the Dirac's notation $|\phi)(\psi|$ for the rank one operator $\chi \to \phi(\psi|\chi)$. The relation (2.8) implies

$$(\psi_0|\Phi_t[|\phi_0)(\phi_0|]\psi_0) = \mathbf{M}|(\phi(t)|\psi_0)|^2 = \mathbf{M}|(\phi_0|\psi(t))|^2 . \tag{3.5}$$

By using Eq. (1.1) and the Ito product formula, we obtain the equation

$$\frac{d}{dt}(\psi_0|\Phi_t[|\phi_0)(\phi_0|]\psi_0) = \mathbf{M}\left[\sum_j |(L_j^*\phi_0|\psi(t))|^2 - 2\operatorname{Re}(K^*\phi_0|\psi(t))(\psi(t)|\phi_0)\right] ,$$

which is equivalent to the forward equation (A.6) with $\rho = |\psi_0)(\psi_0|$, and by using (2.1),

$$\frac{d}{dt}(\psi_0|\Phi_t[|\phi_0)(\phi_0|]\psi_0) = \mathbf{M}\left[\sum_j |(\phi(t)|L_j\psi_0)|^2 - 2\operatorname{Re}(\phi(t)|K\psi_0)(\psi_0|\phi(t))\right] ,$$

which is the backward equation (A.2) with $X = |\phi_0)(\phi_0|$. Assuming that $\mathscr{D}, \mathscr{D}^*$ are invariant domains of the corresponding contraction semigroups in $\mathscr{H}$, one can see from the proofs of Theorems A.1, A.2 that these equations are equivalent to the corresponding Markovian master equations with arbitrary ρ, X, and to their integral versions (A.8), (A.4).

According to these theorems, there exists the minimal solution Φ_t^∞ for both equations. Thus

$$(\psi_0|\Phi_t[|\phi_0)(\phi_0|]\psi_0) \geqq (\psi_0|\Phi_t^\infty[|\phi_0)(\phi_0|]\psi_0) .$$

On the other hand, by Lemma A.3

$$(\psi_0|\Phi_t^\infty[|\phi_0)(\phi_0|]\psi_0) \geqq (\psi_0|\Phi_t^\eta[|\phi_0)(\phi_0|]\psi_0) ,$$

where Φ_t^η is the unique solution of the backward equation (A.10) with $\eta < 1$. Consider the stochastic equation

$$d\phi^{\eta *}(t) = \eta \sum_j \phi^{\eta *}(t) L_j \, dW_j(t) - \phi^{\eta *}(t) K \, dt; \quad \phi^{\eta *}(0) = \phi_0^* ,$$

then

$$(\psi_0|\Phi_t^\eta[|\phi_0)(\phi_0|]\psi_0) = \mathbf{M}|(\phi^\eta(t)|\psi_0)|^2$$

by the uniqueness of the solution of Eq. (A.10). On the other hand, by (2.8) this is equal to $\mathbf{M}|(\phi_0|\psi^\eta(t))|^2$, where $\psi^\eta(t)$ is the solution of

$$d\psi^\eta(t) = \eta \sum_j L_j \psi^\eta(t) \, dW_j(t) - K\psi^\eta(t) \, dt, \quad \psi^\eta(0) = \psi_0 .$$

Arguing as in the proof of Theorem 1, we can find a sequence $\eta_k \uparrow 1$, such that $\psi^{\eta_k}(t) \to \psi(t)$ weakly in $\mathscr{H} \otimes \mathscr{L}_T^2(W)$ uniformly in t. Then

$$\lim_{k\to\infty} \inf \mathbf{M}|(\phi_0|\psi^{\eta_k}(t))|^2 \geqq \mathbf{M}|(\phi_0|\psi(t))|^2 = (\psi_0|\Phi_t[|\phi_0)(\phi_0|]\psi_0)\,.$$

Finally $(\psi_0|\Phi_t[|\phi_0)(\phi_0|]\psi_0) \leqq (\psi_0|\Phi_t^\infty[|\phi_0)(\phi_0|]\psi_0)$, that is Φ_t is the minimal solution. □

The following example, inspired by [6], shows that contrary to the case of bounded operators, the conservativity condition (0.3) does not imply (0.4); thus the dynamical semigroup $\Phi_t = \Phi_t^\infty$, defined by (3.2), need not be unital.

Example 1. Let $\mathscr{H} = l^2$ and let $\{|n);\ n = 0, 1, \ldots\}$ be the canonical basis in $\mathscr{H}$. Let $\mathscr{D} = \{\psi \in \mathscr{H} \colon \Sigma_{n=0}^\infty n^4 |(\psi|n)|^2 < \infty\}$ and consider the equation

$$\psi(t) = \psi_0 + \int_0^t L\psi(s)\,dW(s) - \int_0^t K\psi(s)\,ds\,, \tag{3.6}$$

where $L = (a^\dagger)^2$, $K = \frac{1}{2}a^2 a^{\dagger 2}$ with $a^\dagger = \Sigma_{n=0}^\infty \sqrt{n+1}|n+1)(n|$, $a = \Sigma_{n=1}^\infty \sqrt{n}|n-1)(n|$ being the "creation" and "annihilation" operators. The hypotheses of Theorem 1 are satisfied, so that the equation has the unique wt-solution. The condition (0.3) holds on $\mathscr{D}$, but the solution $\psi(t)$ satisfies $\mathbf{M}\|\psi(t)\|^2 < \|\psi_0\|^2$ for $t > 0$. To see this take $\|\psi_0\| = 1$ and observe that $\mathbf{M}\|\psi(t)\|^2 = \Sigma_{n=0}^\infty p_n(t)$, where $p_n(t) = \mathbf{M}|(n|\psi(t))|^2$ satisfy

$$\frac{d}{dt}p_n(t) = -(n+1)(n+2)p_n(t) + (n-1)n p_{n-2}(t); \quad n \geqq 2\,,$$

which is just the Kolmogorov forward equation for an exploding pure birth process, so that $\Sigma_{n=0}^\infty p_n(t) < 1,\ t > 0$. □

Thus the minimal dynamical semigroup Φ_t^∞ may not be unital; however if it is, then Φ_t^∞ is the unique solution of both Markovian master equations [4, 14]. The situation is similar to that for the Kolmogorov–Feller equations in the theory of Markov processes [10]. The noncommutative analog of the problem of non-explosion for quantum dynamical semigroups was investigated by operator-theoretic methods by Davies [6] and later by Chebotarev and Fagnola [4], who gave some sufficient conditions for the unitality. In the next section we develop a different view on this problem, based on the stochastic representation (3.2) and on Ito's stochastic calculus in a Hilbert space.

4 Hyperdissipativity

Let R be a positive self-adjoint operator in $\mathscr{H}$ defined on $\mathscr{D} = \mathscr{D}(R)$, and let H be a symmetric operator with $\mathscr{D}(H) \supset \mathscr{D}$. Defining

$$K = \tfrac{1}{2}R + iH, \quad \mathscr{D}(K) = \mathscr{D}\,,$$

we have $K^* \supset \frac{1}{2}R - iH$. We shall assume

(II.1) *$\mathscr{D}$ is a core for $\bar{K}$ and K^*.*

This condition will allow us to take $\mathscr{D}^* = \mathscr{D}$.

(II.2) $\|H\psi\| \leqq c_1\|R\psi\| + c_2\|\psi\|$, $\psi \in \mathscr{D}$, *for some* $c_1, c_2 \geqq 0$.

If $c_1 \leqq 1$, then by Wüst's theorem (see [16]) (II.2) $\Rightarrow$ (II.1) and K, K^* are m-accretive operators, which implies (I.2) with $\mathscr{D} = \mathscr{D}^*$.

Consider Eq. (0.1) with L_j satisfying (I.1), (I.2) and the conservativity condition (0.3) written in the form

$$\sum_j \|L_j\psi\|^2 = (R\psi|\psi), \quad \psi \in \mathscr{D}\,. \tag{4.1}$$

By Theorem 1 it has the unique wt-solution satisfying (1.2) with $c = 0$.

Theorem 3 *Let the conditions* (I.1), (I.2), (II.1), (II.2) *and*
(II.3) $\psi \in \mathscr{D}$ *implies* $L_j\psi \in \mathscr{D}(R^{1/2})$ *and*

$$\sum_j \|R^{1/2}L_j\psi\|^2 - 2\,\mathrm{Re}(K\psi|R\psi) \leqq c[(\psi|R\psi) + \|\psi\|^2], \quad \psi \in \mathscr{D}\,,$$

be fulfilled. Then (4.1) *implies mean-square norm conservation* (0.4).

Remark. 2. As shown in [12], similar result can be derived from Theorem 3.2.2 of [19] by taking special Hilbert scale associated with R. The condition (II.3) expresses dissipativity in this scale and therefore we call it *hyperdissipativity*. The condition (II.2) is rather technical and certainly can be relaxed (for example if $R = 0$, H essentially self-adjoint, then (0.1) is the Schroedinger equation, which always has unique solution satisfying (0.4)).

Proof. We use the normal triple $\mathscr{X} \subset \mathscr{H} \subset \mathscr{X}^*$ of Hilbert spaces, where $\mathscr{X} = \mathscr{D}(R^{1/2})$ with the norm $\|\psi\|_{\mathscr{X}}^2 = \|R^{1/2}\psi\|^2 + \|\psi\|^2$. The canonical bilinear form between $\mathscr{X}$ and $\mathscr{X}^*$ will be denoted $[\,\cdot\,,\,\cdot\,]$ (see e.g. [19, Sect. 3.2]).

Let us take $R_n = (I + (1/n)R)^{-1}$ instead of $(I - (1/n)\tilde{K}_c)^{-1}$ in the proof of Theorem 1. The condition (II.2) insures that L_j^n, K^n have the same properties as operators in $\mathscr{H}$ and are bounded in $\mathscr{X}$. Take $\psi_0 \in \mathscr{X}$ and consider (1.5) as equation in the Hilbert space $\mathscr{X}$. From (II.2) and (II.3) one can deduce dissipativity in $\mathscr{X}$:

$$\sum_j \|L_j^n\psi\|_{\mathscr{X}}^2 = 2\,\mathrm{Re}(\psi|K^n\psi)_{\mathscr{X}} \leqq c\|\psi\|_{\mathscr{X}}^2, \quad \psi \in \mathscr{D}\,,$$

with a constant c independent of n. This implies that the solution $\psi^n(t)$ lies in $\mathscr{X}$ and

$$\mathbf{M}\|\psi^n(t)\|_{\mathscr{X}}^2 \leqq \|\psi_0\|_{\mathscr{X}}^2 e^{ct}\,.$$

The sequence $\psi^n(t)$ is norm-bounded in $\mathscr{X} \otimes \mathscr{L}_T^2(W)$ and weakly converges to $\psi(t)$ in $\mathscr{H} \otimes \mathscr{L}_T^2(W)$, hence in $\mathscr{X} \otimes \mathscr{L}_T^2(W)$. Therefore

$$\mathbf{M}\|\psi(t)\|_{\mathscr{X}}^2 \leqq \|\psi_0\|_{\mathscr{X}}^2 e^{ct}\,. \tag{4.2}$$

Taking into account that $\psi(t) \in \mathscr{X}$, the relation (1.1) can be rewritten as

$$(\phi|\psi(t)) - (\phi|\psi_0) = \int_0^t \sum_j (\phi|L_j\psi(s))\,dW_j(s) - \int_0^t [\phi, K\psi(s)]\,ds, \quad \phi \in \mathscr{D}\,,$$

where K is bounded operator from $\mathscr{X}$ to $\mathscr{X}^*$. Indeed by (II.2) it follows that

$$|(\psi|K\psi)| \leqq c[(\psi|R\psi) + \|\psi\|^2], \quad \psi \in \mathscr{D}$$

(see [16, Theorem VI-1.38]), whence the operator $(R+I)^{-1/2}K(R+I)^{-1/2}$ is bounded on a dense domain. Now

$$\|K\psi\|_{\mathscr{X}^*} = \|(R+I)^{-1/2}K\psi\| \leqq \|(R+I)^{-1/2}K(R+I)^{-1/2}\| \cdot \|\psi\|_{\mathscr{X}} .$$

Applying the Ito formula for the square of norm in the form given in [19, Theorem 2.4.2] one obtains

$$\|\psi(t)\|^2 = \|\psi_0\|^2 + \int_0^t \sum_j 2\,\mathrm{Re}(\psi(s)|L_j\psi(s))\,dW_j(s)\,, \tag{4.3}$$

where the integral with respect to ds vanishes because of the conservativity condition (4.1) extended to $\mathscr{X}$. Denoting $p(t) = \|\psi(t)\|^2$,

$$\hat{\psi}(t) = \begin{cases} \psi(t)p(t)^{-1} & \text{if } p(t) > 0\,, \\ \text{a fixed unit vector from } \mathscr{X} & \text{otherwise} \end{cases}$$

and

$$a_j(t) = 2\,\mathrm{Re}(\hat{\psi}(t)|L_j\hat{\psi}(t))\,, \tag{4.4}$$

we can rewrite (4.3) as the exponential equation

$$dp(t) = p(t)\,dZ(t)\,, \tag{4.5}$$

where $Z(t) = \int_0^t \Sigma_j a_j(s)\,dW_j(s)$ is a local martingale. To establish (0.4) it is sufficient to prove that $p(t)$ is martingale, and this can be shown by appropriate modification of an argument of Gatarek and Gisin [11].

Consider the stopping times

$$\tau_n = \inf\{t \geqq 0\colon \|\hat{\psi}(t)\|_{\mathscr{X}}^2 \leqq n\}\,.$$

Since by Cauchy–Schwarz inequality and by (4.1) the quadratic variance $\langle Z(t)\rangle$ of $Z(t)$ is evaluated as

$$\langle Z(t)\rangle = \int_0^t \sum_j a_j(s)^2\,ds \leqq 4\int_0^t \sum_j \|L_j\hat{\psi}(s)\|^2\,ds = 4\int_0^t \|\hat{\psi}(s)\|_{\mathscr{X}}^2\,ds\,,$$

we have

$$\langle Z(t\wedge\tau_n)\rangle \leqq 4tn\,.$$

This is sufficient for $p(t\wedge\tau_n)$ to be uniformly integrable martingale [17], in particular,

$$\mathbf{M}p(t\wedge\tau_n) = p(0)\,.$$

Take $\|\psi_0\| = 1$, then $p(t\wedge\tau_n)$ is the density of the new probability measure $\hat{\mathbf{P}}_n$ with respect to the basic measure $\mathbf{P}$. From (4.2) one has

$$\hat{\mathbf{M}}_n\|\hat{\psi}(t\wedge\tau_n)\|_{\mathscr{X}}^2 = \mathbf{M}\|\psi(t\wedge\tau_n)\|_{\mathscr{X}}^2 \leqq \|\psi_0\|_{\mathscr{X}}^2 e^{ct}\,.$$

It follows that

$$\hat{\mathbf{P}}_n(\tau_n \leqq t) \leqq n^{-2}\hat{\mathbf{M}}_n\|\hat{\psi}(t \wedge \tau_n)\|^2_{\mathscr{X}} \to 0\,,$$

as $n \to \infty$. Therefore

$$\mathbf{M}\|\psi(t)\|^2 \geqq \mathbf{M}\|\psi(t \wedge \tau_n)\|^2 1_{\{\tau_n \geqq t\}} = \hat{\mathbf{P}}_n(\tau_n \geqq t) \to 1\,,$$

whence $\mathbf{M}\|\psi(t)\|^2 = 1 = \|\psi_0\|^2$. □

Example 2. This example shows that (II.3) is not a necessary condition for the mean-square norm conservation. Consider Eq. (3.6) in $\mathscr{H} = \mathscr{L}^2(0,\infty)$ with

$$L = \frac{d}{dx} - \frac{1}{x}; \quad \mathscr{D}(L) = \{\psi\colon \psi(0) = 0,\ \psi' \in \mathscr{L}^2\}\,;$$

$$K = \tfrac{1}{2}R = \tfrac{1}{2}L^*L = -\frac{1}{2}\frac{d^2}{dx^2}; \quad \mathscr{D}(R) = \{\psi\colon \psi(0) = 0,\ \psi'' \in \mathscr{L}^2\}$$

and $H = 0$. In defining L we have in mind the inequality

$$\int_0^\infty x^{-2}|\psi(x)|^2\,dx \leqq 4\int_0^\infty |\psi'(x)|^2\,dx\,,$$

valid for $\psi \in \mathscr{D}(L)$ (see [16, p. 345]). With $\mathscr{D} = \mathscr{D}^* = \mathscr{D}(R)$ one can check the conditions (I.1), (I.2) ((II.1), (II.2) are trivially satisfied) and the conservativity condition (4.1).

The condition (II.3) does not hold, since $\psi \in \mathscr{D}$ does not imply $L\psi \in \mathscr{D}(R^{1/2}) = \mathscr{D}(L)$. However (0.4) holds for solutions of Eq. (3.6). We sketch the argument without going into detail. The function $p(t,x) = \mathbf{M}|\psi(t,x)|^2$ satisfies the forward Kolmogorov equation

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(x^{-1}p) + \frac{1}{2}\frac{\partial^2}{\partial x^2}p; \quad 0 < x < \infty\,,$$

and the initial condition $p(0,x) = |\psi_0(x)|^2$. This is diffusion on $(0,\infty)$ with unit coefficient and with the drift x^{-1}, for which both 0 and ∞ are non-attainable boundaries, hence

$$\mathbf{M}\|\psi(t)\|^2 = \int_0^\infty p(t,x)\,dx = \int_0^\infty p(0,x)\,dx = \|\psi_0\|^2\,. \quad □$$

Now let us consider the wt-solution of a conservative equation (0.1) and assume that (0.4) holds (for example the conditions of Theorem 3 are fulfilled). Let $\|\psi_0\| = 1$, then (0.4) implies that the process $p(t) = \|\psi(t)\|^2$ can be regarded as the Radon–Nikodym derivative of a new probability measure $\hat{\mathbf{P}}$ with respect to the initial $\mathbf{P}$. By the Ito formula for the function $\psi \to \|\psi\|^2$ the process $p(t)$ satisfies the exponential equation (4.5), and is the uniformly integrable exponential martingale

$$p(t) = \exp[Z(t) - \tfrac{1}{2}\langle Z(t)\rangle] .$$

According to Girsanov's theorem, the processes

$$\hat{W}_j(t) = W_j(t) - \int_0^t a_j(s)\,ds; \quad j = 1,2,\ldots,\ t > 0 ,$$

are independent standard Wiener processes with respect to the new probability measure $\hat{\mathbf{P}}$. The normalized process $\hat{\psi}(\,\cdot\,)$ satisfies the nonlinear stochastic equation

$$\hat{\psi}(t) = \hat{\psi}(0) + \int_0^t \sum_{j=1}^d \hat{L}_j(s)\hat{\psi}(s)\,d\hat{W}_j(s) - \int_0^t \hat{K}(s)\hat{\psi}(s)\,ds , \qquad (4.6)$$

where $d \leqq \infty$,

$$\hat{L}_j(s) = L_j - \tfrac{1}{2}a_j(s)I, \quad \hat{K}(s) = K - \frac{1}{2}\sum_{j=1}^d a_j(s)L_j + \frac{1}{8}\sum_{j=1}^d a_j(s)^2 I ,$$

and $a_j(s)$ are the quadratic functions of $\hat{\psi}(s)$, given by (4.4). Equation (4.6) can be derived by applying the Ito product formula to the function $\psi \to \psi \cdot \|\psi\|^{-1}$ and by taking into account the exponential equation for the process $\|\psi(t)\|^{-1}$, which is just square root of the Radon–Nikodym derivative of $\mathbf{P}$ with respect to $\hat{\mathbf{P}}$ (cf. [1]). This is the nonlinear equation for normalized posterior wave function in a continuous measurement process (see [7,2]). The linear equation (0.1) plays the same role for this equation as Zakai's equation in the classical nonlinear filtering theory.

In this way one obtains a generalization of the result of Gatarek and Gisin [11] on the existence of solution of Eq. (4.6) (in the sense of proof of the Theorem 3) with *a priori* given Wiener processes $\hat{W}_j(t)$, in the case $d = 1$, $L = L_1$ self-adjoint and $K = \frac{1}{2}L^2$. Note that the generalization to the case $d > 1$ (or even $d = 1$, L nonselfadjoint) is by no means straightforward, since a condition of the type (II.3) must appear (which is fulfilled automatically in the situation of [11]).

The uniqueness of the strong solution of (4.6) can then be proved along the same lines as in [11]. This solution gives a different stochastic representation for the dynamical semigroup from Sect. 3:

$$(\phi_0|\Phi_t[X]\psi_0) = \hat{\mathbf{M}}(\hat{\phi}(t)|X\hat{\psi}(t)) ,$$

which has important applications in the theory of quantum measurement (see [1,2,7,11] for further details and references).

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Appendix

The minimal solution of Markovian master equations

Let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded operators in a Hilbert space $\mathscr{H}$. Since it is dual of the Banach space of trace-class operators $\mathscr{T}(\mathscr{H})$, it is supplied with the weak* topology, defined by the family of seminorms

$$X \to \operatorname{Tr} \rho X, \quad \rho \in \mathscr{T}(\mathscr{H}) . \tag{A.1}$$

Let Ψ be a linear bounded map in $\mathscr{T}(\mathscr{H})$, then the adjoint map $\Phi = \Psi^*$ in $\mathscr{B}(\mathscr{H})$ is weak* continuous, and every weak* continuous map Φ arises in this way [3]. Ψ is called preadjoint of Φ and denoted Φ_*. If Φ_t is a dynamical semigroup in $\mathscr{B}(\mathscr{H})$, then $\Psi_t = (\Phi_t)_*$ is a strongly continuous semigroup in $\mathscr{T}(\mathscr{H})$, called *preadjoint semigroup.*

Let L_j, K be operators satisfying the dissipativity condition (3.1), then we can consider the *backward quantum Markovian master equation*

$$\begin{aligned} &\frac{d}{dt}(\psi|\Phi_t[X]\phi) \\ &\quad = \sum_j (L_j\psi|\Phi_t[X]\phi) - (K\psi|\Phi_t[X]\phi) - (\psi|\Phi_t[X]K\phi) , \end{aligned} \tag{A.2}$$

for $\phi, \psi \in \mathscr{D}$. We assume the following regularity properties for a solution of the backward equation (A.2): this should be a family (not necessarily a semigroup) Φ_t: $t \geqq 0$, of normal completely positive maps in $\mathscr{B}(\mathscr{H})$, uniformly bounded in norm, satisfying $\Phi_0 = \mathrm{Id}$, and such that all functions $t \to \Phi_t[X]$, $X \in \mathscr{B}(\mathscr{H})$, are weak* continuous.

The following results may be considered as extensions of ideas of Feller [10] to the noncommutative situation.

Theorem A.1 *Let* $\bar{K}$ *be m-accretive and* $\mathscr{D}$ *be an invariant domain of the semigroup* $\exp(-\bar{K}t)$, $t \geqq 0$. *Then there exists the minimal solution* Φ_t^∞ *of Eq.* (A.2), *which is a dynamical semigroup.*

Proof (*Sketch*). Introducing the semigroup $\check{\Psi}_t[\rho] = e^{-\bar{K}t}\rho e^{-K^*t}$ in $\mathscr{T}(\mathscr{H})$, we see that the dense domain

$$\mathsf{D} = \operatorname{lin}\{\rho\colon\ \rho = |\phi)(\psi|,\ \phi, \psi \in \mathscr{D}\} \tag{A.3}$$

is an invariant domain for $\check{\Psi}_t$. Defining $\Lambda[\rho] = \Sigma_j |L_j\phi)(L_j\psi|$ for $\rho = |\phi)(\psi| \in \mathsf{D}$, one can show [14] that (A.2) is equivalent to the integral equation

$$\operatorname{Tr} \rho\Phi_t[X] = \operatorname{Tr} \check{\Psi}_t[\rho]X + \int_0^t \operatorname{Tr} \Lambda[\check{\Psi}_s[\rho]]\Phi_{t-s}[X]\, ds, \quad \rho \in \mathsf{D},\ X \in \mathscr{B}(\mathscr{H}) . \tag{A.4}$$

Indeed, both equations are equivalent to

$$\frac{d}{ds}\operatorname{Tr} \check{\Psi}_s[\rho]\Phi_{t-s}[X] = -\operatorname{Tr} \Lambda[\check{\Psi}_s[\rho]]\Phi_{t-s}[X], \quad 0 \leqq s \leqq t .$$

Note that in (A.4) $\mathscr{B}(\mathscr{H})$ may be replaced by any weak* dense subspace.

The existence of the minimal solution is proved by considering iterations [4, 9]:

$$\operatorname{Tr} \rho \Phi_t^{n+1}[X] = \operatorname{Tr} \check{\Psi}_t[\rho]X + \int_0^t \operatorname{Tr} \Lambda[\check{\Psi}_s[\rho]]\Phi_{t-s}^n[X]\, ds \tag{A.5}$$

with $\Phi_t^1[X] = \check{\Phi}_t[X] = e^{-K^*t}Xe^{-\bar{K}t}$. Complete positivity of Λ implies that $\Phi_t^{n+1} - \Phi_t^n$ is completely positive, and (3.1) implies $\Phi_t^n[I] \leqq I$, by induction. By bounded monotone convergence there exists $\lim_{n\to\infty}\Phi_t^n = \Phi_t^\infty$, satisfying (A.4). Since for any other solution Φ_t the difference $\Phi_t - \Phi_t^n$ is completely positive by induction, Φ_t^∞ is the minimal solution. For detailed proof of properties of Φ_t^∞ see [4, 9, 14]. □

Assuming the condition (I.1), we can consider the *forward equation*:

$$\frac{d}{dt}(\phi|\Psi_t[\rho]\psi) = \sum_j (L_j^*\phi|\Psi_t[\rho]L_j^*\psi) - (K^*\phi|\Psi_t[\rho]\psi) - (\phi|\Psi_t[\rho]K^*\psi)\,. \tag{A.6}$$

For a solution Ψ_t; $t \geqq 0$, of (A.6) we demand that Ψ_t^* should satisfy the regularity properties of solution of the backward equation. Defining

$$\mathsf{D}^* = \operatorname{lin}\{X = |\psi)(\phi|;\ \phi,\psi \in \mathscr{D}^*\}$$

and $\Lambda^*[X] = \Sigma_j |L_j^*\psi)(L_j^*\phi|$ for $X = |\psi)(\phi| \in \mathsf{D}^*$, we have

$$\operatorname{Tr} \Lambda[\rho]X = \operatorname{Tr} \rho\Lambda^*[X]; \quad \rho \in \mathsf{D},\ X \in \mathsf{D}^*\,. \tag{A.7}$$

Theorem A.2 *Let $\bar{K}$ (hence K^*) be m-accretive and $\mathscr{D}^*$ an invariant domain of the semigroup* $\exp(-K^*t)$; $t \geqq 0$. *Then $\Psi_t^\infty = (\Phi_t^\infty)_*$ is the minimal solution of the forward equation* (A.6).

Proof. Like in the previous theorem, one can prove that (A.6) is equivalent to

$$\operatorname{Tr} \Psi_t[\rho]X = \operatorname{Tr} \rho\check{\Phi}_t[X] + \int_0^t \operatorname{Tr} \Psi_{t-s}[\rho]\Lambda^*[\check{\Phi}_s[X]]\, ds, \quad \rho \in \mathscr{T}(\mathscr{H}),\ X \in \mathsf{D}^*\,, \tag{A.8}$$

where $\mathscr{T}(\mathscr{H})$ can be replaced by any norm-dense subspace. The proof then proceeds along the same lines as for the classical Kolmogorov–Feller equation [10]. Consider the iterations

$$\operatorname{Tr} \Psi_t^{n+1}[\rho]X = \operatorname{Tr} \rho\check{\Phi}_t[X] + \int_0^t \operatorname{Tr} \Psi_{t-s}^n[\rho]\Lambda^*[\check{\Phi}_s[X]]\, ds\,, \tag{A.9}$$

with $\Psi_t^1[\rho] = \check{\Psi}_t[\rho]$. If these iterations converge, then the limit is the minimal solution, by the same argument as in previous theorem. We shall prove by induction that $\Psi_t^n = (\Phi_t^n)_*$; then the convergence will follow from the proof of Theorem A.2. In our proof we take $\rho \in \mathsf{D}$ and $X \in \mathsf{D}^*$.

Assuming that $(\Phi_t^k)_* = \Psi_t^k$ for $k = n-1, n$, we have from (A.9) with $n+1$ replaced with n:

$$\operatorname{Tr} \rho\Phi_t^n[X] = \operatorname{Tr} \rho\check{\Phi}_t[X] + \int_0^t \operatorname{Tr} \rho\Phi_{t-s}^{n-1}[\Lambda^*[\check{\Phi}_s[X]]]\, ds\,.$$

By substituting into (A.5),

$$\operatorname{Tr}\rho\Phi_t^{n+1}[X] = \operatorname{Tr}\check{\Psi}_t[\rho]X + \int_0^t \operatorname{Tr}\Lambda[\check{\Psi}_s[\rho]]\check{\Phi}_{t-s}[X]\,ds$$
$$+ \int_0^t \int_0^{t-s} \operatorname{Tr}\Lambda[\check{\Psi}_s[\rho]]\Phi_{t-s-u}^{n-1}[\Lambda^*[\check{\Phi}_u[X]]]\,du\,ds\,.$$

Changing variables and using (A.7), this is transformed to

$$\operatorname{Tr}\rho\check{\Phi}_t[X] + \int_0^t \operatorname{Tr}\rho\check{\Phi}_{t-u}[\Lambda^*[\check{\Phi}_u[X]]]\,du$$
$$+ \int_0^t \int_0^{t-u} \operatorname{Tr}\Lambda[\check{\Psi}_s[\rho]]\Phi_{t-s-u}^{n-1}[\Lambda^*[\check{\Phi}_u[X]]]\,ds\,du$$
$$= \operatorname{Tr}\rho\check{\Phi}_t[X] + \int_0^t \operatorname{Tr}\rho\check{\Phi}_{t-u}[\Lambda^*[\check{\Phi}_u[X]]]\,du\,,$$

by (A.5) with $n+1$ replaced with n. By the induction assumption, we obtain (A.9). □

Consider also the backward equation with L_j replaced with ηL_j $(0 \leqq \eta < 1)$. We shall write it already in the integral form

$$\operatorname{Tr}\rho\Phi_t[X] = \operatorname{Tr}\check{\Psi}_t[\rho]X + \eta^2 \int_0^t \operatorname{Tr}\Lambda[\check{\Psi}_s[\rho]]\Phi_{t-s}[X]\,ds, \quad \rho \in \mathsf{D}\,. \tag{A.10}$$

Lemma A.3 *Equation* (A.10) *has unique solution* Φ_t^η. *Moreover,* $\Phi_t^\infty - \Phi_t^\eta$ *is completely positive for all* t.

Proof (*Sketch*). Let Φ_t be a solution of (A.10) and Φ_t^η be the minimal solution, which exists by Theorem A.1. Denoting $\Delta_t[X] = \Phi_t[X] - \Phi_t^\eta[X]$, $X \geqq 0$, and taking $\rho = |\psi)(\psi|$, $\psi \in \mathscr{D}$, we have

$$0 \leqq (\psi|\Delta_t[X]\psi) \leqq \eta^2 \lim_{0\leqq s\leqq t} \|\Delta_s[X]\| \int_0^t \operatorname{Tr}\Lambda[\check{\Psi}_s[|\psi)(\psi|]\,ds\,,$$

and the integral is evaluated as

$$\int_0^t \sum_j \|L_j e^{-Ks}\psi\|^2\,ds = \int_0^t \frac{d}{ds}\|e^{-Ks}\|^2\,ds \leqq \|\psi\|^2\,.$$

It follows that $\Delta_t \equiv 0$. Second assertion follows by induction from considering (A.5) and similar iterations for (A.10) (see [9]). □

Note added in proof. Recently the author was able to improve Theorem 3, by replacing operator R in the condition (II.3) with arbitrary strictly positive self-adjoint operator A, having the core $\mathcal{D}$. The restrictive technical condition (II.2) is the relaxed to $\| H\psi \| \leqq \| A\psi \|$, $\| R\psi \| \leqq \| A\psi \|$ for all $\psi \in \mathcal{D}$.

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