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Probability Theory and Related Fields

Reinforced Random Walk

Burgess Davis*

Department of Statistics, Purdue University, West Lafayette, IN 47907, USA

Summary. Let a_i, $i \geqq 1$, be a sequence of nonnegative numbers. Define a nearest neighbor random motion $\vec{X} = X_0, X_1, \ldots$ on the integers as follows. Initially the weight of each interval $(i, i+1)$, i an integer, equals 1. If at time n an interval $(i, i+1)$ has been crossed exactly k times by the motion, its weight is $1 + \sum_{j=1}^{k} a_j$. Given $(X_0, X_1, \ldots, X_n) = (i_0, i_1, \ldots, i_n)$, the probability that X_{n+1} is $i_n - 1$ or $i_n + 1$ is proportional to the weights at time n of the intervals $(i_n - 1, i_n)$ and $(i_n, i_n + 1)$. We prove that $\vec{X}$ either visits all integers infinitely often a.s. or visits a finite number of integers, eventually oscillating between two adjacent integers, a.s., and that $\lim_{n \to \infty} X_n/n = 0$ a.s. For much more general reinforcement schemes we prove $P(\vec{X}$ visits all integers infinitely often$) + P(\vec{X}$ has finite range$) = 1$.

1. Introduction

In this paper we study a class of stochastic processes driven by simple dynamics which depend on the entire process history. Although these processes are in general not Markov, we begin our discussion with a comment about Markov processes. Let $\vec{w} = \{w_i, -\infty < i < \infty\}$, where each w_i is a positive number, called the weight of the interval $(i, i+1)$. An integer valued stochastic process $X_0, X_1, \ldots$ which satisfies

$$
\begin{aligned}
(1.1) \qquad & P(X_{n+1} = i_n + 1 \mid (X_0, X_1, \ldots, X_n) = (i_0, i_1, \ldots, i_n)) \\
& = 1 - P(X_{n+1} = i_n - 1 \mid (X_0, X_1, \ldots, X_n) = (i_0, i_1, \ldots, i_n)) \\
& = \frac{w_{i_n}}{w_{i_n} + w_{i_n - 1}}
\end{aligned}
$$

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is an integer valued Markov process with stationary transition probabilities $p_{i,j}$ satisfying $p_{k,k-1}>0$, $p_{k,k+1}>0$, $p_{k,k-1}+p_{k,k+1}=1$, $-\infty<k<\infty$. Conversely, any such Markov process arises from an appropriate $\vec{w}$, unique up to multiplication by constants. This observation is probably due to T.E. Harris.

Now we describe a process which has recently been introduced by Diaconis. This process is integer valued, and will be designated by $Y_0, Y_1, \ldots$ At time n the weight of the interval $(i, i+1)$ is one plus the number of those integers $k<n$ such that (Y_k, Y_{k+1}) is either $(i, i+1)$ or $(i+1, i)$, that is, the weight of an interval is initially 1 and is increased (reinforced) by 1 each time it is crossed. If $w(n, i)$ stands for the weight of $(i, i+1)$ at time n, then the version of (1.1), in which X is replaced by Y and $w_{.}$ by $w(n, \cdot)$, holds. This process remembers where it has been and prefers to cross familiar intervals, that is, those already often crossed. Diaconis studies this motion by showing it is equivalent to having an independent Polya's urn at each integer which directs the motion up or down. de Finetti's theorem, applied to each urn, then shows the motion is equivalent to a random walk in a random environment, and the results of this subject are used. Especially, almost sure recurrence follows almost immediately. (We will say a sequence of integers is recurrent if each integer occurs infinitely often in the sequence, and say the sequence has finite range if only a finite number of integers occur.)

Our introduction to this subject came in two talks Diaconis gave at the 1987 Midwest Probability Conference. His main emphasis was on the limiting distributions for related walks on finite graphs, distributions related to the limiting distributions of Polya's urns. This work, joint with Coppersmith, is not yet written down. For an exposition of the method described in the previous paragraph as well as very interesting results about related processes on trees, see Pemantle [8].

The present paper considers only integer valued processes. Diaconis' method can not be adapted to study many reinforcing schemes other than the one given above, that is, the one which increases the weight of each interval by 1 (or by the same constant amount) each time it is crossed. For example, suppose the initial weights of all intervals are 1, and are increased by 1 only the first time the interval is crossed. The resulting process could be called fair random walk with partial reflection at the prior maximum and prior minimum, and is not too hard to study directly, but Diaconis' approach is inapplicable.

We mainly study one dimensional lattice valued reinforcing walks $\vec{X}=(X_0, X_1 \ldots)$, in which the initial weights are all 1, an assumption in force throughout this paragraph. We prove, for very general reinforcing schemes, that $P(\vec{X}$ is recurrent$)+P(\vec{X}$ has finite range$)=1$. Under less general schemes, still broad enough to include the situation where the nonnegative number a_k (not depending on i) is added to the weight of $(i, i+1)$ the k-th time it is crossed, and also broad enough to cover iid reinforcement, we prove $X_n/n \to 0$ a.s. as $n \to \infty$. Parenthetically, we do not find the weak law of large numbers any easier to prove than the strong law. We also study the analog of gambler's ruin problems, and show that as $\lambda \to \infty$, the order of magnitude of the probability that a reinforced walk, started at 1, hits λ before it hits zero, can be as large as $1/\sqrt{\lambda}$, but no larger.

In the final section we present Herman Rubin's elegant solution of a conjecture we showed him involving generalized Polya urns. Our treatment is self contained, and this part of the paper may be read independently of the rest. We then apply this result to reinforcing walks, and state several open problems.

2. Notation and Definitions

We define *reinforced random walk* (usually just called walk here, sometimes abbreviated RRW) to be a sequence $\vec{X}=\{X_i, i\geqq 0\}$ of integer valued random variables and a matrix $[w]=\{w(n,j), 0\leqq n<\infty, -\infty<j<\infty\}$ of positive random variables, all defined on the same probability space, such that if $\mathscr{G}_n$ is the σ-field $\sigma(\{X_i, 0\leqq i\leqq n, w(i,j), -\infty<j<\infty, 0\leqq i\leqq n\})$ then the following hold.

i) $w(n+1,j)-w(n,j)\geqq 0$, with equality if (X_n, X_{n+1}) is not either $(j, j+1)$ or $(j+1, j)$.

ii)
$$P(X_{n+1}=j+1 \mid X_n=j, \mathscr{G}_n)=1-P(X_{n+1}=j-1 \mid X_n=j, \mathscr{G}_n) = \frac{w(n,j)}{w(n,j)+w(n,j-1)} \text{ a.s.}$$

For brevity we often designate this walk by $\vec{X}$ instead of $(\vec{X}, [w])$. The random variables $w(0,j)$, $-\infty<j<\infty$, are called the *initial weights* of $\vec{X}$, and we say $\vec{X}$ is *initially fair* if all the initial weights are 1. We say the reinforcement is *nonrandomized* if $w(n,j)$ is measurable with respect to $\sigma(X_0, X_1, \ldots, X_n)$, $n\geqq 0$, and it is said to be *up only* [*down only*] if $w(n,j)=w(n-1,j)$ whenever (X_{n-1}, X_n) is $(j+1, j)$ [$(j, j+1)$]. If $X_n=j+1$, $X_{n+1}=j$ or $X_n=j$, $X_{n+1}=j+1$, we say the walk crosses $(j, j+1)$ between times n and $n+1$, and we say the walk starts at k if $X_0=k$ a.s. The distribution of the walk is the distribution of $(\vec{X}, [w])$. By a *reinforcement scheme* we mean a rule which, together with the distribution of $(X_0, w(0,k), -\infty<k<\infty)$, determines the distribution of the walk. There is no need to be more precise than this, since when we use the term it will always be in the context of a specific scheme.

We say a walk is of *sequence type* if there is a sequence $\vec{a}=\{a_k, k\geqq 1\}$ of nonnegative numbers, called the sequence of the walk, such that if $\phi(n,j)$ is the number of times that $(X_0, X_1, \ldots, X_n)$ crosses $(j, j+1)$ then $w(n,j)=w(0,j)+\sum_{i=1}^{\phi(n,j)} a_i$ a.s. (almost surely). That is, the k-th time $\vec{X}$ crosses $(j, j+1)$, the weight of this interval is increased by a_k a.s. Often in situations like this (and in fact in any situation) we omit a.s.

We say a walk is a *Diaconis walk* if it is an initially fair sequence type walk, and all coordinates of the sequence are 1.

Matrices in this paper are always infinite matrices of the form $\{a_{j,i}, -\infty<j<\infty, 1\leqq i<\infty\}=[a]$. A walk is said to be of *matrix type* if there is a matrix of nonnegative terms $[a]$, called the *reinforcing matrix* (or just matrix) of the walk, such that $w(n,j)=w(0,j)+\sum_{i=1}^{\phi(n,j)} a_{j,i}$. Finally, a walk is said to have iid *reinforcement* if

$$w(n,j)=w(0,j)+\sum_{\{0\leqq i\leqq n:\, (X_i, X_{i+1})=(j,j+1) \text{ or } (j+1,j)\}} Z_i,$$

where $Z_1, Z_2 \dots$ are iid nonnegative random variables and, if $n \geqq 0$, Z_{n+1} is independent of $\mathscr{G}_n$, $n \geqq 1$, the σ-field defined in the definition of RRW. We use $\mathscr{G}_n$ only to stand for this σ-field and use $\mathscr{F}_n$, $n \geqq 0$, only to stand for $\sigma(X_0, X_1, \dots, X_n)$. For most of this paper we will be working with non-randomized reinforcement, in which case $\mathscr{G}_n$ equals $\mathscr{F}_n$, and in which case there are a countable number of disjoint atoms in $\mathscr{F}_n$ which have probability totaling 1. (If $P(X_0 = k) = 1$, there are exactly 2^n atoms in $\mathscr{F}_n$ having positive probability.) Let A stand for one of these atoms and consider the process $(X_{n+i}, i \geqq 0, w(n+i, j), i \geqq 0, -\infty < j < \infty)$, conditioned on A. This process is itself a RRW. At most n of the initial weights for this walk may differ from the original initial weights, namely those of intervals crossed by $\vec{X}$ between times 0 and n on A. If the original walk was a matrix type walk, so is this conditioned one (not necessarily with the same matrix), but, if the original walk is a sequence type walk this one may not be sequence type.

Let $(\vec{X}, [w])$ be a RRW and let T be a stopping time with respect to $\mathscr{G}_n$, $n \geqq 0$, such that $P(T < \infty) = 1$. Then $(X_{T+i}, i \geqq 0, w(T+i, j), i \geqq 0, -\infty < j < \infty)$ is also a RRW, although not necessarily matrix type even if $\vec{X}$ is; a finite (random) number of the initial weights for this walk may differ from those of the original walk.

We use τ only for $\inf\{k: X_k \leqq 0\}$. Here, as elsewhere, $\inf \emptyset = \infty$. Absolute positive constants are designated by c, C, c_1, etc. The indicator function of a set A is written $I(A)$, and the minimum of a and b is written $a \wedge b$. If $\vec{v} = (v_1, v_2, \dots, v_n)$ is a vector we put $L(\vec{v}) = n$.

3. Recurrence and Maximal Inequalities

To begin this section we will study recurrence properties of reinforced random walk. The proof involves an extension of a martingale argument used by Harris to study recurrence of the Markov processes described in the first paragraph of this paper, which we briefly recall. First note that if a and b are positive, and if Y is a random variable which satisfies $P(Y = a^{-1}) = a/(a+b)$, $P(Y = -b^{-1}) = b/(a+b)$, then $EY = 0$. Let $X_0, X_1, \dots$ be the process described in the first paragraph of this paper, started at $k > 0$. Put $f(j) = \sum_{i=0}^{j-1} w_i^{-1}$, $j > 0$, and $f(0) = 0$. Then $f(X_{i \wedge \tau})$, $i \geqq 0$, is easily seen to be a nonnegative martingale since $P(f(X_{n+1}) - f(X_n) = w_j^{-1} \mid \mathscr{F}_n) = 1 - P(f(X_{n+1}) - f(X_n) = -w_{j-1}^{-1} \mid \mathscr{F}_n) = w_j/(w_j + w_{j-1})$ on $\{n < \tau, X_n = j\}$. Since nonnegative martingales converge a.s., it is easy to conclude that if $\lim_{n \to \infty} f(n) < \infty$, $P(\tau < \infty) < 1$, while if $\lim_{n \to \infty} f(n) = \infty$, $P(\tau < \infty) = 1$. See [5], p. 106 for a more detailed description of this argument. We are going to construct a supermartingale below, and two facts that will be used are:

(3.1) (i) If a and b are positive numbers and if $P(Z = -b^{-1}) = b/(a+b)$ and $P(Z = d) = a/(a+b)$ for some $d \leqq a^{-1}$, then $EZ \leqq 0$.

(ii) If Z is as in i) and if $Y = (a^{-1} - d) I(Z = d)$ then $E(Z + Y) = 0$.

Lemma 3.0. *Let $\vec{X}$ be a RRW such that all but perhaps a finite number of the initial weights $w(0,j)$ equal 1. Then $P(X_k=0$ for some $k)+P(\vec{X}$ has finite range, $X_k \neq 0, k \geqq 0)=1$.*

Proof. Assume with no loss of generality that $X_0>0$. For $0 \leqq n<\infty$ define

$$F^{\vec{X}}(n,k)=F(n,k)=\sum_{j=0}^{k-1} w(n,j)^{-1}, \qquad k \geqq 1,$$
$$=0 \quad \text{if } k \leqq 0.$$

Put

$$M_n^{\vec{X}}=M_n=F(n \wedge \tau, X_{n \wedge \tau}), \qquad n \geqq 0,$$

and

$$H_n^{\vec{X}}=H_n=M_n+\sum_{i=1}^{n} [w(i-1, X_{i-1})^{-1}-w(i, X_{i-1})^{-1}]$$
$$\cdot I(X_i>X_{i-1}, i-1<\tau), \qquad n \geqq 0,$$

where the sum is taken to be zero if $n=0$. Then M_n, $n \geqq 0$, is a nonnegative supermartingale, and H_n, $n \geqq 0$, is a nonnegative martingale. To prove this, first note that nonnegativity is immediate for M_n, and that $w(i-1,j) \leqq w(i,j)$, so that $H_n \geqq M_n$. Now put $d_n=M_n-M_{n-1}$. We will show

$$E(d_n|\mathcal{G}_{n-1}) \leqq 0, \qquad n \geqq 1. \tag{3.2}$$

On $\{n-1 \geqq \tau\}$, $d_n=0$, so it suffices to prove $E(d_n|\mathcal{G}_{n-1})=0$ on $\{n-1<\tau, X_{n-1}=j\}=A_j$, $j>0$. Now, on A_j, $M_{n-1}=\sum_{i=0}^{j-1} w(n-1, i)^{-1}$, and none of the intervals $(i, i+1)$, $i \leqq j-2$, can be crossed by $\vec{X}$ between times $n-1$ and n, so $w(n-1, i)=w(n, i)$, $0 \leqq i \leqq j-2$. Also $w(n-1, j-1)=w(n, j-1)$ on $A_j \cap \{X_n=j+1\}$. Thus, on A_j, (3.2) follows from the conditional (conditioned on $\mathcal{G}_{n-1}$) version of (3.1) i) with $Z=M_n-M_{n-1}$, $b=w(n-1, j-1)$, $a=w(n-1, j)$, and $d=w(n,j)^{-1}$. Furthermore, $E(H_n-H_{n-1}|\mathcal{G}_{n-1})=0$ on A_j by (3.1) ii), and $H_n=H_{n-1}$ on $\{n-1 \geqq \tau\}$, so that H_n, $n \geqq 0$, is a martingale. Paranthetically we observe that the decomposition $M_n=H_n+(M_n-H_n)$ is not the Doob decomposition of a supermartingale.

Now observe that $H_{n+1}-H_n=1$ on $B_n=\{X_{n+1}>X_n, n<\tau, w(n, X_n)=1\}$. Being a nonnegative martingale, H_n converges, so only a finite number of the events B_n occur. Let Γ be all intervals $(i, i+1)$, $i \geqq X_0$, such that the initial weighting of $(i, i+1)$ is 1. Then $B_n \supseteq D_n=\{n<\tau, \vec{X}$ crosses an interval in Γ for the first time between times n and $n+1\}$. Thus only a finite number of the events D_n occur, that is, only a finite number of intervals in Γ are ever crossed before τ. Since by hypothesis only a finite number of the intervals $(i, i+1)$, $i \geqq X_0$, are not in Γ, the number of distinct intervals crossed by $\vec{X}$ before τ is finite, implying the conclusion of Lemma 3.0. □

For later reference, we observe that if the reinforcement is down only then $H_n=M_n$, $n \geqq 0$, so that M_n, $n \geqq 0$ is a martingale.

Theorem 3.1. *Let $\vec{X}$ be an initially fair RRW. Then* $P(\vec{X}$ is recurrent$)+P(\vec{X}$ has finite range$)=1$.

Proof. It suffices to show that for any integer m,

(3.3) $P(X_i=m$ for infinitely many $i)+P(\vec{X}$ has finite range, $X_i=m$ for at most finitely many $i)=1$.

Since the proof is the same for all m, we do this only for $m=0$. An equivalent formulation of (3.3) for $m=0$ is

(3.4) $P(X_i=0$ for some $i\geqq n)+P(\vec{X}$ has finite range, $X_i\neq 0$ for all $i\geqq n)=1$, $n=0, 1, 2, \ldots$

We have already observed in Sect. 2 that, for fixed n, $\{X_{n+i},\ i\geqq 0,\ w(n+i,j),\ i\geqq 0,\ -\infty<j<\infty\}$ is itself a RRW.

Since the initial weights for $\vec{X}$ are by hypothesis all 1, at most n of the weights $w(n,i)$, $-\infty<i<\infty$, can be different from 1. Thus the walk X_{n+i}, $i\geqq 1$, satisfies the hypotheses of Lemma 3.0, and (3.4) follows from Lemma 3.0. □

We observe that the hypothesis that $\vec{X}$ be initially fair in Theorem 3.2 cannot be entirely dispensed with, since there are initial weights which, without any more reinforcement, give rise to transient Markov chains.

In special cases we can characterize the sample path behavior. If $\vec{a}=a_1, a_2, \ldots$ is a sequence of nonnegative numbers, put $\phi(\vec{a})=\sum_{n=1}^{\infty}\left(1+\sum_{i=1}^{n} a_i\right)^{-1}$.

Theorem 3.2. i) *If $\vec{X}$ is an initially fair sequence type RRW, with reinforcement sequence $\vec{a}$, then if $\phi(\vec{a})=\infty$, $\vec{X}$ is recurrent a.s., and if $\phi(\vec{a})<\infty$, $\vec{X}$ has finite range a.s.*

ii) *If $\vec{X}$ is an initially fair RRW with* iid *reinforcement with associated variables $Z_1, Z_2, \ldots=\vec{Z}$ then if $\phi(\vec{Z})=\infty$ a.s., $\vec{X}$ is a.s. recurrent, and if $\phi(\vec{Z})<\infty$ a.s., $\vec{X}$ has finite range a.s.*

iii) *In the finite range case of* i) *and* ii) *above, there are (random) integers N and j such that $X_i\in\{j, j+1\}$ if $i>N$.*

Proof of i). We first consider the case $\phi(\vec{a})<\infty$. Suppose, with no loss of generality, that $P(X_0=j)=1$ for some j. For $n>j$ put $T_n=\inf\{k\colon X_k=n\}$. Then at T_n, the weight of $(n-1,n)$ is $1+a_1$, the weight of $(n,n+1)$ is 1, and the weight of $(n+1,n+2)$ is also 1. Thus, conditioned on $\{T_n<\infty\}$ and on $\mathscr{F}_{T_n}$, the probability that $(X_{T_n}, X_{T_n+1}, X_{T_n+2}, \ldots)$ is the vector $(n, n+1, n, \ldots)$, with odd components n and even components $n+1$, equals the infinite product

$$\left[\frac{\alpha_0}{(1+a_1)+\alpha_0}\right]\left[\frac{\alpha_1}{1+\alpha_1}\right]\left[\frac{\alpha_2}{(1+a_1)+\alpha_2}\right]\left[\frac{\alpha_3}{1+\alpha_3}\right]\ldots=p>0,$$

where $\alpha_j=1+\sum_{i=1}^{j} a_i$ is the weight of $(n,n+1)$ when it has been crossed exactly j times, and $\alpha_0=1$. Especially $P(T_{n+2}<\infty\,|\,T_n<\infty)\leqq(1-p)$, which implies $P(T_n$

$<\infty$ for all $n)=0$, that is, $P(\sup X_i=\infty)=0$, so, by Theorem 3.1, $P(\vec{X}$ has finite range$)=1$.

Next suppose $\phi(\vec{a})=\infty$, and again that $P(X_0=j)=1$. For $n\geqq j$, define $v_0=\inf\{k>0: X_k=n\}$, and $v_{i+1}=\inf\{k>v_i: X_k=n, X_s \neq i+1, 0\leqq s\leqq k\}$, $i\geqq 1$. Then

$$\{v_0<\infty\}=\{\sup_{i\geqq 1} X_i\geqq n\},$$

and

$$\{v_i<\infty, i\geqq 0\}=\{\sup_{i\geqq 1} X_i=n, \varlimsup_{i\to\infty} X_i=n\}.$$

Now if $n>j$, at time v_i the weight of $(n, n+1)$ is 1 and the weight of $(n-1, n)$ is $1+\sum_{k=1}^{2i+1} a_k=\beta_i$, since, on $\{v_j<\infty\}$, $(n-1, n)$ is crossed exactly two times between v_{j-1} and v_j. Thus

$$P(v_{i+1}<\infty \mid v_i<\infty)=P(X_{v_i+1}=n-1 \mid v_i<\infty)=\beta_i/(1+\beta_i)$$

so

$$P(v_i<\infty, i\geqq 0)=P(v_0<\infty)\prod_{i=0}^{\infty} \beta_i/(1+\beta_i)=0.$$

This implies $P(\sup_i X_i=n, \varlimsup_{i\to\infty} X_i=n)=0$ for each $n>j$, and almost identical reasoning yields this result if $n=j$. Similarly $P(\sup_{i\geqq m} X_i=\varlimsup_{i\to\infty} X_i)=0$ for each $m\geqq 0$, so by Theorem 3.1, $\vec{X}$ is recurrent.

The proof of ii) is very similar and is omitted. The proof of iii) will be given in Sect. 5. □

Let $S_0, S_1, \ldots$ be ordinary (unreinforced) fair nearest neighbor random walk started at the positive integer μ. Let λ be an integer exceeding μ. Then it is well known that

$$P(\max_{0\leqq k<\tau} S_k\geqq\lambda)=\mu/\lambda. \tag{3.5}$$

(Recall $\tau=\inf\{k: S_k\leqq 0\}$.) This equality is in fact one of the many ways to prove recurrence of such unreinforced random walk. If M_n, $n\geqq 0$, is a nonnegative martingale started at μ, then (see [3], p. 314)

$$P(\sup_{0\leqq k<\tau} M_k\geqq\lambda)\leqq\mu/\lambda, \qquad \lambda>0.$$

For general initially fair RRW only a much weaker inequality holds.

Theorem 3.3. *There is a constant C such that if $\vec{X}$ is an initially fair RRW started at $\mu>0$ then*

$$P(\sup_{0\leqq k<\tau} X_k\geqq\lambda)\leqq C\mu/\sqrt{\lambda}, \qquad \lambda>0. \tag{3.6}$$

Furthermore, there is a constant c and an initially fair random walk started at μ *such that the analog for this walk of the probability appearing in* (3.6) *exceeds* $c\mu/\sqrt{\lambda}$ *for each* $\lambda > \mu^2$.

Proof. All constants in this proof are absolute constants not depending on λ or μ. First we prove (3.6). Let H_n, $n \geqq 0$, be as in the proof of Lemma 3.0, and let the sets B_n, D_n, $n \geqq 0$, and Γ, also be as in that proof. Note that, since the initial weights for $\vec{X}$ are all 1, Γ is all intervals $(i, i+1)$, $i \geqq \mu$, so

$$\sum_{i=0}^{\infty} I(D_i) = \sup_{0 \leqq k < \tau} X_k - \mu. \tag{3.7}$$

Since $w(n, X_n) = 1$ on D_n, $H_{n+1} - H_n \geqq 1$ on D_n, so (3.7) implies

$$\sum_{n=0}^{\infty} (H_{n+1} - H_n)^2 \geqq \sup_{0 \leqq k < \tau} X_k - \mu. \tag{3.8}$$

Put $S(H) = \left[H_0^2 + \sum_{n=0}^{\infty} (H_{n+1} - H_n)^2\right]^{\frac{1}{2}} = \left[\mu^2 + \sum_{n=0}^{\infty} (H_{n+1} - H_n)^2\right]^{\frac{1}{2}}$. Then $S(H)$ is the so-called square function of H, and by an inequality of Burkholder (Theorem 8 of [1])

$$P(S(H) > \lambda) < C_1 \sup_{n \geqq 0} E|H_n|/\lambda = C_1 \mu/\lambda, \qquad \lambda > 0, \tag{3.9}$$

the last equality since H is a nonnegative martingale, so that $E|H_n| = EH_n = EH_0 = \mu$. Using (3.8) and (3.9), we get

$$P(\mu^2 + \sup_{0 \leqq k < \tau} X_k - \mu > \lambda^2) < C_1 \mu/\lambda, \qquad \lambda > 0.$$

Thus

$$P(\sup_{0 \leqq k < \tau} X_k > \lambda^2/2) < C_1 \mu/\lambda, \qquad \lambda > 2\mu,$$

which is easily seen to imply (3.6).

We preface the formal construction of the example showing the second statement in Theorem 3.3 with a heuristic explanation of what is going on. Suppose we reinforce an interval by adding M the first time it is crossed downwards and let M get *very* large. Then essentially we are contracting that edge after it is crossed downwards. The number of uncontracted edges between 0 and X_i is just about performing simple random walk. The number of steps this walk takes before hitting zero is $\mu + 2(\sup_{i<\tau} X_i - \mu)$, and the tail probabilities of the time for simple random walk to hit zero are of order $\mu/\sqrt{\lambda}$.

Now we provide the example in detail. Let $\vec{X}$ be an initially fair matrix type RRW with matrix $[a]$ started at $\mu > 0$, such that if $m \geqq \mu$, the sequence $a_{m,i}$, $i \geqq 1$, is 0, 2^{m+1}, 0, 0, ... and if $0 < m < \mu$ this sequence is 2^{m+1}, 0, 0, ... That is, reinforce an interval the first time it is crossed downwards. Note that if $\Theta_n=\{i>0: w(n, i)>1\}$, then

$$\sum_{i\in\Theta_n} w(n, i)^{-1}<\tfrac{1}{2}. \tag{3.10}$$

Also note that, since the reinforcement is down only, the process $M_n^{\vec{X}}=M_n$, $n\geqq 0$, constructed in the proof of Lemma 3.0, is a martingale, as was noted just after the proof of that lemma.

Define $\eta_0=0$, $\eta_i=\inf\{k>\eta_{i-1}: |M_k-M_{\eta_{i-1}}|\geqq 1\}$, $i\geqq 1$. If $\eta_{i-1}<\tau$, η_i is the first time after η_{i-1} that $\vec{X}$ crosses an interval of weight 1. Note that on $F_i=\{\eta_i<\tau, M_{\eta_{i+1}}>M_{\eta_i}\}$, (3.10) implies

$$M_{\eta_{i+1}}-M_{\eta_i}=(M_{\eta_{i+1}}-M_{\eta_{i+1}-1})+(M_{\eta_{i+1}-1}-M_{\eta_i})=1+\sum_{j\in\psi} w(n,j)^{-1}<3/2, \tag{3.11}$$

where $\psi=\{j\in\Theta_{\eta_i}: j\geqq X_{\eta_i}$ and $m\in\Theta_{\eta_i}, X_{\eta_i}\leqq m\leqq j\}$.

A similar formula holds on $G_i=\{\eta_i<\tau, M_{\eta_{i+1}}<M_{\eta_i}\}$. Inequality (3.10) now implies

$$1\leqq|M_{\eta_{i+1}}-M_{\eta_i}|<3/2, \quad \text{if } \eta_i<\tau. \tag{3.12}$$

It is easily checked that $\vec{X}$, and in fact any RRW which reinforces each interval only once, cannot have finite range with positive probability, and this, together with Theorem 3.1, implies $P(\tau<\infty)=1$. Since $M_\tau-M_{\tau-1}=-1$, $\sum_{i=1}^{\infty} P(\tau=\eta_i)=1$.

Put $Q_j=M_{\eta_j\wedge\tau}$, $j\geqq 0$, and let $N=\inf\{k: \eta_k=\tau\}$. Then $Q_0, Q_1, \ldots$ is a martingale, by the optional sampling theorem, and (3.12) gives

$$1\leqq|Q_j-Q_{j+1}|<3/2, \quad j<N. \tag{3.13}$$

Now on F_i, $(X_{\eta_{i+1}-1}, X_{\eta_{i+1}-1}+1)$ is the first interval $(m, m+1)$, such that $m\notin\Theta_{\eta_i}$, and which is crossed by $\vec{X}$ after η_i. Thus, on F_i

$$\max_{0\leqq j\leqq\eta_{i+1}} X_j=1+\max_{0\leqq j\leqq\eta_i} X_j,$$

while on G_i,

$$\max_{0\leqq j\leqq\eta_{i+1}} X_j=\max_{0\leqq j\leqq\eta_i} X_j$$

so that putting $\Delta^+=\{j: Q_{j+1}>Q_j, 0\leqq j<N\}$, $\Delta^-=\{j: Q_{j+1}<Q_j, 0\leqq j<N\}$, and N^+ and N^- respectively the number of elements in Δ^+ and Δ^-, we have

$$N=N^++N^-, \tag{3.14}$$

and

$$N^+=\max_{0\leqq i\leqq\tau} X_i-\mu. \tag{3.15}$$

Let $S^2(Q)=Q_0^2+\sum_{i=1}^{N}(Q_i-Q_{i-1})^2=\mu^2+\sum_{i=1}^{N}(Q_i-Q_{i-1})^2$ be the square of the square function for the martingale Q. By (3.13), we see

(3.16) $$\mu^2+N\leqq S^2(Q)\leqq \mu^2+(3/2)^2 N.$$

Shortly, we will establish the existence of a positive constant c_1, such that

(3.17) $$P(S^2(Q)>\lambda^2)>c_1\,\mu/\lambda, \qquad \lambda>\mu.$$

Since $Q_0=\mu$ and $Q_N=0$ we have

$$-\mu=Q_N-Q_0=\sum_{i=0}^{\infty}(Q_{i+1}-Q_i)\,I(i\in\Delta^+)+\sum_{i=0}^{\infty}(Q_{i+1}-Q_i)\,I(i\in\Delta^-),$$

so (3.13), and the definition of Δ^+ and Δ^- imply

(3.18) $$\tfrac{3}{2}N^+-N^-\geqq -\mu.$$

Together with right hand side of (3.16), and (3.14), this yields

$$[\mu^2+(3/2)^2\mu]+[(3/2)^2+(3/2)^3]\,N^+\geqq S^2(Q),$$

which, together with (3.17), gives

(3.19) $$P(N^+>\lambda^2)>c_2\,\mu/\lambda, \qquad \lambda>\mu,$$

where c_2 does not depend on μ or λ. (Note that to show the existence of a c_2 for which (3.19) holds it suffices to produce constants c_3, c_4, c_5 such that $P(N^+>c_3\,\lambda^2)>c_4\,\mu/\lambda$, $\lambda>c_5\,\mu$.)

Now we prove (3.17). Very roughly, think of Q as a fair random walk, and $S^2(Q)$ as the number of steps it takes before it hits zero. Now the probability of Q getting to λ before τ is about μ/λ by gambler's ruin. Given this event, Q must get to either 0 or 2λ, and the number of steps it takes to go from λ to 0 or 2λ is on the order of λ^2. Now the details are provided. This argument is a routine application of the methods of [2]. Assume WLOG $\lambda>3\mu$. Put $\phi=\inf\{k\colon Q_k\geqq\lambda\}\wedge N$, $A=\{Q_\phi\geqq\lambda\}=\{Q_\phi\neq 0\}$. Then (3.13) implies

(3.20) $$\lambda I(A)\leqq Q_\phi\leqq(\lambda+3/2)\,I(A)<2\,\lambda I(A),$$

and since $Q_{k\wedge\phi}$, $k\geqq 0$, is a bounded martingale,

$$EQ_\phi=EQ_0=\mu,$$

so, taking expectations in (3.20) yields

(3.21) $$P(A)>\mu/2\,\lambda.$$

Now let $\xi=\inf\{k>\phi\colon |Q_k-Q_\phi|\geqq\lambda\}$. Using (3.13) again, we have

(3.22) $$\lambda I(A)\leqq|Q_\xi-Q_\phi|\,I(A)<2\,\lambda I(A).$$

Put $g_i = Q_{(\phi+i)\wedge\xi} I(A)$. Then g_i, $0 \leqq i < \infty$, is a martingale, and putting $e_i = g_{i+1} - g_i$, $i \geqq 0$, we have

$$\sum_{i=0}^{\infty} e_i = Q_\xi - Q_\phi, \quad \text{and} \quad \sum_{i=0}^{\infty} e_i^2 \leqq S^2(Q).$$

The orthogonality of e_i, $i \geqq 0$, implies

$$E\left(\sum_{i=0}^{\infty} e_i^2\right) = E\left(\sum_{i=0}^{\infty} e_i\right)^2 \geqq \lambda^2 P(A), \tag{3.23}$$

while

$$E\left[\left(\sum_{i=0}^{\infty} e_i^2\right)^2\right] \leqq c_6 E\left[\left(\sum_{i=0}^{\infty} e_i\right)^4\right] \leqq c_6 (2\lambda)^4 P(A), \tag{3.24}$$

the first inequality in (3.24) following from a result of Burkholder [1] and the second by the right hand side of (3.22). Put $Z = \sum_{i=0}^{\infty} e_i^2$. Let P_A and E_A denote conditional probability and expectation given A. Then (3.23) yields $E_A Z \geqq \lambda^2$. By (3.24), $E_A Z^2 \leqq c_7 \lambda^4$, $c_7 = 2^4 c_6$. Thus $E_A\, 2c_7 \lambda^2 Z I(Z > 2c_7 \lambda^2) \leqq E_A Z^2 \leqq c_7 \lambda^4$, so $E_A Z I(Z > 2c_7 \lambda^2) \leqq \lambda^2/2$, implying $E_A Z I(Z \leqq 2c_7 \lambda^2) \geqq \lambda^2/2$, and an easy argument now gives $P_A(Z \geqq \lambda^2/4) > c_8$, so that $P(Z \geqq \lambda^2/4) \geqq c_8 P(A) \geqq c_9 \lambda/\mu$, using (3.21) for the last inequality, completing the proof of (3.17).

Now (3.17) has been shown to imply (3.19), which, together with (3.15), shows that this example has the desired properties. □

To conclude this section we prove the following.

Proposition 3.4. *Let $\vec{X}$ be a Diaconis walk, started at* 1. *Then*

$$E \sup_{0 \leqq k < \tau} X_k \leqq 3.$$

Proof. Taking expectations on both sides of the equality in (3.2), and summing over $j > 0$, yields

$$EM_{n-1} - EM_n = -E\{[w(n, X_{n-1})^{-1} - w(n-1, X_{n-1})^{-1}]\, I(X_n > X_{n-1}, n-1 < \tau)\}.$$

Now if $X_n > \max_{0 \leqq k \leqq n-1} X_k$, we have $w(n, X_{n-1}) = 2$ and $w(n-1, X_{n-1}) = 1$ under Diaconis reinforcement, so

$$\begin{aligned} EM_{n-1} - EM_n &\geqq (1/2)\, P(n-1 < \tau, X_n > \max_{0 \leqq k \leqq n-1} X_k) \\ &= (1/2)\, EI(n-1 < \tau, X_n > \max_{0 \leqq k \leqq n-1} X_k). \end{aligned} \tag{3.25}$$

Now $\sum_{n=1}^{N}(EM_{n-1}-EM_n)=EM_0-EM_N\leqq 1$, since $EM_0=1$ and $EM_N\geqq 0$. Also $\sum_{n=1}^{\infty} I(n-1<\tau, X_n>\max_{0\leqq k\leqq n-1} X_k)=\sup_{0\leqq k<\tau} X_k-1$. Thus summing (3.25) from $n=1$ to ∞, we get

$$1\leqq (E\sup_{0\leqq k<\tau} X_k-1)/2,$$

proving the proposition. □

4. The Strong Law

In this section we prove the following theorem.

Theorem 4.0. *Let $\vec{X}$ be an initially fair RRW which is either of sequence type or* iid *reinforced. Then*

$$\lim_{n\to\infty} X_n/n=0 \text{ a.s.}$$

First note that if $P(\vec{X}$ has finite range$)=1$ then $X_n/n\to 0$ a.s. Thus, by virtue of Theorem 3.2, the proof of Theorem 4.0 will be completed upon showing its truth for $\vec{X}$ satisfying $P(\vec{X}$ is recurrent$)=1$.

A word about notation in this section. For a while we use P and E to denote probability and expectation for whatever walk we are talking about, then, to distinguish between several walks discussed in the same sentence or equation, superscripts make their appearance on P and E, and towards the end of the section we switch back to just P and E.

If k is an integer, and if $n_0, n_1, \ldots=\vec{n}$ is a recurrent sequence of integers such that $|n_i-n_{i-1}|=1$, $i>0$, we define $\tau_i(k)=\tau_i$, $i\geqq 0$, by $\tau_0=\inf\{i: X_i=k\}$, $\tau_j=\inf\{i>\tau_{j-1}: X_i=k\}$, $j\geqq 1$, and call $(n_{\tau_i+1}, n_{\tau_i+2}, \ldots, n_{\tau_{i+1}})$ the $(i+1)$st *excursion* of $\vec{n}$ from k, and classify excursions as *up* or *down* from k in the obvious way.

Let $\mathscr{D}(k)=\mathscr{D}$ stand for the collection of all vectors $(v_1, v_2, \ldots, v_n)$ of finite length which satisfy $v_1=k-1$, $v_n=k$, $v_i<k$, $i<n$, and $|v_i-v_{i-1}|=1$, $1<i\leqq n$. That is, $\mathscr{D}$ is the collection of all possible down excursions. Similarly let $\mathscr{U}(k)=\mathscr{U}$ be the collection of all possible up excursions.

For a recurrent walk $\vec{X}$, let $\vec{D}^{\vec{X},k}=\vec{D}=(\vec{D}_1, \vec{D}_2, \ldots)$ be the down excursions, in order, made by $\vec{X}$, and let $\vec{U}=(\vec{U}_1, \vec{U}_2, \ldots)$ be the up excursions. Let $\vec{S}$ be the infinite sequence, each entry either d or u, such that the j-th component of $\vec{S}$ is d or u depending on whether the j-th excursion is up or down.

The following lemma does not generalize to all (including non-matrix) RRW. The proof is somewhat long but easy.

Lemma 4.1. *Let $\vec{X}$ be a recurrent matrix type walk* ([a]), *with constant initial weighting $\vec{w}$, which starts at k. Then $\vec{X}$ is determined by $(\vec{S}, \vec{D}, \vec{U})$, and $\vec{S}$, $\vec{D}$, and $\vec{U}$ are independent.*

Proof. The first statement is immediate. Too see that $\vec{S}$ is independent of $(\vec{D}, \vec{U})$ we put $s_0 = w_{k-1}$, $s_i = w_{k-1} + \sum_{j=1}^{2i} a_{k-1,j}$, $i \geqq 1$, and $t_0 = w_k$, $t_i = w_k + \sum_{j=1}^{2i} a_{k,j}$, so that s_i is the weight of $(k-1, k)$ after this interval has been crossed exactly $2i$ times and t_i is the weight of $(k, k+1)$ after it has been crossed exactly $2i$ times. Then at τ_n, the end of the n-th excursion, if j of the first n excursions have been up and $n-j$ have been down, the weight of $(k, k+1)$ is t_j and the weight of $(k-1, k)$ is s_{n-j}, and the probability that the $n+1$st excursion is up is $t_j/(t_j + s_{n-j}) = \alpha(j, n-j)$. Thus the probability that the first n entries in $\vec{S}$ are, say, all up, given that the first n elements in $\vec{D}$, in order, are $(\vec{d}_1, \ldots, \vec{d}_n)$ and the first n elements in $\vec{U}$ are $(\vec{u}_1, \ldots, \vec{u}_n)$ is just $\prod_{j=0}^{n-1} \alpha(j, 0)$, and the probability of any other possibility for the first n entries in $\vec{S}$ could similarly be computed independently of the first n entries in $\vec{D}$ and $\vec{U}$. Thus $\vec{S}$ is independent of $(\vec{D}, \vec{U})$.

To complete the proof we will show that $\vec{U}$ is independent of $\vec{D}$ by showing that given $\vec{u}_1, \vec{u}_2, \ldots$ elements of $\mathscr{U}$ and $\vec{d}_1, \vec{d}_2, \ldots$ elements of $\mathscr{D}$, there are sequences $p_1, p_2 \ldots$ of numbers depending only on $\vec{u}_1, \vec{u}_2, \ldots$ and $q_1, q_2, \ldots$ of numbers depending on $\vec{d}_1, \vec{d}_2, \ldots$ such that if $\vec{v}$ is a vector of length n with each coordinate u or d with x of the entries u and $y = n - x$ of these entries d, and if F is the event that the first n coordinates of $\vec{S}$ are those of $\vec{v}$, then

(4.1) P(The first x up excursions are $\vec{u}_1, \ldots, \vec{u}_x$ in order, and the first y down excursions are $\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_y$, in order $|F$)

$$= \left(\prod_{i=1}^{x} p_i\right)\left(\prod_{i=1}^{y} q_i\right).$$

The numbers p_i are the probabilities a certain matrix type RRW started at $k+1$ has its first $L(\vec{u}_i)$ coordinates the coordinates of $\vec{u}_i$.

For $0 \leqq j$ let $\vec{\omega}_j = \{\omega(j, i) : i \geqq k\}$ be the weighting of the intervals $(i, i+1)$, $i \geqq k$, given as follows. Let $\psi(j, i)$ be the number of times $k\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_j(k+1)$ crosses $(i, i+1)$, where $k\vec{u}_1, \ldots, \vec{u}_j(k+1)$ stands for the finite vector starting at k, with first j excursions all up and exactly $(\vec{u}_1, \ldots, \vec{u}_j)$, and with the first step after $\vec{u}_j$ to $(k+1)$. (If $j=0$, it stands for the vector $(k, k+1)$.) Put

$$\omega(j, i) = w_i + \sum_{s=1}^{\psi(j,i)} a_{i,s}, \quad j \geqq 1, \ \omega(0, i) = w_i,$$

where the sum is to be taken as zero if $\psi(j, i) = 0$. Let $[b^j]$ be a matrix with entries

$$b^j_{i,s} = a_{i, s+\psi(j,i)}, \quad k \leqq i, \ s \geqq 1.$$

(What the other entries of $[b^j]$ are is irrelevant.) For $j \geqq 1$ let p_j be the probability that a matrix walk with initial weight $\vec{\omega}_{j-1}$ and matrix $[b^{j-1}]$, started at $k+1$, has first n_j coordinates equal to $\vec{u}_j$, if n_j is the number of components of $\vec{u}_j$ (that is, the first n_j states visited by the walk are the coordinates of $\vec{u}_j$). Then if N is a positive integer, given that j of the first N coordinates in $\vec{S}$ are u

and $N-j$ are d, and given that $(\vec{U}_1, \ldots, \vec{U}_j)=(\vec{u}_1, \ldots, \vec{u}_j)$ and that $(\vec{D}_1, \ldots, \vec{D}_{N-j})=(\vec{d}_1, \ldots, \vec{d}_{N-j})$, and also given that the $N+1$st entry in $\vec{S}$ is u, that is, given $X_{\tau_N+1}=k+1$, the probability that the $N+1$st excursion is $\vec{u}_{j+1}$ is p_{j+1}.

This conditioning is on an atom of $\mathscr{F}_{\tau_N+1}$. We have discussed such conditioning towards the end of Sect. 2. Especially, we indicated it was a matrix type walk. The $w(j, i)$ and $b^j_{i,s}$ given above are the initial weights of this walk and those matrix entries of the walk which have bearing on the probabilities in question. The numbers q_i are defined similarly, and similarly the probability of the j-th downward excursion being $\vec{d}_j$, given that the first $j-1$ excursions are, in order, $\vec{d}_1, \ldots, \vec{d}_{j-1}$, can be computed to be q_j regardless of what the upward excursions before the j-th downward excursion are. This establishes (4.1). □

Lemma 4.2. *Let $\vec{X}$ satisfy the conditions of Lemma* 4.1.

i) *The distribution of $\vec{U}$ depends only on $a_{j,i}$, $k+1\leqq j$, $0<i<\infty$, w_i, $i\geqq k$, and $\sum_{s=1}^{2i+1} a_{k,s}$, $0\leqq i<\infty$.*

ii) *The distribution of $\vec{D}$ depends only on $a_{j,i}$, $j<k$, $0<i<\infty$ and w_i, $i<k$.*

Proof. We use the notation of Lemma 4.1. Part i) follows from the fact that $\psi(i, k)$ is always an odd number, so that $\omega(i, k)$, equals $w_k+\sum_{s=1}^{2i+1} a_{k,s}$, $i\geqq 0$, while the other quantities involved in the definition of the probabilities p_j depend only on $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_j)$, w_i, $i\geqq k$, and $a_{j,i}$, $k+1\leqq j<\infty$, $1\leqq i<\infty$.

Part ii) is similar, but simpler; or, we could solve it by reflecting $\vec{X}$ about k and using i). □

Let $m>k$. If $\vec{u}=(\xi_1, \ldots, \xi_n)\in\mathscr{U}$, let $\vec{u}^m=(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_\theta})$, $\theta\leqq n$, be those entries ξ_i of $\vec{u}$, in order, which satisfy either $\xi_i\in[k, m)$ or $\xi_i=m$ and $\xi_{i-1}<m$. Then $L(\vec{u}^m)$ is the number of jumps made by $k\vec{u}$ between two adjacent integers in $[k, m]$. Similarly define, for $\vec{d}=(\gamma_1, \gamma_2, \ldots, \gamma_n)\in\mathscr{D}$ and $\lambda<k$, $\vec{d}^\lambda=(\gamma_1, \ldots, \gamma_{i_s})$, where the entries are those components of $\vec{d}$ either in $(\lambda, k]$ or else equal to λ and with immediate predecessor greater than λ. Let $\mathscr{U}^m=\{\vec{u}^m\colon \vec{u}\in\mathscr{U}\}$ and $\mathscr{D}^\lambda=\{\vec{d}^\lambda\colon \vec{d}\in\mathscr{D}\}$. Note that if a vector in $\mathscr{U}^m$ has i-th component m, the next component must be $m-1$. Thus the distribution of $\vec{U}^m=(\vec{U}^m_1, \vec{U}^m_2, \ldots)$ does not involve w_i, $i\geqq m$ or $a_{j,i}$, $j\geqq m$, $i>0$, and so the following lemma holds for essentially the same reasons as Lemma 4.2.

Lemma 4.3. *Let $m>k$, and let $\vec{X}$ satisfy the conditions of Lemma* 4.1.

i) *The distribution of $\vec{U}^m$ depends only on w_i, $k\leqq i<m$, $a_{j,i}$, $k<j<m$, $i>0$, and $\sum_{s=1}^{2i+1} a_{k,s}$, $i\geqq 0$.*

ii) *Let $\lambda<k$. The distribution of $\vec{D}^\lambda$ depends only on w_i, $\lambda\leqq i<k$, and $a_{j,i}$, $\lambda\leqq j<k$, $i>0$.* □

The proof of the following lemma is virtually identical to that of Theorem 3.2, and is omitted. For the matrix $[a]$, put $\vec{a}_j=\{a_{j,i}, i\geqq 1\}$, $-\infty<j<\infty$.

Lemma 4.4. i) *If $\vec{X}$ is a matrix type walk with matrix $[a]$ and $P(\vec{X}$ is recurrent$)=1$ then $\phi(\vec{a}_j)=\infty, -\infty<j<\infty$.*

ii) *If $\vec{X}$ is a matrix type walk with matrix $[a]$, satisfying $\phi(\vec{a}_j)=\infty, -\infty<j<\infty$, and if all but a finite number of the initial weights are 1, then $\vec{X}$ is recurrent.* □

Now let $0<k$ and let $M>k$. Define

$$T_M=\sum_{i=0}^{\infty} I(X_i\in(0, M], X_{i+1}\in[0, M], i<\tau),$$

so that T_M is the number of jumps $\vec{X}$ makes in $[0, M]$ before τ. Divide these jumps into those made in $[k, M]$ and those made in $[0, k]$ by putting

$$T_{M,k}^{+}=\sum_{i=0}^{\infty} I(X_i\in[k, M], X_{i+1}\in[k, M], i<\tau)$$

and

$$T_{M,k}^{-}=\sum_{i=0}^{\infty} I(X_i\in[0, k], X_{i+1}\in[0, k], i<\tau).$$

Define a matrix $[a'^{,k}]=[a']$ associated with the matrix $[a]$ by $a'_{j,i}=a_{j,i}$, $j\neq k$, $i>0$, $a'_{k,1}=a_{k,1}$, $a'_{k,2i}=0$, $i\geqq 1$, $a'_{k,2i+1}=a_{k,2i}+a_{k,2i+1}$, $i\geqq 1$.

Lemma 4.5. *Let $k>0$, and $M>k$. Let P and E be probability and expectation associated with the recurrent matrix type walk, started at k, with matrix $[a]$ and initial weights $\vec{w}=w_i$, $-\infty<i<\infty$, all but a finite number of which are 1. Let P' and E' be associated with the walk with matrix $[a']$, all other conditions the same. Then*

$$E\,T_M\geqq E'\,T_M. \tag{4.2}$$

Proof. Our straightforward argument will in fact show

$$P(T_M\geqq y\,|\,\vec{D}, \vec{U})\geqq P'(T_M\geqq y\,|\,\vec{D}, \vec{U}), \tag{4.3}$$

which, since $(\vec{D}, \vec{U})$ has the same distribution under both P and P', using Lemma 4.3, implies (4.2). Let γ_n be the number of u appearing before the n-th d in $\vec{S}$. We note that γ_n is independent of $(\vec{D}, \vec{U})$, and that if $\Gamma=\inf\{i: \vec{D}_i$ has a zero entry$\}$, so that τ occurs in the Γ-th down excursion from k, then, given $(\vec{D}, \vec{U})$, the distribution of T_M is determined by the distribution of γ_Γ. In fact, given $(\vec{D}, \vec{U})$,

$$T_M=\text{const.}+\sum_{i=1}^{\gamma_\Gamma} \theta_i, \tag{4.4}$$

where θ_i is $L(\vec{U}_i^M)$ and the sum is taken to be zero if γ_Γ equals zero, and the constant is determined by $\vec{D}$. Thus to prove (4.3) it suffices to show

$$P(\gamma_\Gamma \geqq j \mid \vec{D}, \vec{U}) \geqq P'(\gamma_\Gamma \geqq j \mid \vec{D}, \vec{U}), \tag{4.5}$$

which is implied by

$$P(\gamma_n \geqq j \mid \vec{D}, \vec{U}, \Gamma = n) \geqq P'(\gamma_n \geqq j \mid \vec{D}, \vec{U}, \Gamma = n) \tag{4.6}$$

since Γ is determined by $\vec{D}$. Now γ_n is determined by $\vec{S}$, which by Lemma 4.1 is independent of $(\vec{D}, \vec{U})$ and thus $(\vec{D}, \vec{U}, \Gamma)$, so to prove (4.6) it suffices to prove

$$P(\gamma_n \geqq j) \geqq P'(\gamma_n \geqq j), \quad j \geqq 0, \ n \geqq 1. \tag{4.7}$$

We prove (4.7) by induction on n. Recall the definitions of s_i, t_i, and $\alpha(j, n-j)$ made in the proof of Lemma 4.1, and let s_i', t_i', $\alpha'(j, n-j)$ be the analogous quantities for $[a']$. Now $s_i = s_i'$, $i \geqq 1$, but $t_i \geqq t_i'$, $i \geqq 1$, so that $\alpha(j, n-j) \geqq \alpha'(j, n-j)$, $n \geqq 0$, $0 \leqq j \leqq n$. That is, given the first n entries in $\vec{S}$, the probability that the $n+1$st entry is u under P is always greater than or equal to the probability of this event under P'. Especially since $\{\gamma_1 \geqq j\}$ is the set where the first j entries of $\vec{S}$ are u, we have $P(\gamma_1 \geqq j) \geqq P'(\gamma_1 \geqq j)$. Now suppose (4.7) holds for $n = m \geqq 1$. We will show it also holds for $n = m+1$. We first note, for $\delta \geqq 0$, $z \geqq 0$,

$$\begin{aligned} P(\gamma_{m+1} \geqq \delta + z \mid \gamma_m = \delta) &= \prod_{i=0}^{z-1} \alpha(\delta + i, m) \\ &\geqq \prod_{i=0}^{z-1} \alpha'(\delta + i, m) \\ &= P'(\gamma_{m+1} \geqq \delta + z \mid \gamma_m = \delta). \end{aligned} \tag{4.8}$$

Furthermore, since $\alpha(j, m) \leqq \alpha(j+1, m)$, $j \geqq 0$, $m \geqq 0$, we have

$$P(\gamma_{m+1} \geqq \delta + z \mid \gamma_m = \delta) \leqq P(\gamma_{m+1} \geqq \delta + z \mid \gamma_m = \delta + 1). \tag{4.9}$$

Thus

$$\begin{aligned} P(\gamma_{m+1} \geqq y) &= \sum_{i=0}^{\infty} P(\gamma_{m+1} \geqq y \mid \gamma_m = i)\, P(\gamma_m = i) \\ &\geqq \sum_{i=0}^{\infty} P(\gamma_{m+1} \geqq y \mid \gamma_m = i)\, P'(\gamma_m = i) \\ &\geqq \sum_{i=0}^{\infty} P'(\gamma_{m+1} \geqq y \mid \gamma_m = i)\, P'(\gamma_m = i) \\ &= P'(\gamma_{m+1} \geqq y), \end{aligned}$$

the second inequality following from (4.8), and the first from (4.9) and the induction hypothesis in the following manner: Put $e_i = P(\gamma_{m+1} \geqq y \mid \gamma_m = i)$. Then by (4.9), e_i is increasing in i. Thus

$$\begin{aligned}\sum_{i=0}^{\infty} e_i P(\gamma_m = i) &= \sum_{i=1}^{\infty} (e_i - e_{i-1}) P(\gamma_m \geqq i) + e_0 P(\gamma_m \geqq 0) \\ &\geqq \sum_{i=1}^{\infty} (e_i - e_{i-1}) P'(\gamma_m \geqq i) + e_0 P'(\gamma_m \geqq 0) \\ &= \sum_{i=0}^{\infty} e_i P'(\gamma_m = i).\end{aligned}$$

This completes the proof of (4.7) and thus Lemma 4.5. □

Define the matrix $[a'']$ associated with $[a]$ by $a''_{j,i} = a_{j,i}$ if $j \neq k$ and by $a''_{k,2i-1} = 0$, $i \geqq 1$, $a''_{k,2i} = a_{k,2i-1} + a_{k,2i}$, $i \geqq 1$.

Lemma 4.6. *Let $0 < k < M$. Let P^1 and E^1 be probability and expectation associated with a recurrent matrix type walk, started at M, with matrix $[a]$ and initial weights $\vec{w}$, all of which, except perhaps a finite number, equal* 1. *Let P^2 and E^2 be associated with the walk in which everything is the same except that $[a'']$ replaces $[a]$. Then*

$$E^1 T_M \geqq E^2 T_M.$$

Proof. Let $v = \inf\{i: X_i = k\}$. The distribution of $(\vec{X}_{i \wedge v}, i \geqq 0, w(j, i \wedge v), -\infty < j < \infty, i \geqq 0)$ is identical under P^1 and P^2, since the distribution of these variables involves only quantities the same under P^1 and P^2. Let A be an atom of $\mathscr{F}_v$, of the form $\{X_0 = i_0, \ldots, X_n = i_n, v = n\}$. Of course $i_0 = M$ and $i_n = k$ here. Then as previously remarked, conditioned on A, $\{X_{n+i}, i \geqq 0, w(j, n+i), i \geqq 0, -\infty < j < \infty\}$ is a matrix type walk, and under both P^1 and P^2 it has the same initial weights, and starts at k. Especially note $w(n, k) = w(0, k) + a_{k,1}$ under both P^1 and P^2. Now if $[b]$ and $[b']$ are the respective matrices for these walks, we have $b_{j,i} = b'_{j,i}$, $j \neq k$, $i \geqq 1$, while $b_{k,i} = a_{k,i+1}$ and $b'_{k,i} = a''_{k,i+1} = b_{k,i} + b_{k,i-1}$ if i is odd, $= 0$ if i is even. Thus Lemma 4.6 follows from Lemma 4.5. □

The next lemma is an easy consequence of Lemma 4.6.

Lemma 4.7. *Let P and E be probability and expectation associated with an initially fair matrix type walk started at $M > 0$. There is an initially fair matrix type walk with up only reinforcement such that, if P' and E' are probability and expectation of this walk stated at M,*

$$E T_M \geqq E' T_M.$$

Proof. By Lemma 4.3 ii), only the entries of the matrices corresponding to $(i, i+1)$, $0 \leqq i < M$, have bearing on this. Thus the result follows from Lemma 4.6, upon changing the M rows of the matrix corresponding to each of these intervals, one at a time. □

For $\vec{z} = z_0, z_1, \ldots$, a sequence of real numbers and (a, b) an interval, put $n_1 = \inf\{k: z_k \leqq a\}$, $m_1 = \inf\{k \geqq n_1: z_k \geqq b\}$, and for $i > 1$, $n_i = \inf\{k > m_{i-1}: z_k \leqq a\}$, $m_i = \inf\{k > n_i: z_k \geqq b\}$, and let $u_{\vec{z}}(a, b) = \sup\{i: m_i < \infty\}$ $(\sup \phi = 0)$. Then $u_{\vec{z}}(a, b)$ is called the number of upcrossings of (a, b) by $\vec{z}$. The basic idea of the following lemma goes back to Neveu [7]. See Dubins [4].

Lemma 4.8. *Let γ and $0 < a < b$ be real numbers. Let $\vec{Z} = Z_0, Z_1, \ldots$ be a sequence of integrable random variables, which are all bounded below by the same constant, and put $\mathscr{H}_n = \sigma(Z_i, 0 \leqq i \leqq n)$. Suppose*

i) $P(Z_0 = \gamma) = 1$,

ii) $P(Z_i < a, Z_{i+1} > a) = 0$, $i \geqq 0$, *and* $P(Z_i > a, Z_{i+1} < a) = 0$, $i \geqq 0$,

iii) $P(Z_i < b, Z_{i+1} > b) = 0$, $i \geqq 0$, *and* $P(Z_i > b, Z_{i+1} < b) = 0$, $i \geqq 0$,

iv) $E(Z_{n+1} | \mathscr{H}_n) I(Z_n < b) = Z_n I(Z_n < b)$, $n \geqq 0$,

v) $P(Z_n \in (a, b)) = 0$, $n \geqq 0$, *and*

vi) $\lim_{n \to \infty} Z_n = Z_\infty$ *exists, and* $Z_\infty < a$.

Then $E u_{\vec{Z}}(a, b) = [(\gamma \wedge a) - E Z_\infty]/(b - a)$.

Proof. Suppose first that $\gamma \leqq a$. Let

$$v_1 = \inf\{k: Z_k = a\} \qquad (\inf \phi = \infty)$$

$$\eta_1 = \inf\{k > v_1: Z_k = b\},$$

$$v_i = \inf\{k \geqq \eta_{i-1}: Z_k = a\}, \qquad i \geqq 1,$$

$$\eta_i = \inf\{k \geqq v_i: Z_k = b\}, \qquad i \geqq 1.$$

By property ii), $\{v_1 < \infty\} = \{Z_i \geqq a \text{ for some } i\}$. Let M be a positive integer. Now $\int_{\{v_1 \leqq M\}} Z_{v_1 \wedge M} = a P(v_1 \leqq M)$, and since $Z_{k \wedge v_1}$, $k \geqq 0$, is a martingale, by property iv) and the definition of v_1, we have $\int Z_{v_1 \wedge M} = \int Z_0 = \gamma$. Thus

$$\int_{\{v_1 > M\}} Z_M = \gamma - a P(v_1 \leqq M). \tag{4.10}$$

For $i \geqq 1$, $Z_{v_i} = a$ on $\{v_i \leqq M\}$, and $Z_{(v_i + k) \wedge \eta_i} I(v_i \leqq M)$, $k \geqq 0$, is a bounded martingale, so that

$$\int_{\{v_i \leqq M\}} Z_{M \wedge \eta_i} = \int_{\{v_i \leqq M\}} Z_{M \wedge v_i} = a P(v_i \leqq M).$$

Also, since $Z_{\eta_i} = b$ on $\{\eta_i < \infty\}$, $\int_{\{\eta_i \leqq M\}} Z_{M \wedge \eta_i} = b P(\eta_i \leqq M)$, yielding

$$\int_{\{v_i \leqq M, \eta_i > M\}} Z_{M \wedge \eta_i} = a P(v_i \leqq M) - b P(\eta_i \leqq M). \tag{4.11}$$

Now $\{v_1 > M\} \cup \bigcup_{i=1}^{\infty} \{v_i \leqq M, \eta_i > M\} = \{Z_M \leqq a\}$, using property v). Thus adding (4.10) and (4.11) we get

$$\int_{\{Z_M \leqq a\}} Z_M = \gamma + (a-b) \sum_{i=1}^{\infty} P(\eta_i \leqq M) - a \sum_{i=1}^{\infty} [P(\eta_i \leqq M) - P(v_{i+1} \leqq M)],$$

and now letting M approach infinity yields, with the aid of vi), the bounded below property of Z_i, $i \geqq 0$, and the fact the second sum above equals $P(Z_M \geqq b)$, which goes to 0 as $M \to \infty$, the equality $\int Z_\infty = \gamma + (a-b) \sum_{i=1}^{\infty} P(\eta_i < \infty) = \gamma + (a-b) E u_{\vec{Z}}(a, b)$.

If $\gamma \geqq a$, let $\xi = \inf\{j: X_j = a\}$, and apply the result above to the process $X_{\xi+i}$, $i \geqq 0$. □

Lemma 4.9. *Let P^0 and E^0 be associated with an initially fair recurrent RRW of matrix type, started at $M \geqq 1$. There is an absolute positive constant C such that*

$$E^0 \tau \geqq C M^{\frac{3}{2}}.$$

Proof. We will actually show $E^0 T_M \geqq C M^{\frac{3}{2}}$. Invoking Lemma 4.7, we assume with no loss of generality that $\vec{X}$ under P^0 has up only reinforcement on $(i, i+1)$, $1 \leqq i < M$. Let u_i be short for $u_{\{X_{j \wedge \tau}, 0 \leqq j < \infty\}}(i, i+1)$. Put $U = \sum_{i=1}^{M-1} u_i$. Then $U < T_M \leqq \tau$, and we will prove

$$E^0 U \geqq C M^{\frac{3}{2}}.$$

Let $[a^0]$ be the matrix corresponding to P^0, and for $1 \leqq n \leqq M-1$ let $[a^n]$ be the matrix which satisfies $a^n_{i,j} = a^0_{i,j}$ if $i \notin [M-n, M-1]$, and $a^n_{i,j} = 0$ if $M-n \leqq i < M$. Let P^n and E^n be probability and expectation associated with the initially fair RRW with matrix $[a^n]$, started at M.

Define

$$\begin{aligned} f^n(i,j) &= 0 \quad \text{if } j = M-n \\ &= -\sum_{\alpha=j}^{M-n-1} w(i,\alpha)^{-1}, \quad 0 \leqq j < M-n \\ &= \sum_{\alpha=M-n}^{j-1} w(i,\alpha)^{-1}, \qquad M-n < j < \infty. \end{aligned}$$

Note $f^n(i,j) = F(i, M-n) - F(i,j)$, where F is as in the proof of Lemma 3.0. Let $Q^n_i = f^n(i \wedge \tau, X_{i \wedge \tau})$, $i \geqq 0$. Then under P^n, Q^n_i, $i \geqq 0$, satisfies the conditions i)–vi) required of the process Z in the statement of Lemma 4.8, with (a, b) in this statement replaced by any of the intervals $(\lambda, \lambda+1)$, λ an integer, $0 \leqq \lambda < n$. This is immediate except for condition iv). The proof of iv) follows with reasoning similar to that which led to the comment after the proof of Lemma 3.1, since under P^n the reinforcement of $(i, i+1)$, $0<i\leqq M-1$, is up only, and there is no reinforcement of the interval $(i, i+1)$ if $M-n\leqq i<M$.

The behavior of U for P^{M-1} is particularly easy to analyze, since under P^{M-1} none of the intervals $(i, i+1)$, $0\leqq i\leqq M-1$, are reinforced before τ, so that U has exactly the same distribution under P^{M-1} that it would for (unreinforced) fair random walk started at M. We have $P^{M-1}(Q_0^{M-1}=M-1)=1$ and $P^{M-1}(Q_\infty^{M-1}=-1)=1$, and $Q_i^{M-1}=X_i-1$ if $i<\tau$. Thus Lemma 4.8 implies

$$E^{M-1}u_i=i, \quad 1\leqq i\leqq M-1,$$

and so

$$E^{M-1}U=\sum_{i=1}^{M-1} i=M(M-1)/2. \tag{4.12}$$

Now put $W_n=w(\tau, M-n)^{-1}$, $1\leqq n\leqq M-1$. We will prove, for $1\leqq n\leqq M-1$, $\varepsilon>0$,

$$E^{n-1}u_i=E^n u_i, \quad i<M-n, \tag{4.13}$$

$$E^n u_i-E^{n-1}u_i=E^{n-1}(1-W_n), \quad i>M-n, \tag{4.14}$$

and

$$E^{n-1}u_{M-n}-E^n u_{M-n}\geqq(\varepsilon^{-1}-1)\,P^{n-1}(W_n\leqq\varepsilon). \tag{4.15}$$

To prove (4.13) we first note that the distribution of $(w(\tau, i), 0\leqq i<M-n)$ is the same under both P^n and P^{n-1}, and use Lemma 4.2 ii), applied to the process $X_\gamma, X_{\gamma+1}, \ldots$ conditioned on an atom in $\mathscr{F}_\gamma$, where $\gamma=\inf\{i\colon X_i=M-n\}$, in the same manner as the stopping time v was used in the proof of Lemma 4.6.

This same observation gives the first step in the proof of (4.14), namely

$$E^n\left(\sum_{i=0}^{M-n-1} w(\tau, i)^{-1}\right)=E^{n-1}\left(\sum_{i=0}^{M-n-1} w(\tau, i)^{-1}\right)\stackrel{\text{def}}{=}\delta_n. \tag{4.16}$$

Since $Q_\tau^n=-\sum_{i=0}^{M-n-1} w(\tau, i)^{-1}$,

$$E^n Q_\tau^n=-\delta_n$$

so for $i>M-n$, Lemma 4.8 implies

$$E^n u_i=[i-(M-n)]+\delta_n. \tag{4.17}$$

Now $Q_\tau^{n-1}=-\sum_{i=0}^{M-n} w(\tau, i)^{-1}$, so

$$E^{n-1}Q_\tau^{n-1}=-\delta_n-E^{n-1}w(\tau, M-n)^{-1}=-\delta_n-E^{n-1}W_n,$$

so for $i>M-n$, Lemma 4.8 gives

$$E^{n-1}u_i=[i-(M-(n-1))]+\delta_n+E^{n-1}W_n, \tag{4.18}$$

which, together with (4.17), gives (4.14).

Next we prove (4.15). Put $\psi^n=\sum_{0\leqq s<M-n} w(\tau, s)^{-1}$. Then it follows from the discussion of the next to last paragraph that

$$P^n(\psi^n\geqq t)=P^{n-1}(\psi^n\geqq t), \quad 0\leqq t<\infty. \tag{4.19}$$

Furthermore, $w(\tau, 0)=1$, since reinforcement is up only, so we have

$$P^{n-1}(\psi^n\geqq 1)=1. \tag{4.20}$$

In addition, recalling that $W_n=w(\tau, M-n)^{-1}$, we have

$$Q^n_\tau=-\psi^n, \quad Q^{n-1}_\tau=-\psi^n-W_n. \tag{4.21}$$

Now under P^n, $Q^n_{i\wedge\tau}$, $0\leqq i<\infty$, upcrosses (0, 1) exactly when $X_{i\wedge\tau}$, $0\leqq i<\infty$, upcrosses $(M-n, M-n+1)$, so that, by Lemma 4.8, and (4.21), we have

$$E^n u_{M-n}=E^n\psi^n. \tag{4.22}$$

To estimate $E^{n-1}u_{M-n}$, put

$$\gamma_1=\inf\{i: X_i=M-n\},$$

$$\xi_1=\inf\{i>\gamma_1: X_i=M-n+1\},$$

$$\gamma_i=\inf\{i>\xi_{i-1}: X_i=M-n\} \quad i\geqq 1,$$

and

$$\xi_i=\inf\{i>\gamma_i: X_i=M-n+1\}, \quad i\geqq 1.$$

Let $\delta_i=\left(1+\sum_{k=1}^{2i-1} a_{M-n,k}\right)^{-1}=w(\gamma_i, M-n)^{-1}$, $i\geqq 1$. Then under P^{n-1}, $Q^{n-1}_{\gamma_i}=-\delta_i$ on $\{\gamma_i<\tau\}$, and $Q^{n-1}_{\xi_i}=0$ on $\{\xi_i<\tau\}$. Put $A_i=\{\xi_i<\tau\}$, $B_i=\{\gamma_i<\tau<\xi_i\}$. We have

$$u_{M-n}=\sum_{i=1}^{\infty} I(A_i). \tag{4.23}$$

For $i\geqq 1$, put

$$g^i_j=Q^{n-1}_{(\gamma_i+j)\wedge\xi_i\wedge\tau}I(\gamma_i<\tau), \quad 0\leqq j<\infty, \; g^i=\{g^i_j, j\geqq 0\}.$$

Then, under P^{n-1}, g^i is a martingale which is bounded above by 0 and below by $-M$. Furthermore, under P^{n-1}

$$g_0^i = -\delta_i I(\gamma_i < \tau),$$

while

$$\begin{aligned} g^i_{\xi_i \wedge \tau} &= 0 && \text{on } A_i \\ &= -\delta_i - \psi^n && \text{on } B_i. \end{aligned}$$

Now $A_i \cup B_i = \{\gamma_i < \tau\}$. Since g_i is a martingale under P^{n-1},

$$E^{n-1} g_0^i = E^{n-1} g^i_{\xi_i \wedge \tau} I(\gamma_i < \tau),$$

so that

$$-\delta_i P^{n-1}(\gamma_i < \tau) = \int_{B_i} (-\delta_i - \psi^n)\, dP^{n-1} = -\delta_i P^{n-1}(B_i) - \int_{B_i} \psi^n\, dP^{n-1},$$

yielding

$$\delta_i P^{n-1}(A_i) = \int_{B_i} \psi^n\, dP^{n-1}.$$

Especially,

$$P^{n-1}(A_i) \geqq \varepsilon^{-1} \int_{B_i} \psi^n\, dP^{n-1} \text{ if } \delta_i^{-1} \geqq \varepsilon^{-1}.$$

Thus if $m = \inf\{i\colon \delta_i^{-1} \geqq \varepsilon^{-1}\}$.

$$\sum_{i=m}^{\infty} P^{n-1}(A_i) \geqq \varepsilon^{-1} \int_{\bigcup_{i=m}^{\infty} B_i} \psi^n\, dP^{n-1} = \varepsilon^{-1} \int_{\{\gamma_m < \tau\}} \psi^n\, dP^{n-1},$$

while, noting $\delta_i \leqq 1$, $i \geqq 1$, and $\{\xi_i < \tau\} = \{\gamma_{i+1} < \tau\}$, we have

$$\sum_{i=1}^{m-1} P^{n-1}(A_i) \geqq \int_{\bigcup_{i=1}^{m-1} B_i} \psi^n\, dP^{n-1} = \int_{\{\gamma_m > \tau\}} \psi^n\, dP^{n-1}.$$

Together with the previous inequality and the fact $\psi^n \geqq w(0, \tau) \geqq 1$, this gives

$$\begin{aligned} \sum_{i=1}^{\infty} P^{n-1}(A_i) &\geqq E^{n-1} \psi^n + (\varepsilon^{-1} - 1) \int_{\{\gamma_m < \tau\}} \psi^n\, dP^{n-1} \\ &\geqq E^{n-1} \psi^n + (\varepsilon^{-1} - 1) P^{n-1}(\gamma_m < \tau). \end{aligned}$$

Since $\{\gamma_m < \tau\} = \{W_n \leqq \varepsilon\}$, this together with (4.22) and (4.23) completes the proof of (4.15).

Now we conclude the proof of the lemma. Using (4.13)–(4.15), we have

$$\begin{aligned} E^{n-1} U - E^n U &= E^{n-1} u_{M-n} - E^n u_{M-n} + \sum_{i=M-n+1}^{M-1} E^{n-1}(W_n - 1) \\ &\geqq (\varepsilon^{-1} - 1) P^{n-1}(W_n \leqq \varepsilon) + (n-1) E^{n-1}(W_n - 1). \end{aligned}$$

Using (4.12), we have

$$(4.24)\qquad E^0 U = \sum_{n=1}^{M-1} (E^{n-1} U - E^n U) + E^{M-1} U$$

$$\geqq (\varepsilon^{-1} - 1) \sum_{n=1}^{M-1} P^{n-1}(W_n \leqq \varepsilon) + \sum_{n=1}^{M-1} (n-1) E^{n-1} W_n$$

$$- \sum_{n=1}^{M-1} (n-1) + M(M-1)/2$$

$$\geqq \varepsilon^{-1} \sum_{n=1}^{M-1} P^{n-1}(W_n \leqq \varepsilon) + \sum_{n=1}^{M-1} (n-1) E^{n-1} W_n.$$

Now let $M \geqq 12$. Then if $[\,]$ (not be confused with a matrix) denotes the greatest integer function, we have $M/4 \leqq [M/3] \leqq (M-1)/2$, so that either $[M/3]$ of the probabilities $P^{n-1}(W_n \leqq \varepsilon)$ exceed 1/2 or $[M/3]$ of the probabilities $P^{n-1}(W_n \geqq \varepsilon)$ exceed 1/2. In the first case,

$$\varepsilon^{-1} \sum_{n=1}^{M-1} P^{n-1}(W_n \leqq \varepsilon) \geqq \varepsilon^{-1} M/8, \qquad \text{if } M \geqq 12,$$

while in the second $E^{n-1} W_n \geqq \varepsilon/2$ for at least $M/4$ integers n, and so, for $M \geqq 12$

$$\sum_{n=1}^{M-1} (n-1) E^{n-1} W_n \geqq \sum_{n=1}^{[M/3]} (n-1)\, \varepsilon/2$$

$$\geqq (M/4 - 1)(M/4)\, \varepsilon/4 \geqq (M/6)(M/4)\, \varepsilon/4.$$

Thus taking $\varepsilon = M^{-\frac{1}{2}}$, we get

$$E^0 U \geqq M^{3/2}/96, \qquad M \geqq 12,$$

which, together with the fact that $E^0 U > 1$, finishes the proof. □

Now we complete the proof of Theorem 4.0. Recall that just after the statement of this theorem we observed that it is trivial except in the case that $\vec{X}$ is recurrent, an assumption we make from now on. Let $\varepsilon > 0$ and pick $M = M(\varepsilon)$ so large that $CM^{3/2} > M/\varepsilon$, where C is as in the statement of the last lemma. Suppose with no loss of generality that $P(X_0 = 0) = 1$. Let $v_k = \inf\{i \colon X_i = -kM\}$, $0 \leqq k < \infty$, and put

$$\phi_k = \sum_{i=v_k}^{v_{k+1}-1} I(-kM \leqq X_i, X_{i+1} \leqq -(k+1)M).$$

Then $v_k \geqq \sum_{i=0}^{k-1} \phi_i$.

First, suppose $\vec{X}$ is initially fair sequence type with associated sequence $\vec{a}$. Then the distribution of ϕ_i conditioned on $\mathscr{F}_{v_{i-1}}$ is exactly the distribution of ϕ_1, for any $i \geqq 0$. This follows from Lemma 4.3 ii), together with the comment about conditioning made in Sect. 2. Thus ϕ_i, $i \geqq 1$, are iid. Furthermore Lemma 4.9 and our choice of M give $Ev_1 > M/\varepsilon$. Thus $\varliminf_{k\to\infty} v_k/k > M/\varepsilon$, so $v_k > Mk/\varepsilon$ for all but finitely many k, that is $X_i > -Mk$, $0 \leqq i < Mk/\varepsilon$ for all but finitely many k, which implies $\varliminf_{i\to\infty} X_i/i \geqq -\varepsilon$. Similarly $\varlimsup_{i\to\infty} X_i/i \leqq \varepsilon$. This proves Theorem 4.0 for sequence type walk.

Now we treat the case of iid reinforcement in Theorem 4.0. Let μ be a distribution on $[0, \infty)$ such that the initially fair walk with iid reinforcement in which the reinforcing variables have distribution μ is recurrent. Another way to construct a RRW with the same distribution is as follows. Let $[Z] = Z_{j,i}$, $-\infty < j < \infty$, $1 \leqq i < \infty$, be iid variables each with distribution μ. The initially fair RRW which reinforces $(j, j+1)$ by $Z_{j,i}$ the i-th time it is crossed has exactly the same distribution as the original walk, so it suffices to show that this walk satisfies the strong law. Let P and E be probability and expectation associated with this walk, started at 0. Now conditioned on $[Z] = [r]$, where r is a matrix of positive numbers for which this conditioning makes sense, the walk under P is an initially fair matrix type walk, and so the last lemma implies

$$E(\phi_k | [Z]) \geqq CM^{3/2}.$$

Thus $E\phi_k \geqq CM^{3/2}$. It is easily checked that the ϕ_k are iid and thus the same proof used in the previous paragraph can be used to prove the strong law here. □

5. Appendix: Herman Rubin's Generalized Polya Urn Theorem, the Proof of Theorem 3.2 iii), and Two Open Problems

In the classical Polya urn, an urn contains both red and white balls, one is drawn at random and replaced together with another ball of the same color, and this procedure is repeated indefinitely. It is easy to show that, with probability one, infinitely many balls of each color are drawn, regardless of the initial distribution. Here is a much tougher problem: Is it still true that infinitely many balls of each color are drawn probability 1, if now the k-th time a red (white) ball is drawn it is replaced together with k additional red (white) balls? This question is, as will be seen later, related to the proof of Theorem 3.2 iii). The proof is, as we mentioned earlier, due to Herman Rubin. Such urn models have been studied by learning theorists (see Luce [6]) but to our knowledge questions of this type have not been addressed.

To state Rubin's theorem in the generality necessary, we disregard urns and just give a rule for generating an infinite sequence of letters, each r or w. Let $\vec{r} = (r_0, r_1, \ldots)$ and $\vec{w} = (w_0, w_1, \ldots)$ be two sequences of nonnegative numbers such that $r_0>0$ and $w_0>0$. Put $R_k=\sum_{i=0}^{k} r_i$ and $W_k=\sum_{i=0}^{k} w_i$. The first entry of the infinite sequence is r with probability $R_0/(R_0+W_0)$, w with probability $W_0/(R_0+W_0)$. Given the first n entries consist of x r's and $y=n-x$ w's, in a given order, the probability that the $n+1$st entry is r equals $R_x/(R_x+W_y)$, and the probability it is w equals $W_y/(R_x+W_y)$. A sequence so generated will be called a generalized Polya sequence (corresponding to $\vec{r}$ and $\vec{w}$). Let $p_r=P$(all but finitely many elements of the sequence are red) and $p_w=P$(all but finitely many elements of the sequence are white). Put $\phi(\vec{r})=\sum_{i=0}^{\infty} R_i^{-1}$, $\phi(\vec{w})=\sum_{i=0}^{\infty} W_i^{-1}$.

Rubin's Theorem. i) *If* $\phi(\vec{r})<\infty$ *and* $\phi(\vec{w})<\infty$ *then* $p_r>0$, $p_w>0$, *and* $p_r+p_w=1$.

ii) *If* $\phi(\vec{r})<\infty$ *and* $\phi(\vec{w})=\infty$, $p_r=1$.

iii) *If* $\phi(\vec{r})=\infty$ *and* $\phi(\vec{w})=\infty$, *both* p_r *and* p_w *equal* 0.

Proof. Let $Y_0, Y_1, \ldots$ be independent exponential random variables such that $EY_i=R_i^{-1}$. Let $Z_0, Z_1, \ldots$ be independent exponential random variables which are also independent of the sequence Y_i, $i\geqq 0$, and such that $EZ_i=W_i^{-1}$. Put $A=\left\{\sum_{i=0}^{k} Y_i, k\geqq 0\right\}$, $B=\left\{\sum_{i=1}^{k} Z_i, k\geqq 0\right\}$, and $G=A\cup B$. Let ξ_1 be the smallest number in G, and in general let ξ_i be the i-th smallest number in G. Define a random sequence of r's and w's, called the random variable sequence, by making the k-th element of the sequence r if $\xi_k\in A$, w if $\xi_k\in B$.

The sequence just constructed above has exactly the same distribution as the generalized Polya sequence corresponding to $\vec{r}$ and $\vec{w}$. The proof of this relies on the lack of memory property of the exponential as well as the fact that if U and V are independent exponentials with expectations u and v, respectively, $P(U<V)=u^{-1}/(u^{-1}+v^{-1})$ and $P(V<U)=v^{-1}/(u^{-1}+v^{-1})$. Thus the probability that the first entry in the random variable sequence is r is given by $P(\xi_1\in A)=P(Y_0<Z_0)=R_0/(R_0+W_0)$, as it should be, that is, agreeing with the probabilities defining the generalized Polya sequence. Instead of giving a proof that the conditional probabilities are also what they should be, to avoid complicated notation we will just treat a representative case, by calculating the probability that the fourth component of the random variable sequence is r given $H=\{$the first three components are $rwr\}=\{\xi_1\in A, \xi_2\in B, \xi_3\in A\}$. On H, the distance α from ξ_3 to the smallest element of A greater than ξ_3 is Y_2, and conditioned on H, α has the distribution of Y_2. On H, the distance β from ξ_3 to the smallest element of B greater than ξ_3 is $Z_1+\xi_2-\xi_3=Z_1+Z_0-(Y_1+Y_0)$, and the lack of memory property of Z_1 implies that, conditioned on H, β has the distribution of Z_1, noting $H=\{Y_0<Z_0<Y_0+Y_1<Z_0+Z_1\}$, so that even given H and the values of Y_0, Y_1, Z_0, β still has the distribution of Z_1. The independence of the random variables $\{Y_i, Z_i; i\geqq 0\}$ guarantees that α and β are conditionally independent given H. Thus

$$P(\xi_4\in A\,|\,H)=P(\alpha<\beta)=R_2/(R_2+W_1).$$

This agrees with the relevant conditional probability for the generalized Polya sequence, and completes our justification that the random variable sequence has the same distribution as the generalized Polya sequence.

The rest of the proof is almost immediate. We note $P\left(\sum_{i=0}^{\infty} Y_i < \infty\right) = 1$ if $\sum_{i=0}^{\infty} R_i^{-1} < \infty$, $P\left(\sum_{i=0}^{\infty} Y_i = \infty\right) = 1$ if $\sum_{i=0}^{\infty} R_i^{-1} = \infty$, and that if $\sum_{i=0}^{\infty} R_i^{-1} < \infty$, $\sum_{i=0}^{\infty} Y_i$ has a density which is positive on $(0, \infty)$. Analogs hold for $\sum_{i=0}^{\infty} Z_i$. Finally, we note that $p_r = P\left(\sum_{i=0}^{\infty} Y_i < \sum_{i=0}^{\infty} Z_i\right)$, and $p_w = P\left(\sum_{i=0}^{\infty} Z_i < \sum_{i=0}^{\infty} Y_i\right)$, and it is easy to use these, together with the remarks just made, to finish the proof. □

Proof of theorem 3.2 iii). Let j be an integer, and let $T_1 = \inf\{k \geqq 0: X_k = j\}$, and $T_i = \inf\{k > T_{i-1}: X_k = j\}$, $i > 1$. Write down a sequence of r's and w's by making the i-th entry r if $X_{T_i+1} = j+1$, and making the i-th entry w if $X_{T_i+1} = j-1$. This may be a finite sequence, but the probability we generate an infinite sequence of r and w with an infinite number of both r and w appearing is less than or equal to the probability the same event occurs in a generalized Polya sequence, with the corresponding $\vec{r}$ and $\vec{w}$ depending slightly on whether X_0 equals j, exceeds j, or is smaller than j. For example, if $X_0 = j$, both r_0 and w_0 equal 1, and both r_i and w_i equal $1 + \sum_{s=1}^{2i} a_{j,s}$, $i \geqq 1$. Essentially this observation was the basis of Diaconis' approach to Diaconis reinforcement. Now under the hypothesis of Theorem 3.2 iii), $\phi(\vec{a}) < \infty$, which implies that $\phi(\vec{r}) < \infty$ and $\phi(\vec{w}) < \infty$ for the $\vec{r}$ and $\vec{w}$ corresponding to the urn that would yield our sequences of r's and w's. Now part i) of Rubin's Theorem gives that $p_r + p_w = 1$, so the probability of infinitely many r and infinitely many w equals zero, which translates in our situation to the statement that, with probability one, both $j-1$ and $j+1$ are not visited infinitely often. This, together with the fact that $\vec{X}$ has finite range with probability one, completes the proof of Theorem 3.2 iii). □

Finally, we mention two questions we have been unable to solve. The first is whether every reinforced random walk $\vec{X}$, as defined in Sect. 2, satisfies $\lim_{n \to \infty} X_n/n = 0$. The second is to decide if the two dimensional analog, on the standard two dimensional lattice, of the reinforced random walk described in the fourth paragraph of the first section is recurrent. (For this walk each line segment of length 1 connecting lattice points (i, j) initially has weight 1, and the weights determine the jump probabilities, so the first jump our walk makes is equally likely to be in any of the four directions. The first time a segment is crossed its weight increases to 2, and this is never increased further.) The same question was asked by Diaconis for the analog of Diaconis reinforcement on the lattice. Our question, which involves a simpler reinforcement scheme, should be easier than Diaconis', which is also unsettled, but we cannot handle it. Of course, our conjecture is that the walk is recurrent.

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