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Probability Theory and Related Fields

White noise driven SPDEs with reflection*

C. Donati-Martin and E. Pardoux

Mathématiques, URA 225, Université de Provence, 3, place Victor Hugo, F-13331 Marseille Cedex 3, France

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Summary. We study reflected solutions of a nonlinear heat equation on the spatial interval [0, 1] with Dirichlet boundary conditions, driven by space–time white noise. The nonlinearity appears both in the drift and in the diffusion coefficient. Roughly speaking, at any point (t, x) where the solution $u(t, x)$ is strictly positive it obeys the equation, and at a point (t, x) where $u(t, x)$ is zero we add a force in order to prevent it from becoming negative. This can be viewed as an extension both of one-dimensional SDEs reflected at 0, and of deterministic variational inequalities. Existence of a minimal solution is proved. The construction uses a penalization argument, a new existence theorem for SPDEs whose coefficients depend on the past of the solution, and a comparison theorem for solutions of white–noise driven SPDEs.

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0 Introduction

We want to study reflected solutions of parabolic SPDEs driven by a space time white noise. More precisely, we are looking for a continuous random field $\{u(x, t), 0 \leqq x \leqq 1, t \geqq 0\}$ which is a solution of an SPDE at any point (x, t) where $u(x, t)$ is strictly positive and which is constrained to be non negative everywhere. Furthermore, we require that the force needed to keep u non negative is minimal.

There is a vast literature both on reflected solutions of (deterministic) PDEs, which are usually called "variational inequalities" and are motivated by applications in stochastic optimal stopping time problems (see Bensoussan and Lions [1]) and in mechanics (see Duvaut and Lions [2]), and on reflected solutions of finite dimensional SDEs (see e.g. Saisho [11]). The problem we are interested in in this paper may be considered as a combination of the two above ones. In a sense, it is

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a variational inequality where one of the coefficients is taken to be a space–time white noise.

The same problem has been studied by Haussmann and Pardoux [4] in the case of a SPDE driven by a Wiener process with nuclear covariance. On the other hand, in the particular case of a constant diffusion coefficient, Nualart and Pardoux [8] proved the existence and uniqueness of a solution to a reflected SPDE driven by a white noise.

As in [4] and [8], we shall construct a solution as the limit of a sequence $\{u^\varepsilon\}$ of solutions to penalized equations. In order to establish the monoticity of the approximating sequence, we use a comparison theorem for solutions of parabolic SPDEs with different drift functions.

Let us explain our problem. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and W a space–time white noise on $[0,1]\times\mathbb{R}_+$ i.e. $\{W(A), A\in\mathscr{B}([0,1]\times\mathbb{R}_+)\}$ is a centered gaussian process defined on $(\Omega, \mathscr{F}, P)$ whose covariance function is given by $E[W(A)W(B)] = \lambda(A\cap B)$ where λ denotes the Lebesgue measure on $[0,1]\times\mathbb{R}_+$ and $\mathscr{B}(R)$ denotes the Borel field of subsets of the topological space R. f and σ are coefficients satisfying assumptions to be specified in Sect. 1. Set $\mathscr{F}_t = \sigma\{W(A), A\in\mathscr{B}([0,1]\times[0,t])\}\vee\mathscr{N}$, where $\mathscr{N}$ is the class of P-null sets of $\mathscr{F}$.

Suppose that u_0 is a positive continuous function on $[0,1]$ which vanishes at 0 and 1. We are looking for a pair (u,η) such that:

(i) u is a continuous process on $[0,1]\times\mathbb{R}_+$; $u(x,t)$ is $\mathscr{F}_t$ measurable and $u(x,t)\geqq 0$ a.s.

(ii) η is a random measure on $(0,1)\times\mathbb{R}_+$ such that

a) $\eta((0,1)\times\{t\}) = 0, \quad \forall t\geqq 0$;

b) $\int_0^t\int_0^1 x(1-x)\eta(dx,ds) < \infty$, $t\geqq 0$.

c) η is adapted in the sense that for any measurable mapping ψ: $[0,1]\times\mathbb{R}_+\to\mathbb{R}_+$,

$$\int_0^t\int_0^1 \psi(x,s)\eta(dx,ds) \text{ is } \mathscr{F}_t \text{ measurable .}$$

(iii) (u,η) solves the parabolic SPDE:

$$\left.\begin{aligned}&\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t)) = \sigma(u(x,t))\dot W(x,t) + \eta(x,t)\\ &u(\cdot,0) = u_0;\quad u(0,t) = u(1,t) = 0\end{aligned}\right\} \tag{1}$$

in the following sense ($(\cdot,\cdot)$ denotes the scalar product in $L^2[0,1]$):

$\forall t\in\mathbb{R}_+$, $\varphi\in C^2([0,1])$ with $\varphi(0)=\varphi(1)=0$,

$$\begin{aligned}&(u(t),\varphi) - \int_0^t (u(s),\varphi'')ds + \int_0^t (f(u(s)),\varphi)ds = (u_0,\varphi)\\ &+ \int_0^t\int_0^1 \varphi(x)\sigma(u(x,s))W(dx,ds) + \int_0^t\int_0^1 \varphi(x)\eta(dx,ds) \quad \text{a.s.}\end{aligned} \tag{2}$$

(iv) $\int_Q u\,d\eta = 0$,

where $Q = (0,1)\times\mathbb{R}_+$.

Note that $C^2([0,1])$ denotes the restriction to $[0,1]$ of functions $f \in C^2(\mathbb{R})$. The stochastic integral in (2) is an Itô type integral (it is a particular case of an integral with respect to a martingale measure developed in Walsh [13]).

We say that the reflected parabolic equation (RPE) has a solution (u, η) if the pair (u, η) satisfies (i), (ii), (iii) and (iv).

The organization of this paper is as follows: our assumptions are stated in Sect. 1; in Sect. 2, we prove a comparison theorem for solutions of parabolic SPDEs driven by a space–time white noise, which is analogous to results obtained independently by other authors. Kotelenez[1] [5] and Shiga [12] prove a similar result using different approximations. By a discretization procedure, Mueller [7] obtains a comparison theorem for a slightly different SPDE when the initial condition varies. We nevertheless include a complete proof, since the result we need is not exactly contained in these references, and we believe that our proof, which uses an approximation by an SPDE driven by a finite dimensional Wiener process and some techniques from [9], is interesting in itself. Indeed, comparison theorems are rather easily proved for SPDEs driven by a Wiener process with nuclear covariance, since we can use Itô calculus to analyse the solutions of such equations, see [9, 10]. In Sect. 3, we prove an existence and uniqueness theorem for an SPDE whose coefficients depend on the whole past of the solution, which is needed in Sect. 4, which is devoted to the construction of a solution (u, η) by a penalization procedure using the results of Sect. 2. In Sect. 5, we prove the minimality of the above solution. The uniqueness problem remains open; classical methods based on the Itô formula to prove uniqueness of reflection problems seem to be useless in our context. Finally, an Appendix is devoted to the proof of some technical lemmas which are needed in Sect. 4.

Notations. Let $Q = (0,1) \times \mathbb{R}_+$ and for any $T > 0$ $Q_T = (0,1) \times (0,T)$, $\bar{Q}_T = [0,1] \times [0,T]$, $H = L^2(0,1)$, $V = \{u \in H^1(0,1), u(0) = u(1) = 0\}$ where $H^1(0,1)$ denotes the usual Sobolev space of absolutely continuous functions defined on $(0,1)$ whose derivative belongs to $L^2(0,1)$, and $A = -\dfrac{\partial^2}{\partial x^2}$.

1 Assumptions on the coefficients

The coefficients f and σ will be $\mathscr{B}([0,1]) \otimes \mathscr{P} \otimes \mathscr{B}(\mathbb{R})$ measurable functions from $[0,1] \times \Omega \times \mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{R}$, where $\mathscr{P}$ denotes the σ-field of $\mathscr{F}_t$-progressively measurable subsets of $\Omega \times \mathbb{R}_+$. When we shall say that f and σ are *locally Lipschitz*, we shall mean that

$$f(x, \omega, t; z) = f_1(x, \omega, t) + f_2(x, \omega, t; z)$$

with $f_1 \in \bigcap_{T>0} L^2((0,1) \times \Omega \times (0,T); dx \times dP \times dt)$, $f_2(x, \omega, t; 0) \equiv 0$; for each $T, M > 0$, there exists $c(T, M)$ such that

$$|f_2(x, \omega, t; z) - f_2(x, \omega, t; r)| + |\sigma(x, \omega, t; z) - \sigma(x, \omega, t; r)| \leq c(T, M)|z - r|$$

[1] As far as we know, this author was the first one to circulate a preprint containing such a comparison result

for all $(x, \omega, t) \in (0, 1) \times \Omega \times (0, T)$, $z, r \in [-M, M]$; and moreover for each $T > 0$ there exists $\bar{c}(T)$ such that

$$|f_2(x, \omega, t; z)| + |\sigma(x, \omega, t; z)| \leqq \bar{c}(T)(1 + |z|)$$

for all $(x, \omega, t, z) \in (0, 1) \times \Omega \times (0, T) \times \mathbb{R}$. Note that locally Lipschitz really means locally Lipschitz with at most linear growth.

When we shall say that f and σ are *globally Lipschitz*, we shall mean that the above assumption is satisfied with a constant $c(T, M)$ depending only on T, and not on M.

We shall omit the variable ω and moreover we shall write $f(u(x, t))$ (resp. $\sigma(u(x, t))$) instead of $f(x, t; u(x, t))$ (resp. $\sigma(x, t; u(x, t))$). $f(u_t)$ (resp. $\sigma(u_t)$) will stand for the mapping $x \to f(x, t; u(x, t))$ (resp. $x \to \sigma(x, t; u(x, t))$).

Let f and σ be globally Lipschitz, u_0 be a continuous function on $[0, 1]$ which vanishes at 0 and 1. We consider the parabolic equation:

$$\left.\begin{aligned} &\frac{\partial u(x, t)}{\partial t} - \frac{\partial^2 u(x, t)}{\partial x^2} + f(u(x, t)) = \sigma(u(x, t))\dot{W}(x, t) \\ &u(\cdot, 0) = u_0; \quad u(0, t) = u(1, t) = 0 . \end{aligned}\right\} \tag{3}$$

Several authors, including Walsh [13], have shown that (3) has a unique continuous solution, in the sense that u is the unique continuous adapted process which satisfies:

$$\forall t \in \mathbb{R}_+, \varphi \in C^2([0, 1]) \text{ with } \varphi(0) = \varphi(1) = 0,$$

$$(u(t), \varphi) + \int_0^t (u(s), A\varphi)ds + \int_0^t (f(u(s)), \varphi)ds = (u_0, \varphi)$$
$$+ \int_0^t \int_0^1 \varphi(x)\sigma(u(x, s))\, W(dx, ds) \quad \text{a.s.},$$

or equivalently $u(x, t)$ satisfies the integral equation

$$u(x, t) = \int_0^1 u_0(y) G_t(x, y) dy - \int_0^t \int_0^1 f(u(y, s)) G_{t-s}(x, y) dy\, ds$$
$$+ \int_0^t \int_0^1 \sigma(u(y, s)) G_{t-s}(x, y)\, W(dy, ds) ,$$

where G is the Green's function associated to the operator $\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions.

It follows from standard localization arguments that the same existence and uniqueness result holds in the case of locally Lipschitz coefficients.

We shall prove our results under the locally Lipschitz assumption. Using a localization argument, it will be sufficient to do some of the proofs under the globally Lipschitz assumption. Note that, using the results in Gyöngy and Pardoux [3], the results of this paper can be extended to more general drift coefficients f, which could be the sum of a locally Lipschitz and an increasing function of its third argument z.

2 A comparison theorem for solutions of parabolic SPDEs

The aim of this section is to prove a comparison theorem for white noise driven SPDEs, which will be used in Sect. 4.

Theorem 2.1 *Let the two pairs of coefficients f, σ and g, σ be globally Lipschitz, with $f \geqq g$. We denote by u (resp. v) the solution of* (3) *corresponding to f (resp. g) with the same initial condition. Then,* a.s., *for all $(x, t) \in [0, 1] \times \mathbb{R}_+$, $u(x, t) \leqq v(x, t)$.*

Proof. Let (e_k) be an orthonormal basis of H such that $\| e_k \|_\infty \leqq C$ for all $k \in \mathbb{N}$. We denote

$$W_t^k = \int_0^t \int_0^1 e_k(x) W(dx, ds) .$$

$(W^k)_{k \in \mathbb{N}}$ is a family of mutually independent Brownian motions. For $n \geqq 1$, let B^n be the H-valued Wiener process defined by

$$B_t^n = \sum_{k=1}^n W_t^k e_k .$$

Let u^n be the unique adapted solution in $L^2(\Omega \times (0, T); V)$ of the evolution equation

$$\left.\begin{aligned} &du_t^n + Au_t^n dt + f(u_t^n)dt = \Sigma(u_t^n)dB_t^n \\ &u(\cdot, 0) = u_0 \end{aligned}\right\} \tag{4}$$

where Σ is the operator of multiplication by σ i.e. for any $u \in H$, $\Sigma(u)$ is the element of $\mathscr{L}(L^\infty(0, 1); H)$ defined by:

$$\Sigma(u)(h)(x) = \sigma(u(x))h(x) , \quad h \in H .$$

For existence, uniqueness and properties of the solution of (4), see for example Pardoux [9] or [10].

Step 1 Comparison of solutions of (4).

We denote by v^n the solution of (4) with f replaced by g, and $\omega^n = u^n - v^n$. Let $p \in \mathbb{N}^*$,

$$\psi_p : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto \begin{cases} 0 & \text{for} \quad x \leqq 0 \\ 2px & \qquad x \in \left[0, \dfrac{1}{p}\right] \\ 2 & \qquad x \geqq \dfrac{1}{p} , \end{cases}$$

$$\varphi_p(x) = \mathbf{1}_{\{x \geqq 0\}} \int_0^x dy \int_0^y dz \psi_p(z) .$$

Then

- φ_p is a C^2 function on $\mathbb{R}$,
- $0 \leqq \varphi'_p(x) \leqq 2x^+; 0 \leqq \varphi''_p(x) \leqq 2.1_{x \geqq 0}$,
- $\varphi_p(x) \nearrow (x^+)^2$, $x \in \mathbb{R}$, as $p \to \infty$.

We define

$$\Phi_p: H \to \mathbb{R}$$

$$h \mapsto \int_0^1 \varphi_p(h(x))dx .$$

Φ_p is twice Fréchet differentiable; $\Phi'_p(h) \in \mathscr{L}(H, \mathbb{R})$ is given by

$$(\Phi'_p(h), k) = \int_0^1 \varphi'_p(h(x))k(x)dx .$$

$\Phi''_p(h)$ is a symmetric continuous bilinear form on $H \times H$ given by

$$\Phi''_p(h)(k_1, k_2) = \int_0^1 \varphi''_p(h(x))k_1(x)k_2(x)dx .$$

The Itô formula (cf. [9, p. 62] or [10, Part 1, Theorem 3.2]) implies

$$\Phi_p(w_t^n) + \int_0^t \langle Aw_s^n, \varphi'_p(w_s^n)\rangle ds + \int_0^t (\varphi'_p(w_s^n), f(u_s^n) - g(v_s^n))\, ds$$

$$= \sum_{k=0}^n \int_0^t (\varphi'_p(w_s^n), [\sigma(u_s^n) - \sigma(v_s^n)]e_k)dW_s^k$$

$$+ \frac{1}{2}\int_0^t \mathrm{Tr}(\Phi''_p(w_s^n)d\langle\!\langle M^n \rangle\!\rangle_s) \tag{5}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between V and $V' \simeq H^{-1}(0, 1)$, $M_s^n = \int_0^t (\Sigma(u_s^n) - \Sigma(v_s^n))dB_s^n$ is an H valued martingale and $\langle\!\langle M^n \rangle\!\rangle$ is the unique continuous process with values in the space of nuclear operators such that $(M_t^n, h)(M_t^n, k) - (\langle\!\langle M^n \rangle\!\rangle_t h, k)$ is a martingale for all $h, k \in H$.

$$\int_0^t \mathrm{Tr}(\Phi''_p(w_s^n)d\langle\!\langle M^n \rangle\!\rangle_s)$$

$$= \sum_{k=0}^n \int_0^t (\varphi''_p(w_s^n)[\sigma(u_s^n) - \sigma(v_s^n)]e_k, [\sigma(u_s^n) - \sigma(v_s^n)]e_k)ds$$

$$(\varphi'_p(w_s^n), Aw_s^n) = \int_0^1 \varphi''_p(w_s^n)\left(\frac{d}{dx}w_s^n(x, s)\right)^2 dx \geqq 0$$

$$(\varphi'_p(w_s^n), f(u_s^n) - g(v_s^n)) = (\varphi'_p(w_s^n), f(u_s^n) - f(v_s^n)) + (\varphi'_p(w_s^n), f(v_s^n) - g(v_s^n)) .$$

The last term in the right-hand side of the above equation is positive by assumption ($f \geqq g$). Taking expectation in (5) yields:

$$E[\Phi_p(w_t^n)] + E\left[\int_0^t (\varphi_p'(w_s^n), f(u_s^n) - f(v_s^n))\,ds\right]$$
$$\leqq \frac{1}{2}\sum_{k=0}^{n} E\left[\int_0^t\int_0^1 \varphi_p''(w^n(x,s))[\sigma(u^n(x,s)) - \sigma(v^n(x,s))]^2 e_k^2(x)\,dx\,ds\right].$$

Using properties of φ_p', φ_p'' and e_k, we obtain

$$E[\Phi_p(w_t^n)] \leqq (2K + nK^2C^2)E\left[\int_0^t ((w_s^n)^+, (w_s^n)^+)\,ds\right] \tag{6}$$

where K is a Lipschitz constant for f and σ.

Let $\Phi(h) = \int_0^1 (h^+(x))^2 dx$; *then* $\Phi_p(w_t^n) \nearrow \Phi(w_t^n)$ a.s., as $p \to \infty$.
From (6), $E[\Phi(w_t^n)] \leqq K' E[\int_0^t \Phi(w_s^n) ds]$.
Hence from Gronwall's lemma, $E[\Phi(w_t^n)] = 0$ for $t \geqq 0$.
By continuity of w^n,

$$\text{a.s.,}\quad \forall (x,t) \in [0,1] \times \mathbb{R}_+,\ w^n(x,t) \leqq 0$$

i.e.

$$\text{a.s.,}\quad \forall (x,t) \in [0,1] \times \mathbb{R}_+,\ u^n(x,t) \leqq v^n(x,t)\,. \tag{7}$$

Step 2 Convergence of u^n (resp. v^n) to u (resp. v).

The solution u^n of (4) satisfies the integral equation

$$u^n(x,t) = \int_0^1 u_0(y)G_t(x,y)dy - \int_0^t\int_0^1 f(u^n(y,s))G_{t-s}(x,y)\,dy\,ds$$
$$+ \sum_{k=0}^{n}\int_0^t\left(\int_0^1 \sigma(u^n(y,s))G_{t-s}(x,y)e_k(y)dy\right)dW_s^k\,.$$

Lemma 2.1 *Let* $p \geqq 1$, u *(resp.* u^n*) be the solution of* (3) *(resp.* (4)*), for any* $T > 0$,

$$\sup_{(x,t)\in \bar{Q}_T} E[|u(x,t) - u^n(x,t)|^p] \xrightarrow[n\to\infty]{} 0\,. \tag{8}$$

Proof.

$$u(x,t) - u^n(x,t) = A_n(x,t) + B_n(x,t) + C_n(x,t)$$

where

$$A_n(x,t) = \int_0^t\int_0^1 [f(u^n(y,s)) - f(u(y,s))]G_{t-s}(x,y)\,dy\,ds$$

$$B_n(x,t) = \sum_{k=0}^{n}\int_0^t\left(\int_0^1 [\sigma(u(y,s)) - \sigma(u^n(y,s))]G_{t-s}(x,y)e_k(y)dy\right)dW_s^k$$

$$C_n(x,t) = \int_0^t\int_0^1 (\Psi_{x,t}(y,s) - \Psi_{x,t}^n(y,s))\,W(dy\,ds)$$

with

$$\Psi_{x,t}(y,s) = \sigma(u(y,s))G_{t-s}(x,y)$$

$$\Psi^n_{x,t}(y,s) = \sum_{k=0}^{n}\left(\int_0^1 \sigma(u(z,s))G_{t-s}(x,z)e_k(z)dz\right)e_k(y)\,.$$

We use the following estimate on the Green's function (see [13]): let $0 < r < 3$, then

$$\sup_{(x,t)\in\bar{Q}_T}\int_0^t\int_0^1 G_s^r(x,y)dy\,ds < \infty\,. \tag{9}$$

Let $p > 6$, $q = \dfrac{p}{p-1}$, and set $F_n(t) = \sup_{x\in[0,1]} E[|u(x,t) - u^n(x,t)|^p]$.

- $E[|A_n(x,t)|^p] \leqq C\left(\int_0^t\int_0^1 G^q_{t-s}(x,y)dy\,ds\right)^{\frac{p}{q}} E\left[\int_0^t\int_0^1 |u(x,t) - u^n(x,t)|^p dy\,ds\right].$

  By (9),

$$E[|A_n(x,t)|^p] \leqq C\int_0^t F_n(s)ds\,. \tag{10}$$

- $E[|B_n(x,t)|^p]$

$$\leqq C_pE\left[\left\{\int_0^t\sum_{k=0}^{n}\left(\int_0^1[\sigma(u(y,s)) - \sigma(u^n(y,s))]G_{t-s}(x,y)e_k(y)\,dy\right)^2 ds\right\}^{p/2}\right]$$

$$\sum_{k=0}^{n}\left\{\int_0^1[\sigma(u(y,s)) - \sigma(u^n(y,s))]G_{t-s}(x,y)e_k(y)dy\right\}^2$$

$$= \sum_{k=0}^{n}([\sigma(u(\cdot,s)) - \sigma(u^n(\cdot,s))]G_{t-s}(x,\cdot), e_k)^2_H$$

$$\leqq |[\sigma(u(\cdot,s)) - \sigma(u^n(\cdot,s))]G_{t-s}(x,\cdot)|^2_H$$

$$E[|B_n(x,t)|^p]$$

$$\leqq C_pE\left[\left\{\int_0^t\int_0^1[\sigma(u(y,s)) - \sigma(u^n(y,s))]^2G^2_{t-s}(x,y)dy\,ds\right\}^{p/2}\right]$$

$$\leqq C_p\left(\int_0^t\int_0^1 G^{2q'}_{t-s}(x,y)dy\,ds\right)^{\frac{1}{2q'}} E\left[\int_0^t\int_0^1|\sigma(u(y,s)) - \sigma(u^n(y,s))|^p dy\,ds\right]$$

where $q' = \dfrac{p/2}{p/2-1}$, $2q' < 3$ since $p > 6$. By (9),

$$E[|B_n(x,t)|^p] \leqq C\int_0^t F_n(s)ds\,. \tag{11}$$

- $$E[|C_n(x,t)|^p] \leqq C_p E\left[\left\{\int_0^t \int_0^1 (\Psi_{x,t}(y,s) - \Psi^n_{x,t}(y,s))^2\, dy\, ds\right\}^{p/2}\right]$$

$$\Psi^n_{x,t}(\cdot, s) = \sum_{k=0}^{n} (\Psi_{x,t}(\cdot, s), e_k)_H e_k$$

$$\int_0^1 (\Psi_{x,t}(y,s) - \Psi^n_{x,t}(y,s))^2 dy = |\Psi_{x,t}(\cdot, s) - \Psi^n_{x,t}(\cdot, s)|^2_H \underset{n\to\infty}{\searrow} 0 \quad \text{a.s.}\,.$$

Moreover, $|\Psi_{x,t}(\cdot, s) - \Psi^n_{x,t}(\cdot, s)|^2_H \leqq |\Psi_{x,t}(\cdot, s)|^2_H$, and

$$E\left[\left\{\int_0^t |\Psi_{x,t}(\cdot, s)|^2_H\right\}^{p/2}\right] < \infty\,.$$

By the dominated convergence theorem,

$$E[|C_n(x,t)|^p] \underset{n\to\infty}{\longrightarrow} 0\,. \tag{12}$$

Set $\gamma_n(x,t) = E[\{\int_0^t \int_0^1 (\Psi_{x,t}(y,s) - \Psi^n_{x,t}(y,s))^2 dy\, ds\}^{p/2}]$. γ_n is a sequence of continuous functions on the compact $\bar{Q}_T$ which converges pointwise to the function 0. Moreover, γ_n is a decreasing sequence; by Dini's theorem, (γ_n) converges uniformly to 0. The convergence in (12) is therefore uniform in x. Let $\varepsilon > 0$, there exists N such that:

$$n \geqq N \Rightarrow \sup_x E[|C_n(x,t)|^p] \leqq \varepsilon\,. \tag{13}$$

Now, using (10), (11), (13); for $n \geqq N$,

$$F_n(t) \leqq C_p \int_0^t F_n(s) ds + C_p \varepsilon\,.$$

By Gronwall's lemma, $\lim_{n\to\infty} \sup_{t \leqq T} F_n(t) = 0$.

Step 3

By Lemma 2.1, for all $(x,t) \in [0,1] \times \mathbb{R}_+$, there exists a sequence n_k such that

$$u(x,t) = \lim_{k\to\infty} u^{n_k}(x,t) \quad \text{a.s.}$$

$$v(x,t) = \lim_{k\to\infty} v^{n_k}(x,t) \quad \text{a.s.}$$

By step 1, $u^{n_k}(x,t) \leqq v^{n_k}(x,t)$ a.s., so the same inequality holds for u and v. By continuity of u and v, a.s.,

$$\forall (x,t) \in [0,1] \times \mathbb{R}_+,\ u(x,t) \leqq v(x,t)\,.$$

This ends the proof of the theorem. □

It follows from a standard localization procedure that Theorem 2.1 remains true if we replace the global Lipschitz conditions by local Lipschitz conditions. Indeed, if that would not be the case, then there would exist $R, T > 0$ such that:

$$P\left(\sup_{0\leqq x\leqq 1, 0\leqq t\leqq T} u(x,t) - v(x,t) > 0;\ \sup_{0\leqq x\leqq 1, 0\leqq t\leqq T} |u(x,t)| \vee |v(x,t)| \leqq R\right) > 0$$

and the result would not be true for any pair of globally Lipschitz coefficients which agree with f, σ on $[0,1]\times[0,T]\times[-R,R]$, which contradicts the theorem.

3 An existence and uniqueness result for SPDEs with coefficients depending on the past of the solution

In this section, we prove an existence and uniqueness result for an SPDE whose coefficients are allowed to depend on the whole past of the solution. Note that this kind of equation has already been considered by Manthey and Stieve in [6], but the assumptions there are a bit too strong for our purpose.

For any $u\in C([0,1]\times\mathbb{R}_+)$, $t\geqq 0$, we denote by u^t the element of $C([0,1]\times[0,t])$ which is the restriction of u to $[0,1]\times[0,t]$.

In this section, we are given two mappings:

$$f,\sigma\colon [0,1]\times\mathbb{R}_+\times C([0,1]\times\mathbb{R}_+)\to\mathbb{R}$$

such that:

(i) For any $u, v\in C([0,1]\times\mathbb{R}_+)$, $(x,t)\in[0,1]\times\mathbb{R}_+$ such that $u^t=v^t$,

$$f(t,x,u)=f(t,x,v)$$

$$\sigma(t,x,u)=\sigma(t,x,v)\,.$$

(ii) For any $T, M>0$, there exists $C(T,M)$ such that for any $x\in[0,1]$, $t\in[0,T]$, $u, v\in C([0,1]\times\mathbb{R}_+)$ satisfying $\sup_{x,t\leqq T}|u(x,t)|\leqq M$, $\sup_{x,t\leqq T}|v(x,t)|\leqq M$,

$$|f(x,t,u)-f(x,t,v)|+|\sigma(x,t,u)-\sigma(x,t,v)|\leqq C(T,M)\sup_{y,s\leqq t}|u(y,s)-v(y,s)|\,.$$

(iii) For any $T>0$, there exists a constant $C(T)$ such that for any $x\in[0,1]$, $t\in[0,T]$, $u\in C([0,1]\times\mathbb{R}_+)$,

$$|f(x,t,u)|+|\sigma(x,t,u)|\leqq C(T)\left(1+\sup_{y,s\leqq t}|u(y,s)|\right).$$

We consider the SPDE

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}(x,t)-\frac{\partial^2 u}{\partial x^2}(x,t)+f(x,t,u)=\sigma(x,t,u)\dot W(t,x)\,,\\ &u(0,x)=u_0(x);\ u(t,0)=u(t,1)=0\end{aligned}\right\}\tag{14}$$

which again must be interpreted as

$$\left.\begin{aligned}u(x,t)=&\int_0^1 G_t(x,y)u_0(y)dy-\int_0^t\int_0^1 G_{t-s}(x,y)f(y,s,u)dy\,ds\\ &+\int_0^t\int_0^1 G_{t-s}(x,y)\sigma(y,s,u)W(dy,ds)\,.\end{aligned}\right\}\tag{15}$$

Theorem 3.1 *Let* $u_0\in C_0([0,1])$, *and suppose that* f,σ *satisfy conditions* (i), (ii) *and* (iii). *Then Eq.* (15) *has a unique continuous and* $\mathscr{F}_t$*-adapted solution* $\{u(x,t);\,0\leqq x\leqq 1,\,t\geqq 0\}$.

The proof of the theorem will rely on some estimates from Walsh [13], and in particular on the following version of the famous Kolmogorov Lemma which is a consequence of a result of Garsia–Rodemich–Rumsey (see Corollary 1.2, p. 273 in Walsh [13] for example).

Lemma 3.1 *Let R be a cube in $\mathbb{R}^n$ and $\{X_\alpha, \alpha \in R\}$ be a real valued stochastic process. Suppose there exist constants $k > 1$, $K > 0$, $\eta > 0$ such that*

$$E[|X_\alpha - X_\beta|^k] \leqq K|\alpha - \beta|^{n+\eta}$$

then

(i) *X has a continuous version,*

(ii) *there exist constants a, γ depending only on n, k and η, and a* r.v. *Y such that* a.s., *for all $(\alpha, \beta) \in R^2$,*

$$|X_\alpha - X_\beta| \leqq Y|\alpha - \beta|^{\eta/k}\left(\log\left(\frac{\gamma}{|\alpha - \beta|}\right)\right)^{2/k} \tag{16}$$

and

$$E(Y^k) \leqq aK\,. \tag{17}$$

Proof of Theorem 3.1 The proof is divided in three steps.

Step 1 Non explosion

Suppose τ is a stopping time such that $\tau > 0$ a.s. and $\{u(x, t); x \in [0, 1], t \in [0, \tau]\}$ is a continuous and adapted solution of equation (15) on the random interval $[0, \tau]$. We show that this implies that for any $t > 0$, $p \geqq 1$, there exists a constant $c(t, p)$ such that:

$$E\left(\sup_{x, s \leqq t \wedge \tau} |u(x, s)|^p\right) \leqq c(t, p)\,. \tag{18}$$

Let $p > 6, \frac{1}{p} + \frac{1}{q} = 1$

$$E(|u(x, t \wedge \tau)|^p) \leqq c\left\{\left|\int_0^1 G_t(x, y)u_0(y)dy\right|^p\right.$$

$$+\left(\int_0^t\int_0^1 G^q_{t-s}(x, y)dy\,ds\right)^{p/q} E\int_0^{t\wedge\tau}\int_0^1 |f(x, s, u)|^p dx\,ds$$

$$\left.+\left(\int_0^t\int_0^1 G^r_{t-s}(x, y)dy\,ds\right)^{(p-2)/2} E\int_0^{t\wedge\tau}\int_0^1 |\sigma(x, s, u)|^p dy\,ds\right\},$$

where $r = 2p/(p-2) < 3$, hence the integrals of powers of G above converge, see (9). We deduce, with the help of the assumption (iii), that for $t \leqq T$,

$$E(|u(x, t \wedge \tau)|^p) \leqq \bar{C}(T)\left(1 + E\int_0^{t\wedge\tau}\sup_{y, r \leqq s}|u(y, r)|^p dr\right).$$

From the above inequalities and the computations made in Walsh [13, Corollary 3.4], we can show the following estimate, again with $p > 6$, $s, t \leqq T$.

$$E(|u(x, t \wedge \tau) - u(y, s \wedge \tau)|^p)$$
$$\leqq \bar{C}(T)|(x, t) - (y, s)|^{\frac{p}{4}-3}\left(1 + E\int_0^{(t \vee s) \wedge \tau} \sup_{z, \alpha \leqq r} |u(z, \alpha)|^p dr\right).$$

Now choose $p > 20$. By the Kolmogorov Lemma 3.2, there exists a random variable X_p such that:

$$|u(x, t \wedge \tau) - u(y, s \wedge \tau)|^p \leqq X_p^p|(x, t) - (y, s)|^{\frac{p}{4}-5}$$
$$\times\left(\log\left(\frac{\gamma}{|(x, t) - (y, s)|}\right)\right)^2, \tag{19}$$

and

$$E[X_p^p] \leqq ac\left(1 + E\int_0^{(t \vee s) \wedge \tau} \sup_{z, \alpha \leqq r} |u(z, \alpha)|^p dr\right). \tag{20}$$

We now choose $s = 0$ in (19), and deduce from (19), (20) that there exists C_T such that for any $t \leqq T$,

$$E\left(\sup_{x, s \leqq t} |u(x, s \wedge \tau)|^p\right) \leqq C_T\left(1 + E\int_0^{t \wedge \tau} \sup_{y, r \leqq s} |u(y, r)|^p ds\right)$$
$$\leqq C_T\left(1 + E\int_0^t \sup_{y, r \leqq s} |u(y, r \wedge \tau)|^p ds\right)$$

(18) now follows from Gronwall's lemma. It now remains to prove the rest of the Theorem under a globally Lipschitz assumption on f and σ (i.e. $C(T, M)$ in (ii) does not depend on M), which we assume from now on.

Step 2 Uniqueness

Let $\{u(x, t); x \in [0, 1], t \in \mathbb{R}_+\}$ and $\{v(x, t); x \in [0, 1], t \in \mathbb{R}_+\}$ denote two continuous and adapted solutions, and define $\bar{u} = u - v$.

$$\bar{u}(x, t) = \int_0^t \int_0^1 G_{t-s}(x, y)[f(y, s, v) - f(y, s, u)]dy\, ds$$
$$+ \int_0^t \int_0^1 G_{t-s}(x, y)[\sigma(y, s, u) - \sigma(y, s, v)]\, W(dy, ds).$$

So if $t \leqq T$, $p > 6$, we deduce from (ii) that there exists C_T such that

$$E|\bar{u}(x, t)|^p \leqq C_T E\int_0^t \sup_{y, r \leqq s} |\bar{u}(y, r)|^p ds.$$

Moreover, similarly as above,

$$E(|\bar{u}(x, t) - \bar{u}(y, s)|^p) \leqq C_T|(x, t) - (y, s)|^{\frac{p}{4}-3} E\int_0^{t \vee s} \sup_{z, \alpha \leqq r} |\bar{u}(z, \alpha)|^p dr.$$

The same arguments as in step 1 now yield, for $p > 20$,

$$E\left(\sup_{x,s\leqq t}|\bar{u}(x,s)|^p\right)\leqq \bar{C}_T E\int_0^t \sup_{y,r\leqq s}|\bar{u}(y,r)|^p ds\,,$$

hence $\bar{u}=0$ from Gronwall's lemma.

Step 3 Existence

Consider the following mapping S, which maps continuous adapted random fields into continuous adapted random fields:

$$S(u)(x,t)\triangleq\int_0^1 G_t(x,y)u_0(y)dy-\int_0^t\int_0^1 G_{t-s}(x,y)f(y,s,u)dy\,ds$$
$$-\int_0^t\int_0^1 G_{t-s}(x,y)\sigma(y,s,u)W(dy,ds)\,.$$

From the arguments in step 2, there exists $p > 20$ and a constant $c(T)$ such that for any $t\in[0,T]$:

$$E\left(\sup_{x,s\leqq t}|S(u)(x,s)-S(v)(x,s)|^p\right)\leqq c(T)\int_0^t E\left(\sup_{x,r\leqq s}|u(x,r)-v(x,r)|^p\right)ds\,.$$

Now we construct the solution by successive approximations. Define $u^{n+1}(x,t)=S(u^n)(x,t)$; $u^0(x,t)=u_0(x)$. Then, using the above equation, we can prove that the sequence $u^n(x,t)$ converges in $L^p(\Omega)$, locally uniformly with respect to (x,t). Then, $u=\lim u^n$ is a solution of (15) (see Walsh [13] for details).

4 Existence of a solution

In this section, we suppose that f, σ are locally Lipschitz, u_0 is a positive continuous function on $[0,1]$ such that $u_0(0)=u_0(1)=0$.

We consider the penalized SPDE:

$$\left.\begin{aligned}&\frac{\partial u^\varepsilon(x,t)}{\partial t}-\frac{\partial^2 u^\varepsilon(x,t)}{\partial x^2}+f(u^\varepsilon(x,t))\\&\quad=\sigma(u^\varepsilon(x,t))\dot{W}(x,t)+\frac{1}{\varepsilon}(u^\varepsilon(x,t))^-\\&u^\varepsilon(\cdot,0)=u_0;\,u^\varepsilon(0,t)=u^\varepsilon(1,t)=0\,.\end{aligned}\right\}\tag{21}$$

For each $\varepsilon>0$, (21) admits a unique continuous solution u^ε which satisfies

$$\sup_{(x,t)\in\bar{Q}_T} E[|u^\varepsilon(x,t)|^p]<\infty\,,\quad\forall p\geqq 1\,,\ T>0\,.$$

Moreover, according to Theorem 2.1, if $\varepsilon<\varepsilon'$, then $u^{\varepsilon'}\leqq u^\varepsilon$ a.s. .

Theorem 4.1 *The solution u^ε of* (21) *converges* a.s. *to a continuous process u on $[0,1]\times\mathbb{R}_+$, as $\varepsilon\to 0$.*

Proof. u^ε is an increasing sequence as $\varepsilon \to 0$, set $u = \lim_{\varepsilon\to 0} u^\varepsilon = \sup_{\varepsilon \to 0} u^\varepsilon$.

Step 1 $u < \infty$ a.s.

We shall compare u^ε with two other quantities v^ε and w^ε to be defined below. From a standard localization procedure, it is sufficient to prove the comparison results under the assumption that f is globally Lipschitz, which we assume for most of step 1.

Let v^ε be the unique continuous solution of

$$\left.\begin{aligned} &\frac{\partial v^\varepsilon(x,t)}{\partial t} - \frac{\partial^2 v^\varepsilon(x,t)}{\partial x^2} + f(v^\varepsilon(x,t)) = \sigma(u^\varepsilon(x,t))\dot{W}(x,t) \\ &v^\varepsilon(\cdot,0) = u_0;\ v^\varepsilon(0,t) = v^\varepsilon(1,t) = 0\,. \end{aligned}\right\} \tag{22}$$

For any $T > 0$, $z^\varepsilon = v^\varepsilon - u^\varepsilon$ is a.s. the unique solution in $L^2((0,T)\times(0,1))$ of

$$\left.\begin{aligned} &\frac{\partial z_t^\varepsilon}{\partial t} + Az_t^\varepsilon = f(u_t^\varepsilon) - f(v_t^\varepsilon) - \frac{1}{\varepsilon}(u_t^\varepsilon)^- \\ &z^\varepsilon(\cdot,0) = 0;\ z^\varepsilon(0,t) = z^\varepsilon(1,t) = 0 \end{aligned}\right\} \tag{23}$$

which satisfies $z^\varepsilon \in L^2(0,T;V)$. According to Bensoussan and Lions [1, Lemma 6.1, p. 132], $(z^\varepsilon)^+ \in L^2(0,T;V) \cap C([0,T];H)$ a.s.,

$$\int_0^t \left(\frac{\partial}{\partial s} z_s^\varepsilon, (z_s^\varepsilon)^+\right) ds = \frac{1}{2}|(z_t^\varepsilon)^+|_H^2$$

and similarly

$$\int_0^t \left(\frac{\partial}{\partial x} z_s^\varepsilon, \frac{\partial}{\partial x}(z_s^\varepsilon)^+\right) ds = \int_0^t \left|\frac{\partial}{\partial x}(z_s^\varepsilon)^+\right|^2 ds \geqq 0\,.$$

We can multiply Eq. (23) by $(z^\varepsilon)^+$ to obtain

$$\int_0^t \left(\frac{\partial}{\partial s} z_s^\varepsilon, (z_s^\varepsilon)^+\right) ds + \int_0^t \left(\frac{\partial}{\partial x} z_s^\varepsilon, \frac{\partial}{\partial x}(z_s^\varepsilon)^+\right) ds + \int_0^t (f(v_s^\varepsilon) - f(u_s^\varepsilon), (z_s^\varepsilon)^+) ds$$

$$= -\frac{1}{\varepsilon}\int_0^t ((u_s^\varepsilon)^-, (z_s^\varepsilon)^+) ds \leqq 0\,.$$

Since

$$\int_0^t (f(v_s^\varepsilon) - f(u_s^\varepsilon), (z_s^\varepsilon)^+) ds \geqq -c\int_0^t |(z_s^\varepsilon)^+|_H^2\, ds\,,$$

we deduce from Gronwall's lemma that $|(z_t^\varepsilon)^+|^2 = 0$ a.s., and by continuity, a.s.

$$\forall (x,t) \in [0,1]\times \mathbb{R}_+,\quad u^\varepsilon(x,t) \geqq v^\varepsilon(x,t)\,. \tag{24}$$

From Theorem 3.1, the following equation has a unique solution $\{w^\varepsilon(x,t);\ x\in[0,1],\ t\geqq 0\}$:

$$\left.\begin{aligned} &\frac{\partial w^\varepsilon(x,t)}{\partial t} - \frac{\partial^2 w^\varepsilon(x,t)}{\partial x^2} + f(w^\varepsilon(x,t) + \sup_{s\leqq t,\, y\in[0,1]} (w^\varepsilon(y,s))^-) \\ &= \sigma(u^\varepsilon(x,t))\dot{W}(x,t) \\ &w^\varepsilon(\cdot,0) = u_0;\ w^\varepsilon(0,t) = w^\varepsilon(1,t) = 0\,. \end{aligned}\right\} \tag{25}$$

We set

$$\bar{w}^\varepsilon(x,t) = w^\varepsilon(x,t) + \sup_{s \leqq t,\, y \in [0,1]} (w^\varepsilon(y,s))^-$$

$$\triangleq w^\varepsilon(x,t) + \Phi_t^\varepsilon$$

$\bar{w}^\varepsilon(x,t) \geqq 0$ a.s., Φ^ε is an increasing process.

For any $T > 0$, $\bar{z}^\varepsilon \triangleq u^\varepsilon - \bar{w}^\varepsilon$ is the unique solution in $L^2(0,T; H^1(0,1))$ of

$$\left.\begin{aligned} &\frac{d\bar{z}_t^\varepsilon}{dt} + A\bar{z}_t^\varepsilon + f(u_t^\varepsilon) - f(\bar{w}_t^\varepsilon) + \frac{d\Phi_t^\varepsilon}{dt} = \frac{1}{\varepsilon}(u_t^\varepsilon)^- \\ &\bar{z}^\varepsilon(\cdot, 0) = 0;\ \bar{z}^\varepsilon(0,t) = \bar{z}^\varepsilon(1,t) = -\Phi_t^\varepsilon \end{aligned}\right\}$$

(since $A(u_s^\varepsilon - w_s^\varepsilon) = A\bar{z}_s^\varepsilon$). Multiplying that equation by $(\bar{z}^\varepsilon)^+ \in L^2(0,T;V)$ a.s., we obtain by the same arguments as above:

$$\int_0^t \left(\frac{\partial}{\partial s}\bar{z}_s^\varepsilon, (\bar{z}_s^\varepsilon)^+\right) ds + \int_0^t \left(\frac{\partial \bar{z}_s^\varepsilon}{\partial x}, \frac{\partial (\bar{z}_s^\varepsilon)^+}{\partial x}\right) ds$$

$$+ \int_0^t (f(u_s^\varepsilon) - f(\bar{w}_s^\varepsilon), (\bar{z}_s^\varepsilon)^+) ds + \int_0^t \int_0^1 (\bar{z}^\varepsilon(x,s))^+ \, dx \, d\Phi_s^\varepsilon = \frac{1}{\varepsilon}\int_0^t ((u^\varepsilon)^-, (\bar{z}_s^\varepsilon)^+) ds\,.$$

The right-hand side of the above equality is zero because $(\bar{z}_s^\varepsilon)^+ > 0$ implies $u_s^\varepsilon > \bar{w}_s^\varepsilon \geqq 0$. Hence we again deduce from Gronwall's lemma

$$u^\varepsilon(x,t) \leqq \bar{w}^\varepsilon(x,t) \quad \text{a.s.}\,. \tag{26}$$

By (24), (26),

$$|u^\varepsilon(x,t)| \leqq |v^\varepsilon(x,t)| + 2 \sup_{s \leqq t,\, y \in [0,1]} |w^\varepsilon(y,s)|\,. \tag{27}$$

We now return to the assumption of locally Lipschitz coefficients. From Lemma 6.1 of the Appendix, for arbitrarily large p and any $T > 0$,

$$\sup_\varepsilon E\left[\sup_{(x,t)\in \bar{Q}_T} |v^\varepsilon(x,t)|^p\right] < \infty\,,$$

and

$$\sup_\varepsilon E\left[\sup_{(x,t)\in \bar{Q}_T} |w^\varepsilon(t,x)|^p\right] < \infty$$

implying

$$\sup_\varepsilon E\left[\sup_{(x,t)\in \bar{Q}_T} |u^\varepsilon(x,t)|^p\right] < \infty\,. \tag{28}$$

So $u = \sup_\varepsilon u^\varepsilon$ is a.s. bounded on $\bar{Q}_T$.

It follows from (27), again by a standard localization procedure, that the rest of the proof can be done under the assumption of globally Lipschitz coefficients.

Step 2 Continuity of u

We first note that for any real number c, $u_t^{\varepsilon,c} := e^{-ct}u_t^\varepsilon$ satisfies the same equation as u^ε, but with f (resp. σ) replaced by f_c (resp. σ_c), where

$$f_c(x, t; z) = e^{-ct}f(x, t; e^{ct}z) + cz$$

$$\sigma_c(x, t; z) = e^{-ct}\sigma(x, t; e^{ct}z) .$$

Now, since f is assumed to be uniformly Lipschitz in z, we can choose c such that

$$z \to f_c(\cdot\,; z)$$

is non decreasing (at least for $(x, t) \in [0, 1] \times [0, T]$, T arbitrary). Hence, we can and will assume that $z \to f(\cdot, z)$ is non decreasing in this second part of the proof.

Let $\bar{v}^\varepsilon$ be the solution of

$$\left.\begin{aligned} &\frac{\partial \bar{v}^\varepsilon(x, t)}{\partial t} + A\bar{v}^\varepsilon(x, t) = \sigma(u^\varepsilon(x, t))\dot{W}(x, t) \\ &\bar{v}^\varepsilon(\cdot, 0) = u_0;\, \bar{v}^\varepsilon(0, t) = \bar{v}^\varepsilon(1, t) = 0 \end{aligned}\right\} \tag{29}$$

and $\bar{v}$ be the solution of

$$\left.\begin{aligned} &\frac{\partial \bar{v}(x, t)}{\partial t} + A\bar{v}(x, t) = \sigma(u(x, t))\dot{W}(x, t) \\ &\bar{v}(\cdot, 0) = u_0;\, \bar{v}(0, t) = \bar{v}(1, t) = 0 . \end{aligned}\right\} \tag{30}$$

(29), (30) admit unique continuous solutions (continuity can be proved as in Walsh [13]). Let $z^\varepsilon = u^\varepsilon - \bar{v}^\varepsilon$, z^ε is the solution of

$$\left.\begin{aligned} &\frac{\partial z_t^\varepsilon}{\partial t} + Az_t^\varepsilon + \left(f(z_t^\varepsilon + \bar{v}_t^\varepsilon) - \frac{1}{\varepsilon}(z_t^\varepsilon + \bar{v}_t^\varepsilon)^-\right) = 0 \\ &z^\varepsilon(\cdot, 0) = 0;\, z^\varepsilon(0, t) = z^\varepsilon(1, t) = 0 . \end{aligned}\right\}$$

We denote by f_ε the function $f_\varepsilon(r) = f(r) - \frac{1}{\varepsilon}r^-$, so z^ε is a solution of

$$\left.\begin{aligned} &\frac{\partial z_t^\varepsilon}{\partial t} + Az_t^\varepsilon + f_\varepsilon(z_t^\varepsilon + \bar{v}_t^\varepsilon) = 0 \\ &z^\varepsilon(\cdot, 0) = 0;\, z^\varepsilon(0, t) = z^\varepsilon(1, t) = 0 . \end{aligned}\right\}$$

Let $\bar{z}^\varepsilon$ be the solution of

$$\left.\begin{aligned} &\frac{\partial \bar{z}_t^\varepsilon}{\partial t} + A\bar{z}_t^\varepsilon + f_\varepsilon(\bar{z}_t^\varepsilon + \bar{v}_t) = 0 \\ &\bar{z}^\varepsilon(\cdot, 0) = 0;\, \bar{z}^\varepsilon(0, t) = \bar{z}^\varepsilon(1, t) = 0 . \end{aligned}\right\}$$

Following Nualart and Pardoux [8], we deduce from the fact that f_ε is non decreasing:

$$\|\bar{z}^\varepsilon - z^\varepsilon\|_{T,\infty} \leqq \|\bar{v} - \bar{v}^\varepsilon\|_{T,\infty} , \tag{31}$$

where $\|\ \|_{T,\infty}$ denotes the uniform norm on $C(\bar{Q}_T)$ ($T > 0$ is arbitrary).

The proof of (31) uses the same arguments as in step 1: if we denote $w = \bar{z}^\varepsilon - z^\varepsilon - \|\bar{v} - \bar{v}^\varepsilon\|_{T,\infty}$, we can prove that $w^+ = 0$. So, $\bar{z}^\varepsilon - z^\varepsilon \leqq \|\bar{v} - \bar{v}^\varepsilon\|_{T,\infty}$ and by symmetry, $z^\varepsilon - \bar{z}^\varepsilon \leqq \|\bar{v} - \bar{v}^\varepsilon\|_{T,\infty}$.

$\bar{u}^\varepsilon = \bar{z}^\varepsilon + \bar{v}$ solves

$$\frac{\partial \bar{u}^\varepsilon(x,t)}{\partial t} + A\bar{u}^\varepsilon(x,t) + f(\bar{u}^\varepsilon(x,t)) = \sigma(u(x,t))\dot{W}(x,t) + \frac{1}{\varepsilon}(\bar{u}^\varepsilon)^-(x,t)\,,$$

or equivalently

$$\bar{u}^\varepsilon(x,t) + G * f(\bar{u}^\varepsilon)(x,t) - \frac{1}{\varepsilon} G * (\bar{u}^\varepsilon)^-(x,t) = v(x,t) \tag{32}$$

where

$$v(x,t) = \int_0^t G_t(x,y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u(y,s))W(dy,ds) \tag{33}$$

is a continuous process on $\bar{Q}_T$, and we have used the notation

$$G * \varphi(x,t) = \int_0^t \int_0^1 G_{t-s}(x,y)\varphi(y,s)dy\,ds\,.$$

Now, it has been shown in Nualart and Pardoux [8] that the solution of (32) converges as ε tends to zero to a continuous function $\bar{u}$ on the compact set $\bar{Q}_T$.

Moreover, the sequence $(\bar{u}^\varepsilon)_\varepsilon$ being increasing (see [8]), the convergence is uniform on $\bar{Q}_T$. So $\bar{z}^\varepsilon$ converges uniformly to a continuous function $\bar{z}$. Using $z^\varepsilon = u^\varepsilon - \bar{v}^\varepsilon$ and (31), it will follow from the convergence $\|\bar{v} - \bar{v}^\varepsilon\|_{T,\infty} \xrightarrow[\varepsilon \to 0]{} 0$ a.s. that $u = \bar{v} + \bar{z}$ is a continuous function.

By Lemma 6.2 of the Appendix, $E[\|\bar{v} - \bar{v}^\varepsilon\|_{T,\infty}] \xrightarrow[\varepsilon \to 0]{} 0$.

The proof of the theorem is now complete. □

Theorem 4.2 *Let σ, f be locally Lipschitz. The reflected problem* (RPE) *has a solution* (u, η).

Proof. Let u^ε be the solution of the penalized equation and $u = \lim_{\varepsilon \to 0} u^\varepsilon$. Let φ be a C^∞ function with compact support contained in $(0,1) \times \mathbb{R}_+$.

$$\begin{aligned}(u_t^\varepsilon, \varphi_t) &- \int_0^t \left(u_s^\varepsilon, \frac{\partial}{\partial s}\varphi_s\right) + \int_0^t (A\varphi_s, u_s^\varepsilon)ds + \int_0^t (f(u_s^\varepsilon), \varphi_s)ds = (u_0, \varphi_0) \\ &+ \int_0^t \int_0^1 \sigma(u^\varepsilon(x,s))\varphi(x,s)W(dx,ds) + \frac{1}{\varepsilon}\int_0^t ((u_s^\varepsilon)^-, \varphi_s)ds \quad \text{a.s.}\,.\end{aligned} \tag{34}$$

We denote by η_ε the random measure $\eta_\varepsilon(dx,dt) = \frac{1}{\varepsilon}(u^\varepsilon)^-(x,t)dx\,dt$ on $[0,1] \times \mathbb{R}_+$.

All the terms on the left-hand side of (34) converge a.s. when ε tends to 0.

$$\int_0^t \int_0^1 \sigma(u^\varepsilon(x,s))\varphi(x,s)dW(x,s) \xrightarrow[\varepsilon \to 0]{L^2} \int_0^t \int_0^1 \sigma(u(x,s))\varphi(x,s)dW(x,s)\,.$$

Indeed, from a.s. convergence, (28) and linear growth of σ,

$$E \int_0^t \int_0^1 |\sigma(u(x,s)) - \sigma(u^\varepsilon(x,s))|^2 dx\, ds \to 0 .$$

So we may suppose that the above convergence holds a.s. (taking a subsequence). We deduce from (34) that η_ε converges in the distributional sense to a distribution η on $(0,1) \times \mathbb{R}_+$ a.s.. η is a positive distribution and hence a measure on $(0,1) \times \mathbb{R}_+$. For all $T > 0$ and $\varphi \in C^\infty(Q)$ with support in $(0,1) \times [0,T]$, we have for all $t \geqq 0$:

$$(u_t, \varphi_t) - \int_0^t \left(u_s, \frac{\partial}{\partial s}\varphi_s \right) + \int_0^t (A\varphi_s, u_s)ds + \int_0^t (f(u_s), \varphi_s)ds = (u_0, \varphi_0)$$

$$+ \int_0^t \int_0^1 \sigma(u(x,s))\varphi(x,s) W(dx,ds) + \int_0^t \int_0^1 \varphi(x,s)\eta(dx,ds) \quad \text{a.s.} . \tag{35}$$

Multiplying (34) by ε and letting $\varepsilon \to 0$, we obtain

$$\int_0^t (u_s^-, \varphi_s)ds = 0 \quad \text{a.s.}$$

for all C^∞ functions φ with compact support in $(0,1) \times [0, \mathrm{T}]$.
This implies $u(x,t) \geqq 0$ a.e. on $(0,1) \times [0,T]$, a.s.; u being continuous, $u(x,t) \geqq 0$ on $(0,1) \times [0,T]$ a.s..
For $\varepsilon \leqq \varepsilon'$, $u^\varepsilon \geqq u^{\varepsilon'}$ a.s. therefore $\operatorname{supp} \eta^\varepsilon \subset \operatorname{supp} \eta^{\varepsilon'}$ and $\operatorname{supp} \eta \subset \operatorname{supp} \eta^\varepsilon$.
Now, $u^\varepsilon \leqq 0$ on $\operatorname{supp} \eta^\varepsilon$ therefore $\int_Q u^\varepsilon d\eta \leqq 0$ a.s. (it can be $-\infty$).
Let $Q_{T,n} = \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \times [0,T]$.

$$-\infty < \int_{Q_{T,n}} u^\varepsilon d\eta \leqq 0 \quad \text{a.s.} .$$

By monotone convergence, $\int_{Q_{T,n}} u d\eta \leqq 0$, and $\int_{Q_{T,n}} u d\eta = 0$.
Letting $n \to \infty$, $\int_{Q_T}^t u d\eta = 0$ by monotone convergence.

The pair (u, η) is a solution of the reflected problem (RPE) in a weak sense that is (2) is satisfied for any C^∞ function φ with compact support in $(0,1)$.

Let v be the continuous stochastic process defined by (33). v satisfies:

$$(v_t, \varphi) + \int_0^t (A\varphi, v_s)ds = (u_0, \varphi_0) + \int_0^t \int_0^1 \sigma(u(x,s))\varphi(x,s) W(dx,ds)$$

for φ a C^∞ function with compact support in $(0,1)$.
(u, η) satisfies (I):

(i′) u is a continuous process on $[0,1] \times \mathbb{R}_+$, $u(x,t) \geqq 0$; η is a measure on $(0,1) \times \mathbb{R}_+$ such that $\int_{[0,1] \times \mathbb{R}_+} u d\eta = 0$, $u_0 = v_0$, $(u - v)(0,t) = (u - v)(1,t) = 0$,

(ii′) $\forall t \geqq 0$, $\varphi \in C_K^\infty(\mathbb{R})$ with $\operatorname{supp} \varphi \subset (0,1)$,

$$(u(t), \varphi) + \int_0^t (u(s), A\varphi)ds + \int_0^t (f(u(s)), \varphi)ds$$

$$= (v(t), \varphi) + \int_0^t (v(s), A\varphi)ds + \int_0^t \int_0^1 \varphi(x)\eta(dx,ds) \quad \text{a.s.} .$$

Now, following Nualart and Pardoux [8], (I) has a unique solution for a given $v \in \bigcap_{T>0} C(\bar{Q}_T)$. We can therefore apply the results of Nualart and Pardoux. It follows that $\eta((0,1)\times\{t\}) = 0$, $\int_{Q_t} x(1-x)\eta(dx, ds) < \infty$ a.s. for all $t > 0$, and for all $\varphi \in C_K^\infty(\mathbb{R})$ with $\varphi(0) = \varphi(1) = 0$,

$$(u(t), \varphi) + \int_0^t (u(s), A\varphi)ds + \int_0^t (f(u(s)), \varphi)ds = (u_0, \varphi)$$

$$+ \int_0^t \int_0^1 \varphi(x)\sigma(u(x,s))W(dx\,ds) + \int_0^t \int_0^1 \varphi(x)\eta(dx, ds) \quad \text{a.s.} . \qquad \square$$

Remarks. (i) $\int_0^t \int_0^1 G_{t-s}(x,y)\eta(dy, ds) < \infty$ a.s., and

$$u(x,t) = \int_0^t G_t(x,y)u_0(y)dy - \int_0^t \int_0^1 G_{t-s}(x,y)f(u(y,s))dy\,ds$$

$$+ \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u(y,s))W(dy,ds) + \int_0^t \int_0^1 G_{t-s}(x,y)\eta(dy,ds) . \tag{36}$$

(ii) Conversely, let $(v, \bar{\eta})$ be a solution of (RPE). The weak form (2) can be extended to smooth functions $\varphi(x,t)$ which satisfy $\varphi(0,t) = \varphi(1,t) = 0$: a.s.

$$(u(t), \varphi(t)) + \int_0^t \left(u(s), A\varphi(s) - \frac{\partial}{\partial s}\varphi(s) \right) ds + \int_0^t (f(u(s), \varphi(s))ds$$

$$= (u_0, \varphi(0)) + \int_0^t \int_0^1 \varphi(x,s)\sigma(u(x,s))W(dx\,ds) + \int_0^t \int_0^1 \varphi(x,s)\eta(dx,ds) . \tag{37}$$

Then, applying (37) to $\varphi(y,s) = G_{t-s}(\Psi, y)$ for $\Psi \in C_K^\infty(\mathbb{R})$, we can prove that $\int_0^t \int_0^1 G_{t-s}(x,y)\bar{\eta}(dy,ds) < \infty$ a.s., and the integral equation (36) is true for $(v, \bar{\eta})$.

5 Minimality of the solution

The results of this section need only be proved in case of globally Lipschitz coefficients, again from a standard localization procedure. Hence we assume below that the coefficients are globally Lipschitz.

Theorem 5.1 *The pair* (u, η) *constructed in the above section is minimal in the following sense: if* $(v, \bar{\eta})$ *is a solution of the reflected problem* (RPE), *then* $u \leqq v$ a.s..

Proof. $u = \lim_{\varepsilon\to 0} u^\varepsilon$ where u^ε is the solution of

$$\frac{\partial u^\varepsilon(x,t)}{\partial t} + Au^\varepsilon(x,t) + f(u^\varepsilon(x,t)) = \sigma(u^\varepsilon(x,t))\dot{W}(x,t) + F_\varepsilon(u^\varepsilon(x,t)) \tag{38}$$

with $F_\varepsilon(x) = \frac{1}{\varepsilon}x^-$; F_ε is a positive, Lipschitz decreasing function.

Let $(v, \bar{\eta})$ be a solution of (RPE). Since $v \geqq 0$ a.s., $F_\varepsilon(v(x,t)) \equiv 0$ a.s., and v is also a solution of

$$\frac{\partial v(x,t)}{\partial t} + Av(x,t) + f(v(x,t)) = \sigma(v(x,t))\dot{W}(x,t) + \bar{\eta} + F_\varepsilon(v(x,t)) . \tag{39}$$

As in Sect. 2, we want to compare the two solutions of (38) and (39). Let us first prove the:

Lemma 5.1 *Given* $\bar{\eta}$ *satisfying* (ii), *there exists a unique continuous and adapted solution* v *of*

$$\left.\begin{aligned} &\frac{\partial v(x,t)}{\partial t} + Av(x,t) + f(v(x,t)) \\ &\quad = \sigma(v(x,t))\dot{W}(x,t) + \bar{\eta} + F_\varepsilon(v(x,t)) \\ &v(\cdot,0) = u_0;\ v(0,t) = v(1,t) = 0 \end{aligned}\right\} \tag{40}$$

in the following sense

$$\begin{aligned} v(x,t) = {}& \int_0^t G_t(x,y)u_0(y)dy - \int_0^t\int_0^1 G_{t-s}(x,y)f(v(y,s))dy\,ds \\ &+ \int_0^t\int_0^1 G_{t-s}(x,y)\sigma(v(y,s))W(dy,ds) \\ &+ \int_0^t\int_0^1 G_{t-s}(x,y)\bar{\eta}(dy,ds) + \int_0^t\int_0^1 G_{t-s}(x,y)F_\varepsilon(v(y,s))dy\,ds\,. \end{aligned}$$

Proof. It suffices to prove uniqueness of the solution of (40), since existence will follow from the proof of the theorem. Now uniqueness is proved exactly as in Walsh [13], since F_ε is Lipschitz. □

We now return to the proof of Theorem 5.1. Let

$$\bar{\eta}_n(x,t) = n\int_{t-2/n}^{t-1/n}\int_0^1 G_{t-s}(x,y)\bar{\eta}(dy,ds)\,.$$

Since for some c_n, $G_{t-s}(x,y) \leqq c_n\ y(1-y)$ for $0 \leqq s \leqq t - 1/n$, $0 \leqq t \leqq T$, $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, $\bar{\eta}_n$ is a bounded function. It follows from Theorem 2.1 that $u^\varepsilon \leqq v^n$, where v^n is the unique solution of

$$\frac{\partial v^n(x,t)}{\partial t} + Av^n(x,t) + f(v^n(x,t)) = \sigma(v^n(x,t))\dot{W}(x,t) + \bar{\eta}_n(x,t) + F_\varepsilon(v^n(x,t))\,.$$

It remains to show that $v^n \to v$ in probability as $n \to \infty$, where v is the unique solution of (40). This will follow from:

$$A_n \to \int_0^t\int_0^1 G_{t-s}(x,y)\bar{\eta}(dy,ds) \quad \text{a.s.}\,,$$

where

$$\begin{aligned} A_n &= \int_0^t\int_0^1 G_{t-s}(x,y)\bar{\eta}_n(y,s)dy\,ds \\ &= n\int_0^t\left(\int_{s-2/n}^{s-1/n}\int_0^1 G_{t-r}(x,z)\bar{\eta}(dz,dr)\right)ds\,. \end{aligned}$$

Hence

$$\int_0^{t-2/n}\int_0^1 G_{t-r}(x,z)\bar{\eta}(dz,dr) \leqq A_n \leqq \int_0^{t-1/n}\int_0^1 G_{t-r}(x,z)\bar{\eta}(dz,dr),$$

and the above convergence follows from $\bar{\eta}((0,1)\times\{t\})=0$, Remark (ii) at the end of Sect. 3, and dominated convergence. □

6 Appendix

Lemma 6.1 *Let* v^ε (resp. w^ε) *be the solution of* (22), (resp. *of* (25)) *then for all* $p \geqq 1$, $T>0$,

$$\sup_\varepsilon E\left[\sup_{(x,t)\in\bar{Q}_T}|v^\varepsilon(x,t)|^p\right]<\infty\,,\quad \sup_\varepsilon E\left(\sup_{(t,x)\in\bar{Q}_T}|w^\varepsilon(t,x)|^p\right)<\infty\,.$$

Proof. In this proof, which exploits similar techniques as those used in the proof of Theorem 3.1, c will stand for a constant (independent of ε, but depending on the exponent p) which may vary from place to place. First choose $(x,t)\in[0,1]\times\mathbb{R}_+$.

$$v^\varepsilon(x,t)=\int_0^1 G_t(x,y)u_0(y)dy-\int_0^t\int_0^1 G_{t-s}(x,y)f(v^\varepsilon(y,s))dy\,ds$$
$$+\int_0^t\int_0^1 G_{t-s}(x,y)\sigma(u^\varepsilon(y,s))W(dy\,ds)\,,$$

$$w^\varepsilon(x,t)=\int_0^1 G_t(x,y)u_0(y)dy$$
$$-\int_0^t\int_0^1 G_{t-s}(x,y)f\left(w^\varepsilon(y,s)+\sup_{r\leqq s,\,z\in[0,1]}(w^\varepsilon(z,r))^-\right)dy\,ds$$
$$+\int_0^t\int_0^1 G_{t-s}(x,y)\sigma(u^\varepsilon(y,s))W(dy\,ds)\,.$$

Let $p>6$, $\dfrac{1}{p}+\dfrac{1}{q}=1$,

$$E[|v^\varepsilon(x,t)|^p]\leqq c\left\{\left|\int_0^1 u_0(y)G_t(x,y)dy\right|^p\right.$$
$$+\left(\int_0^t\int_0^1 G^q_{t-s}(x,y)dy\,ds\right)^{p/q}E\left[\int_0^t\int_0^1|f(v^\varepsilon(y,s))|^p dy\,ds\right]$$
$$\left.+\left(\int_0^t\int_0^1 G^r_{t-s}(x,y)\,dy\,ds\right)^{(p-2)/2}\left(1+E\int_0^t\int_0^1|u^\varepsilon(y,s)|^p dy\,ds\right)\right\}$$

$$E(|w^\varepsilon(x,t)|^p] \leqq c\left\{\left|\int_0^1 u_0(y)G_t(x,y)dy\right|^p + \left(\int_0^t\int_0^1 G^q_{t-s}(x,y)dy\,ds\right)^{p/q}\right.$$
$$\times E\left[\int_0^t\int_0^1 \left|f\left(w^\varepsilon(y,s)+\sup_{z,r\leqq s}(w^\varepsilon(z,r))^-\right)\right|^p dy\,ds\right]$$
$$\left.+\left(\int_0^t\int_0^1 G^r_{t-s}(x,y)dy\,ds\right)^{(p-2)/2}\left(1+E\int_0^t\int_0^1|u^\varepsilon(y,s)|^p dy\,ds\right)\right\}$$

where $r = 2p/(p-2) < 3$, hence all integrals in the above right hand side are finite, from (9). It follows from the last inequality and (27) that

$$E[|v^\varepsilon(x,t)|^p] \leqq C\left(1+\int_0^t E\left(\sup_{y,r\leqq s}|v^\varepsilon(y,r)|^p + \sup_{y,r\leqq s}|w^\varepsilon(y,r)|^p\right)ds\right) \tag{41}$$

$$E[|w^\varepsilon(x,t)|^p] \leqq C\left(1+\int_0^t E\left(\sup_{y,r\leqq s}|v^\varepsilon(y,r)|^p + \sup_{y,r\leqq s}|w^\varepsilon(y,r)|^p\right)ds\right). \tag{42}$$

We now estimate the moments of the increments of v^ε. Let again $p > 6$. From the above inequalities and the computations made in Walsh [13, Corollary 3.4], we can show the following estimate

$$E[|v^\varepsilon(x,t)-v^\varepsilon(y,s)|^p] \leqq c\left(1+E\int_0^{t\vee s}\left(\sup_{z,\theta\leqq r}|v^\varepsilon(z,\theta)|^p + \sup_{z,\theta\leqq r}|w^\varepsilon(z,\theta)|^p\right)dr\right)\times|(x,t)-(y,s)|^{p/4-3}. \tag{43}$$

$$E[|w^\varepsilon(x,t)-w^\varepsilon(y,s)|^p] \leqq c\left(1+E\int_0^{t\vee s}\left(\sup_{z,\theta\leqq r}|v^\varepsilon(z,\theta)|^p + \sup_{z,\theta\leqq r}|w^\varepsilon(z,\theta)|^p\right)dr\right)\times|(x,t)-(y,s)|^{p/4-3}. \tag{44}$$

Now choose $p > 20$. By the Kolmogorov Lemma 3.1, there exists a random variable $Y_{\varepsilon,p}$ such that:

$$|v^\varepsilon(x,t)-v^\varepsilon(y,s)|^p \leqq Y^p_{\varepsilon,p}|(x,t)-(y,s)|^{p/4-5}\left(\log\left(\frac{\gamma}{|(x,t)-(y,s)|}\right)\right)^2 \tag{45}$$

$$|w^\varepsilon(x,t)-w^\varepsilon(y,s)|^p \leqq Z^p_{\varepsilon,p}|(x,t)-(y,s)|^{p/4-5}\left(\log\left(\frac{\gamma}{|(x,t)-(y,s)|}\right)\right)^2 \tag{46}$$

and by (17),

$$E[(Y_{\varepsilon,p})^p] \leqq ac\left(1+E\int_0^{t\vee s}\left(\sup_{z,\alpha\leqq r}|v^\varepsilon(z,\alpha)|^p + \sup_{z,\alpha\leqq r}|w^\varepsilon(z,\alpha)|^p\right)dr\right). \tag{47}$$

$$E[(Z_{\varepsilon,p})^p] \leqq ac\left(1+E\int_0^{t\vee s}\left(\sup_{z,\alpha\leqq r}|v^\varepsilon(z,\alpha)|^p + \sup_{z,\alpha\leqq r}|w^\varepsilon(z,\alpha)|^p\right)dr\right). \tag{48}$$

Choosing $s = 0$, $y = 0$ in (45), (46), we deduce that for any $T > 0$ there exists C_T such that

$$\sup_{x,s \leqq t} |v^\varepsilon(x,s)|^p \leqq C_T(1 + Y^p_{\varepsilon,p}), \quad \sup_{x,s \leqq t} |w^\varepsilon(x,s)|^p \leqq C_T(1 + Z^p_{\varepsilon,p}),$$

with $Y_{\varepsilon,p}$ (resp. $Z_{\varepsilon,p}$) satisfying (47) (resp. (48)). The result now follows from Gromvall's Lemma. □

Lemma 6.2 *Let* $\bar{v}^\varepsilon$ (resp. $\bar{v}$) *be the solution of* (29) (resp. (30)), *then for any* $T > 0$

$$E[\|\bar{v}^\varepsilon - \bar{v}\|_{T,\infty}] \xrightarrow[\varepsilon \to 0]{} 0 .$$

where $\|\cdot\|_{T,\infty}$ *is the norm in* $L^\infty(\bar{Q}_T)$.

Proof. (i) Set $w^\varepsilon = \bar{v} - \bar{v}^\varepsilon$.

$$w^\varepsilon(x,t) = \int_0^t \int_0^1 [\sigma(u^\varepsilon(y,s)) - \sigma(u(y,s))] G_{t-s}(x,y) W(dy\,ds) .$$

Let $p > 6$,

$$E[|w^\varepsilon(x,t)|^p] \leqq C_p\left(\int_0^t \int_0^1 G^{2q}_{t-s}(x,y)dy\,ds\right)^{p/2q}$$

$$\times E\left[\int_0^t \int_0^1 |\sigma(u^\varepsilon(y,s)) - \sigma(u(y,s))|^p dy\,ds\right]$$

with $q = \dfrac{p}{p-2}(2q < 3)$.
By (9), $\sup_{(x,t)} \int_0^t \int_0^1 G^{2q}_{t-s}(x,y)dy\,ds < \infty$.
Since $u^\varepsilon(y,s) \to_{\varepsilon\to 0} u(y,s)$ a.s. and σ has linear growth, it follows from (28) that for any $T > 0$

$$\sup_{(x,t)\in\bar{Q}_T} E[|w^\varepsilon(x,t)|^p] \xrightarrow[\varepsilon\to 0]{} 0 .$$

(ii) Following again some computations of Walsh [13, Corollary 3.4], we can find, using (i), a constant $K_{\varepsilon,p}$ such that:

$$E[|w^\varepsilon(x,t) - w^\varepsilon(y,s)|^p] \leqq K_{\varepsilon,p}|(x,t) - (y,s)|^{p/4-3}$$

with $K_{\varepsilon,p} \to_{\varepsilon\to 0} 0$. Let $p > 20$, by the Kolmogorov lemma, there exists a r.v. $U_{\varepsilon,p}$ such that:

$$|w^\varepsilon(x,t) - w^\varepsilon(y,s)| \leqq U_{\varepsilon,p}|(x,t) - (y,s)|^{1/4-5/p}\left(\log\left(\frac{\gamma}{|(x,t)-(y,s)|}\right)\right)^{2/p}$$

and $E[U^p_{\varepsilon,p}] \leqq aK_{\varepsilon,p}$; thus $E[U_{\varepsilon,p}] \to_{\varepsilon\to 0} 0$.
Therefore, for any $T > 0$

$$E\left[\sup_{(x,t)\in\bar{Q}_T} |w^\varepsilon(x,t)|\right] \xrightarrow[\varepsilon\to 0]{} 0 ,$$

that is $E[\|\bar{v}^\varepsilon - \bar{v}\|_{T,\infty}] \xrightarrow[\varepsilon\to 0]{} 0.$ □

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