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The critical value for some non-attractive long range nearest particle systems*

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Summary. We consider the nearest particle system which gives birth rate λ to each vacant interval, concentrated on the interval's midpoint(s). We prove that a critical value for λ exists and equals one. The proof extends to a large class of nearest particle systems. This paper solves a problem suggested by Liggett (1985).

In the following we deal with nearest particle systems $\{\eta_t : t \geqq 0\}$. These can be described as particle systems with the following flip rates:

$$\text{for } \eta(x)=1, \quad c(x,\eta)=1$$
$$\text{for } \eta(x)=0, \quad c(x,\eta)=\lambda\beta(l_x, r_x)$$

where l_x for a configuration η is the smallest positive 1 so that $\eta(x-l)=1$ and r_x is similarly the smallest positive r so that $\eta(x+r)=1$. For a survey of the important properties and results see Liggett (1985), Chap. 7. In this paper we will be concerned with the question of survivability of infinite systems. In this paper we say the process survives if $P^1(\eta_t(0)=1)$ is bounded away from 0 (here and throughout the paper $P^1(\ )$ will refer to probabilities starting with $\eta(x)\equiv 1$).

Our ideas will apply to a wide range of nearest particle systems, but for concreteness we will consider the "midpoint" nearest particle system for which

$$\beta(l,r) = \begin{matrix} \lambda & \text{for } l=r \\ \lambda/2 & \text{for } |l-r|=1 \end{matrix}$$

and $\beta(l,r)$ equals zero in all other cases. For notational simplicity we will assume that we happen to only be dealing with intervals of even length so that the birth rate is always concentrated at a single point in an interval. This tidying assumption will not affect the validity of the proofs. We prove

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Theorem 1. *The midpoint nearest particle system survives for $\lambda > 1$, but not for $\lambda \leqq 1$.*

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Remark. It will be clear that the methods used are equal to proving that if a nearest particle system has birth function $\lambda\beta(l, r)$ where, for each $\delta > 0$, there exists a $\chi > 0$ with

$$\liminf_{n \to \infty} \sum_{l+r=n, l, r > \chi n} \beta(l, r) > 1 - \delta ,$$

then the particle survives for $\lambda > 1$.

It is worth noting that in the non-attractive case such as with the "mid-point" nearest particle system it is not apriori obvious that a critical value of λ exists. Another difference between our "midpoint" nearest particle system and attractive nearest particle systems is that our process is not Feller. That is given a continuous function f, the function $P_t f(\eta)$ is not necessarily also continuous. (Here P_t denotes the particle system semi-group.) This is because the flip rate at a site x is sensitive to the configuration spins a long distance away if it is contained in a large vacant interval. Given that our process is non-Feller it is not immediate from Theorem One that a non-trivial invariant measure exists. For instance the usual step of taking the limit of some sequence of Cesaro averages as in Liggett (1985), is not open to us. Nonetheless we can prove

Theorem 2. *Under the conditions of Theorem One there exists a non-trivial invariant measure.*

We define $b(n)$ as $\sum_{l+r=n} \beta(l, r)$, the total growth rate over an interval of length n divided by λ. Theorem 5.5 of Liggett (1985), Chap. 7 states that if $b(n) \leqq 1$ for each n, then the NPS does not survive for $\lambda \leqq 1$. This completely general fact ensures that the particle system of Theorem One cannot survive if $\lambda \leqq 1$, and so it only remains for us to establish survival for $\lambda > 1$.

A natural question arising from Theorem 5.5 of Liggett (1985), and raised there, was whether this bound was the best possible or, equivalently, whether $b(n) \leqq 1$ for each n implied that the NPS does not survive for all $\lambda < \lambda_0$ for some λ_0 strictly greater than 1. This question was resolved in a recent paper, Bramson (1989), where it was shown that for each $\lambda' > 1$ there is a NPS, with $b(n) \leqq 1$ for each n, which survives for $\lambda > \lambda'$. Using some of the ideas of this paper, Mountford (1991) showed that the "uniform" nearest particle system, for which $\beta(l, r) = \lambda/l + r - 1$, survived if and only if $\lambda > 1$. This work used the attractive property of the "uniform" nearest particle system extensively. This paper is an attempt to adapt the ideas therein to a non-attractive setting.

Throughout the paper we will treat a configuration η_t alternately as a subset of Z, as a member of $\{0, 1\}^Z$ or as a mapping from Z to $\{0, 1\}$. When thinking of a configuration as a subset we will use phrases such as "there is x in η_t . . ", we hope $\eta \cap [x, y]$ and $|\eta|$ will cause no confusion.

We will be dealing with λ strictly greater than 1. We will reserve ε as a shorthand for $\lambda - 1$.

Section one

In this section we introduce "Integer Splitting Processes" (ISPs) which may be thought of as an offshoot of the interval splitting processes studied in Peyriere (1979) and used in Mountford (1991).

An ISP with index α for $\alpha < 1$, (ISP(α)) is a Markov chain $\underline{N}_n : n \geqq 0$, which takes values in $\bigcup_{n>0}(Z_+)^n$. Its transition probabilities are as follows:

a. If $\underline{n} \in A = \left\{ \bigcup_{r>0} \{(z_1, z_2, \ldots z_r) \in (Z^+)^r \text{ and } \min_i z_i \leqq 5\} \right\}$, then $p(\underline{n}, \underline{n}) = 1$. That is all points in A are trap points.

b. If n is in $(Z^+)^r$ but not in A then for $j \in \{1, 2, 3, \ldots r\}$, $p(\underline{n}, \underline{n}^j) = 1/r$, where for $\underline{n} = (n_1, n_2, \ldots n_r)$, $\underline{n}^j = (n'_1, n'_2, \ldots, n'_{r+1})$ where

$$\text{for } i < j,\ n'_i = n_i$$

$$n'_j = n'_{j+1} = [\alpha n_j] + 1$$

$$\text{for } i > j + 1,\ n'_i = n_{i-1}.$$

Throughout this paper T will denote the first hitting time of the trap set A defined above. From the definitions given above we note that if $\underline{N}_0$ is in $(Z^+)^r$ and T is at least n then $\underline{N}_n \in (Z^+)^{r+n}$.

For any positive integer r and $\underline{n} = (n_1, n_2, \ldots n_r) \in (Z^+)^r$ we define $|\underline{n}^{-1}|$ to be $\sum_{j=1}^{r} \frac{1}{n_j}$. The following lemma has a simple proof which is entirely along the lines of the proof of Lemma 2.4 of Peyrière and Lemma 2.2 of Mountford (1991).

Lemma 1.1 *Let $\underline{N}_n$ be an* ISP(α) *process and let T be the first hitting time of the trap set A described above. If β is at least $2/\alpha - 1$ then for $n \geqq 1$*

$$M_n = \left(\prod_{i=1}^{T \wedge n} (1 + \beta/i) \right)^{-1} |\underline{N}_n^{-1}|$$

is a super-martingale.

A continuous time Markov chain is a continuous integer splitting process with parameters α, λ or CISP (α, λ) if its embedded discrete time Markov chain is an ISP(α) and its transition rates at points in $(Z^+)^r$ equals $r\lambda$. We can think of an ISP as a collection of integers such that at integer times a randomly chosen integer splits into two smaller integers. We can then view a CISP as a collection of integers, each one of which independently of the others splits into two smaller integers at a fixed rate constant over all integers.

Lemma 1.2 below is this paper's analogue of Lemma 2.4 of Mountford (1991).

Lemma 1.2 *Let $\underline{N}_t$ be a* CISP(α, λ) *process and let T be the first hitting time of the trap set A of Lemma* 1.1. *Suppose $\underline{N}_0 = (n_1, n_2, \ldots n_r)$ and $n_i \geqq 2^m$ for each i. Then $P[T < \alpha_1 m] \leqq Cr2^{-\alpha_1 m}$ for some C and α_1 depending on α, λ but not on m or r.*

Proof. We can consider a CISP(α, λ), $\underline{N}_t$ with $\underline{N}_0 \in (Z_+)^r$ as r independent processes with $\underline{N}_0 \in Z_+$. So without loss of generality we will suppose that $r = 1$ in the proof of this lemma.

Fix $\beta > 2/\alpha - 1$, $v < 1/\beta$, and $c < \log(2)v/\lambda$.

Let $\underline{N}'_m$ be the ISP(α) process embedded in $\underline{N}_t$. Let T' be the first hitting time of the trap set A by this embedded process. If T is less than cm then either $T' < 2^{mv}$ or the number of jumps of $\underline{N}_t$ by time cm is greater than 2^{mv}. On the event $\{T' < 2^{mv}\}$, $M_{T'}$ must assume a value greater than $\frac{1}{5}\left(\prod_{i=1}^{2^{mv}} (1 + \beta/i) \right)^{-1} > H2^{-mv\beta}$ for some constant H not depending on m. Using Doob's optional sampling theorem and the supermartingale of Lemma 1.1, we see that for fixed constants C and K

$$P[T' < 2^{mv}] < P\left[\sup_n M_n > \frac{C}{5} 2^{-mv\beta}\right] < K2^{mv\beta}/2^m$$

while

$$P[\# \text{ of jumps of } \underline{N}_t \text{ by time } cm > 2^{mv}] \leqq E[\# \text{ of jumps of } \underline{N}_t \text{ by time } cm]/2^{mv} < e^{cm\lambda}/2^{mv}.$$

It is clear that any $\alpha_1 < \min(1 - v\beta, v - c\lambda/\log(2))$ will satisfy the requirements of the lemma. □

Lemma 1.3 *Let $\underline{N}_t$ be a* CISP(α, λ) *process and let S be the first hitting time of the set $\bigcup_r \{(n_1, n_2, \ldots n_r) \in (Z^+)^r\colon \min_i n_i \leqq 2^{3n/4}\}$. Suppose $\underline{N}_0 = (n_1, n_2, \ldots n_r)$ and $n_i \geqq 2^n$ for each i. Then $P[S < \alpha_1 m] \leqq Cr2^{-\alpha_1 m}$ for some C and α_1 depending on α, λ but not on m or r.*

Sketch of Proof

As with the previous lemma we may assume that $r = 1$. Again following Lemma 1.2 we take S' to be the first hitting time of the set $\bigcup_r \{(n_1, n_2, \ldots n_r) \in (Z^+)^r : \min_i n_i \leqq 2^{3n/4}\}$.

$$P[S' < 2^{mv}] < P\left[\sup_n M_n > \frac{C}{5} 2^{-mv\beta}\right] < K2^{mv\beta}/2^{m/4}.$$

So the previous argument works with c and v reduced by a factor of 4.

Section two

In this section we introduce the coupling which will enable us to relate the behaviour of CISPs to that of various NPSs. We first require some definitions.

We are dealing with a specific nearest particle system which is defined for configurations containing infinitely many 1*s*. A *finite* NPS will be a particle system taking its value from the set of configurations with finitely many 1*s* and with flip rates equal to zero on infinite vacant intervals. Such a process must eventually be trapped by the identically zero configuration. The following lemma is obvious.

Lemma 2.1 *Consider an initial configuration η_0 (finite or infinite) for the* NPS η_t. *Let $x_1 < y_1 < x_2 < \cdots < x_n < y_n$ be integers. Then there exist independent finite* NPSs η_t^j *so that*
(i) *For each j $\eta_0^j = \eta_0 \cap [x_j, y_j]$.*
(ii) *For all time $\bigcup_{j=1}^n \eta_t^j \subset \eta_t$.*

We shall introduce for comparison purposes the continuous time birth and death process $\{B(t) : t \geqq 0\}$ which has transition rates $q(n, n+1) = \lambda(n-1)$, $q(n, n-1) = n$ for n greater than one, $q(n, n+1) = 0$ for $n = 0$ or 1 and $q(1, 0) = 1$.

Note that if η_t is a finite NPS, then until the first time that η_t contains adjacent points, $|\eta_t|$ can be thought of as a version of $B(t)$. This is one of the key ideas used in Bramson (1989).

The lemma underneath is precisely Lemma 3.2 of Mountford (1991) and so no proof will be given.

Lemma 2.2 *For each* $0 < \delta < \varepsilon = \lambda - 1$ *there exists a positive constant* c_δ *such that for all t sufficiently large*

$$P[B(t) > e^{\delta t}] > c_\delta$$

The following coupling is fundamental to our proof of Theorem one.

Lemma 2.3 *Consider an initial configuration* η_0 *so that* $\eta_0(x) = \eta_0(y) = 1$ *(assume* $x < y$*). There exists a coupling between* $\{B(t): t \geqq 0\}$, *a* CISP$(1/4, \lambda)$ *process* $\underline{N}_t$ *and a* NPS-*like process* ψ_t *so that*

1 *For all* $t < T$, *the stopping time for* $\underline{N}_t$ *referred to in Lemma* 2.1, $\psi_t \subset \eta_t$.
2 $\psi_0 = \{x, y\}$, $\underline{N}_0 = y - x$
3 *For* $t < T$, $B(t) = |\psi(t)|$.
4 *For* $t < T$, *we can write* ψ_t *as* $\{z_1 < z_2 < \cdots < z_{B(t)}\}$ *so that there exist distinct* n_{i_j} *in* $\underline{N}_t$ *so that* $z_{j+1} - z_j \geqq n_{i_j}$.

Proof. We first describe the process ψ_t. We will define ψ in such a way that if $\eta_0 = \{x, y\}$ (as opposed to simply $\{x, y\} \subset \eta_0$), then for all t, $\psi_t = \eta_t$. The process ψ is introduced because this may not be the case. Suppose that many sites around $(x + y)/2$ were occupied by η_0 or even that $[x, y]$ itself was occupied. Then there is no birth rate at all for η in the interval $[x, y]$, or there is a birth rate but far away from $(x + y)/2$. Our solution to this is (adjoining an independent Poisson process of rate λ if necessary) to say that if there is no birth rate for η on $[3x/4 + y/4, x/4 + 3y/4]$, then there must be a site in this interval occupied by η but not by ψ. So at a "birth" time for ψ we just pretend that a particle has been born at a site in $[3x/4 + y/4, x/4 + 3y/4]$ which was previously occupied by η but not by ψ. We maintain this approach for larger configurations of ψ. We hope this preamble will help the reader make sense of the formal definition for ψ. Let ψ_t equal $\{x_1 < x_2 < \cdots x_n\}$ for some n. We then define for $i < n$

$$y_i = \sup\left\{ k \leqq \frac{x_i + x_{i+1}}{2} : \eta_t(k) = 1 \right\}$$

$$z_i = \inf\left\{ k \geqq \frac{x_i + x_{i+1}}{2} : \eta_t(k) = 1 \right\}.$$

We now define w_i. There are three cases to consider: if $y_i > 3/4x_i + 1/4x_{i+1}$ then w_i is defined to be equal to y_i. If y_i is too small for w_i to be defined in this way but $z_i < 3/4x_{i+1} + 1/4x_i$ then we define w_i to be equal to z_i. If both these conditions fail we define $w_i = 1/2y_i + 1/2z_i$. Here we are assuming for notational convenience that $y_i + z_i$ is even.

We give our ψ_t process transition rates

$$q(\psi_t, \psi_t \cup \{w_i\}) = \lambda \quad \text{for each } i$$

$$q(\psi_t, \psi_t/\{x_i\}) = 1 \quad \text{for each } i$$

It must be stressed that for each i, w_i depends on η_t and that w_i may change while ψ does not. Given these rates it is clear that ψ, η and B may be coupled in the way demanded by the lemma. We now detail how the CISP process $\underline{N}_t$ is constructed:

At all times $\underline{N}_t$ will consist of a list of integers $(n_1, n_2, \ldots, n_m)$ each integer n_i will either be identified or unidentified. The unidentified integers evolve (split) at rates independent of ψ, the identified integers correspond in a one to one and onto way to the vacant intervals between two occupied sites of ψ, so that if n_i corresponds to the interval $[x_j, x_{j+1}]$, then $n_i \leqq x_{j+1} - x_j$. We couple the split rates of n_i with the birth rates on $[x_j, x_{j+1}]$ so that n_i splits into $n_{i,a}$ and $n_{i,b}$ at the identical time that $[x_j, x_{j+1}]$ splits into $[x_j, x]$ and $[x, x_{j+1}]$. The new integers are identified and correspond with the new intervals in the obvious manner. When a particle x_j dies, so two vacant intervals of ψ are merged into $[x_{j-1}, x_{j+1}]$, we simply let n_i be identified with the new interval and make the identified integer of $\underline{N}_t$ corresponding to $[x_{j-1}, x_j]$ into an unidentified integer. This specifies our coupling of ψ and $\underline{N}_t$. □

Corollary 2.4 *Let x and $y \in Z^1$ satisfy $y - x \geqq 2^m$. Consider a finite* NPS *ϕ_t so that $\phi_0(x) = \phi_0(y) = 1$. Then for some positive α_1 not depending on m, $P[\phi_s \neq 0$ for each s in $[0, \alpha_1 m]] \geqq c(\varepsilon)$ for some strictly positive function c.*

Proof. Clearly it serves to establish the existence of an α_1 which works for all m large. We use the coupling of the previous lemma.

$$P[\phi_s \neq \emptyset \forall s \in [0, \alpha_1 m]] \geqq P[\psi_s \neq \emptyset \forall s \in [0, \alpha_1 m]]$$

$$\geqq P[B(s) > 0 \forall s] - P[T < \alpha_1 m] .$$

The result follows from Lemma 1.2 and Lemma 2.2. □

Definition A_n will denote the interval $[-2^n, -3/4\, 2^n]$, B_n will denote the interval $[3/4\, 2^n, 2^n]$. For one of the intervals, I_n defined above, we say η is *in state* 1 *with respect to I_n* if $\eta \cap I_n$ contains a subset C of cardinality $2 \log^2 n$ so that every point in C is at least $2^{3n/4}$ apart from the rest of C.

Notation. From now on I_n will refer to either of the above two intervals. The following plays the same role as Lemma 3.7 in Mountford (1991).

Corollary 2.5 *Let η_0 be in state* 1 *with respect to I_n. Then for α_1 the constant of Corollary* 2.4, *$P[\eta_t \neq 0$ for each t in $[0, \alpha_1 3n/4]] > 1 - e^{-k(\varepsilon)\log^2 n}$.*

Proof. By the definition of state 1, there exist $x_1 < y_1 < x_1 \cdots < x_{\log^2 n} < y_{\log^2 n}$ so that for each i $\eta_0(x_i) = \eta_0(y_i) = 1$, $y_i \geqq x_i + 2^{3n/4}$ and $x_{i+1} > y_i + 2^{3n/4}$. By Lemma 2.1 we can couple independent processes η_t^i so that

a. For each $t \cup \eta_t^i \subset \eta_t$ and
b. For each i, $\eta_0^i \subset [x_i, y_i]$.

Now η_t will be non-zero for all t in $[0, \alpha_1 3n/4]$ if the same is true for one of the η^is. Given the independence of these processes and Corollary 2.4, it follows that

$$P[\text{there exists } t \text{ in } [0, \alpha_1 3n/4] \text{ so that } \eta_t = 0] < (1 - c(\varepsilon))^{\log^2 n}$$

and the result follows. □

Section three

Define the stopping time T_{I_n} to be the first time that ϕ_t is in state 1 with respect to I_n.

Lemma 3.1 *Let $\eta_0(x) = \eta_0(y) = 1$ for $x, y \in I_n$ with $y - x \geqq 2^n/32$. Then there exists $k > 0$, not depending on x, y, n so that $P[\eta_{5\log\log(n)/\varepsilon - 1}$ is in State $1] > k$.*

Proof. This result follows from the coupling of Lemma 2.3 with Lemma 2.2 and Lemma 1.3. □

Lemma 3.2 *The events*

$$C_n = \{\text{for each } t \text{ in } (0, \sqrt{n})\ \eta_t \cap A_{n+1} \neq \emptyset\}$$
$$\cap \{\text{for each } t \text{ in } (0, \sqrt{n})\ \eta_t \cap B_{n+1} \neq \emptyset\} \cap \{T_{A_n} > \sqrt{n}\}$$

and

$$D_n = \{\text{for each } t \text{ in } (0, \sqrt{n})\ \eta_t \cap A_{n+1} \neq \emptyset\}$$
$$\cap \{\text{for each } t \text{ in } (0, \sqrt{n})\ \eta_t \cap B_{n+1} \neq \emptyset\} \cap \{T_{B_n} > \sqrt{n}\}$$

have probability less than $e^{-k(\varepsilon)\sqrt{n}/\log\log n}$ for n large enough. This bound is uniform over all initial configurations.

Proof. Obviously it suffices to prove the bound for C_n only. It is clear that given only $\eta_0 \cap A_{n+1}$ and $\eta_0 \cap B_{n+1} \neq \emptyset$, there is a constant $c > 0$ so that

$$P[\eta_1 \text{ contains two points in } A_n \text{ more than } 2^n/32 \text{ apart}] > c\ .$$

By Lemma 3.1 and the Markov property this gives $P[\eta_{5\log\log(n)/\varepsilon}$ is in State $1] > ck$. Using the Markov property again, we see that $P[C_n] \leqq (1 - ck)^{\varepsilon\sqrt{n}/5\log\log(n) - 1}$. □

Using the strong Markov property and Corollary 2.5 we obtain the following corollary

Corollary 3.2 *For all t and for all n large enough, the event*

$$D_{n,t} = \{\text{for each } s \text{ in } (t, t + \sqrt{n})\ \eta_s \cap A_{n+1} \neq \emptyset\}$$
$$\cap \{\text{for each } s \text{ in } (t, t + \sqrt{n})\ \eta_s \cap B_{n+1} \neq \emptyset\}$$
$$\cap \{\text{there exists } s \in (t + \sqrt{n}, t + \sqrt{n} + \sqrt{n-1}) \text{ s.t.}$$
$$\eta_s \cap A_n = \emptyset \text{ or } \eta_s \cap B_n = \emptyset\}$$

has probability bounded by $e^{-k(\varepsilon)\log^2 n}$. Again the bound is uniform over all initial configurations.

Section four (Proof of Theorem one)

Recall the definition of the event $D_{n,t}$ given in the statement of Corollary 3.2. Our strategy is to use (and reuse) Corollary 3.2 to complete the proof of Theorem one. Corollary 3.2 says that if there are occupied sites in A_{n+1} and B_{n+1} during a time period $(s, s + \sqrt{n})$, then with large probability there will be occupied sites in A_n and B_n throughout the interval $(s + \sqrt{n}, s + \sqrt{n} + \sqrt{n-1})$. Iterating this we will show that with non-negligible probability there is at time of the order $s + \sum_{i=1}^{n} \sqrt{i}$, a particle of η close to the origin. Let us fix N so that $\sum_{m \geqq N} e^{-k(\varepsilon)\log^2 m} < 1/3$ for $k(\varepsilon)$ the constant given by Corollary 3.2. Now for t large enough we can find an n ($> N$) so that

$$t = \sum_{m=N}^{n} m^{1/2} + s + 1$$

and $\sqrt{n} < s < (n+1)/28 - \sqrt{n}$. Define the constants

$$l_i = \sum_{j=n+1-i}^{n} \sqrt{j}\,.$$

Suppose $\eta_{t-1}^1 \cap A_N = \emptyset$ or $\eta_{t-1}^1 \cap B_N = \emptyset$. Then one of the following must have occurred:

a. There exist a time v before $(s + \sqrt{n})$ at which either $\eta_v^1 \cap A_{n+1} = \emptyset$ or $\eta_v^1 \cap B_{n+1} = \emptyset$ or

b. There exists a j among $\{1, 2, \ldots n - N + 1\}$ so that $D_{n+1-j, s+l_j}$ occurs.

Now it is easy to see that the probability that $\eta_v^1 \cap A_{n+1}$ is empty for some $v \leqq s + \sqrt{n} \leqq (n+1)/28$ is less than the probability that $2^{n+1}/4$ independent mean one exponential random variables are all less than $(n+1)/28$. This probability is less than $(1 - e^{-(n+1)/28})^{2^{n+1}/4}$. Thus the probability of the event "a" is more than exponentially small. On the other hand by Corollary 3.2 the probability of "b" is less than $\sum_{m=N}^{n} e^{-k(\varepsilon)\log^2 m}$, which by our choice of N is less than 1/3. Thus (for t large enough) the probability that at time $t-1$ η_{t-1}^1 has occupied sites in A_N and B_N is greater than 2/3. It is clear that if $\eta_0(x) = \eta_0(y) = 1$, then with nonzero probability, depending only on $|x - y|$, all sites in $[x, y]$ are occupied by η_1. Therefore by the strong Markov property we can find $c(N)$ so that for all t large enough $P[\eta_t^1(0) = 1] > c(N)$. This completes the proof of Theorem one. □

Section five (Proof of Theorem two)

In this section we seek to establish the existence of a non-trivial invariant measure. In our previous paper we dealt with the attractive case and survival in the sense of Theorem one of this paper guaranteed the existence of an invariant measure. We have shown that the measures P_t^1 on $\{0, 1\}^Z$ are such that there exists a strictly positive c so that $P_t^1[\eta(0)] > c$. Thus any weak cluster point μ, of the measures

$$\mu_t(A) = \frac{1}{t} \int_0^t P_s^1[A]\, ds$$

must have $\mu[\eta(0)] > c$, and hence must be non-trivial. Unfortunately it cannot be immediately deduced that any such measure is invariant for our process, since the semi-group is not Feller, as noted in the introduction. We will now fix a sequence t_i tending to infinity so that μ, the weak limit of μ_{t_i} exists. The rest of the paper will be devoted to showing that μ must be invariant, which by the foregoing discussion will establish Theorem two.

We will be done if we can show that for any positive integer r and any cylinder function f, depending on $\{0, 1\}^{[-r,r]}$ we have

$$\mu(P_t(f)) = \mu(f) . \qquad (*)$$

We now fix r and f. We shall now without loss of generality assume that the L^∞ norm of f is bounded by 1. Our problem is that $P_t(f)$ is not necessarily continuous so that while

$$\lim_{i \to \infty} \mu_{t_i}(P_t(f)) = \lim_{i \to \infty} \mu_{t_i}(f) = \mu(f)$$

it is no longer obvious that the first limit above should equal $\mu(P_t(f))$. We are forced to use the following "link" functions (defined for n greater than r)

$$f_{n,t}(\eta) = E_\eta[f(\eta_t); \exists i \in [-2n, -n] \text{ s.t. }$$
$$\eta_s(i) = 1 \forall s \in [0, t]; \exists i \in [n, 2n] \text{ s.t. } \eta_s(i) = 1 \forall s \in [0, t]] .$$

These linking functions are useful since they are continuous and also they are close to the function $P_t(f)$ in the following sense.

Lemma 5.1 *For any measure* v,

$$|\int f_{n,t}(\eta) v(d\eta) - \int P_t(f)(\eta) v(d\eta)| \leqq \int P^\eta[\forall i \in [-2n, -n] \exists s \in [0, t] \text{ s.t. }$$
$$\eta_s(i) = 0 \text{ or } \forall i \in [n, 2n] \exists s \in [0, t] \text{ s.t. } \eta_s(i) = 0] v(\eta) .$$

If we define $n_1(\eta)$ to equal $\sum_{j=-2n}^{-n} \eta(j)$ and $n_2(\eta)$ to be $\sum_{j=n}^{2n} \eta(j)$, then the right most integral above is majorized by

$$\int (1 - e^{-t})^{n_1(\eta)} + (1 - e^{-t})^{n_2(\eta)} v(d\eta) .$$

Proof is clear.

Lemma 5.2 *Let* $n_k(\eta)$ *be as defined above then*

$$\sup_s \int (1 - e^{-t})^{n_1(\eta)} + (1 - e^{-t})^{n_2(\eta)} P_s^1(d\eta). \to 0 \quad \text{as} \quad n \to \infty .$$

Proof. It is sufficient to establish that for any $\delta > 0$ and all s sufficiently large, there is n_0 such that for all s sufficiently large and $n \geqq n_0$

$$\int (1 - e^{-t})^{n_1(\eta)} + (1 - e^{-t})^{n_2(\eta)} P_s^1(d\eta) < \delta .$$

Recall that $n_i(\eta)$ depends implicitly on n.

Fix $\delta > 0$, and integer r. We choose N so large that the asymptotics of Corollary 3.2 hold and $\sum_{n=N} e^{-k(\varepsilon)\log^2(n)} < \delta/2r$. Then it is easy to see that the arguments of Section four show that for s sufficiently large, $P^1[\exists \in A_N$, s.t. $(\eta_s(i) = 1] > 1 - \delta/2r$. The measures μ_s are all translation invariant so it follows that for s large enough for the above to hold and $n > r2^N$, we have $P^1[n_1(\eta_s)$, $n_2(\eta_s) > r] > 1 - \delta$. Therefore for s and n as above

$$\int (1 - e^{-t})^{n_1(\eta)} + (1 - e^{-t})^{n_2(\eta)} P_s^1(d\eta) \leqq \delta + 2(1 - e^{-t})^r .$$

Since δ are r are arbitrary, the proof is complete. $\square$

We are now ready to show $(*)$ and thus complete the paper:

Proof of Theorem two

As was stated in the opening paragraphs of the section we merely must show that $\mu(P_t(f)) = \mu(f)$. It follows from Proposition 5.1 that uniformly for v in $\{\mu_s, s \geqq 0, \mu\}$ we have $\int (1 - e^{-t})^{n_1(\eta)} + (1 - e^{-t})^{n_2(\eta)} v(d\eta) \to 0$. In particular we have

$$\mu(P_t(f)) = \lim_{n \to \infty} \mu(f_{n,t})$$

by the continuity of the latter integrand this is equal to

$$\lim_{n \to \infty} \lim_{i \to \infty} \mu_{t_i}(f_{n,t})$$

invoking the uniform convergence once more we see that the above equals

$$\lim_{i \to \infty} \lim_{n \to \infty} \mu_{t_i}(f_{n,t}) = \lim_{i \to \infty} \mu_{t_i}(P_t(f)) .$$

As noted above this limit equals (by the definition of the μ_ts and weak convergence) $\mu(f)$ and we are done. □

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