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Probability Theory and Related Fields

Black holes on the plane drawn by a Wiener process

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Summary. We say that the disc $D(\alpha) \subset R^2$ of radius α, located around the origin is ρ-covered in time T by a Wiener process $W(\cdot)$ if for any $z \in D(\alpha)$ there exists a $0 \leqq t \leqq T$ such that $W(t)$ is a point of the disc of radius ρ, located around z. The supremum of those α's ($\alpha \geqq 0$) is studied for which $D(\alpha)$ is ρ-covered in T.

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1 Introduction

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random vectors taking values from Z^2 with distribution

$$\mathbf{P}\{X_1 = (0,1)\} = \mathbf{P}\{X_1 = (0,-1)\} = \mathbf{P}\{X_1 = (1,0)\} = \mathbf{P}\{X_1 = (-1,0)\} = 1/4$$

and let

$$S_0 = 0 = (0,0) \quad \text{and} \quad S_n = S(n) = X_1 + X_2 + \cdots + X_n \quad (n = 1, 2, \ldots),$$

i.e. $\{S_n\}$ is the simple, symmetric random walk on the plane. Further let

$$\xi(x, n) = \#\{k : 0 < k \leqq n, S_k = x\}$$

($n = 1, 2, \ldots; x = (i,j);\ i,j = 0, \pm 1, \pm 2, \ldots$) be the local time of the random walk. We say that the disc

$$Q(N) = \{x = (i,j) : \|x\| = (i^2 + j^2)^{1/2} \leqq N\}$$

is covered by the random walk in time n if

$$\xi(x, n) > 0 \quad \text{for every } x \in Q(N) .$$

Let $R(n)$ be the largest integer for which $Q(R(n))$ is covered in n. We quote a few known properties of $R(n)$.

Theorem A. ([2] Theorem 22.1) *For any* $0 < \varepsilon < 1$ *and* $C > 0$ *we have*

(a) $$R(n) \leqq \exp(2(\log n)^{1/2} \log_3 n) \quad a.s.$$

for all but finitely many n,

(b) $$R(n) \geqq \exp\left(\frac{1-\varepsilon}{\sqrt{120}} (\log n \log_3 n)^{1/2}\right) \quad i.o.\ a.s.$$

(c) $$R(n) \leqq \exp(C(\log n)^{1/2}) \quad i.o.\ a.s.$$

(d) $$R(n) \geqq \exp((\log n)^{1/2} (\log_2 n)^{-1/2-\varepsilon}) \quad a.s.$$

for all but finitely many n.

Here and in what follows $\log_p$ is the p-th iterated of log.

It is natural to study the analogue properties of a Wiener process $\{W(t) \in \mathbf{R}^2, t \geqq 0\}$. We say that the disc

$$D(\alpha) = \{z = (x, y) : \|z\| = (x^2 + y^2)^{1/2} \leqq \alpha\} \subset \mathbf{R}^2$$

is ρ-covered by $W(\cdot)$ in time T if for every $z \in D(\alpha - \rho)$ there exists a $0 \leqq t = t(z, \rho, \omega) \leqq T$ such that

$$W(t) \in D(z, \rho) = \{u : \|z - u\| \leqq \rho\} .$$

Let $R = R(T, \rho) = R(T, \rho, \omega)$ be the supremum of those r's ($r \geqq 0$) for which $D(r)$ is ρ-covered in T. We prove the following analogue of Theorem A.

Theorem 1. *For any* $\rho > 0$ *and* $\delta > 0$ *there exists a* $T_0 = T_0(\rho, \delta, \omega) > 0$ *such that*

$$\exp((\log T)^{1/2} (\log_2 T)^{-1/2-\delta}) \leqq R(T, \rho) \leqq \exp(2(\log T)^{1/2} \log_3 T)$$

for all $T \geqq T_0$.

It is also natural to ask whether the above lower estimate of R remains true if ρ is replaced in it by a function $\rho(T) \downarrow 0$ $(T \to \infty)$. The following two theorems give answers of this question.

Theorem 2. *Let*

$$\rho(T) = \rho(T, \gamma) = \exp(-(\log T)^{1/2-\gamma}) \quad (0 < \gamma < 1/2) .$$

Then for any $\delta > 0$ *and* $0 < \gamma < 1/2$ *there exists a* $T_0 = T_0(\delta, \gamma, \omega) > 0$ *such that*

$$R(T, \rho(T)) \geqq \exp((\log T)^{1/2} (\log_2 T)^{-1/2-\delta})$$

for all $T \geqq T_0$.

Theorem 3. *Let*

$$\rho_1(T) = \rho_1(T, \gamma) = \exp(-(\log T)^{1/2+\gamma}) \quad (\gamma > 0) .$$

Then there exist a $T_0 = T_0(\gamma, \omega) > 0$ *and a*

$$z = z(T, \gamma, \omega) \in D(1)$$

such that

$$\mathbf{P}\{W(t) \notin D(z, \rho_1(T)) \quad \textit{for all } 0 \leqq t \leqq T \textit{ and } T \geqq T_0\} = 1 .$$

In order to illuminate the meanings of Theorems 2 and 3 it is worthwhile to recall a theorem of Spitzer (1958).

Theorem B. *Let $g(t)$ be a positive, nonincreasing function. Then*

$$\| W(T) \| \geqq T^{1/2} g(T) \quad a.s.$$

for any T big enough if and only if

$$\sum_{k=1}^{\infty} (k|\log g(k)|)^{-1} < \infty .$$

Note that the function

$$g(T) = T^{-(\log_2 T)^{1+\beta}}$$

satisfies the above conditions if and only if $\beta > 0$. Hence a Wiener process will meet the disc $D(T^{1/2}g(T))$ in time T i.o. if and only if $\beta \leqq 0$. However Therorem 3 claims that within the unit circle $D(1)$ there exists a disc of radius $\rho_1(T) \gg T^{1/2}g(T)$, ($\beta = 0$) not visited by $W(\cdot)$ till T. At the same time Theorem 2 tells us that any disc of radius $\rho(T)$ within $D(1)$ or even within

$$D(\exp((\log T)^{1/2}(\log_2 T)^{-1/2-\delta}))$$

will be visited by $W(\cdot)$ before T.

In [3] and [4] we investigated the radius of the largest disc (not necessarily around the origin) covered by the random walk in time n. Formally speaking let $u = (u_1, u_2) \in Z^2$ and define

$$Q(u, N) = \{x = (i, j) : \| x - u \|^2 = (i - u_1)^2 + (j - u_2)^2 \leqq N^2\} .$$

Let $r(n)$ be the largest integer for which there exists a random vector $u = u(n) \in Z^2$ such that $Q(u, r(n))$ is covered by the random walk in time n i.e.

$$\xi(x, n) \geqq 1 \quad \text{for every} \quad x \in Q(u, r(n)) .$$

On the limit behaviour of $r(n)$ we proved

Theorem C.

$$n^{1/50} \leqq r(n) \leqq n^{0.42} \quad a.s.$$

for all but finitely many n.

In order to study the analogous properties of a Wiener process we say that the disc

$$D(\alpha, u) = \{z : \| z - u \| \leqq \alpha\}$$

is ρ-covered by $W(\cdot)$ in time T if for every $z \in D(\alpha - \rho, u)$ there exists a $0 \leqq t = t(z, \rho, \omega) \leqq T$ such that

$$W(t) \in D(z, \rho) .$$

Let $r(T, \rho)$ be the supremum of those r's ($r \geqq 0$) for which there exists a random vector $u = u(T, \omega)$ such that $Q(u, r)$ is ρ-covered by $W(\cdot)$ in time T. On the limit behaviour of $r(T, \rho)$ we present our

Theorem 4. *Let* $\kappa = \bar{\rho} = 0.005$. *Then*

$$r(T, T^{-\bar{\rho}}) \geqq T^{\kappa} \quad a.s.$$

if T *is big enough.*

Unfortunately I cannot characterize the domain of the points of $(\kappa, \bar{\rho})$ for which the above inequality holds true. The choice $\kappa = \bar{\rho} = 0.005$ is just one of the possibilities.

Comparing Theorem 4 with our previous results we can say that the "largest black hole" is not only much larger than the "black hole" around the origin but it is also "much more black".

Theorem 5. *For any fixed* $\vartheta > 0$ *we have*

$$r(T, \vartheta) \leqq T^{\chi} \quad a.s.$$

if T *is big enough and* $\chi > \sqrt{6} - 2$.

In [3] and [4] we presented the

Conjecture. *There exists a*

$$\frac{1}{50} \leqq \alpha \leqq 0.42$$

such that

$$\lim_{n \to \infty} \frac{\log r(n)}{\log n} = \alpha \quad a.s.$$

Now, we present the following analogue

Conjecture. *There exists a*

$$0.005 \leqq \beta \leqq \sqrt{6} - 2$$

such that

$$\lim_{T \to \infty} \frac{\log r(T, \vartheta)}{\log T} = \beta \quad a.s.$$

where β *does not depend on* ϑ.

2 Proof of Theorem 2

Lemma A. (Knight 1981, Theorem 4.3.8) *Let* $0 < a < b < c < \infty$. *Then*

$$\mathbf{P}\{\inf\{s : s > 0, \| W(t+s) \| = a\} < \inf\{s : s > 0, \| W(t+s) \| = c\} \mid B\} = \frac{\log c - \log b}{\log c - \log a}$$

where $B = \{\| W(t) \| = b\}$.

Definitions. The Wiener path $\{W(t); 0 < t_1 \leqq t \leqq t_2\}$ is called a ϑ-excursion if

(a) $$\| W(t_1) \| = \| W(t_2) \| = \vartheta ,$$

(b) $$\| W(t) \| > \vartheta \quad (t_1 < t < t_2) ,$$

(c) $$\exists t : t_1 < t < t_2, \| W(t) \| \geqq 2\vartheta .$$

Let $e(T) = e(T, \vartheta)$ be the number of ϑ-excursions completed before T.
Let $l = l(\vartheta) = t_2 - t_1$ be the length of the first ϑ-excursion ($\{W(t); 0 < t_1 \leqq t \leqq t_2\}$ is the first ϑ-excursion).

Let $v = v(\vartheta) = t_3 - t_2$ be the waiting time between the first and the second ϑ-excursion. (t_3 is the starting point of the second ϑ-excursion.)
Finally, let

$$\bar{D}(u, r) = \{x : \| x - u \| = r\} ,$$

$$\tau(u,r) = \inf\{s : W(s) \in \bar{D}(u, r)\} .$$

The proof of Theorem 2 is based on quite a few lemmas. Before presenting them in detail we give the basic idea of the proof.
Step 1. We prove that the number of ϑ-excursions completed before T cannot be very small. In fact Lemma 6 tells us that for any $\varepsilon > 0$

$$e(T, \vartheta) \geqq (\log T)(\log_2 T)^{-1-\varepsilon} \quad \text{a.s.}$$

if T is big enough.
Step 2. We prove that the probability that a ϑ-excursion meets a disc $D(x, \vartheta)$ cannot be very small (Lemma 7).

These two facts will imply Theorem 2.

Lemma 1. *Let $u \in \mathbf{R}^2$ satisfying the inequality*

$$e^{-T} < r < 2r \leqq \| u \| < T^{1/2-\delta} .$$

Then for any $0 < \varepsilon < \delta < 1/2$ there exists a $T_0 = T_0(\varepsilon, \delta) > 0$ and an absolute constant $C > 0$ such that

$$1 - (1 + \varepsilon)\frac{\log(\| u \| r^{-1})}{\log(T^{1/2-\varepsilon} r^{-1})} \leqq \frac{\log T^{1/2-\varepsilon} - \log \| u \|}{\log T^{1/2-\varepsilon} - \log r} - \exp(-CT^{2\varepsilon})$$

$$\leqq \mathbf{P}\{\exists t : 0 \leqq t \leqq T, W(t) \in D(u, r)\}$$

$$\leqq \frac{\log T^{1/2+\varepsilon} - \log \| u \|}{\log T^{1/2+\varepsilon} - \log r} + \exp(-CT^{2\varepsilon}) \leqq 1 - (1 - \varepsilon)\frac{\log(\| u \| r^{-1})}{\log(T^{1/2+\varepsilon} r^{-1})}$$

if $T \geqq T_0$.

Proof. Clearly we have

$$\mathbf{P}\{\exists t : 0 \leqq t \leqq T, W(t) \in D(u, r)\} = \mathbf{P}\{\tau(u, r) \leqq T\}$$

$$\geqq \mathbf{P}\{\tau(u, r) \leqq \tau(u, T^{1/2-\varepsilon}) \leqq T\}$$

$$\geqq \mathbf{P}\{\tau(u, r) \leqq \tau(u, T^{1/2-\varepsilon})\} - \mathbf{P}\{\tau(u, T^{1/2-\varepsilon}) > T\}$$

$$\geqq \frac{\log T^{1/2-\varepsilon} - \log \| u \|}{\log T^{1/2-\varepsilon} - \log r} - \exp(-CT^{2\varepsilon}) .$$

Further

$$\mathbf{P}\{\exists t: 0 \leqq t \leqq T,\ W(t) \in D(u, r)\} \leqq \mathbf{P}\{\tau(u, r) \leqq \tau(u, T^{1/2+\varepsilon}) \text{ or } \tau(u, T^{1/2+\varepsilon}) \leqq T\}$$

$$\leqq \frac{\log T^{1/2+\varepsilon} - \log\|u\|}{\log T^{1/2+\varepsilon} - \log r} + \exp(-CT^{2\varepsilon}) .$$

Hence we have Lemma 1.

Lemma 2. *For any* $\varepsilon > 0$ *there exist a* $T_0 = T_0(\varepsilon) > 0$ *and an absolute constant* $C > 0$ *such that*

$$(1 - 2\varepsilon)\frac{2\log 2}{\log(T\vartheta^{-1})} \leqq (1 - \varepsilon)\frac{\log 2}{\log(T^{1/2+\varepsilon}\vartheta^{-1})} - \exp(-CT\vartheta^{-2})$$

$$\leqq \mathbf{P}\{e(T, \vartheta) = 0\}$$

$$\leqq (1 + \varepsilon)\frac{\log 2}{\log(T^{1/2+\varepsilon}\vartheta^{-1})} + \exp(-CT\vartheta^{-2})$$

$$\leqq (1 + 2\varepsilon)\frac{2\log 2}{\log(T\vartheta^{-1})}$$

if $T \geqq T_0$ *and* $e^{-T} < \vartheta < 1$.

Proof. Applying Lemma 1 with $\|u\| = 2\vartheta$, $r = \vartheta$ we obtain

$$\mathbf{P}\{e(T, \vartheta) = 0\} = \mathbf{P}\{W(t)\ (0 \leqq t \leqq T) \text{ does not meet } \bar{D}(0, 2\vartheta) \text{ or}$$

$$\text{it does but from } \bar{D}(0, 2\vartheta) \text{ it does not return to } \bar{D}(0, \vartheta)\}$$

$$\leqq \exp(-CT\vartheta^{-2}) + (1 + \varepsilon)\frac{\log 2}{\log(T^{1/2-\varepsilon}\vartheta^{-1})}$$

which implies the upper part of our inequality. Similarly

$$\mathbf{P}\{e(T) = 0\} \geqq \mathbf{P}\{W(t) + (2\vartheta, 0)\ (0 \leqq t \leqq T) \text{ does not meet } \bar{D}(0, \vartheta)\}$$

$$\geqq (1 - \varepsilon)\frac{\log 2}{\log(T^{1/2+\varepsilon}\vartheta^{-1})} .$$

Hence we have Lemma 2.

Lemma 3. *There exists an absolute constant* $C > 0$ such that

$$\mathbf{P}\{v(\vartheta) > u\} \leqq \exp(-Cu\vartheta^{-2}) \quad (u > 0, \vartheta > 0) .$$

Proof.

$$\mathbf{P}\{v(\vartheta) > u\} \leqq \mathbf{P}\{\text{the waiting time to arrive } \bar{D}(0, 2\vartheta) \text{ from } \bar{D}(0, \vartheta) > u\}$$

$$\leqq \mathbf{P}\left\{\sup_{t \leqq u} \|W(t)\| < 3\vartheta\right\} \leqq \exp(-Cu\vartheta^{-2}) .$$

Hence we have Lemma 3.

Lemma 4. *For any* $0 < \varepsilon < 1/2$ *there exists a* $u_0 = u_0(\varepsilon) > 0$ *such that*

$$(1-\varepsilon)\frac{2\log 2}{\log(u\vartheta^{-1})} \leqq \mathbf{P}\{l + v > u\} \leqq (1+\varepsilon)\frac{2\log 2}{\log(u\vartheta^{-1})}$$

if $u \geqq u_0$ *and* $\exp(-u/2) < \vartheta < 1$.

Proof. By Lemmas 2 and 3 we have

$$\begin{aligned}\mathbf{P}\{l + v > u\} &\leqq \mathbf{P}\left\{l > \frac{u}{2}\right\} + \mathbf{P}\left\{v > \frac{u}{2}\right\} \\ &\leqq \mathbf{P}\left\{e\left(\frac{u}{2}, \vartheta\right) = 0\right\} + \mathbf{P}\left\{v > \frac{u}{2}\right\} \\ &\leqq (1+\varepsilon)\frac{\log 2}{\log(u^{1/2+\varepsilon}2^{-1/2-\varepsilon}\vartheta^{-1})} + \exp(-Cu\vartheta^{-2}) \\ &\leqq (1+2\varepsilon)\frac{2\log 2}{\log(u\vartheta^{-1})}\end{aligned}$$

and we have the upper part of Lemma 4. Since by Lemma 2

$$\begin{aligned}(1-\varepsilon)\frac{2\log 2}{\log(u\vartheta^{-1})} &\leqq \mathbf{P}\{e(2u, \vartheta) = 0\} \\ &\leqq \mathbf{P}\left\{\sup_{t \leqq u} \|W(t)\| \leqq \vartheta\right\} + \mathbf{P}\{l > u\} \\ &\leqq \exp(-Cu\vartheta^{-2}) + \mathbf{P}\{l + v > u\}\end{aligned}$$

and we obtain the lower part as well.

Lemma 5. *For any* $\varepsilon > 0$ *and* $0 < \psi < 1/4$ *there exists a* $T_0 = T_0(\varepsilon, \psi) > 0$ *such that*

$$\begin{aligned}1 - \exp(-(1-\varepsilon)2(\log 2)x) &\leqq \mathbf{P}\{e(T, \vartheta) < x\log(T\vartheta^{-1})\} \\ &\leqq 1 - \exp(-(1+\varepsilon)2(\log 2)x)\end{aligned}$$

if $T \geqq T_0$, $0 \leqq x \leqq (\log(T\vartheta^{-1}))^2$ *and* $\exp(-T^{1/4-\psi}) \leqq \vartheta < e^{-1}$.

Proof. Let

$$q = [x\log(T\vartheta^{-1})] + 1, \quad u = Tq^{-1}$$

and observe that

$$\begin{aligned}\exp\left(-\frac{u}{2}\right) = \exp\left(-\frac{T}{2q}\right) &\leqq \exp\left(-(1-\varepsilon)\frac{T}{2x\log(T\vartheta^{-1})}\right) \\ &\leqq \exp\left(-(1-\varepsilon)\frac{T}{2(\log(T\vartheta^{-1}))^3}\right) \leqq \exp(-T^{1/4-\psi}) \leqq \vartheta .\end{aligned}$$

Then by Lemma 4 we have

$$\mathbf{P}\{e(T,\vartheta)\geqq q\}\geqq(\mathbf{P}\{l+v<Tq^{-1}\})^q\geqq\left(1-(1+\varepsilon)\frac{2\log 2}{\log(Tq^{-1}\vartheta^{-1})}\right)^q$$

$$\geqq\exp(-(1+2\varepsilon)2(\log 2)x)$$

if T is big enough. Which, in turn, implies the upper part of the inequality of Lemma 5.

Let $l_1, l_2, \ldots$ be the lengths of the first, second, ... ϑ-excursions and let $v_1, v_2, \ldots$ be the waiting times between the first and second, the second and third, ... ϑ-excursions. Let E_k $(k=1,2,\ldots,q)$ be the event that precisely k of the variables l_i+v_i $(i=1,2,\ldots,q)$ are greater than or equal to T, while $q-k$ of them are less than T. Then

$$\{e(T,\vartheta)\geqq x\log(T\vartheta^{-1})\}\subset\prod_{k=1}^{q}\bar{E}_k,$$

where $\bar{E}_k$ is the complement of E_k. Hence

$$\mathbf{P}\{e(T,\vartheta)<x\log(T\vartheta^{-1})\}\geqq\sum_{k=1}^{q}\mathbf{P}\{E_k\}$$

$$=\sum_{k=1}^{q}\binom{q}{k}(\mathbf{P}\{l_1+v_1\geqq T\})^k(1-\mathbf{P}\{l_1+v_1\geqq T\})^{q-k}$$

$$=1-(1-\mathbf{P}\{l_1+v_1\geqq T\})^q\geqq 1-\left(1-(1-\varepsilon)\frac{2\log 2}{\log(T\vartheta^{-1})}\right)^q$$

$$\geqq 1-\exp(-(1-2\varepsilon)2(\log 2)x)$$

which implies the lower part of Lemma 5.

Lemma 6. *For any $\varepsilon>0$ there exists a $T_0=T_0(\varepsilon,\omega)>0$ such that*

$$(\log T)(\log_2 T)^{-1-\varepsilon}\leqq e(T,\vartheta)\leqq(1+\varepsilon)(2\log 2)^{-1}(\log T)\log_3 T$$

if $T\geqq T_0$.

Proof. Let

$$T_k=\exp((1+\varepsilon)^k)\quad k=1,2,\ldots$$

Then by Lemma 5 we have

$$\mathbf{P}\{e(T_k,\vartheta)\geqq(1+2\varepsilon)(2\log 2)^{-1}(\log T_k)\log_3 T_k\}$$

$$\leqq\exp\left(-\left(1+\frac{\varepsilon}{2}\right)\log_3 T_k\right)\leqq O(k^{-1-\varepsilon/2})$$

and by Borel-Cantelli lemma

$$e(T_k,\vartheta)<(1+2\varepsilon)(2\log 2)^{-1}(\log T_k)\log_3 T_k\quad\text{a.s.}$$

for all but finitely many k. Let $T_k \leqq T \leqq T_{k+1}$. Then

$$e(T, \vartheta) \leqq e(T_{k+1}, \vartheta) \leqq (1 + 2\varepsilon)(2\log 2)^{-1}(\log T_{k+1})\log_3 T_{k+1}$$
$$\leqq (1 + 3\varepsilon)(2\log 2)^{-1}(\log T)\log_3 T .$$

Hence we have the upper part of Lemma 6.
In order to prove its lower part, let

$$T'_k = \exp(e^k) .$$

Then

$$\mathbf{P}\left\{e(T'_k, \vartheta) < \frac{\log T'_k}{(\log_2 T'_k)^{1+2\varepsilon}}\right\} \leqq O(k^{-1-\varepsilon})$$

and by Borel-Cantelli lemma

$$e(T'_k, \vartheta) \geqq \frac{\log T'_k}{(\log_2 T'_k)^{1+2\varepsilon}} \quad \text{a.s.}$$

for all but finitely many k. Let $T'_k \leqq T \leqq T'_{k+1}$. Then

$$e(T, \vartheta) \geqq e(T'_k, \vartheta) \geqq \frac{\log T'_k}{(\log_2 T'_k)^{1+2\varepsilon}} \geqq \frac{\log T}{(\log_2 T)^{1+3\varepsilon}} .$$

Hence we have the Lemma.

Lemma 7. *Let $\{W(t); 0 < t_1 \leqq t \leqq t_2\}$ be a ϑ-excursion. Then*

$$\frac{1}{6}\frac{\log 2}{\log(\|x\|\vartheta^{-1})} \leqq \mathbf{P}\{\exists t : t_1 \leqq t \leqq t_2,\, W(t) \in D(x, \vartheta)\} \leqq \frac{\log 2}{\log(\|x\|\vartheta^{-1})}$$

provided that $2\vartheta \leqq \|x\|$.

Proof. Let $x = \|x\|e^{i\varphi}$. Then by Lemma A and symmetry reasons the probability that the ϑ-excursion $\{W(t); 0 < t_1 \leqq t \leqq t_2\}$ meets the arc $\|x\|e^{i\alpha}\left(\varphi - \frac{\pi}{3} < \alpha < \varphi + \frac{\pi}{3}\right)$ is larger than or equal to

$$\frac{1}{3}\frac{\log 2\vartheta - \log\vartheta}{\log\|x\| - \log\vartheta} = \frac{1}{3}\frac{\log 2}{\log(\|x\|\vartheta^{-1})} .$$

(Note that if $\tau = \inf\{t : t > 0, \|W(t)\| = 2\vartheta\}$ then $W(\tau)$ is uniformly distributed on the circle $2\vartheta e^{i\varphi}$ $(-\pi \leqq \varphi \leqq \pi)$.) Since starting from any point of the arc $\|x\|e^{i\alpha}\left(\varphi - \frac{\pi}{3} < \alpha < \varphi + \frac{\pi}{3}\right)$ the probability that $W(t)$ meets $\bar{D}(x, \vartheta)$ before $\bar{D}(0, \vartheta)$ is larger than $1/2$, we obtain our lower estimate.

The upper estimate is trivial by Lemma A.

Proof of Theorem 2. Let $0 < t_1 < t_2 \leqq t_3 < t_4 \leqq \cdots \leqq t_{2N-1} < t_{2N}$ be the start- resp. endpoints of the first, second, ..., N-th ϑ-excursion of $W(\cdot)$.

Assume that

$$\|x\| \leqq \exp(N^{1/2}(\log N)^{-\varepsilon})$$

and

$$\exp(-N^{1/2}(\log N)^{-2\varepsilon}) \leqq \vartheta < 1 .$$

Then by Lemma 7 we have

$$\begin{aligned} &\mathbf{P}\{\nexists t: 0 \leqq t \leqq t_{2N},\ W(t) \in D(x, \vartheta)\} \\ &\leqq \left(1 - \frac{1}{12} \frac{\log 2}{\log(\|x\| \vartheta^{-1})}\right)^N \leqq \exp\left(-\frac{1}{12} \frac{(\log 2)N}{\log(\|x\| \vartheta^{-1})}\right) \\ &\leqq \exp\left(-\frac{1}{20} N^{1/2}(\log N)^{\varepsilon}\right). \end{aligned}$$

Let $x_1, x_2, \ldots, x_{K(N)}$ be a sequence of points satisfying:

$$K(N) = [2\exp(2N^{1/2}(\log N)^{-\varepsilon} + 2N^{1/2}(\log N)^{-2\varepsilon})] ,$$

$$\|x_i\| \leqq \exp(N^{1/2}(\log N)^{-\varepsilon}) \quad (i = 1, 2, \ldots, K(N)),$$

$$D(0, \exp(N^{1/2}(\log N)^{-\varepsilon}) \subset \bigcup_{i=1}^{K(N)} D(x_i, \exp(-N^{1/2}(\log N)^{-2\varepsilon})) .$$

It is easy to see the existence of such a sequence.
Then

$$\mathbf{P}\left\{\bigcup_{i=1}^{K(N)} \{\nexists s_i: 0 \leqq s_i \leqq 2t_{2N},\ W(s_i) \in D(x_i, \exp(-N^{1/2}(\log N)^{-2\varepsilon}))\}\right\}$$
$$\leqq K(N) \exp\left(-\frac{1}{20} N^{1/2}(\log N)^{\varepsilon}\right).$$

Hence by Borel-Cantelli lemma we have: for any

$$x \in D(0, \exp(N^{1/2}(\log N)^{-\varepsilon})$$

until t_{2N} the disc

$$D(x, \exp(-N^{1/2}(\log N)^{-2\varepsilon}))$$

is visited for all but finitely many N with probability 1.

Let $N = (\log T)(\log_2 T)^{-1-\varepsilon}$. Then by Lemma 6 $t_{2N} \leqq T$ with probability one for all but finitely many N. Consequently for any

$$x \in D(0, \exp((\log T)^{1/2}(\log_2 T)^{-1/2-\varepsilon})$$

till T the disc

$$D(x, \exp(-(\log T)^{1/2}(\log_2 T)^{-1/2-\varepsilon}))$$

is visited.

Hence Theorem 2 and the lower inequality in Theorem 1 is proved.

3 Proof of Theorem 3

Let $x_1, x_2, \ldots, x_k \in D(1)$ be such points that the discs $D_i = D(x_i, \delta)$ have no points in common. Denote by

$$m_k(D_1, D_2, \ldots, D_k; T) = \mathbf{P}\{\forall i = 1, 2, \ldots, k\ \exists t_i, \text{ such that } 0 \leqq t_i \leqq T,\ W(t_i) \in D_i\}$$

the probability that all discs are visited until time T.

Lemma 8. *Let* $\varepsilon > 0$ *and*

$$\delta = \delta(T) = \delta(T, \varepsilon) = \exp(-(\log T)^{1/2+\varepsilon}),$$
$$\|x_i - x_j\| \geqq \exp(-(\log T)^{1/2}) \quad (i, j = 1, 2, \ldots, k; i \neq j).$$

Then

$$m_k(D_1, D_2, \ldots, D_k; T) \leqq \exp\left(-\frac{(1-\varepsilon)2}{(\log T)^{1/2-\varepsilon}} \log k\right).$$

Proof. It is easy to see that

$$\begin{aligned} &m_k(D_1, D_2, \ldots, D_k; T) \\ &= \mathbf{P}\left\{\bigcup_{i=1}^{k} \{\text{all } D_j \text{ are visited before } T \text{ and } D_i \text{ is the last visited disc}\}\right\} \\ &\leqq \mathbf{P}\left\{\bigcup_{i=1}^{k} \{\text{the discs } D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_k \text{ are visited before } T\}\right\} \\ &\quad \times \max_{\substack{i \neq j \\ x \in D_i}} \mathbf{P}\{D_j \text{ is visited before } T \mid W(0) = x\}. \end{aligned}$$

Clearly

$$\begin{aligned} &\mathbf{P}\left\{\bigcup_{i=1}^{k} \{\text{the discs } D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_k \text{ are visited before } T\}\right\}. \\ &\leqq \sum_{i=1}^{k} m_{k-1}(D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_k; T) - (k-1) m_k(D_1, \ldots, D_k; T) \end{aligned}$$

and by Lemma 1 (with $\|u\| = \exp(-(\log T)^{1/2})$, $r = \delta$)

$$\begin{aligned} \max_{\substack{i \neq j \\ x \in D_i}} \boldsymbol{P}\{D_j \text{ is visited before } T \mid W(0) = x\} &\leqq 1 - (1-\varepsilon) \frac{\log(\exp(-(\log T)^{1/2})\delta^{-1})}{\log(T^{1/2+\varepsilon}\delta^{-1})} \\ &\leqq 1 - (1-2\varepsilon) \frac{2}{(\log T)^{1/2-\varepsilon}}. \end{aligned}$$

Hence

$$\begin{aligned} m_k(D_1, D_2, \ldots, D_k; T) \leqq \Bigg(&\sum_{i=1}^{k} m_{k-1}(D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_k; T) \\ &- (k-1) m_k(D_1, \ldots, D_k; T) \Bigg) \left(1 - (1-2\varepsilon) \frac{2}{(\log T)^{1/2-\varepsilon}}\right) \end{aligned}$$

and

$$\begin{aligned} &m_k(D_1, D_2, \ldots, D_k; T) \\ &\leqq \frac{(1 - (1-2\varepsilon)2(\log T)^{-1/2+\varepsilon}) \sum_{i=1}^{k} m_{k-1}(D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_k; T)}{1 + (k-1)(1 - (1-2\varepsilon)2(\log T)^{-1/2+\varepsilon})}. \end{aligned}$$

Since

$$\frac{1-a}{1+(k-1)(1-a)} \leqq \frac{1-a/k}{k} \leqq k^{-1}\exp(-a/k) \quad (0 \leqq a \leqq 1, k \geqq 1),$$

we have

$$m_k(D_1, D_2, \ldots, D_k; T) \leqq k^{-1}\exp(-(1-2\varepsilon)2k^{-1}(\log T)^{-1/2+\varepsilon}) \times \sum_{i=1}^{k} m_{k-1}(D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_k; T)$$

and by induction

$$m_k(D_1, D_2, \ldots, D_k; T) \leqq \exp(-(1-2\varepsilon)2(\log T)^{-1/2+\varepsilon}\log k),$$

and we have Lemma 8.

Proof of Theorem 3. Let $x_1, x_2, \ldots, x_k$ be a sequence of points of $D(1)$ satisfying the conditions of Lemma 8. We can also assume that $k = k(T) \geqq \exp((\log T)^{1/2-\varepsilon/2})$. Then we obtain

$$m_k(D_1, D_2, \ldots, D_k; T) \leqq \exp(-(1-2\varepsilon)2(\log T)^{\varepsilon/2}).$$

Let $T_l = e^l$. The Borel-Cantelli lemma implies that at least one D_i is not visited until time T_l for all but finitely many l.

Assume that $T_l \leqq T \leqq T_{l+1}$. With probability 1 there is an $x \in D(1)$ such that the disc $D(x, \delta(T_{l+1}))$ is not visited before T. Since $\delta(T_{l+1}, \varepsilon) \geqq \delta(T_l, \varepsilon/2)$ we obtain the Theorem.

4 Proof of the upper part of Theorem 1

Let $x_1, x_2, \ldots, x_k \in D(\exp((\log T)^{1/2}\log_3 T))$ be such points that the discs $D_i = D_i(x_i, \delta)$ have no points in common ($\delta > 0$ is fixed). We also assume that

$$k = k(T) = \exp((\log T)^{1/2}),$$

$$\|x_i - x_j\| \geqq \exp((1-\varepsilon)(\log T)^{1/2}\log_3 T) \quad (i, j = 1, 2, \ldots, k; i \neq j)$$

with some $0 < \varepsilon < 1$ and

$$\|x_i\| = \exp((\log T)^{1/2}\log_3 T).$$

Then repeating the proof of Lemma 8 in the present case we obtain

$$m_k(D_1, D_2, \ldots, D_k; T) \leqq \exp(-(1-\varepsilon)2(\log_3 T)(\log T)^{-1/2}\log k) = \exp(-(1-\varepsilon)2(\log_3 T)).$$

Let $T_l = \exp(e^l)$. The Borel-Cantelli lemma implies that at least one D_i is not visited until time T_l for all but finitely many l. Since

$$\exp((\log T_{l+1})^{1/2}\log_3 T_{l+1}) \leqq \exp(2(\log T_l)^{1/2}\log_3 T_l)$$

we obtain our statement.

5 Proof of Theorem 4

At first we describe the main idea of the proof.

Step 1. Let $W_1(\cdot), W_2(\cdot), \ldots, W_{[T^\alpha]}(\cdot)$ $(\alpha > 0)$ be independent Wiener processes and let $e_i(T, \vartheta)$ be the number of ϑ-excursions of W_i completed before T. Then Lemma 9 tells us that $\max_{1 \leqq i \leqq T^\alpha} e_i(T^\beta, \vartheta)$ is about $(\log T)^2$. (Remember that by Lemma 6 $e_1(T^\beta, \vartheta)$ is about $\log T$.)

Step 2. Consider the independent Wiener processes

$$W_i(t) = W(iT^{1/2} + t) - W(iT^{1/2}) \quad (i = 1, 2, \ldots, [T^{1/2}]; 0 \leqq t \leqq T^{1/2}) .$$

By Step 1 there exists a (random) $i_0 \in [1, [T^{1/2}]]$ such that $e_{i_0}([T^{1/2}], \vartheta)$ is about $(\log T)^2$. Then using a somewhat stronger form of Lemma 7 (Lemma 10) and repeating the proof of Theorem 2 for the process $W_{i_0}(t)$ we obtain Theorem 4.

Lemma 9. *Let $W_1(t), W_2(t), \ldots$ be a sequence of independent Wiener processes on $\mathbf{R}^2$. Let $e_i(T, \vartheta)$ be the number of ϑ-excursions of $W_i(\cdot)$ completed before T. Then for any $\alpha, \beta, u, \varepsilon > 0$ and $0 < \psi < \beta/4$ there exists a $T_0 = T_0(\alpha, \beta, u, \varepsilon, \psi)$ such that*

$$\mathbf{P}\left\{ \max_{1 \leqq i \leqq T^\alpha} e_i(T^\beta, \vartheta) < u(\log(T^\beta \vartheta^{-1}))^2 \right\} \leqq \exp(-T^\alpha (T^\beta \vartheta^{-1})^{-2u(1+\varepsilon)\log 2})$$

if $T \geqq T_0$ *and* $\exp(-T^{\beta/4-\psi}) \leqq \vartheta < e^{-1}$.

Proof. By Lemma 5 we have

$$\begin{aligned}\mathbf{P}\left\{ \max_{1 \leqq i \leqq T^\alpha} e_i(T^\beta, \vartheta) < u(\log(T^\beta \vartheta^{-1}))^2 \right\} &\leqq (1 - \exp(-(1+\varepsilon)2(\log 2)u \log(T^\beta \vartheta^{-1})))^{T^\alpha} \\ &\leqq \exp(-T^\alpha \exp(-(1+\varepsilon)2(\log 2)u \log(T^\beta \vartheta^{-1}))) \\ &= \exp(-T^\alpha (T^\beta \vartheta^{-1})^{-2u(1+\varepsilon)\log 2}) .\end{aligned}$$

Hence we have the Lemma.

Let $W_1(t), W_2(t), \ldots$ be a sequence of independent Wiener processes on $\mathbf{R}^2$. Let $t_{2N}(i) = t_{2N}(i, \vartheta)$ be the endpoint of the N-th ϑ-excursion of $W_i(\cdot)$. Further let $x_1, x_2, \ldots, x_{K(N)}$ be a sequence of points satisfying:

$$K(N) = [2\exp(2(\mu + \nu)\sqrt{N})] \quad (\mu > 0, \nu > 0) ,$$

$$\|x_j\| \leqq \exp(\mu\sqrt{N}) \quad (j = 1, 2, \ldots, K(N)) ,$$

$$D(0, \exp(\mu\sqrt{N})) \subset \bigcup_{j=1}^{K(N)} D(x_j, \exp(-\nu\sqrt{N})) .$$

It is easy to see that for any $\mu > 0$, $\nu > 0$ there exists such a sequence.

Lemma 10. *For any $\mu > 0$, $\nu > 0$, $\lambda > 0$ we have*

$$\mathbf{P}\left\{ \bigcup_{i=1}^{L(N)} \bigcup_{j=1}^{K(N)} \{\nexists s(i,j) : 0 \leqq s(i,j) \leqq t_{2N}(i), W_i(s(i,j)) \in D(x_j, \vartheta)\} \right\} \leqq 2\exp\left(-\left(\frac{\log 2}{6(\mu+\nu)} - 2(\mu+\nu) - \lambda\right)\sqrt{N}\right)$$

where $\vartheta \geqq \exp(-\nu\sqrt{N})$ *and* $L(N) = [\exp(\lambda\sqrt{N})]$.

Proof. By Lemma 7

$$\mathbf{P}\{\nexists s: 0 \leqq s \leqq t_{2N}(i), W_i(s) \in D(x_j, \vartheta)\} \leqq \exp\left(-\frac{\log 2}{6}\frac{N}{\log(\|x_j\|\vartheta^{-1})}\right)$$

$$\leqq \exp\left(-\frac{\log 2}{6(\mu+\nu)}\sqrt{N}\right).$$

Hence

$$\mathbf{P}\left\{\bigcup_{i=1}^{L(N)}\bigcup_{j=1}^{K(N)}\{\nexists s(i,j): 0 \leqq s(i,j) \leqq t_{2N}(i), W_i(s(i,j)) \in D(x_j, \vartheta)\}\right\}$$

$$\leqq L(N)K(N)\exp\left(-\frac{\log 2}{6(\mu+\nu)}\sqrt{N}\right)$$

$$= 2\exp\left(-\left(\frac{\log 2}{6(\mu+\nu)} - 2(\mu+\nu) - \lambda\right)\sqrt{N}\right)$$

and we have the Lemma.
Apply Lemma 9 with

$$W_i(t) = W(i\sqrt{T} + t) - W(i\sqrt{T}) \quad (i = 0, 1, 2, \ldots, [\sqrt{T}]; 0 \leqq t \leqq \sqrt{T}),$$

$$\alpha = \beta = 1/2, \quad \vartheta = T^{-\kappa}, \quad \kappa > 0,$$

$$u = \frac{1-\varepsilon}{2(1+2\kappa)\log 2}.$$

Then by the Borel-Cantelli lemma we obtain that (with probability 1 if T is big enough) there exists a $0 \leqq i_0 = i_0(\omega) \leqq \sqrt{T}$ such that

$$t_{2N}(i_0) \leqq \sqrt{T}$$

where

$$N = [u(\log(T^{1/2}\vartheta^{-1}))^2] = \left[\frac{(1-\varepsilon)(1+2\kappa)}{8\log 2}(\log T)^2\right]$$

i.e.

$$T = \exp\left(\left(\frac{8\log 2}{(1-\varepsilon)(1+2\kappa)}\right)^{1/2}\sqrt{N}\right).$$

Apply Lemma 10 with the above N. Then we obtain that for any $0 \leqq i \leqq L(N)$, $0 \leqq j \leqq K(N)$ (with probability 1 if N is big enough) there exists a $0 \leqq s(i,j) \leqq t_{2N}(i)$ such that

$$W_i(s(i,j)) \in D(x_j, \vartheta)$$

provided that

$$\frac{\log 2}{6(\mu+\nu)} > 2(\mu+\nu) + \lambda.$$

Lemmas 9 and 10 combined claim that there exists a $0 \leqq i_0 = i_0(\omega) \leqq \sqrt{T}$ such that the disc $D(W(i\sqrt{T}), \exp(\mu\sqrt{N}))$ is $\vartheta = \exp(-\nu\sqrt{N})$-covered by $W(\cdot)$ provided that the above condition holds.

Choose

$$\nu = \kappa\left(\frac{8\log 2}{(1-\varepsilon)(1+2\kappa)}\right)^{1/2}$$

(in order to get $\vartheta = \exp(-\nu\sqrt{N}) = T^{-\kappa}$),

$$\lambda = \frac{1}{2}\left(\frac{8\log 2}{(1-\varepsilon)(1+2\kappa)}\right)^{1/2}$$

(in order to get $T^{1/2} = \exp(\lambda\sqrt{N})$). Let

$$\exp(\mu\sqrt{N}) = T^{\bar{\rho}}$$

i.e.

$$\mu = \bar{\rho}\left(\frac{8\log 2}{(1-\varepsilon)(1+2\kappa)}\right)^{1/2}.$$

Let

$$A = \left(\frac{8\log 2}{(1-\varepsilon)(1+2\kappa)}\right)^{1/2} = 2\lambda .$$

Then

$$\mu + \nu = (\kappa + \bar{\rho})A$$

and our condition formulated with ρ and κ (instead of μ, ν, λ) is:

$$\frac{\log 2}{6}\frac{1}{(\kappa+\bar{\rho})A} \geqq 2(\kappa+\bar{\rho})A + \frac{1}{2}A ,$$

i.e.

$$2(\kappa+\bar{\rho})^2 A^2 + \frac{\kappa+\bar{\rho}}{2}A^2 - \frac{\log 2}{6} \leqq 0 .$$

Since $\kappa = \bar{\rho} = 0.005$ satisfy the above inequality (provided that $0 < \varepsilon < 1/2$) we proved Theorem 4.

6 Proof of Theorem 5

Let $0 < \delta < 1/2$, $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$, $k = [T^{\alpha(1/2-\delta)}]$ and $u_i = iT^{\beta(1/2-\delta)}$ $(i = 0, 1, 2, \ldots, k)$. Then clearly

$$0 \leqq u_i \leqq T^{1/2-\delta},$$

$$u_j - u_i \geqq [T^{\beta(1/2-\delta)}] \quad (0 \leqq i < j \leqq k).$$

Consider the set

$$\mathscr{A} = \mathscr{A}(T) = \bigcup_{i=0}^{k} D(x_i, \vartheta) \subset D(0, T^{1/2-\delta})$$

where $x_i = (u_i, 0)$ $(i = 0, 1, 2, \ldots, k)$.

Then we prove the following analogue of Lemma 8.

Lemma 11. *Let* $D_i = D(x_i, \vartheta)$ $(i = 1, 2, \ldots, k)$. *Then for any* $\varepsilon > 0$ *we have*

$$m_k(D_1, D_2, \ldots, D_k; T) \leqq T^{-(1-\varepsilon)\alpha\beta(1-2\delta)^2/2}\,.$$

Proof. Just like in the proof of Lemma 8 we obtain

$$m_k(D_1, D_2, \ldots, D_k; T) \leqq \left(\sum_{i=1}^{k-1} m_{k-1}(D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_k; T) - (k-1)m_k(D_1, \ldots, D_k; T)\right) \times \max_{\substack{i \neq j \\ x \in D_i}} \mathbf{P}\{D_j \text{ is visited before } T \mid W(0) = x\}\,.$$

By Lemma 1 we have

$$\max_{\substack{i \neq j \\ x \in D_i}} \mathbf{P}\{D_j \text{ is visited before } T \mid W(0) = x\} \leqq -1 - (1-\varepsilon)\frac{\log(T^{(1/2-\delta)\beta}\vartheta^{-1})}{\log(T^{1/2+\varepsilon}\vartheta^{-1})}$$

$$\leqq 1 - (1-2\varepsilon)\frac{(1-2\delta)\beta}{1+2\varepsilon}\,.$$

Hence

$$m_k(D_1, D_2, \ldots, D_k; T) \leqq \exp(-\psi \log k) = k^{-\psi}$$

where

$$\psi = (1-2\varepsilon)\frac{1-2\delta}{1+2\varepsilon}\beta$$

and we have Lemma 11.

Let

$$\mathscr{B} = \mathscr{B}(T) = \bigcup_{i=0}^{k} D((0, u_i), \vartheta) \subset D(0, T^{1/2-\delta})\,,$$

$$\mathscr{C} = \mathscr{C}(T) = \bigcup_{i=0}^{k} D((-u_i, 0), \vartheta) \subset D(0, T^{1/2-\delta})\,,$$

$$\mathscr{D} = \mathscr{D}(T) = \bigcup_{i=0}^{k} D((0, -u_i), \vartheta) \subset D(0, T^{1/2-\delta})\,.$$

Clearly, we say that the set $\mathscr{A}$ (resp. $\mathscr{B}, \mathscr{C}, \mathscr{D}$) is ϑ-covered in T if: for any $0 \leqq i \leqq k$ there exists a $0 \leqq t_i \leqq T$ such that $W(t_i) \in D((u_i, 0), \vartheta)$ (resp. $D((0, u_i), \vartheta)$, $D((-u_i, 0)\vartheta)$, $D((0, -u_i), \vartheta)$). Then Lemma 11 implies

$$\mathbf{P}\{\text{at least one of } \mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D} \text{ is } \vartheta\text{-covered}\} \leqq 4T^{-K}$$

where

$$K = \frac{(1-\varepsilon)\alpha\beta(1-2\delta)^2}{2} = (1-\varepsilon)\frac{(1-2\delta)^2}{8}$$

if $\alpha = \beta = 1/2$.

Observe that if $x \in D(0, T^{1/2-\delta})$ (or equivalently $0 \in D(x, T^{1/2-\delta})$) then the disc $D(x, T^{1/2-\delta})$ contains at least one of the sets $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}$. Hence

$$\mathbf{P}\{\text{at least one of the discs } D(u, T^{1/2-\delta})(u \in D(0, T^{1/2-\delta}) \text{ is } \vartheta\text{-covered}\} \leqq 4T^{-K}.$$

Choose

$$\delta < \frac{5}{2} - \sqrt{6} = 0.05\ldots$$

and observe that

$$2\delta < K = (1-\varepsilon)\frac{(1-2\delta)^2}{2}$$

if ε is small enough. Let

$$2\delta < \gamma < K$$

and consider the point $W(iT^{1-\gamma})$ $(i = 0, 1, 2, \ldots, [T^\gamma])$.

Let $\mathbf{D}_T$ be the set of those discs $D(x, T^{1/2-\delta})$ for which there exists an i $(i = 0, 1, 2, \ldots, [T^\gamma])$ such that

$$W(iT^{1-\gamma}) \in D(x, T^{1/2-\delta}).$$

Then we have

$$\mathbf{P}\{\text{at least one element of } \mathbf{D}_T \text{ is } \vartheta\text{-covered}\} \leqq 4T^\gamma T^{-K}.$$

Let τ be a real number for which

$$\tau(K-\gamma) > 1$$

and consider the sequence $T_j = j^\tau (j = 1, 2, \ldots)$. Then we obtain with probability 1 for all but finitely many j all $D(u, T_j^{1/2-\delta}) \in \mathbf{D}_{T_j}$ will not be ϑ-covered. Observe also that for all $iT^{1-\gamma} \leqq t \leqq (i+1)T^{1-\gamma}$ $(i = 0, 1, 2, \ldots, [T^\gamma] - 1)$ we have

$$W(t) \in D(W(iT^{1-\gamma}), T^{1/2-\delta}) \quad \text{a.s.}$$

for all T big enough (since $\delta < \gamma/2$). Consequently

$$r(T_j, \vartheta) \leqq T_j^{1/2-\delta} \quad \text{a.s.}$$

for all but finitely many j. This implies the Theorem immediately.

Note added in proof. As the referee pointed out the ϑ-excursions $\{W(t);\ t_{2i+1} \leqq t \leqq t_{2i+2}\} i = 0, 1, 2, \ldots$ are non-independent. Hence the application of Lemma 7 in the proof of Theorem 2 requires some explanation. Observe that the probability that the second ϑ-excursion meets $D(x + W(t_2), \vartheta)$ (given the first ϑ-excursion) is also larger than or equal to $(\log 2)(6\log(\|x\|\vartheta^{-1}))^{-1}$. Hence replacing ϑ by 2ϑ the proof is going smoothly.

Prof. J.F. Le Gall was kind enough to inform me that his student, T. Meyre, proved

$$\limsup_{T\to\infty} \frac{(\log R(T, \rho))^2}{\log T(\log_3 T)^2} \geqq \frac{1}{32} \quad \text{a.s.}$$

References

1. Knight, F.B.: Essentials of Brownian motion and diffusion. Providence, R.I.: Am. Math. Soc. 103 (1981)
2. Révész, P.: Random walk in random and non-random environments. Singapore: World Scientific 1990
3. Révész, P.: Clusters of a random walk on the plane. Ann. Probab. (to appear) (1992)
4. Révész, P.: Clusters of a random walk on the plane II. (Manuscript)
5. Spitzer, F.: Some theorems concerning 2-dimensional Brownian motion. Ann. Math. Soc. **87**, 187–197 (1958)