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Dirichlet forms on fractals: Poincaré constant and resistance

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Summary. We study Dirichlet forms associated with random walks on fractal-like finite graphs. We consider related Poincaré constants and resistance, and study their asymptotic behaviour. We construct a Markov semi-group on fractals as a subsequence of random walks, and study its properties. Finally we construct self-similar diffusion processes on fractals which have a certain recurrence property and plenty of symmetries.

Mathematics Subject Classification: 60 J 60

0 Introduction

There are many works on diffusion processes on fractals, but most of them are confined to finitely ramified fractals (e.g. nested fractals). Exceptions are the works by Barlow and Bass (e.g. [1, 2]) on diffusion processes in a 2-dimesional Sierpinski carpet. Barlow and Bass [2] have shown a deep estimate of the resistance, and the arguments there will work even in the case of fractals embedded in higher dimensional spaces. However, it seems that the method to show the Harnack inequality in Barlow and Bass [1] works only for fractals embedded in a 2-dimensional space.

The aim of this paper is to give a different approach to random walks in (not necessarily finitely ramified) fractals and to give some complements to the results of Barlow and Bass. We will mainly consider the "Poincaré constant". Several conditions are introduced and we study the relationship between them.

We are thinking of quite general self-similar fractals. However, using a regularity property of harmonic functions, we can handle the case when the fractal has plenty of symmetry and "recurrence" properties. In that case, partly supported by Barlow-Bass' "Knight Moves" argument, we can prove that there exists a non-degenerate self-similar diffusion process on such a fractal (Sect. 7).

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Concerning the 2-dimensional Sierpinski carpet, our results are almost contained in Barlow-Bass' results. The only new point is the existence of a *self-similar* diffusion process on the fractal. However, we believe that our approach also works for carpets with holes in higher-dimensional spaces, provided the holes of a carpet are large enough (so that the "spectral dimension" is less than 2). This will be demonstrated in Sect. 8, Example 3. Since we start from random walks, which are 0-dimensional objects, we can keep the advantage that the "spectral dimension" is less than 2.

We are strongly influenced by Barlow and Bass [1, 2] and Moser [6]. In [6], Moser considered the opposite case when the dimension is greater than 2. However, we cannot show any kind of regularity property of harmonic functions in this case.

Since we introduce a large number of assumptions and show many results, we summarize the relations between them here in the remainder of this Introduction. The basic assumptions (A-1)–(A-4) are introduced in Sect. 1. These assumptions are geometrical, and it is easy to check them in each fractal. We keep these assumptions throughout the paper. The results in Sects. 2 and 3 follow from those assumptions alone. In Sect. 4 we introduce assumption (B-1), which will be assumed until Sect. 7. This assumption is analytic and is not easy to check in general. However, in Sect. 8 we show that it can be verified for a fractal which has enough symmetries (Proposition 8.1).

In Sect. 5 we introduce assumption (B-2), which is rather bad because we do not know how it can be verified in general. In Sect. 6 we introduce assumption (GB), which is geometrical and easy to check. We show that there exists a self-similar local Dirichlet form under assumptions (A-1)–(A-4), (B-1), (B-2) and (GB) (Theorem 6.9). The problems with this theorem are the difficulty to verify assumption (B-2) and to prove the regularity of the Dirichlet form.

A partial solution to the above problems is given in Sect. 7, where we introduce assumptions (R), (KM) and (LS), and show that assumption (B-2) and the regularity of the Dirichlet form follow from them (Theorem 7.19). Assumption (R) is very strong. Roughly speaking it means that "the spectral dimension" is less than 2, so it is not satisfied for Sierpinski carpets in more than 2 dimensions (see Example 2 in Sect. 8). Assumption (LS) is essentially geometrical and therefore easy to check. Assumption (KM) is also strong and not so easy to check in general. However, Barlow and Bass [1] used a clever idea to verify it for the Sierpinski carpet in 2 dimensions. We believe that their idea works even in other fractals, though this is not shown in the present paper.

1 Basic assumptions, Poincaré constant and other constants

Let $\alpha > 1$ and $I = \{1, \ldots, N\}$. Let $\{\psi_i; i \in I\}$ be a family of α-similitudes in $\mathbb{R}^D$, i.e., ψ_i's are maps on $\mathbb{R}^D$ satisfying $|\psi_i(x) - \psi_i(y)| = \alpha^{-1}|x - y|$ for any $x, y \in \mathbb{R}^D$. Then it is well-known that there is a unique non-void compact set E in $\mathbb{R}^D$ satisfying $E = \bigcup_{i \in I} \psi_i(E)$. For example, if $D = 2$, $N = 3$ and $\psi_1(x) = \frac{1}{2}x$, $\psi_2(x) = \frac{1}{2}(x - (1, 0))$, $\psi_3(x) = \frac{1}{2}(x - (\frac{1}{2}, \frac{\sqrt{3}}{2}))$, $x \in \mathbb{R}^2$, then ψ_i's are 2-similitudes and the set E is called the Sierpinski gasket.

Let us assume the following.

(A-1) (The open set condition) There exists a non-void open set V such that $\bigcup_{i \in I} \psi_i(V) \subset V$, and $\psi_i(V) \cap \psi_j(V) = \phi$ for any $i, j \in I$ with $i \neq j$.

Then ψ_i's, $i \in I$, have distinct fixed points, and the Hausdorff dimension of the set E is d_f, where $d_f = (\log \alpha)^{-1} (\log N)$. Moreover, d_f-Hausdorff measure of E is finite and non-zero. Let ν be the normalized d_f-Hausdorff measure on E. For any $x = (x_1, \ldots, x_n) \in I^n$, let ψ_x denote the map $\psi_{x_1} \circ \ldots \circ \psi_{x_n}$ in $\mathbb{R}^D$.

Then we have the following.

(1.1) Proposition. $M_0 = \sup\{\#(\{y \in I^n; \psi_y(E) \cap \psi_x(E) \neq \phi\}); n \geqq 1, x \in I^n\} < \infty$.

Proof. Let V be a non-void open set in $\mathbb{R}^D$ as in the assumption (A-1), Let d be the diameter of the closure of V. Also, let $B(r, \xi)$, $r > 0$, $\xi \in \mathbb{R}^D$ be a ball of radius r with a center ξ in $\mathbb{R}^D$. It is easy to see that if $\psi_x(E) \cap \psi_y(E) \neq \phi$, $x, y \in I^n$, then the diameter of $\psi_x(V) \cap \psi_y(V)$ is less than or equal to $2d\alpha^{-n}$. So for any $x \in I^n$, $\cup \{\psi_y(V); y \in I^n, \psi_y(E) \cap \psi_x(E) \neq \phi\}$ is contained in $B(2d\alpha^{-n}, \xi)$ for some $\xi \in \mathbb{R}^D$. So we see that

$$\begin{aligned} &\#\{y \in I^n; \psi_y(E) \cap \psi_x(E) \neq \phi\} \\ &= (\alpha^{-nD} |V|)^{-1} |\cup \{\psi_y(V); y \in I^n, \psi_y(E) \cap \psi_x(E) \neq \phi\}| \\ &\leqslant |V|^{-1} B(2d; 0)|. \end{aligned}$$

This proves our assertion.

Let $\gamma_0 \in [0, d_f)$. This γ_0 is fixed throughout this paper. For each $n \geqq 1$, we introduce a relation $\underset{n}{\sim}$ by $x \underset{n}{\sim} y$ if the Hausdorff dimension of the set $\psi_x(E) \cap \psi_y(E)$ is greater than or equal to γ_0. Also, we define $q_{xy}^{(n)}$, $x, y \in I^n$, by $q_{xy}^{(n)} = 1$, if $x \underset{n}{\sim} y$, and $q_{xy}^{(n)} = 0$ otherwise. Also, we define $q_{xy}^{(n)}$, $x, y \in I^n$, by $q_{xy}^{(n)} = 1$, if $x \underset{n}{\sim} y$, and $q_{xy}^{(n)} = 0$ otherwise.

Remark. It is obvious that the relation $\underset{n}{\sim}$ depends on the choice of γ_0. The assumption (GB) to be introduced in Sect. 6 depend on the relation $\underset{n}{\sim}$ strongly, and so we sometimes have to choose γ_0 cleverly.

We assume the following furthermore.

(A-2) The matrix $(q_{xy}^{(n)})_{x,y \in I^n}$ is irreducible for any $n \geqq 1$.

(A-3) There are $m_0 \geqq 1$ and $\varepsilon_0 > 0$ satisfying the following.

(1) If $\xi, \xi' \in E$ and $|\xi - \xi'| \leqq \varepsilon_0 \cdot \alpha^{-n}$, $n \geqq 1$, then there are $x_0, \ldots, x_{m_0} \in I^n$ such that $\xi \in \psi_{x_0}(E)$, $\xi' \in \psi_{x_{m_0}}(E)$ and $x_{i-1} \underset{n}{\sim} x_i$, $i = 1, \ldots, m_0$.

(2) If $x, y \in I^n$ and $\psi_x(E) \cap \psi_y(E) = \phi$, then $|\xi - \xi'| > \varepsilon_0 \cdot \alpha^{-n}$ for any $\xi \in \psi_x(E)$ and $\xi' \in \psi_y(E)$.

From the assumption (A-2), we see that the set E is connected.

For any $x = (x_1, \ldots, x_n) \notin I^n$ and $y = (y_1, \ldots, y_m) \in I^m$, $x \cdot y$ denotes the element $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ in I^{n+m}. Then we have the following.

(1.2) Proposition. *Let $x, y \in I^n$ and $z, w \in I^m$.*

(1) *If $x \cdot z \underset{n+m}{\sim} y \cdot w$, then $x \underset{n}{\sim} y$.*

(2) *$x \cdot z \underset{n+m}{\sim} x \cdot w$, if and only if $z \underset{m}{\sim} w$.*

For any subsets A, B of I^n, we write $A \underset{n}{\sim} B$ if there are $x \in A$ and $y \in B$ such that $x \underset{n}{\sim} y$. We denote by $A \cdot B$ the subset $\{x \cdot y; x \in A, y \in B\}$ of I^{n+m} for any subset A of I^n and subset B of I^m. Let $\mathscr{B}_{n,m} = \{\{x \cdot I^m\}; x \in I^{n-m}\}$, $0 \leqq m \leqq n$. For any

$k = m, \ldots, n$, and $B \in \mathscr{B}_{n,m}$, $B_{n,k}(B)$ denotes the set in $\mathscr{B}_{n,k}$ which contains B. For each $n \geqq 1$, ∂I^n denotes the set of points x in I^n such that there are $m \geqq 1$ and $y, z \in I^m$ with $y \neq z$ and $\psi_{y \cdot x}(E) \cap \psi_z(E) \neq \phi$. It is obvious that $\partial I^n \neq \phi$, $n \geq 1$.

Now we assume the following assumption.

(A-4) There is an $n \geq 1$ such that $\partial I^n \neq I^n$.

This assumption can be verified by the following condition.

(1.3) Proposition. *Suppose that there is a non-void open set V such that $\bigcup_{i \in I} \psi_i(V) \subset V$, $\psi_i(V) \cap \psi_j(V) = \phi$, $i, j \in I$, $i \neq j$, and $V \cap E \neq \phi$. Then the assumption* (A-4) *holds.*

Proof. By the assumption, we see that there is an $n \geqq 1$ and $x \in I^n$ such that $\psi_x(E) \subset V$. Let $y, z \in I^m$ with $y \neq z$. Then we see that $\psi_y(V) \cap \psi_z(V) = \phi$ and so $\psi_y(V) \cap \psi_z(E) = \phi$. Since $\psi_{y \cdot x}(E) \subset \psi_y(V)$, we have $\psi_{y \cdot x}(E) \cap \psi_z(E) = \phi$. This proves that $x \in I^n \backslash \partial I^n$.

For any subset A of I^n, let $\mathscr{E}_{n,A}$ be a symmetric bilinear form in $C(I^n; \mathbb{R})$ defined by

$$\mathscr{E}_{n,A}(u, v) = \sum_{x, y \in A} q_{xy}^{(n)} (u(x) - u(y))(v(x) - v(y)), \quad u, v \in C(I^n; \mathbb{R}).$$

We denote $\mathscr{E}_{n,I^n}$ by $\mathscr{E}_n$. Let $\langle u \rangle_A$ denote $\#(A)^{-1} \cdot \sum_{x \in A} u(x)$ for any finite set A and $u \in C(A; \mathbb{R})$.

Now we introduce the following Poincare constant which plays a key role in this paper.

$$\text{(1.4)} \qquad \lambda_n = \sup \left\{ \sum_{x \in I^n} (u(x) - \langle u \rangle_{I^n})^2;\ u \in C(I^n; \mathbb{R}),\ \mathscr{E}_n(u, u) = 1 \right\}, \quad n \geqq 1.$$

Then we have the following.

(1.5) Proposition. (1) *$\sum_{x \in B} (u(x) - \langle u \rangle_B)^2 \leq \lambda_m \cdot \mathscr{E}_{n,B}(u, u)$ for any $n \geqq m \geqq 1$, $B \in \mathscr{B}_{n,m}$ and $u \in C(I^n; \mathbb{R})$*

(2) *$(\langle u \rangle_B - \langle u \rangle_{I^n})^2 \leqq N^{-n+1} \cdot \lambda_n \cdot \mathscr{E}_n(u, u)$ for any $B \in \mathscr{B}_{n,n-1}$ and $u \in C(I^n; \mathbb{R})$.*

(3) *$|\langle u \rangle_{B_{n,k}(x)} - \langle u \rangle_{B_{n,k-1}(x)}| \leq \{N^{-k+1} \cdot \lambda_k \cdot \mathscr{E}_{n,B_{n,k}(x)}(u, u)\}^{1/2}$*

for any $n \geqq 1$, $k = 1, \ldots, n$, $x \in I^n$ and $u \in C(I^n; \mathbb{R})$.

Proof. The assertion (1) is obvious. Let $u \in C(I^n; \mathbb{R})$ and $B \in \mathscr{B}_{n,n-1}$. Let $f \in C(I^n; \mathbb{R})$ such that $f(x) = 1$, $x \in B$, and $f(x) = 0$, $x \in I^n \backslash B$. Then by Schwarz' inequality we have

$$N^{2n-2}(\langle u \rangle_B - \langle u \rangle_{I^n})^2 = \left\{ \sum_{x \in I^n} f(x) \cdot (u(x) - \langle u \rangle_{I^n}) \right\}^2 \leqq N^{n-1} \cdot \lambda_n \mathscr{E}_n(u, u).$$

This implies the assertion (2). The assertion (3) is an easy consequence of the assertion (2).

This completes the proof.

For any subsets A and B of I^n with $A \cap B = \phi$, let

$$\text{(1.6)} \qquad R_n(A, B) = \min\{\mathscr{E}_n(u, u);\ u \in C(I^n, \mathbb{R}),\ u|_A = 0,\ u|_B = 1\}^{-1}.$$

This is the effective resistance between A and B in the network constructed by using the relation $\underset{n}{\sim}$. For any subset A of I^n, $C_{n,m}(A)$, $m = 0,\dots, n$, denotes $\cup\{B\in\mathscr{B}_{n,m};\ (\bigcup_{x\in A}\psi_x(E))\cap(\bigcup_{x\in B}\psi_x(E)) \neq \phi\}$. Also, let us define R_m, $m \geqq 0$, by

$$R_m = \inf\{R_n(B, I^n\backslash C_{n,m}(B));\ n \geqq m,\ B\in\mathscr{B}_{n,m}\}\ . \tag{1.7}$$

For any B, $B'\in\mathscr{B}_{n,m}$, with $B\underset{n}{\sim}B'$, let $\sigma_{n,m}(B, B')$ be given by

$$\sigma_{n,m}(B, B') = \sup\{N^m(\langle u\rangle_B - \langle u\rangle_{B'})^2;\ u\in C(I^n; \mathbb{R}),\ \mathscr{E}_{n,B\cup B'}(u, u) = 1\}\ . \tag{1.8}$$

Also, let σ_m, $m \geqq 0$, be given by

$$\sigma_m = \sup\{\sigma_{n,m}(B, B');\ n \geq m \vee 1,\ B, B'\in\mathscr{B}_{n,m},\ B\underset{n}{\sim}B'\}\ , \tag{1.9}$$

and let $\lambda_n^{(D)}$, $n \geqq 1$, be given by

$$\lambda_n^{(D)} = \sup\{N^n\cdot\langle u\rangle_{I^n}^2;\ u\in C(I^n; \mathbb{R}),\ u|_{\partial I^n} = 0,\ \mathscr{E}_n(u, u) = 1\}\ . \tag{1.10}$$

The quantities σ_n and $\lambda_n^{(D)}$ are kinds of Poincare constants.

2 Basic estimates

The purpose in this section is to prove the following.

(2.1) Theorem. *There is a constant $C > 0$ such that*

$$\lambda_n^{(D)} \leqq C\cdot\lambda_n\ ,\quad n \geqq 1\ , \tag{2.2}$$

$$\lambda_n\cdot(N^m R_m) \leqq C\cdot\lambda_{n+m}\ ,\quad n, m \geqq 1\ , \tag{2.3}$$

$$\lambda_{n+m} \leqq C\cdot\lambda_n\cdot\sigma_m\ ,\quad n, m \geqq 1\ , \tag{2.4}$$

$$\sigma_{n+m} \leqq C\cdot\sigma_n\cdot\sigma_m\ ,\quad n, m \geqq 1\ , \tag{2.5}$$

and

$$(\alpha^2 N^{-1})^n \leqq C\cdot R_n\ ,\quad n \geqq 1\ . \tag{2.6}$$

To prove this theorem, we have to make some preparations.

(2.7) Proposition. *There is a constant $C > 0$ such that $R_m \geqq C\cdot(\alpha^2 N^{-1})^m$ for any $m \geqq 0$.*

Proof. Let $B\in\mathscr{B}_{n,m}$, $K_0 = \cup\{\psi_x(E);\ x\in B\}$ and $K_1 = \cup\{\psi_x(E);\ x\in I^n\backslash C_{n,m}(B)\}$. Then by the assumption (A-3) (2), we see that $\mathrm{dis}(K_0, K_1) \geqq \varepsilon_0\cdot\alpha^{-(n-m)}$, where $\mathrm{dis}(K, K') = \inf\{|x - y|;\ x\in K,\ y\in K'\}$ for any subsets K and K' of E. Let $f\colon\mathbb{R}^D\to\mathbb{R}$ be given by $f(\xi) = \{\mathrm{dis}(K_0, K_1)^{-1}\ \mathrm{dis}(\xi, K_0)\} \wedge 1$, $\xi\in\mathbb{R}^D$. Let $u\colon I^n\to\mathbb{R}$ be given by $u(x) = N^{-n}\cdot\int_{\psi_x(E)} f(\xi)\nu(d\xi)$. Then it is easy to see that $u|_B = 0$ and $u|_{I^n\backslash C_{n,m}(B)} = 1$. Moreover, we see that for $x, y\in I^n$ with $x\underset{n}{\sim}y$,

$$\begin{aligned}|u(x) - u(y)| &\leqq \alpha^{-n}(\mathrm{dis}(K_0, K_1)^{-1}\cdot 2\cdot\mathrm{diameter}\ (E))\\ &\leqq \alpha^{-m}(2\cdot\mathrm{diameter}(E)\cdot\varepsilon_0^{-1})\ .\end{aligned}$$

So we have

$$\mathscr{E}_n(u,u) \leqq 2 \cdot \sum_{x \in C_{n,m}(B)} \sum_{y \in I^n} q_{xy}^{(n)} (u(x) - u(y))^2$$
$$\leqq \alpha^{-2m} N^m (8M_0^2 \cdot \text{diameter}(E)^2 \cdot \varepsilon_0^{-2}) .$$

This proves our assertion.

(2.8) Lemma. *Let* $\varphi_x \in C(I^{n+m}; [0,1])$, $x \in I^n$, *such that* $\sum_{x \in I^n} \varphi_x = 1$, $\varphi_x(z) = 0$, $z \in I^{n+m} \setminus C_{n+m,m}(x \cdot I^m)$, $x \in I^n$. *For each* $u \in C(I^n; \mathbb{R})$, *let* $\tilde{u} \in C(I^{n+m}; \mathbb{R})$ *be given by* $\tilde{u}(z) = \sum_{x \in I^n} u(x) \varphi_x(z)$, $z \in I^{n+m}$. *Then we have*

$$\mathscr{E}_{n+m}(\tilde{u}, \tilde{u}) \leqq (2m_0 + 1)^2 (M_0)^{4m_0+2} \cdot \left(\max_{x \in I^n} \mathscr{E}_{n+m}(\varphi_x, \varphi_x) \right) \cdot \mathscr{E}_n(u,u) .$$

Proof. For any $z, z' \in I^{n+m}$, let $S(z,z') = \{x \in I^n;\ \varphi_x(z) + \varphi_x(z') > 0\}$ and $v(z,z') = \langle u \rangle_{S(z,z')}$. Then we see that

$$\begin{aligned}
&\mathscr{E}_{n+m}(\tilde{u}, \tilde{u}) \\
&= \sum_{x \in I^n} \sum_{z \in x \cdot I^m} \sum_{z' \in I^{n+m}} q_{zz'}^{(n+m)} (\tilde{u}(z) - \tilde{u}(z'))^2 \\
&= \sum_{x \in I^n} \sum_{z \in x \cdot I^m} \sum_{z' \in I^{n+m}} \left\{ \sum_{y \in S(z,z')} q_{zz'}^{(n+m)} (u(y) - v(z,z'))(\varphi_y(z) - \varphi_y(z')) \right\}^2 . \\
&\leq \sum_{x \in I^n} \sum_{z \in x \cdot I^m} \sum_{z' \in I^{n+m}} \left[\left\{ \sum_{y \in S(z,z')} q_{zz'}^{(n+m)} (u(y) - v(z,z'))^2 \right\} \right. \\
&\qquad \left. \times \left\{ \sum_{y \in S(z,z')} q_{zz'}^{(n+m)} (\varphi_y(z) - \varphi_y(z'))^2 \right\} \right] .
\end{aligned}$$

From the assumptions and the assumption (A-3), we see that if $z \in x \cdot I^m$, $z \underset{n+m}{\sim} z'$, and $y \in S(z,z')$, there are $x_0, \ldots, x_{2m_0+1} \in I^n$ such that $x = x_0$, $y = x_{2m_0+1}$ and $x_{i-1} \underset{n}{\sim} x_i$, $i = 1, \ldots, 2m_0 + 1$. So we have for $z \in x \cdot I^m$ and $z' \in I^{n+m}$,

$$\begin{aligned}
\sum_{y \in S(z,z')} q_{zz'}^{(n+m)} (u(y) - v(z,z'))^2 &\leqq q_{zz'}^{(n+m)} \cdot \sum_{y \in S(z,z')} (u(y) - u(x))^2 \\
&\leqq (2m_0 + 1) \cdot f(x)
\end{aligned}$$

where $f(x_0) = \displaystyle\sum_{x_1, \ldots, x_{2m_0+1} \in I^n} \left(\prod_{i=1}^{2m_0+1} q_{x_{i-1}x_i}^{(n+m)} \right) \left(\sum_{i=1}^{2m_0+1} (u(x_i) - u(x_{i-1}))^2 \right)$.

By Proposition 1.1, we have

$$\sum_{x \in I^n} f(x) \leqq (2m_0 + 1)(M_0)^{2m_0} \mathscr{E}(u,u) .$$

So we have

$$\begin{aligned}
&\mathscr{E}_{n+m}(\tilde{u}, \tilde{u}) \\
&\leqq (2m_0 + 1) \cdot \sum_{x_0 \in I^n} f(x_0) \left\{ \sum_{z \in x_0 \cdot I^m} \sum_{z' \in I^{n+m}} \sum_{x_1, \ldots, x_{2m_0+1} \in I^n} \left(\prod_{i=1}^{2m_0+1} q_{x_{i-1}x_i}^{(n+m)} \right) \right. \\
&\qquad \left. \times q_{zz'}^{(n+m)} (\varphi_{x_{2m_0+1}}(z) - \varphi_{x_{2m_0+1}}(z'))^2 \right\}
\end{aligned}$$

$$\leqq (2m_0+1)\cdot \sum_{x_0 \in I^n} f(x_0)\left\{\sum_{x_1,\dots,x_{2m_0+1}\in I^n}\left(\prod_{i=1}^{2m_0+1} q^{(n+m)}_{x_{i-1}x_i}\right)\right.$$
$$\left.\times \mathscr{E}_{n+m}(\varphi_{x_{2m_0+1}},\varphi_{x_{2m_0+1}})\right\}$$
$$\leqq (2m_0+1)^2(M_0)^{4m_0+2}\cdot \mathscr{E}_n(u,u)\cdot\left(\max_{x\in I^n}\mathscr{E}_{n+m}(\varphi_x,\varphi_x)\right).$$

This proves our assertion.

(2.9) Lemma. *There are a constant C and a map* $T_{n,m,k}: C(I^n;\mathbb{R})\to C(I^{n+m+k};\mathbb{R})$, $n, m, k \geqq 1$, *such that*

$$\mathscr{E}_{n+m+k}(T_{n,m,k}u,\ T_{n,m,k}u)\leqq C\cdot N^{2k}\cdot R_m^{-1}\cdot \mathscr{E}_n(u,u)$$

and

$$(T_{n,m,k}u)(z)=u(x),\quad x\in I^n,\ y\in I^k\backslash \partial I^k,\ z\in (x\cdot y)\cdot I^m,$$

for any $u\in C(I^{n+m+k};\mathbb{R})$, $n, m, k\geqq 1$.

Proof. For any $x\in I^{n+k}$, let $v_x\in C(I^{n+k+m};[0,1])$ such that

$$v_x|_{x\cdot I^m}=1,\quad v_x|_{I^{n+k+m}\backslash C_{n+k+m,m}(x\cdot I^m)}=0,$$

and

$$\mathscr{E}_{n+k+m}(v_x,v_x)=R_{n+k+m}(x\cdot I^m, I^{n+k+m}\backslash C_{n+k+m,m}(x\cdot I^m))^{-1}.$$

Then we see that $\mathscr{E}_{n+k+m}(v_x,v_x)\leqq R_m^{-1}$.

Let $w=\sum_{x\in I^{n+k}} v_x$. Then $w\geqq 1$. Let $\varphi_x\in C(I^{n+k+m};\mathbb{R})$ be given by $\varphi_x = w^{-1}\cdot v_x$. Note that

$$(\varphi_x(z)-\varphi_x(z'))=(w(z)w(z'))^{-1}\{w(z)(v_x(z)-v_x(z'))-v_x(z)(w(z)-w(z'))\}.$$

Let $S(x)=\{y\in I^{n+k};\ v_x(z)q^{(n+k+m)}_{z,z'}(v_y(z)+v_y(z'))>0$ for some $z,z'\in I^{n+k+m}\}$. Then we see that $\#(S(x))\leqq M_0^{3(m_0+1)}$. So we have

$$\mathscr{E}_{n+k+m}(\varphi_x,\varphi_x)$$
$$\leqq 2\left\{\mathscr{E}_{n+k+m}(v_x,v_x)+\sum_{z,z'\in I^{n+k+m}} q^{(n+k+m)}_{z,z'}v_x(z)^2(w(z)-w(z'))\right\}$$
$$=2\left\{\mathscr{E}_{n+k+m}(v_x,v_x)+\sum_{z,z'\in I^{n+k+m}} q^{(n+k+m)}_{z,z'}v_x(z)^2\left(\sum_{y\in S(x)}(v_y(z)-v_y(z'))\right)^2\right\}$$
$$\leqq 2(M_0^{3(m_0+1)}+1)R_m^{-1}.$$

Now let $\tilde{T}_{n,k}: C(I^n;\mathbb{R})\to C(I^{n+k};\mathbb{R})$ be given by $\tilde{T}_{n,k}u(z)=u(x)$, $z\in x\cdot I^k$, $x\in I^n$, for any $u\in C(I^n;\mathbb{R})$. Then we see that

$$\mathscr{E}_{n+k}(\tilde{T}_{n,k}u,\tilde{T}_{n,k}u)\leqq N^{2k}\cdot\mathscr{E}_n(u,u),\ u\in C(I^n;\mathbb{R}).$$

Let $T_{n,m,k}: C(I^n;\mathbb{R})\to C(I^{n+m+k};\mathbb{R})$ be given by

$$(T_{n,m,k}u)(z)=\sum_{x\in I^{n+k}}(\tilde{T}_{n,k}u)(x)\varphi_x(z),\ z\in I^{n+k+m}.$$

Then we have by Lemma 2.8

$$\mathscr{E}_{n+m+k}(T_{n,m,k}u,\ T_{n,m,k}u)\leqq 8(m_0+1)^2M_0^{4m_0+2}(M_0^{3(m_0+1)}+1)N^{2k}R_m^{-1}\mathscr{E}_n(u,u).$$

On the other hand, it is easy to see that if $x \in I^n$, $y \in I^k \setminus \partial I^k$ and $z \in (x \cdot y) \cdot I^m$, then $T_{n,m,k} u(z) = u(x)$.

This proves our assertion.

(2.10) Proposition. *There is a constant C such that*

$$\lambda_{n+m+k} \geqq C \cdot (N^{-2k}(\#(I^k \setminus \partial I^k))(N^m R_m) \lambda_n$$

for any n, m, k $\geqq 1$.

Proof. Let $u_0 \in C(I^n; \mathbb{R})$, such that $\mathscr{E}_n(u_0, u_0) = 1$ and $\sum_{x \in I^n} (u_0(x) - \langle u_0 \rangle_{I^n})^2 = \lambda_n$. Let $u = T_{n,m,k} u_0$. Then we see that

$$\mathscr{E}_{n+m+k}(u, u) \leqq C \cdot N^{2k} R_m^{-1} .$$

On the other hand, we have

$$\sum_{z \in I^{n+k+m}} (u(z) - \langle u \rangle_{I^{n+k+m}})^2 \geqq N^m \cdot \sum_{x \in I^n} \sum_{y \in I^k \setminus \partial I^k} (u_0(x) - \langle u_0 \rangle_{I^n})^2$$
$$\geqq \#(I^k \setminus \partial I^k) \cdot N^m \cdot \lambda_n .$$

Therefore we have

$$\sum_{z \in I^{n+k+m}} (u(z) - \langle u \rangle_{I^{n+k+m}})^2 \geqq (N^m R_m) \lambda_n \cdot N^{-2k} (\#(I^k \setminus \partial I^k)) C \cdot \mathscr{E}_{n+m+k}(u, u) .$$

So by Proposition 1.5(1), we have our assertion.

(2.11) Proposition. $\lambda_n^{(D)} \leqq \lambda_{n+1}$, $n \geqq 1$.

Proof. Let $u \in C(I^n; \mathbb{R})$ such that $u|_{\partial I^n} = 0$, $\mathscr{E}_n(u, u) = 1$ and $N^n \cdot \langle u \rangle_{I^n}^2 = \lambda_n^{(D)}$. Let $y, y' \in I$ with $y \neq y'$ and let $v \in C(I^{n+1}; \mathbb{R})$ be given by $v(y \cdot x) = u(x)$, $v(y' \cdot x) = -u(x)$ and $u(z \cdot x) = 0$, $z \in I \setminus \{y, y'\}$, $x \in I^n$. Then we see that $\langle v \rangle_{I^{n+1}} = 0$, $\mathscr{E}_{n+1}(v, v) = 2$ and

$$\sum_{x \in I^{n+1}} (v(x) - \langle v \rangle_{I^{n+1}})^2 = 2 \left(\sum_{y \in I^n} u(y)^2 \right) \geqq 2N^{-n} \left(\sum_{y \in I^n} u(x) \right)^2 = 2\lambda_n^{(D)} .$$

This proves our assertion.

(2.12) Lemma. *Let* $S_{n,m}: C(I^{n+m}; \mathbb{R}) \to C(I^n; \mathbb{R})$ *be given by* $(S_{n,m} u)(x) = \langle u \rangle_{x \cdot I^m}$, $x \in I^n$, $u \in C(I^{n+m}; \mathbb{R})$. *Then we have*

$$\mathscr{E}_{n,A}(S_{n,m} u, S_{n,m} u) \leqq 2M_0 \sigma_m \cdot N^{-m} \cdot \mathscr{E}_{n+m, A \cdot I^m}(u, u),\ u \in C(I^{n+m}; \mathbb{R}) ,$$

for any subset A of I^n.

Proof. Note that

$$\mathscr{E}_{n,A}(S_{n,m} u, S_{n,m} u) = \sum_{\substack{B, B' \in \mathscr{B}_{n+m,m} \\ B \cup B' \subset A \cdot I^m, B_n \sim_m B'}} (\langle u \rangle_B - \langle u \rangle_{B'})^2$$
$$\leqq \sigma_m \cdot N^{-m} \left\{ \sum_{\substack{B, B' \in \mathscr{B}_{n+m,m} \\ B \cup B' \subset A \cdot I^m, B_n \sim_m B}} \mathscr{E}_{n+m, B \cup B'}(u, u) \right\}$$
$$\leqq 2M_0 \sigma_m \cdot N^{-m} \cdot \mathscr{E}_{n+m, A \cdot I^m}(u, u) .$$

This proves our assertion.

(2.13) Proposition. (1) $\lambda_{n+m} \leqq \lambda_m + 2M_0 \cdot \lambda_n \cdot \sigma_m$, $n \geqq 1$, $m \geqq 0$.
(2) *There is a constant* $C > 0$ *such that* $\lambda_{n+m} \leqq C \cdot \lambda_n \sigma_m$, $n, m \geqq 1$.
(3) *There is a constant* $C > 0$ *such that* $\sigma_{n+m} \leqq C \cdot \sigma_n \sigma_m$, $n, m \geqq 1$.

Proof. (1) Let $u \in C(I^{n+m}; \mathbb{R})$ such that $\mathscr{E}_{n+m}(u, u) = 1$ and $\sum_{x \in I^{n+m}} (u(x) - \langle u \rangle_{I^{n+m}})^2 = \lambda_{n+m}$. Then we see that

$$\lambda_{n+m} = \sum_{B \in \mathscr{B}_{n+m,m}} \sum_{x \in B} (u(x) - \langle u \rangle_B)^2 + N^m \cdot \sum_{x \in I^n} (S_{n,m} u(x) - \langle S_{n,m} u \rangle_{I^n})^2 .$$

By Proposition 1.5(1), we see that

$$\sum_{B \in \mathscr{B}_{n+m,m}} \sum_{x \in B} (u(x) - \langle u \rangle_B)^2 \leqq \lambda_m \cdot \mathscr{E}_{n+m}(u, u) .$$

So by Lemma 2.12 we have

$$\lambda_{n+m} \leqq \lambda_m + \lambda_n N^m \cdot \mathscr{E}_n(S_{n,m} u, S_{n,m} u) \leqq \lambda_m + 2M_0 \cdot \lambda_n \sigma_m .$$

This implies our assertion (1).
(2) From Propositions 2.7 and 2.10, we see that there is an $n' \geqq 1$ such that $\lambda_{n+m} \geqq 2\lambda_m$, $n \geqq n'$, $m \geqq 1$. So from the assertion (2), we have our assertion.

The assertion (3) follows from Lemma 2.12 by a similar argument to the proof of the assertion (1).

This completes the proof.

Theorem 2.1 follows from Propositions 2.7, 2.10 and 2.13. This completes the proof of Theorem 2.1.

3 Nash type estimates and smoothness of measures

Now let $L^{(n)}$ be a linear operator in $C(I^n; \mathbb{R})$ given by

$$\sum_{x \in I^n} (L^{(n)} u)(x) v(x) = -\mathscr{E}_n(u, v) , \quad u, v \in C(I^n; \mathbb{R}) . \tag{3.1}$$

Also let $P_t^{(n)} = \exp(t \cdot L^{(n)})$, $t \geqq 0$. Then $\{P_t^{(n)}\}_{t \geqq 0}$ is a symmetric Markov semigroup. Let p_n: $[0, \infty) \times I^n \times I^n$ be given by

$$\sum_{y \in I^n} p_n(t, x, y) u(y) = (P_t^{(n)} u)(y) , \quad u \in C(I^n; \mathbb{R}) . \tag{3.2}$$

Then by using the similar arguments in Carlen-Kusuoka-Stroock [3] or in Nash [7] we have the following.

(3.3) Theorem. *There is a constant* $c_0 > 0$ *such that*

$$p_n(c_0 \cdot \lambda_m, x, y) \leqq N^{-m} , \quad x, y \in I^n$$

For any $n \geqq 1$ *and* $m = 0, 1, \ldots, n-1$.

Proof. Let $f \in C(I^n; [0, \infty))$ with $\sum_{x \in I^n} f(x) = 1$. Let $u(t, x) = (P_t^{(n)} f)(x)$ and $g(t) = \sum_{x \in I^n} u(t, x)^2$. Then it is obvious that $\sum_{x \in I^n} u(t, x) = 1$. Also, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} g(t) &= -2 \cdot \mathscr{E}_n(u(t), u(t)) \leqq -2 \cdot \left(\sum_{B \in \mathscr{B}_{n,m}} \mathscr{E}_{n,B}(u(t, \cdot), u(t, \cdot)) \right) \\ &\leqq -2 \cdot \lambda_m^{-1} \left\{ \sum_{B \in \mathscr{B}_{n,m}} \left(\sum_{x \in B} u(t, x)^2 - N^m \langle u(t, \cdot) \rangle_B^2 \right) \right\} \\ &= -2 \cdot \lambda_m^{-1} \cdot \sum_{B \in \mathscr{B}_{n,m}} \sum_{x \in B} u(t, x)^2 + 2 \cdot \lambda_m^{-1} \cdot N^{-m} \left(\left(\sum_{B \in \mathscr{B}_{n,m}} \sum_{x \in B} u(t, x) \right)^2 \right) \\ &= -2 \cdot \lambda_m^{-1} (g(t) - N^{-m}) .\end{aligned}$$

So we see that

$$-\frac{\mathrm{d}}{\mathrm{d}t} \log(g(t) - N^{-m}) \geqq 2 \cdot \lambda_m^{-1}, \text{ if } g(t) > N^{-m} .$$

Let $t_m = \min\{t \geqq 0, g(t) \leqq N^{-m+1}\}$, $m = 0, 1, \ldots, n$. If $t_m > 0$, then we have

$$2 \cdot \lambda_m^{-1}(t_m - t_{m-1}) \leqq -\log(g(t_m) - N^{-m}) + \log(g(t_{m-1}) - N^{-m}) \leqq \log(N + 1) .$$

So we see that

$$t_m - t_{m-1} \leqq (\lambda_m/2) \cdot \log(N + 1) , \quad m = 1, \ldots, n . \tag{3.4}$$

(3.4) holds, even if $t_m = 0$. Also, we see that $t_0 = 0$. By Theorem 2.1, we have

$$\sum_{k=1}^{m} (\lambda_k/\lambda_m) \leqq C \left(\sum_{k=0}^{\infty} (N^k R_k)^{-1} \right) \leqq C^2 \left(\sum_{k=0}^{\infty} \alpha^{-2k} \right) = C^2 (1 - \alpha^{-2})^{-1} .$$

So if we let $c = C^2(1 - \alpha^{-2})^{-1}(\log(N + 1)/2)$, we have

$$t_m \leqq \lambda_m \cdot \left(\sum_{k=1}^{m} (\lambda_k/\lambda_m) \right) \cdot (\log(N + 1)/2) \leqq c \cdot \lambda_m .$$

So we see that $g(c\lambda_m) \leqq N^{-m+1}$, $m = 0, 1, \ldots, n$. This proves that $\| P_{c \cdot \lambda_m}^{(n)} \|_{\ell^1 \to \ell^2}^2 \leqq N^{-m+1}$, where $\| \cdot \|_{\ell^1 \to \ell^2}$ denotes the operator norm for linear operators from $\ell^1(I^n)$ into $\ell^2(I^n)$. Since $P_t^{(n)}$ is symmetric, we see that $\| P_t^{(n)} \|_{\ell^1 \to \ell^\infty} \leqq \| P_t^{(n)} \|_{\ell^1 \to \ell^2} \cdot \| P_t^{(n)} \|_{\ell^2 \to \ell^\infty} = \| P_t^{(n)} \|_{\ell^1 \to \ell^2}^2$. So by Theorem 2.1 we have our assertion.

This completes the proof.

(3.5) Proposition. *For each* $\gamma > 0$, *there is a constant* C_γ *such that*

$$\sup\{\textstyle\int_E \exp(-s|\xi - \eta|^\gamma) \nu(\mathrm{d}\eta); \xi \in E\} \leqq C_\gamma \cdot s^{-d_f \cdot \gamma^{-1}}, s > 0 .$$

Proof. Let $\tilde{E} = \bigcup_{k=1}^{\infty} \psi_1^{-k}(E)$. Note that there is a measure $\tilde{\nu}$ on $\tilde{E}$ such that $\tilde{\nu}(A) = \lim_{k \to \infty} \alpha^k \cdot \nu(E \cap (\psi_1^k(A))$ for any Borel subset A of $\tilde{E}$. Then we see that $C_\gamma = \sup\{\int_{\tilde{E}} \exp(-|\xi - \eta|^\gamma) \tilde{\nu}(\mathrm{d}\eta); \xi \in \tilde{E}\} < \infty$. For any $s \in [\alpha^{k\gamma}, \infty)$, $k \in \mathbb{Z}$, we have

$$\begin{aligned}\int_E \exp(-s|\xi - \eta|^\gamma) \nu(\mathrm{d}\eta) &\leqq \int_{\tilde{E}} \exp(-s|\xi - \eta|^\gamma) \tilde{\nu}(\mathrm{d}\eta) \\ &= N^{-k} \cdot \int \exp(-\alpha^{-k\gamma} s |\alpha^k \xi - \eta|^\gamma) \tilde{\nu}(\mathrm{d}\eta) \leqq C_\gamma \cdot N^{-k} .\end{aligned}$$

So we have our assertion.

By using the argument in Nash [7], we have the following.

(3.6) Lemma. *Let* $r_n(x, y) = \alpha^n \cdot \max\{|\xi - \xi'|;\ \xi \in \psi_x(E),\ \xi' \in \psi_y(E)\}$, $x, y \in I^n$. *Then for any* $\gamma > 0$, *there is a constant* $C_\gamma > 0$ *such that*

$$\sum_{y \in I^n} r_n(x, y)^\gamma p_n(c_0 \cdot \lambda_m, x, y) \geqq C_\gamma \cdot \alpha^{m\gamma}$$

for any $n \geqq 1$, $x \in I^n$, *and* $m = 0, \ldots, n-1$.

Proof. Let $h(t) = h_n(t, x) = -\sum_{y \in I^n} p_n(t, x, y) \cdot \log p_n(t, x, y)$, and $M(t) = \sum_{y \in I^n} r_n(x, y)^\gamma p_n(t, x, y)$. $t \geqq 0$, $n \geqq 1$, $x \in I^n$. Then by Theorem 3.3, we see that

$$h(c_0 \cdot \lambda_m) \geqq \sum_{y \in I^n} \log(N^m) \cdot p_n(c_0 \cdot \lambda_m, x, y) = m \cdot \log N\,. \tag{3.7}$$

Notice that $u \cdot \log u + s \cdot u \geqq -\exp(-(s+1))$, $u, s > 0$, we have

$$\begin{aligned}
&-h(t) + aM(t) + b \\
&= \sum_{y \in I^n} \{p_n(t, x, y) \cdot \log p_n(t, x, y) + (a \cdot r_n(x, y)^\gamma + b) p_n(t, x, y)\} \\
&\geqq -\mathrm{e}^{-(b+1)} \cdot \sum_{y \in I^n} \exp(-a \cdot r_n(x, y)^\gamma) \\
&\geqq -\mathrm{e}^{-(b+1)} N^n \cdot \sup\left\{\int_E \exp(-a \cdot (\alpha^n \cdot |\xi - \eta|)^\gamma) \nu(d\eta);\ \xi \in E\right\} \\
&\geqq -C_\gamma \cdot \mathrm{e}^{-1} \cdot \mathrm{e}^{-b} \cdot a^{-d_f \cdot \gamma^{-1}}
\end{aligned}$$

for any $a, b > 0$. Letting $a = M(t)^{-1}$ and $b = -d_f \gamma^{-1} \cdot \log a$, we have

$$-h(t) + d_f \gamma^{-1} \cdot \log M(t) \geqq -C_\gamma \cdot \mathrm{e}^{-1} - 1\,,$$

which implies that $M(t) \geqq \exp(d_f^{-1} \gamma (h(t) - C_\gamma \cdot \mathrm{e}^{-1} - 1))$. Combining this with (3.7), we have our assertion.

For any signed measure μ in I^n, let $|\mu|_{n,k}$, $0 \leqq k \leqq n$, be given by

$$|\mu|_{n,k} = \left\{\sum_{B \in \mathscr{B}_{n,k}} \mu(B)^2\right\}^{1/2}. \tag{3.8}$$

Then we have the following.

(3.9) Lemma. $\left|\int_{I^n} u \, d\mu - \langle u \rangle_{I^n} \cdot \mu(I^n)\right| \leqq \left\{\sum_{k=1}^{n} (\lambda_k \cdot N^{-k})^{1/2} |\mu|_{n,k-1}\right\} \cdot \mathscr{E}_n(u, u)^{1/2}$

for any signed measure μ *on* I^n *and* $u \in C(I^n; \mathbb{R})$.

Proof. From Proposition 1.5(3), we have

$$\left(\int_{B'} (\langle u \rangle_{B_{n,k}(x)} - \langle u \rangle_{B_{n,k-1}(x)}) \mu(dx)\right)^2 \leqq \lambda_k N^{-k-1} \cdot |\mu(B')|^2 \cdot \mathscr{E}_{n,B}(u, u)$$

for any $u \in C(I^n; \mathbb{R})$, $B \in \mathscr{B}_{n,k}$ and $B' \in \mathscr{B}_{n,k-1}$ with $B' \subset B$. So we have

$$\left(\int_B (\langle u \rangle_{B_{n,k}(x)} - \langle u \rangle_{B_{n,k-1}(x)}) \mu(\mathrm{d}x) \right)^2 \leqq \lambda_k N^{-k} \left(\sum_{\substack{B' \in \mathscr{B}_{n,k-1} \\ B' \subset B}} \mu(B')^2 \right) \cdot \mathscr{E}_{n,B}(u, u)$$

for any $n \geqq m \geqq 1$, $k = 1, \ldots, m$, $B \in \mathscr{B}_{n,m}$, and $u \in C(I^n; \mathbb{R})$. This and Schwarz' inequality imply that

$$\left(\int_{I^n} (\langle u \rangle_{B_{n,k}(x)} - \langle u \rangle_{B_{n,k-1}(x)}) \mu(\mathrm{d}x) \right)^2 \leqq \lambda_k N^{-k} |\mu|_{n,k-1} \cdot \mathscr{E}_n(u, u) .$$

This implies our assertion.

(3.10) Proposition. $|u(x) - u(y)| \leqq 2 \cdot \{\sum_{k=0}^m (\lambda_k \cdot N^{-k})^{1/2}\} \cdot \mathscr{E}_n(u, u)^{1/2}$ *for any* $u \in C(I^n; \mathbb{R})$ *and* $x, y \in I^n$ *such that there is a* $B \in \mathscr{B}_{n,m}$ *with* $x, y \in B$.

Proof. Let μ be a signed measure on I^n given by $\mu = \delta_x - \delta_y$. Then we see that $|\mu|_{n,k} = 0$, $k = m, m+1, \ldots, n$, and $|\mu|_{n,k} \leqq 2$, $k = 0, 1, \ldots, m$. This and Lemma 3.9 implies our assertion.

(3.11) Proposition. *Let A be a subset of I^n. Assume there are $\ell \in \{0, 1, \ldots, n\}$, $\gamma \in [0, 1]$ and $r \geqq 1$ such that*

$$\#(A) \geqq r^{-1} N^{\ell} N^{\gamma(n-\ell)} ,$$

$$\#(A \cap B) = N^{-\ell} \cdot \#(A) , \quad B \in \mathscr{B}_{n,n-\ell} ,$$

$$\#(A \cap B) \leqq r \cdot N^{\gamma m} , \quad B \in \mathscr{B}_{n,m} , \quad m = 0, 1, \ldots, n - \ell ,$$

and

$$\#(\{B \in \mathscr{B}_{n,m}; A \cap B \neq \phi\}) \leqq r \cdot N^{(1-\gamma)\ell + \gamma(n-m)} , \quad m = 0, 1, \ldots, n - \ell .$$

Then we have

$$|\langle u \rangle_A - \langle u \rangle_{I^n}| \leqq (1 + r^3) N^{-\gamma n/2} \cdot \left\{ \sum_{k=0}^{n-\ell} (\lambda_k \cdot N^{-(1-\gamma)k})^{1/2} \right\} \cdot \mathscr{E}_n(u, u)^{1/2}$$

for any $u \in C(I^n; \mathbb{R})$.

Proof. Let μ be a signed measure on I^n given by $\int_{I^n} u \, \mathrm{d}\mu = \langle u \rangle_A - \langle u \rangle_{I^n}$, $u \in C(I^n; \mathbb{R})$. Then we have

$$|\mu|_{n,k} = 0 , \quad k = n - \ell, n - \ell + 1, \ldots, n ,$$

and

$$|\mu|_{n,k} \leqq (\#(A))^{-1} \left\{ \sum_{B \in \mathscr{B}_{n,k}} \#(A \cap B)^2 \right\}^{1/2} + (\#(I^n))^{-1} \left\{ \sum_{B \in \mathscr{B}_{n,k}} \#(B)^2 \right\}^{1/2}$$

$$\leqq r^3 \cdot N^{-\gamma(n-k)/2} + N^{-(n-k)/2}$$

$$\leqq (1 + r^3) N^{-\gamma(n-k)/2} .$$

This and Lemma 3.9 implies our assertion.

4 Tightness of semigroups

In this section and in the next section, we assume the following assumption furthermore.

(B-1) There are $C \in (0, \infty)$ and $k \geqq 0$ such that $\sigma_n \leqq C \cdot \lambda_{n+k}^{(D)}$ for all $n \geqq 1$.

This assumption can be verified for good fractals (see Sect. 8).

By the assumption (B-1) and Theorem 2.1, we have the following.

(4.1) Proposition. *There is a constant C such that*

$$C^{-1}\lambda_n \leqq \lambda_n^{(D)} \leqq C\lambda_n , \quad n \geqq 1 , \tag{4.2}$$

$$C^{-1}\lambda_n \leqq \sigma_n \leqq C\lambda_n , \quad n \geqq 1 , \tag{4.3}$$

and

$$\lambda_{n+m} \leqq C \cdot \lambda_n \lambda_m, \quad n, m \geqq 1 . \tag{4.4}$$

Let $\{P_x^{(n,\cdot)}; x \in I^n\}$ be a Markov process on I^n, whose generator is $L^{(n)}$. Let $\mathbb{Q}_+$ denote $\mathbb{Q} \cap [0, \infty)$. Let us take an $x_0 \in E$ and fix it. Let $Q^{(n)}$ be the probability law of $\{\psi_{w(\lambda_n t)}(x_0), t \in \mathbb{Q}_+\}$ under $N^{-n}\sum_{x \in I^n} P_x^{(n)}(\mathrm{d}w)$. Then $Q^{(n)}$, $n \geqq 1$, are probability measures in $E^{\mathbb{Q}_+}$. Since the space $E^{\mathbb{Q}_+}$ is compact, we see that $\{Q^{(n)}; n \geqq 1\}$ is tight.

Our main result in this section is the following.

(4.5) Theorem. *For each cluster point $\tilde{Q}$ of $\{Q^{(n)}\}_{n=1}^{\infty}$, there is a strongly continuous symmetric Markov semigroup $\{Q_t\}_{t \geqq 0}$ in $L^2(E, \mathrm{d}\nu)$ such that*

$$E^{\tilde{Q}}[f_0(w(t_0))f_1(w(t_1)) \ldots f_n(w(t_n))]$$
$$= (Q_{t_n - t_{n-1}}(f_{n-1}(Q_{t_{n-1}-t_{n-2}}(f_{n-2}(\ldots (Q_{t_1 - t_0} f_0) \ldots), f_n)_{L^2(E, \mathrm{d}\nu)}$$

for any $0 \leqq t_0 \leqq t_1 \leqq \ldots \leqq t_n$, $t_0, \ldots, t_n \in \mathbb{Q}_+$ and $f_0, \ldots, f_n \in C(E; \mathbb{R})$. Moreover, we have

$$\| Q_t f \|_{L^2(E, \mathrm{d}\nu)} \leqq \mathrm{e}^{-t} \| f \|_{L^2(E, \mathrm{d}\nu)} , \quad t \geqq 0 ,$$

for any $f \in L^2(E, \mathrm{d}\nu)$ with $\int_E f \mathrm{d}\nu = 0$.

To prove this theorem, we need some preparations. Let $\varphi_n \in C(I^n; \mathbb{R})$ such that $\varphi_n|_{\partial I^n} = 0$, $\langle \varphi_n \rangle_{I^n} = 1$ and $N^n \cdot \langle \varphi_n \rangle_{I^n}^2 = \lambda_n^{(D)} \mathscr{E}_n(\varphi_n, \varphi_n)$. Then we have the following.

(4.6) Proposition. (1) *$\varphi_n \geqq 0$ and $L^{(n)}\varphi_n|_{I^n \setminus \partial I^n} = -(\lambda_n^{(D)})^{-1}$.*

(2) *$\lambda_n^{(D)} \cdot \varphi_n(x) \leqq \lambda_{n+1}^{(D)} \cdot \varphi_{n+1}(i \cdot x)$ for any $n \geqq 1$, $x \in I^n$, $i \in I$.*

(3) *$\sup\{\varphi_n(x); x \in I^n, n \geqq 1\} < \infty$.*

Proof. It is easy to see that $L^{(n)}\varphi_n|_{I^n \setminus \partial I^n}$ is constant, say $c \in \mathbb{R}$. Then we have $N^n = \lambda_n^{(D)} \cdot \mathscr{E}_n(\varphi_n, \varphi_n) = -\lambda_n^{(D)} \cdot \sum_{x \in I^n} c \cdot \varphi_n(x) = -c \cdot \lambda_n^{(D)} N^n$. So we have the assertion (1).

Let $\tau_A(w) = \min\{t \geqq 0, w(t) \in A\}$, and let $P_t^{D,n}u(x) = E^{P_x}[u(w(t)), t < \tau_{\partial I^n}]$, $t > 0$, $u \in C(I^n; \mathbb{R})$. Then we see that $\lambda_n^{(D)} \cdot \varphi_n(x) = \int_0^\infty (P_t^{D,n}1)(x)\,\mathrm{d}t$. Since the law of $w(\cdot \wedge \tau_{i \cdot \partial I^n})$ under $P_{i \cdot x}^{D,n+1}$ and the law of $i \cdot w(\cdot \wedge \tau_{\partial I^n})$ under $P_x^{D,n}$, $i \in I$, $x \in I^n$ are the same, we have the assertion (2).

Note that for any $u \in C(I^n; \mathbb{R})$ with $u|_{\partial I^n} = 0$,

$$\sum_{x \in I^n} u(x)^2 \leqq 2\left\{\sum_{x \in I^n} (u(x) - \langle u \rangle_{I^n})^2 + N^n \langle u \rangle_{I^n}\right\} \leqq 2(\lambda_n + \lambda_n^{(D)})\mathscr{E}_n(u, u) .$$

So we have

$$\left\{\sum_{x \in I^n} (P_t^{D,n_1})(x)^2\right\}^{1/2} \leqq \exp(-(2(\lambda_n + \lambda_n^{(D)}))^{-1} t) .$$

Then by Theorem 3.3, we have

$$\begin{aligned} \lambda_n^{(D)} \varphi_n &= \int_0^\infty P_t^{D,n_1} \mathrm{d}t \leqq c_0 \cdot \lambda_{n-1} + P^{(n)}_{c_0 \cdot \lambda_{n-1}} \left(\int_0^\infty P_t^{D,n_1} \mathrm{d}t\right) \\ &\leqq c_0 \cdot \lambda_{n-1} + (N^{-n+1})^{1/2} \left(\int_0^\infty \exp(-(2(\lambda_n + \lambda_n^{(D)}))^{-1} t)\, \mathrm{d}t\right) \cdot N^{n/2} \\ &= c_0 \cdot \lambda_{n-1} + 2 \cdot N^{1/2}(\lambda_n + \lambda_n^{(D)}) . \end{aligned}$$

So we have the assertion (3).

(4.7) Proposition. *Let* $g(s) = \sup\{N^{-n} \cdot \#(\{x \in I^n;\ \varphi_n(x) \leqq s\});\ n \geqq 1\},\ s \in (0, \infty)$. *Then we see that there are constants* $C_0 \in (0, 1)$ *and* $C_1, s_0 \in (0, \infty)$ *such that*

$$g(s) \leqq C_0 \cdot g(C_1 \lambda_m s) + C_1 \cdot (R_m N^m)^{-1}, \quad m \geqq 1,\ s \in (0, s_0] .$$

In particular, $g(s) \to 0$ *as* $s \downarrow 0$.

Proof. Since $\mathscr{E}_n(\varphi_n, \varphi_n) = N^n(\lambda_n^{(D)})^{-1}$, by Proposition 1.5(1) we see that

$$N^{-(n+m)}\left(\sum_{x \in I^{n+m}} (\varphi_{n+m}(x) - \langle \varphi_{n+m} \rangle_{B_{n+m,n}(x)})^2\right) \leqq \lambda_n (\lambda_{n+m}^{(D)})^{-1} , \quad n, m \geqq 1 .$$

So we have

$$N^{-n} \cdot \sum_{y \in I^n} (N^{-m} \cdot \sum_{x \in I^m} \varphi_{n+m}(x \cdot y) - 1)^2 \leqq \lambda_n (\lambda_{n+m}^{(D)})^{-1} .$$

This implies that

$$N^{-n} \cdot \#\left(\left\{y \in I^n;\ N^{-m} \cdot \sum_{x \in I^m} \varphi_{n+m}(x \cdot y) \leqq 1/2\right\}\right) \leqq 4 \cdot \lambda_n (\lambda_{n+m}^{(D)})^{-1} .$$

Let $\kappa = \sup\{\varphi_n(x);\ x \in I^n,\ n \geqq 1\}$. Then $\kappa \geqq 1$. Note that if $N^{-m} \cdot \sum_{x \in I^m} \varphi_{n+m}(x \cdot y) \geqq 1/2$, then $N^{-m} \cdot \#(\{x \in I^m;\ \varphi_{n+m}(x \cdot y) \leqq (4\kappa)^{-1}\}) \leqq C_0$. Here $C_0 = 1 - \dfrac{1}{4\kappa}$.

Also, by Proposition 4.6(2), we have

$$\varphi_{n+m}(x \cdot y) \geqq \lambda_n^{(D)} (\lambda_{n+m}^{(D)})^{-1} \varphi_n(y) , \quad x \in I^m,\ y \in I^n,\ n, m \geqq 1 .$$

Therefore we see that for $s \in (0, (4\kappa)^{-1})$,

$$N^{-(n+m)} \cdot \#(\{z \in I^{n+m}; \varphi_{n+m}(z) \leqq s\}$$
$$= N^{-(n+m)} \cdot \#(\{(x, y) \in I^m \times I^n; \varphi_{n+m}(x \cdot y) \leqq s, \varphi_n(y) \leqq \lambda_{n+m}^{(D)}(\lambda_n^{(D)})^{-1} s\}$$
$$\leqq C_0 \cdot N^{-n} \cdot \#(\{y \in I^n; \varphi_n(y) \leqq s \cdot \lambda_{n+m}^{(D)}(\lambda_n^{(D)})^{-1}\})$$
$$+ N^{-(n+m)} \cdot \#(\{x, y) \in I^m \times I^n; N^{-m} \cdot \sum_{x \in I^m} \varphi_{n+m}(x \cdot y) \leqq 1/2,$$
$$\varphi_n(y) \leqq s \cdot \lambda_{n+m}^{(D)}(\lambda_n^{(D)})^{-1}\}$$
$$\leqq C_0 \cdot N^{-n} \cdot \#(\{y \in I^n; \varphi_n(y) \leqq s \cdot \lambda_{n+m}^{(D)}(\lambda_n^{(D)})^{-1}\}) + (4\lambda_n(\lambda_{n+m}^{(D)})^{-1}).$$

So we have the first assertion. From this, we have

$$\lim_{s \downarrow 0} g(s) \leqq (1 - C_0)^{-1} C_1 (R_m N^m)^{-1}.$$

By Theorem 2.1, we see that $R_m N^m \to \infty$ as $m \to \infty$. So we see that $\lim_{s \downarrow 0} g(s) = 0$.
This completes the proof.

(4.8) Proposition. $\lim_{T \downarrow 0} \overline{\lim_{n \to \infty}} N^{-n} \sum_{x \in I^n} P_x^{(n)}[\tau_{\partial I^n} \leqq \lambda_n^{(D)} T] = 0.$

Proof. Let $e_{n,\alpha}(x) = E^{P_x^{(n)}}[\exp(-\alpha(\lambda_n^{(D)})^{-1}\tau_{\partial I^n})]$, $x \in I^n$, $\alpha > 0$, $x \in I^n$. Then we see that $(\alpha(\lambda_n^{(D)})^{-1} I - L^{(n)}) e_{n,\alpha}|_{I^n \setminus \partial I^n} = 0$ and $e_{n,\alpha}|_{\partial I^n} = 1$. So we have

$$\alpha \cdot \lambda_n^{(D)} \cdot \sum_{x \in I^n} e_{n,\alpha}(x) \varphi_n(x) = \sum_{x \in I^n} (L^{(n)}(e_{n,\alpha^{-1}}))(x) \varphi_n(x)$$
$$= \sum_{x \in I^n} (e_{n,\alpha}(x) - 1)(L^{(n)} \varphi_n)(x) = \lambda_n^{(D)} \cdot \sum_{x \in I^n} (1 - e_{n,\alpha}(x)).$$

So we see that

$$N^{-n} \sum_{x \in I^n} P_x^{(n)}[\tau_{\partial I^n} \leqq \lambda_n^{(D)} \alpha^{-1}] \leqq e \cdot N^{-n} \cdot \sum_{x \in I^n} e_{n,\alpha}(x)$$
$$\leqq e \cdot \left\{ (\varepsilon N^n)^{-1} \cdot \sum_{x \in I^n} e_{n,\alpha}(x) \varphi_n(x) + N^{-n} \cdot \#(\{x \in I^n; \varphi_n(x) \leqq \varepsilon\}) \right\}$$
$$\leqq e \cdot \{\varepsilon^{-1} \cdot \alpha^{-1} + N^{-n} \cdot \#(\{x \in I^n; \varphi_n(x) \leqq \varepsilon\})\}$$

for any $\varepsilon, \alpha > 0$ and $n \geqq 1$.

So by Proposition 4.7, we have our assertion.

The following is an easy consequence of Proposition 4.8.

(4.9) Proposition.

$$\lim_{T \downarrow 0} \overline{\lim_{m \to \infty}} \sup \left\{ \#(B)^{-1} \sum_{x \in B} P_x^{(n)}[w(\lambda_m t) \in I^n \setminus B]; t \in (0, T], B \in \mathscr{B}_{n,m}, n \geqq m \right\} = 0.$$

For each $n \geqq 1$, let $\tilde{P}_n : L^1(E, d\nu) \to C(I^n; \mathbb{R})$ and $\iota_n : C(I^n; \mathbb{R}) \to L^\infty(E, d\nu)$ be given by

$$\tilde{P}_n f(x) = \nu(\psi_x(E))^{-1} \int_{\psi_x(E)} f(\xi) \nu(d\xi), \quad x \in I^n, f \in L^1(E, d\nu),$$

and

$$\iota_n u(\xi) = u(x), \text{ if } \xi \in \psi_x(E), \quad x \in I^n, u \in C(I^n; \mathbb{R}).$$

Let $Q_t^{(n)} = \iota_n \circ P_{\lambda_n t}^{(n)} \circ \tilde{P}_n$, $t > 0$, $n \geqq 1$. Then we see that $\{Q_t^{(n)}\}_{t>0}$ is a semigroup of symmetric Markov operators in $L^2(E, \mathrm{d}\nu)$ for each $n \geqq 1$, which is not necessarily strongly continuous. Let $P_n = \iota_n \circ \tilde{P}_n$, $n \geqq 1$. Then P_n is an orthogonal projection in $L^2(E, \mathrm{d}\nu)$ whose range is finite dimensional.

(4.10) Lemma. (1) $\|(I - P_m)\iota_n u\|^2_{L^2(\mathrm{d}\nu)} \leqq \lambda_{n-m} N^{-n} \cdot \mathscr{E}_n(u, u)$, $u \in C(I^n; \mathbb{R})$, $n \geqq m \geqq 1$.

(2) *There is a constant* $C \in (0, \infty)$ *such that*

$$\|(I - P_m)Q_t^{(n)}\|_{L^2(\mathrm{d}\nu) \to L^2(\mathrm{d}\nu)} \leqq C \cdot (N^m R_m)^{-1} \cdot t^{-1/2}\,, \quad t > 0,\ n \geqq m \geqq 1\,.$$

(3) $\overline{\lim_{t \downarrow 0}}\ \overline{\lim_{n \to \infty}}\ \{\|f - Q_t^{(n)} f\|_{L^2(\mathrm{d}\nu)};\ n \geqq 1\} = 0$, *for any* $f \in C(E; \mathbb{R})$.

Proof. Note that $\|(I - P_m)\iota_n u\|^2_{L^2(\mathrm{d}\nu)}$

$= N^{-n} \cdot \left(\sum_{B \in \mathscr{B}_{n,n-m}} \sum_{x \in B} (u(x) - \langle u \rangle_B)^2 \right)$, $u \in C(I^n; \mathbb{R})$. So we have the assertion (1) from Proposition 1.5(1). The assertion (2) follows from Theorem 2.1, Proposition 4.1 and the fact that $\mathscr{E}_n(P_{\lambda_n t}^{(n)} u,\ P_{\lambda_n t}^{(n)} u) \leqq (2t)^{-1} (\sum_{x \in I^n} u(x)^2)$, $n \geqq 1$, $u \in C(I^n; \mathbb{R})$. The assertion (3) follows from Proposition 4.9 and the fact that

$$\begin{aligned} &\|f - Q_t^{(n)} f\|^2_{L^2(\mathrm{d}\nu)} \\ &= N^{-n} \cdot \sum_{x \in I^n} (\tilde{P}_n f)(x) \{(\tilde{P}_n f)(x) - 2 \cdot E^{P_x^{(n)}}[(\tilde{P}_n f)(w(\lambda_n t)] \\ &\qquad + E^{P_x^{(n)}}[(\tilde{P}_n f)(w(2\lambda_n t)]\}\,. \end{aligned}$$

Proof of Theorem 4.5 Suppose that $\{n_k\}$ is a subsequence such that $Q^{(n_k)} \to \tilde{Q}$ as probability measures in $E^{\mathbb{Q}_+}$. By Lemma 4.10, we see that there is a semigroup $\{Q_t; t \in \mathbb{Q}_+\}$ of symmetric Markov operators in $L^2(E, \mathrm{d}\nu)$, if necessary taking a subsequence, such that $Q_t^{(n_k)} \to Q_t$ strongly as $k \to \infty$ for any $t \in \mathbb{Q}_+$. By Lemma 4.10(3), we see that $\lim_{t \downarrow 0, t \in \mathbb{Q}} \|f - Q_t f\|_{L^2(\mathrm{d}\nu)} = 0$, $f \in C(E; \mathbb{R})$. So we can extend the semigroup $\{Q_t;\ t \in \mathbb{Q}_+\}$ to a strongly continuous symmetric Markov semigroup $\{Q_t;\ t \in [0, \infty)\}$. This implies the first assertion of Theorem 4.5.

It is obvious that $\|Q_t^{(n)} f\|_{L^2(E, \mathrm{d}\nu)} \leqq \mathrm{e}^{-t} \|f\|_{L^2(E, \mathrm{d}\nu)}$ for any $f \in L^2(E, \mathrm{d}\nu)$ with $\int_E f \mathrm{d}\nu = 0$. So we have the latter assertion of Theorem 4.5.

This completes the proof of Theorem 4.5.

5 A remark on the domain of Dirichlet forms

In this section, we assume the assumption (B-1) and the following assumption.

(B-2) There is a $\rho > 0$ such that $0 < \inf \rho^{-n} \lambda_n \leqq \sup \rho^{-n} \lambda_n < \infty$.

(5.1) Remark. (1) By Proposition 4.1, we see that $\rho = \lim_{n \to \infty} (\lambda_n)^{1/n}$ exists and $\inf_n \rho^{-n} \lambda_n > 0$, if the condition (B-1) holds.

(2) We do not know how to prove this condition (B-2) in general, even if our fractal has a lot of symmetry as Sierpinski carpets. However, we can prove this condition in the recurrent case (see Sect. 7). So 2-dimensional Sierpinski carpet etc. satisfies this condition. In the case of 2-dimensional carpet, this has been essentially proved by Barlow and Bass [1, 2].

Let $\mathscr{E}^{(n)}$, $n \geqq 1$, be a Dirichlet form in $L^2(E, \mathrm{d}\nu)$ given by

$$\mathscr{E}^{(n)}(f, g) = \rho^n N^{-n} \cdot \mathscr{E}_n(\tilde{P}_n f, \tilde{P}_n g),\ f, g \in L^2(E, \mathrm{d}\nu).$$

Then we have the following.

(5.2) Proposition. *There is a constant C such that $\mathscr{E}^{(n)}(f, f) \leqq C \cdot \mathscr{E}^{(n+m)}(f, f)$ for any $n, m \geqq 1$ and $f \in L^2(E, \mathrm{d}\nu)$.*

Proof. By Lemma 2.12, we see that

$$\mathscr{E}^{(n)}(f,f) \leqq M_0(\rho^{-m}\sigma_m) \cdot \mathscr{E}^{(n+m)}(f,f)\,, \quad f \in L^2(E, \mathrm{d}\nu).$$

This implies our assertion.

Let $\mathscr{D}c\hbar$ be the set of Dirichlet forms associated with the cluster points of $\{Q^{(n)}\}_{n=1}^{\infty}$, and let $\mathscr{D}_0 = \{f \in L^2(E, \mathrm{d}\nu);\ \sup_n \mathscr{E}^{(n)}(f, f) < \infty\}$.

Then we have the following.

(5.3) Proposition. *$f \circ \psi_i \in \mathscr{D}_0$ for any $f \in \mathscr{D}_0$ and $i \in I$.*

Proof. This follows from the fact that

$$\mathscr{E}^{(n+1)}(f, f) \geqq \rho N^{-1} \cdot \sum_{i \in I} \mathscr{E}^{(n)}(f \circ \psi_i, f \circ \psi_i)\,.$$

The following is a main result in this section.

(5.4) Theorem. (1) *$\mathscr{D}om(\mathscr{E}) = \mathscr{D}_0$ for any $\mathscr{E} \in \mathscr{D}c\hbar$.*

(2) *There are constants $c_0, c_1 > 0$ such that*

$$c_0 \cdot \sup_n \mathscr{E}^{(n)}(f, f) \leqq \mathscr{E}(f,f) \leqq c_1 \cdot \varliminf_{n \to \infty} \mathscr{E}^{(n)}(f, f)$$

for any $\mathscr{E} \in \mathscr{D}c\hbar$ and $f \in \mathscr{D}_0$.

Proof. Let $\mathscr{E} \in \mathscr{D}c\hbar$ and $\{Q_t\}_{t \geqq 0}$ be the associated Markov semigroup in $L^2(E, \mathrm{d}\nu)$ with the Dirichlet form $\mathscr{E}$. Also, assume that $Q^{(n_k)}$ converges as $k \to \infty$ to the associated probability measure with $\{Q_t\}_{t \geqq 0}$.

Let $R^{(n)} = \Gamma(1/2) \cdot \int_0^\infty t^{-1/2} \mathrm{e}^{-t} Q_t^{(n)} \mathrm{d}t$. Then we see that $(\rho^{-n}\lambda_n) \cdot \mathscr{E}^{(n)}(R^{(n)}f, R^{(n)}f) \leqq \|f\|^2_{L^2(E,\mathrm{d}\nu)}$, $f \in L^2(E, \mathrm{d}\nu)$. Therefore by Proposition 5.2, we see that there is a constant C such that

$$\mathscr{E}^{(n)}(R^{(n+m)}f, R^{(n+m)}f) \leqq C \cdot \|f\|^2_{L^2(E,\mathrm{d}\nu)} \tag{5.5}$$

for any $n, m \geqq 1$ and $f \in L^2(E, \mathrm{d}\nu)$.

Let $R = \Gamma(1/2)^{-1} \cdot \int_0^\infty t^{-1/2} \mathrm{e}^{-t} Q_t \mathrm{d}t$. Then we see that $R^{(n_k)} \to R$, $k \to \infty$, strongly in $L^2(E, \mathrm{d}\nu)$ and so we have

$$\mathscr{E}^{(n)}(Rf, Rf) \leqq C \cdot \|f\|^2_{L^2(E,\mathrm{d}\nu)}\,, \quad n \geqq 1, f \in L^2(E, \mathrm{d}\nu)\,. \tag{5.6}$$

Therefore we have

$$\sup_n \mathscr{E}^{(n)}(f, f) \leqq C \cdot \{\mathscr{E}(f, f) + \|f\|^2_{L^2(E,\mathrm{d}\nu)}\}, \quad f \in \mathscr{D}om(\mathscr{E})\,.$$

So by Theorem 4.5, we see that

$$\sup_n \mathscr{E}^{(n)}(f, f) \leqq 2C \cdot \mathscr{E}(f, f) \tag{5.7}$$

for any $f \in \mathcal{D}om(\mathcal{E})$ with $\int_E f dv = 0$. Since $\mathcal{E}^{(n)}(1, 1) = 0$ and $\mathcal{E}(1, 1) = 0$, we see that (5.7) holds for all $f \in \mathcal{D}om(\mathcal{E})$.

On the other hand, for any $f \in L^2(E, dv)$, we have

$$t^{-1}(f - Q_t f, f)_{L^2(E, dv)} = \lim_{k \to \infty} t^{-1}(f - Q_t^{(n_k)} f, f)_{L^2(E, dv)}$$
$$\leq \overline{\lim_{n \to \infty}} (\rho^{-n}\lambda_n) \cdot \mathcal{E}^{(n)}(f, f), \quad t > 0 .$$

So letting $C = \sup_n \rho^{-n}\lambda_n$, we see that

(5.8) $$\lim_{t \downarrow 0} t^{-1}(f - Q_t f, f)_{L^2(E, dv)} \leqq C \cdot \sup_n \mathcal{E}^{(n)}(f, f), \quad f \in L^2(E, dv) .$$

By Proposition 5.2, we have

(5.9) $$\sup_n \mathcal{E}^{(n)}(f, f) \leqq C \cdot \underline{\lim}_{n \to \infty} \mathcal{E}^{(n)}(f, f), \quad f \in L^2(E, dv) .$$

(5.7), (5.8) and (5.9) imply our assertion.

6 Fractal with good borders and the existence of self-similar local Dirichlet forms

Let $D_n = \{(x, y) \in I^n \times I^n;\ x \underset{n}{\sim} y,\ x \neq y\}$, $n \geqq 1$. For any $(x, y) \in D_n$, $n \geqq 1$, let $A_{xy}^{n,m} = \{z \in I^m;\ x \cdot z_n \underset{+m}{\sim} y \cdot I^m\}$, $m \geqq 1$. Then the following is obvious.

(6.1) Proposition. (1) $A_{xy}^{n,m+1} \subset A_{xy}^{n,m} \cdot I$, $x, y \in D_n$, $n, m \geqq 1$.
(2) $x \cdot A_{xy}^{n,m+1} = \{\xi \cdot A_{\xi\eta}^{n+m,1};\ \xi \in x \cdot A_{xy}^{n,m}, \eta \in y \cdot A_{yx}^{n,m},\ (\xi, \eta) \in D_{n+m}\}$ *for any* $x, y \in D_n$, $n, m \geqq 1$.

Let $\mathcal{A}_m = \{A_{xy}^{n,m};\ (x, y) \in D_n, n \geqq 1\}_{m \geqq 1}$. In this section, we assume the following.
(GB) (1) There is an $M_1 \geqq 1$ satisfying the following. For any $(x, y) \in D_n$, $n \geqq 1$, $\#(A_{xy}^{n,1}) = M_1$ and there is a unique $z \in A_{yx}^{n,1}$ with $y \cdot z_n \underset{+1}{\sim} x \cdot w$ for each $w \in A_{xy}^{n,1}$.
(2) There is an m_1 satisfying the following.
(i) $\#(\mathcal{A}_m) = \#(\mathcal{A}_{m_1})$, $m \geqq m_1$.
(ii) For any $A \in \mathcal{A}_{m+1}$, there is a unique $A' \in \mathcal{A}_m$ with $A \subset A' \cdot I$, for each $m \geqq m_1$.
(iii) For any $A \in \mathcal{A}_m$, there is a unique $A' \in \mathcal{A}_{m+1}$ with $A' \subset A \cdot I$, for each $m \geqq m_1$.

(6.2) Remark. Nested fractal always satisfy the condition (GB). Also, Sierpinski Carpets satisfy the condition (GB) when we take suitable γ_0 (see Sect. 8).

(6.3) Proposition. *Suppose that the condition* (GB) *holds. Then for any* $(x, y) \in D_n$, $n \geqq 1$, *and* $m \geqq 1$, *we have the following.*

(1) $\#(A_{xy}^{(n,m)}) = M_1^m$, *and* $\#((x \cdot A_{xy}^{n,m}) \cap C) = 0$ *or* M_1^k, $C \in \mathcal{B}_{n+m,k}$, $k = 0, 1, \ldots, m$.

(2) *There is a unique* $z \in A_{yx}^{n,m}$ *with* $y \cdot z_n \underset{+m}{\sim} x \cdot w$ *for any* $w \in A_{xy}^{n,m}$.

(3) $$|\langle u \rangle_{I^m} - \langle u \rangle_{A_{xy}^{n,m}}| \leqq 2 \cdot M_1^{-m/2} \left\{ \sum_{k=0}^{m} (\lambda_k N^{-k} M_1^k)^{1/2} \right\} \cdot \mathcal{E}_m(u, u)^{1/2}$$

for any $u \in C(I^m; \mathbb{R})$.

(4) $|\langle u\rangle_{x\cdot A_{xy}^{n,m}} - \langle u\rangle_{y\cdot A_{yx}^{n,m}}|^2 \leqq M_1^{-m}\cdot\mathscr{E}_{n+m,(x\cdot I^m)\cup(y\cdot I^m)}(u,u)$

for any $u\in C(I^{n+m};\mathbb{R})$.

(5) $|\langle u\rangle_{x\cdot I^m} - \langle u\rangle_{y\cdot I^m}|^2$

$$\leqq [24(\lambda_m N^{-m})\cdot\left\{\sum_{k=0}^{m}[(\lambda_{m-k}N^{-(m-k)}M_1^{m-k})/(\lambda_m N^{-m}M_1^m)]^{1/2}\right\}^2$$

$$+ 3\cdot M_1^{-m}]\cdot\mathscr{E}_{n+m,(x\cdot I^m)\cup(y\cdot I^m)}(u,u)$$

for any $u\in C(I^{n+m};\mathbb{R})$.

Proof. The assertions (1) and (2) hold when $m = 1$ by the assumption (GB). By using Proposition 6.1(2), we can easily show these assertions for $m = \ell + 1$ under the assumption that these assertions hold for $m = \ell$. So by induction we have the assertions (1) and (2).

The assertion (3) follows from the assertion (1) and Proposition 3.11. The assertion (4) is obvious from the assertions (1) and (2). The assertion (5) follows from the assertions (3) and (4).

This completes the proof.

(6.4) Proposition. *Assume that* (GB) *holds and that* $\varliminf_{n\to\infty}\frac{1}{n}\log R_n + \log M_1 > 0$. *Then there is a constant* $C > 0$ *such that* $\sigma_n \leqq C\cdot\lambda_n$ $n \geqq 1$.

Proof. Note that $\sum_{n=1}^{\infty}(R_n M_1^n)^{-1} < \infty$. By Theorem 2.1, we see that there is a constant $C > 0$ such that

$$(\lambda_{m-k}N^{-(m-k)}M_1^{m-k})/(\lambda_m N^{-m}M_1^m) \leqq C\cdot(R_k M_1^k)^{-1}, \quad m \geqq k \geqq 0\,,$$

and

$$(\lambda_m N^{-m})/(M_1^{-m}) \geqq C^{-1}\cdot R_m M_1^m\,.$$

So by Proposition 6.3(5), we have our assertion.

(6.5) Proposition. *Assume that* (GB) *holds and that* $\varliminf_{n\to\infty}\frac{1}{n}\log R_n + \log M_1 > 0$. *Then*

$$\lim_{m,r\to\infty}|\langle\tilde{P}_m f\rangle_{A_{xy}^{n,m}} - \langle\tilde{P}_r f\rangle_{A_{xy}^{n,r}}| = 0$$

and

$$\lim_{m\to\infty}|\langle\tilde{P}_{n+m}f\rangle_{x\cdot A_{xy}^{n,m}} - \langle\tilde{P}_{n+m}f\rangle_{y\cdot A_{yx}^{n,m}}| = 0$$

for any $(x,y)\in D_n$, $n \geqq 1$ and $f\in L^2(E,\mathrm{d}\nu)$ with $\sup_m \lambda_m N^{-m}\mathscr{E}_m(\tilde{P}_m f,\tilde{P}_m f) < \infty$.

Proof. Let $m < r$ and μ be a signed measure on I^r given by

$$\int_{I^r} u\,\mathrm{d}\mu = \langle u\rangle_{A_{xy}^{n,m}\cdot I^{r-m}} - \langle u\rangle_{A_{xy}^{n,r}}\,, \quad u\in C(I^r;\mathbb{R})\,.$$

Then we have $|\langle\tilde{P}_m f\rangle_{A_{xy}^{n,m}} - \langle\tilde{P}_r f\rangle_{A_{xy}^{n,r}}| = |\int_{I^r}\tilde{P}_r f\,\mathrm{d}\mu|$.
Note that $|\mu|_{r,k} = 0$, $k \geqq r - m$, and $|\mu|_{r,k}^2 \leqq 4\cdot M_1^{-(r-k)}$, $k \leqq r - m$.

Therefore by Lemma 3.9, we have

$$\left|\int_{I^r} u\,\mathrm{d}\mu\right|^2 \leqq 4\cdot\left\{\sum_{k=1}^{r-m+1}(\lambda_k N^{-k} M_1^{-(r-k+1)})^{1/2}\right\}^2\cdot\mathscr{E}_r(u,u)$$

$$\leqq 4M_1\left\{\sum_{k=m-1}^{r-1}[(\lambda_{r-k}N^{-(r-k)}M_1^{r-k})/(\lambda_r N^{-r}M_1^r)]^{1/2}\right\}^2\cdot\lambda_r N^{-r}\mathscr{E}_r(u,u).$$

Then by the similar argument of the proof of Proposition 6.4, we have the first assertion. The second assertion follows from Proposition 6.3(4) and the Proof of Proposition 6.4.

This completes the proof.

Let $\mathscr{A}_\infty = \{\{A_m\}_{m=m_1}^{\infty};\ A_m\in\mathscr{A}_m,\ A_{m+1}\subset A_m\cdot I,\ m\geqq m_1\}$. Then the following is an easy consequence of Proposition 6.5.

(6.6) Corollary. *Assume that* (GB) *holds and that*

$\varliminf_{n\to\infty}\frac{1}{n}\log R_n + \log M_1 > 0$. *Let* $\{A_m\}_{m=m_1}^{\infty}\in\mathscr{A}_\infty$. *Then* $\lim_{m\to\infty}\langle\tilde{P}_m f\rangle_{A_m}$ *exists for any* $f\in L^2(E,\mathrm{d}\nu)$ *with* $\sup_m \lambda_m N^{-m}\mathscr{E}_m(\tilde{P}_m f,\tilde{P}_m f)<\infty$.

Now we assume the assumptions (B-1), (B-2) ad (GB), and assume that Y—0.2n,0blim $\varliminf_{n\to\infty}\frac{1}{n}\log R_n + \log M_1 > 0$ throughout this section. We use the notion in Sect. 5.

By Corollary 6.6, we can define $K\colon \mathscr{D}_0\times\mathscr{A}_\infty\to\mathbb{R}$ by $K(f,\{A_m\}_{m=m_1}^{\infty}) = \lim_{m\to\infty}\langle\tilde{P}_m f\rangle_{A_m}$. Then we have the following from Theorem 5.4.

(6.7) Proposition. (1) *There is a* $C\in(0,\infty)$ *such that*

$$\left|\int_E f\,\mathrm{d}\nu - K(f,a)\right|^2 \leqq C\cdot\varliminf_{n\to\infty}\mathscr{E}^{(n)}(f,f)\ \textit{for any}\ f\in\mathscr{D}_0\ \textit{and}\ a\in\mathscr{A}_\infty .$$

(2) $K(f\circ\psi_x,\{A_{xy}^{n,m}\}_{m=m_1}^{\infty}) = K(f\circ\psi_y,\{A_{yx}^{n,m}\}_{m=m_1}^{\infty}),\quad f\in\mathscr{D}_0,\ x,y\in D_n,\ n\geqq 1.$

Now let $\bar{\mathscr{E}}\colon\mathscr{D}_0\times\mathscr{D}_0\to\mathbb{R}$ be a bilinear form given by

$$\bar{\mathscr{E}}(f,g) = \sum_{a\in\mathscr{A}_\infty}\left(K(f,a)-\int_E f\,\mathrm{d}\nu\right)\left(K(g,a)-\int_E g\,\mathrm{d}\nu\right),\quad f,g\in\mathscr{D}_0 .$$

Also let $\bar{\mathscr{E}}^{(n)}\colon\mathscr{D}_0\times\mathscr{D}_0\to\mathbb{R}$, $n\geqq 1$, be a bilinear form given by

$$\bar{\mathscr{E}}^{(n)}(f,g) = \rho^n N^{-n}\cdot\sum_{x\in I^n}\bar{\mathscr{E}}(f\circ\psi_x, g\circ\psi_x),\quad f,g\in\mathscr{D}_0 .$$

Then we have the following.

(6.8) Proposition. *There is a* $C\in(0,\infty)$ *satisfying the following.*

(1) $\bar{\mathscr{E}}^{(n)}(f,f)\leqq C\cdot\sup_m\mathscr{E}^{(n+m)}(f,f),\quad f\in\mathscr{D}_0,\ n\geqq 1.$

(2) $\mathscr{E}^{(n)}(f,f)\leqq C\cdot\bar{\mathscr{E}}^{(n)}(f,f),\quad f\in\mathscr{D}_0,\ n\geqq 1.$

Proof. By Proposition 6.7, we see that

$$\bar{\mathscr{E}}^{(n)}(f,f) \leqq \rho^n N^{-n} \cdot \sum_{x\in I^n} C \cdot \varliminf_{m\to\infty} \mathscr{E}^{(m)}(f\circ\psi_x, f\circ\psi_x)$$
$$\leqq C\cdot \sup_m \mathscr{E}^{(n+m)}(f,f)\,.$$

This proves the assertion (1).

By Proposition 6.7(2), we have

$$\mathscr{E}^{(n)}(f,f) = \frac{1}{2}\cdot\rho^n N^{-n}\cdot \sum_{(x,y)\in D_n} \left(\int_E f\circ\psi_x \,\mathrm{d}\nu - \int_E f\circ\psi_y\,\mathrm{d}\nu\right)^2$$
$$\leqq 2\rho^n N^{-n}\cdot \sum_{(x,y)\in D_n}\left(\int_E f\circ\psi_x\,\mathrm{d}\nu - K(f\circ\psi_x, \{A^{n,m}_{xy}\}^{\infty}_{m=m_1})\right)^2$$
$$\leqq 2\cdot\bar{\mathscr{E}}^{(n)}(f,f).$$

This proves the assertion (2).

The following is the main result in this section.

(6.9) Theorem. *Assume that the assumptions* (GB), (B-1) *and* (B-2) *hold, and that* $\varliminf_{n\to\infty}\frac{1}{n}\log R_n + \log M_1 > 0$. *Then there is a local Dirichlet form* $\mathscr{E}$ *in* $L^2(E,\mathrm{d}\nu)$ *satisfying the following.*

(1) $\mathscr{D}om(\mathscr{E}) = \mathscr{D}_0 = \{f\in L^2(E,\mathrm{d}\nu);\ \sup_n \mathscr{E}^{(n)}(f,f) < \infty\}$.

(2) $\mathscr{E}(f,g) = \rho\cdot N^{-1}\cdot\sum_{i\in I}\mathscr{E}(f\circ\psi_i, g\circ\psi_i),\quad f,g\in\mathscr{D}om(\mathscr{E})$.

Proof. Let us take an $\mathscr{E}_0\in\mathscr{D}ch$ and fix it. Then $\mathscr{D}_0 = \mathscr{D}om(\mathscr{E}_0)$ is regarded as a separable Hilbert space with an inner product $(\cdot,*)_{L^2(E,\mathrm{d}\nu)} + \mathscr{E}_0(\cdot,*)$. Let $\mathscr{G}$ be a $\mathbb{Q}$-vector subspace of $\mathscr{D}_0$ for which $\mathscr{G}$ is dense in $\mathscr{D}_0$. Let $\tilde{\mathscr{E}}_n$ be a bilinear form in $\mathscr{D}_0$ given by

$$\tilde{\mathscr{E}}^{(n)}(f,g) = \frac{1}{n}\cdot\sum_{k=1}^{n}\bar{\mathscr{E}}^{(k)}(f,g)\,,\quad f,g\in\mathscr{D}_0\,.$$

Then $\{\tilde{\mathscr{E}}^{(n)}(g,g)\}_{n\geqq 1}$ is a bounded sequence for any $g\in\mathscr{G}$. So by diagonal argument, we see that there is a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $\tilde{\mathscr{E}}^{(n_k)}(g,g)$ converges as $k\to\infty$ for any $g\in\mathscr{G}$. By Theorem 5.4 and Proposition 6.8, we see that $c_0\cdot\overline{\lim}_{k\to\infty}\tilde{\mathscr{E}}^{(n_k)}(f,f)\leqq C\cdot\mathscr{E}_0(f,f), f\in\mathscr{D}_0$. So we see that $\tilde{\mathscr{E}}^{(n_k)}(f,f)$ converges as $k\to\infty$ for all $f\in\mathscr{D}_0$. Let $\mathscr{E}:\mathscr{D}_0\times\mathscr{D}_0\to\mathbb{R}$ be a bilinear form given by

$$\mathscr{E}(f,g) = \frac{1}{4}\cdot\lim_{k\to\infty}\left(\tilde{\mathscr{E}}^{(n_k)}(f+g,f+g) - \tilde{\mathscr{E}}^{(n_k)}(f-g,f-g)\right)\,,\quad f,g\in\mathscr{D}_0\,.$$

Then by Theorem 5.4 and Proposition 6.8, we see that $c_0\cdot\mathscr{E}(f,f) \leqq C\cdot\mathscr{E}_0(f,f)\leqq C^2\cdot c_1\cdot\mathscr{E}(f,f)$, $f\in\mathscr{D}_0$. Therefore we see that $\mathscr{E}$ is closed. It is obvious that $\mathscr{E}$ has the Markov property. So $\mathscr{E}$ is a Dirichlet form in $L^2(E,\mathrm{d}\nu)$.

Since $\bar{\mathscr{E}}^{(n+1)}(f,f) = \rho N^{-1}\cdot\sum_{i\in I}\bar{\mathscr{E}}^{(n)}(f\circ\psi_i, f\circ\psi_i)$, we have the assertion (2). The locality of the Dirichlet form $\mathscr{E}$ follows from the assertion (2).

This completes the proof.

7 Recurrent case

Theorem 4.4 guarantees that there is a strongly continuous symmetric Markov semigroup on $L^2(E, \mathrm{d}\nu)$. But this does not imply the existence of a good diffusion in E. To prove this, we need strong estimate like a Harnack inequality given by Barlow and Bass [1]. We cannot prove such results in general. But we will give some conditions which lead to such results. First, we think of the following assumption.

(R) $\varliminf_{n\to\infty} \frac{1}{n} \log R_n > 0$.

(*7.1*) *Remark.* By Theorem 2.1, we have $\varliminf_{n\to\infty} \frac{1}{n} \log R_n \geqq 2\cdot\log\alpha - \log N$. So if $d_f = (\log\alpha)^{-1}(\log N) < 2$, the condition (R) is satisfied.

Let $r_n(x, y) = \alpha^n \cdot \max\{|\xi - \xi'|;\ \xi\in\psi_x(E),\ \xi'\in\psi_y(E)\}$, $x, y\in I^n$, as in Lemma 3.5. Then we have the following.

(7.2) Theorem. *Assume the assumption* (R). *Then for any* $\beta\in\left(\left(0, \varliminf_{n\to\infty} \frac{1}{n}\log R_n\right) \Big/ (2\cdot\log\alpha)\right)$, *there is a constant* $C_0 > 0$ *satisfying the following.*

$$|u(x) - u(y)| \leqq C_0\cdot(\alpha^{-n} r_n(x, y))^{\beta}\cdot\{(\lambda_n N^{-n})\cdot\mathscr{E}_n(u, u)\}^{1/2}$$

for any $u\in C(I^n; \mathbb{R})$, $n\geqq 1$, *and* $x, y\in I^n$.

Proof. Let $\delta = \beta(2\cdot\log\alpha)$. By Theorem 2.1, we see that there is a $c > 0$ such that

$$(\lambda_n N^{-n})\mathrm{e}^{m\delta} \leqq c\cdot(\lambda_{n+m} N^{-(n+m)}) \tag{7.3}$$

for any $n, m\geqq 1$. So by Proposition 3.10, we see that

$$\begin{aligned}|u(x) - u(y)| &\leqq 2\left(\sum_{k=0}^{m} (c\cdot\mathrm{e}^{-\delta(n-k)})^{1/2}\right)\{\lambda_n N^{-n}\mathscr{E}_n(u, u)\}^{1/2}\\ &\leqq 2(c(1-\mathrm{e}^{-\delta})^{-1})^{1/2}\cdot\alpha^{-\beta(n-m)}\{\lambda_n N^{-n}\mathscr{E}_n(u, u)\}^{1/2}\end{aligned}$$

for any $u\in C(I^n; \mathbb{R})$ and $x, y\in I^n$ such that there is a $B\in\mathscr{B}_{n,m}$ with $x, y\in B$. Therefore by the assumption (A-3) (1), we have our assertion.

(7.4) Corollary. *Let* $\{Q_t;\ t\in[0, \infty)\}$ *be the symmetric Markov semigroup in* $L^2(E, \mathrm{d}\nu)$ *as in Theorem* 4.5. *If the assumption* (R) *is satisfied, then the image of* Q_t, $t > 0$, *is contained in* $C(E; \mathbb{R})$.

Proof. Since $\lambda_n N^{-n}\cdot\mathscr{E}_n(\tilde{P}_n Q_t^{(n)} f, \tilde{P}_n Q_t^{(n)} f) \leqq (2t)^{-1}\|f\|^2_{L^2(\mathrm{d}\nu)}$, $f\in L^2(E, \mathrm{d}\nu)$, by Theorem 7.2, we see that there is a constant $C > 0$ such that

$$|Q_t^{(n)} f(\xi) - Q_t^{(n)} f(\eta)| \leqq C\cdot t^{-1/2}(|\xi - \eta| + C\cdot\alpha^{-n})^{\beta}\|f\|_{L^2(\mathrm{d}\nu)}$$

for any $n\geqq 1, f\in L^2(E, \mathrm{d}\nu)$, $\xi, \eta\in E$ and $t > 0$. Since there is a subsequence $\{n_k\}$ such that $Q_t^{(n_k)} \to Q_t$ strongly as $k\to\infty$ for any $t > 0$, we have our assertion.

(7.5) Lemma. *Let* $f, g \in C^2((0, \infty); \mathbb{R})$ *such that* $|f'(s)| > 0$, $s \in (0, \infty)$, $g'(s) = |f'(s)|^2$ *and* $\left|\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{g(s)}{f'(s)}\right)\right| \geqq 0$, $s \in (0, \infty)$. *If* $u \in C(I^n; (0, \infty))$, $\eta \in C(I^n; \mathbb{R})$ *and* $\eta(x)^2 (L^{(n)} u)(x) g(u(x)) \geqq 0$, $x \in I^n$, *then*

$$\sum_{x, y \in I^n} q_{xy}^{(n)} (\eta(x)^2 \wedge \eta(y)^2)(f(u(x)) - f(u(y)))^2$$
$$\leqq 4 \cdot \sum_{x, y \in I^n} q_{xy}^{(n)} \cdot \left(\left(\frac{g(u(x))}{f'(u(x))} \right)^2 \vee \left(\frac{g(u(y))}{f'(u(y))} \right)^2 (\eta(x) - \eta(y) \right)^2 .$$

Proof. For any $v \in C(I^n; \mathbb{R})$ and $x, y \in I^n$, let $v_{xy}(t) = t \cdot v(x) + (1-t) v(y)$, $t \in [0, 1]$. Then we see that for any $v \in C(I^n; (0, \infty))$, $\varphi \in C(I^n; \mathbb{R})$ and $x, y \in I^n$

$$\int_0^1 \varphi_{xy}(t)^2 \left| \frac{\mathrm{d}}{\mathrm{d}t} (f(v_{xy}(t)) \right|^2 \mathrm{d}t$$
$$= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} (\varphi_{xy}(t)^2 g(v_{xy}(t))) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} v_{xy}(t) \right) \mathrm{d}t$$
$$- 2 \int_0^1 \varphi_{xy}(t) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} \varphi_{xy}(t) \right) \cdot g(v_{xy}(t)) \left(\frac{\mathrm{d}}{\mathrm{d}t} v_{xy}(t) \right) \mathrm{d}t$$
$$= (v(x) - v(y))(\varphi(x)^2 g(v(x)) - \varphi(y)^2 g(v(y)))$$
$$- 2 \cdot \int_0^1 \varphi_{xy}(t) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} f(v_{xy}(t) \right) \cdot \left(\frac{g(v_{xy}(t))}{f'(v_{xy}(t))} \right) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} \varphi_{xy}(t) \right) \mathrm{d}t.$$

Note that $\left(\frac{g(v_{xy}(t))}{f'(v_{xy}(t)} \right)^2 \leqq \left(\frac{g(v(x))}{f'(v(x))} \right)^2 \vee \left(\frac{g(v(y))}{f'(v(y)} \right)^2$, $t \in [0, 1]$, because $\left| \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{g(s)}{f'(s)} \right| \geqq 0$, $s \in (0, \infty)$. So we have

$$\text{(7.6)} \qquad \sum_{x, y \in I^n} q_{xy}^{(n)} \cdot \left(\int_0^1 \varphi_{xy}(t)^2 \left| \frac{\mathrm{d}}{\mathrm{d}t} (f(v_{xy}(t)) \right|^2 \mathrm{d}t \right) - \mathscr{E}_n(v, \varphi^2 \cdot g \circ v)$$
$$\leqq 2 \cdot \left\{ \sum_{x, y \in I^n} q_{xy}^{(n)} \cdot \int_0^1 \varphi_{xy}(t)^2 \left| \frac{\mathrm{d}}{\mathrm{d}t} f(v_{xy}(t)) \right|^2 \mathrm{d}t \right\}^{1/2}$$
$$\times \left\{ \sum_{x, y \in I^n} q_{xy}^{(n)} \cdot \left(\left(\frac{g(v(x))}{f'(v(x))} \right)^2 \vee \left(\frac{g(v(y))}{f'(v(y))} \right)^2 \right) (\varphi(x) - \varphi(y))^2 \right\}^{1/2}$$

for any $v \in C(I^n; (0, \infty))$ and $\varphi \in C(I^n; \mathbb{R})$. Since

$\mathscr{E}_n(u, \eta^2 g \circ u) = - \sum_{x \in I^n} L^{(n)} u(x) \cdot \eta(x)^2 g(u(x)) \leqq 0$, we have

$$\sum_{x, y \in I^n} \int_0^1 \eta_{xy}(t)^2 \left| \frac{\mathrm{d}}{\mathrm{d}t} f(u_{xy}(t)) \right|^2 \mathrm{d}t$$
$$\leqq 4 \cdot \sum_{x, y \in I^n} q_{xy}^{(n)} \cdot \left(\left(\frac{g(u(x))}{f'(u(x))} \right)^2 \vee \left(\frac{g(u(y))}{f'(u(y))} \right)^2 \right) (\eta(x) - \eta(y))^2 .$$

By Schwarz inequality, we have

$$\sum_{x,y\in I^n} q_{xy}^{(n)}(\eta(x)^2 \wedge \eta(y)^2)(f(u(x)) - f(u(y))^2 \leqq \sum_{x,y\in I^n} \int_0^1 \eta_{xy}(t)^2 \left|\frac{\mathrm{d}}{\mathrm{d}t} f(u_{xy}(t))\right|^2 \mathrm{d}t .$$

So we have our assertion.

(7.7) Corollary. *If* $u\in C(I^n; (0,\infty))$, $\eta\in C(I^n;\mathbb{R})$ *and* $\eta(x)^2(L^{(n)}u)(x) \leqq 0$, $x\in I^n$, *then*

$$\sum_{x,y\in I^n} q_{xy}^{(n)}(\eta(x)^2 \wedge \eta(y)^2)(\log u(x) - \log u(y))^2 \leqq 4\cdot \sum_{x,y\in I^n} q_{xy}^{(n)}\cdot(\eta(x)-\eta(y))^2 .$$

Proof. Let $f(t) = \log t$ and $g(t) = -t^{-1}$, $t\in(0,\infty)$. Then we have our assertion from Lemma 7.5.

We say that a subset G of I^ℓ, $\ell \geqq 1$, is *ℓ-connected,* if there are $n\geqq 1$ and $z_0,\ldots,z_n\in G$ such that $z_0 = x$, $z_n = y$ and $z_{i-1} \underset{\ell}{\sim} z_i$, $i = 1,\ldots,n$, for any $x, y\in G$.

(7.8) Lemma. *Suppose that the assumptions* (R) *and* (B-1) *are satisfied. Let* $\ell \geqq 1$, G_0 *be a ℓ-connected non-void subset in* I^ℓ, *and* G_1 *be non-void subsets of* I^ℓ *with* $G_0\cap G_1 = \emptyset$. *Suppose moreover that there is a* $C_1 > 0$ *such that*

$$\lambda_{n+\ell} \leqq C_1\cdot N^{n+\ell} R_{n+\ell}(G_0\cdot I^n, G_1\cdot I^n) \tag{7.9}$$

for any $n\geqq 1$. *Then there is a* $\delta > 0$ *satisfying the following. If* $n\geqq 1$, $u\in C(I^{n+\ell};[0,\infty))$ *and* $L^{(n+\ell)}u|_{I^{n+\ell}\setminus G_1\cdot I^n} \leqslant 0$, *then*

$$\delta\cdot \max_{x\in G_0\cdot I^n} u(x) \leqq \min_{x\in G_0\cdot I^n} u(x) . \tag{7.10}$$

Proof. Let $\eta_n\in C(I^{n+\ell},[0,1])$, $n\geqq 1$, be such that $\eta_n|_{G_0\cdot I^n} = 1$, $\eta_n|_{G_1\cdot I^n} = 0$ and $\mathscr{E}_{n+\ell}(\eta_n,\eta_n) = R_{n+\ell}(G_0\cdot I^n, G_1\cdot I^n)^{-1}$. Then it is easy to see that $\sup_n \lambda_{n+\ell} N^{-(n+\ell)}\cdot\mathscr{E}_{n+\ell}(\eta_n,\eta_n) < \infty$. Since $L^{(n+\ell)}u|_{I^{n+\ell}\setminus G_1\cdot I^n} \leqq 0$, by Corollary 7.7 we see that for any $\varepsilon > 0$ and $n\geqq n_0$,

$$\begin{aligned} &\mathscr{E}_{n+\ell, G_0\cdot I^n}(\log(u+\varepsilon), \log(u+\varepsilon)) \\ &\leqq \sum_{x,y\in I^n} q_{xy}^{(n)}(\eta_n(x)^2 \wedge \eta_n(y)^2)(\log(u(x)+\varepsilon) - \log(u(y)+\varepsilon))^2 \\ &\leqq 4\cdot\mathscr{E}_{n+\ell}(\eta_n,\eta_n) \leqq 4C\cdot\lambda_{n+\ell}^{-1} N^{n+\ell} . \end{aligned} \tag{7.11}$$

Let $u_z\in C(I^n;[0,\infty))$, $z\in G_0$, be given by $u_z(x) = u(z\cdot x)$, $x\in I^n$. Then by Theorem 2.1, we see that

$$\lambda_n N^{-n}\cdot\mathscr{E}_n(\log(u_z+\varepsilon), \log(u_z+\varepsilon)) \leqq 4\cdot c_1^{-1} C_1 C\cdot(R_\ell^{-1} N^\ell) .$$

So by Theorem 7.2, we have

$$\begin{aligned} &\max_{x\in I^n} \log(u_z(x)+\varepsilon) - \min_{x\in I^n} \log(u_z(x)+\varepsilon) \\ &\leqq 2C_0\cdot\mathrm{diameter}(E)^\beta (c_1^{-1} C_1 C\cdot\sigma_\ell)^{1/2} . \end{aligned} \tag{7.12}$$

Also, if $z_0, z_1\in G_0$ and $z_0 \underset{\ell}{\sim} z_1$, then there are $x_0, x_1\in I^n$ such that $z_0 x_0 \underset{n+\ell}{\sim} z_1 x_1$. Then we have

$$\begin{aligned} &|\log(u_{z_0}(x_0)+\varepsilon) - \log(u_{z_1}(x_1)+\varepsilon)| \\ &\leqq \mathscr{E}_{n+\ell, G_0\cdot I^n}(\log(u+\varepsilon), \log(u+\varepsilon))^{1/2} \leqq 2(C_1\cdot\lambda_{n+\ell}^{-1} N^{n+\ell})^{1/2} . \end{aligned} \tag{7.13}$$

Since $\sup\{\lambda_n^{-1}N^n; n \geqq 1\} < \infty$, we have our assertion from (7.12) and (7.13).
This completes the proof.

(7.14) Lemma. *Suppose that the assumptions* (R) *and* (B-1) *are satisfied. Let* $\ell \geqq 1$, G_0 *be a* ℓ*-connected non-void subset in* I^ℓ, *and* G_1 *be a non-void subset in* I^ℓ *with* $G_0 \cap G_1 = \emptyset$. *Assume that*

$$\inf\{P_x^{(n+\ell+k)}[\tau_{z\cdot I^n} < \tau_{G_1\cdot I^{n+k}}]; z \in G_0 \cdot I^k, x \in G_0 \cdot I^{n+k}, n \geqq 1\} > 0 \tag{7.15}$$

for any $k \geqq 0$. *Then there is a* $C > 0$ *such that*

$$\lambda_{n+\ell} \leqq C \cdot N^{n+\ell} R_{n+\ell}(G_0 \cdot I^n, G_1 \cdot I^n), \quad n \geqq 1 .$$

Proof. Let φ_n, $n \geqq 1$, be as in Sect. 4. LLet $\xi_n \in C(I^{n+\ell}, \mathbb{R})$, $n \geqq 1$, be given by $\xi_n(x \cdot y) = \varphi_n(y)$, $x \in G_0$, $y \in I^n$, and $\xi_n(z) = 0$, $z \in I^{n+\ell} \backslash (G_0 \cdot I^n)$. Then by Proposition 4.6, it is obvious that

$$\langle \xi_n \rangle_{G_0 \cdot I^n} = 1 ,$$

and

$$\sup_n \lambda_{n+\ell} N^{-(n+\ell)} \mathscr{E}_{n+\ell}(\xi_n, \xi_n) = \#(G_0) N^{-\ell} \cdot \sup_n \lambda_{n+\ell} (\lambda_n^{(D)})^{-1} < \infty .$$

So we see that $\max_{x \in G_0 \cdot I^n} \xi_n(x) \geqq 1$. Also, by Theorem 7.2, we see that there are $\beta > 0$ and $C > 0$ such that

$$|\xi_n(x) - \xi_n(y)| \leqq C \cdot (\alpha^{-n-\ell} \cdot r_{n+\ell}(x, y))^\beta, \quad n \geqq 1, \ x, y \in I^{n+\ell} .$$

Let us take $k \geqq 1$ such that $C \cdot \alpha^{-k}(2 \cdot \text{diameter}(E)) < 1/2$. Then we see that there is $z_n \in G_0 \cdot I^k$, for each $n \geqq k$, such that

$$\inf\{\xi_n(x); n \geqq k, x \in z_n \cdot I^{n-k}\} > 1/2 .$$

Let $\eta_n \in C(I^{n+\ell}; [0, 1])$, $n \geqq k$, be given by

$$\eta_n(x) = P_x^{(n+\ell)}[\tau_{z_n \cdot I^{n-k}} < \tau_{G_1 \cdot I^n}], \quad x \in I^{n+\ell} .$$

Since $\xi_n(x) = 0$, $x \in G_1 \cdot I^n$, we see that

$$\mathscr{E}_{n+\ell}(\eta_n, \eta_n) \leqq \mathscr{E}_{n+\ell}(2\xi_n, 2\xi_n) = 4 \cdot \#(G_0) \cdot N^n (\lambda_n^{(D)})^{-1} .$$

By the assumption, we see that

$$c = \inf\{\eta_n(x); x \in G_0 \cdot I^n, n \geqq k\} > 0 .$$

So we have

$$\begin{aligned} R_{n+\ell}(G_0 \cdot I^n, G_1 \cdot I^n)^{-1} &\leqq \mathscr{E}_{n+\ell}(c^{-1}\eta_n, c^{-1}\eta_n) \\ &\leqq 4c^{-2} \cdot \#(G_0) \cdot N^n (\lambda_n^{(D)})^{-1} \end{aligned}$$

for any $n \geqq k$. This implies our assertion.
This completes the proof.

Now we assume the following which can be proved by "Knight Moves" argument in Barlow and Bass [1].
(KM) For any $\ell \geqq 1$, any ℓ-connected non-void subset G_0 of I^ℓ and any non-void subset G_1 of I^ℓ, if $\text{dis}(\bigcup_{x \in G_0} \psi_x(E), \bigcup_{x \in G_1} \psi_x(E)) > 0$, then

$$\inf\{P_x^{(n+\ell)}[\tau_{z \cdot I^n} < \tau_{G_1 \cdot I^n}]; z \in G_0, x \in G_0 \cdot I^n, n \geqq 1\} > 0 .$$

Also, we assume the following local similarity assumption.
(LS) There is an $k_0 \geqq 0$ such that

$$R_m = \min\{R_{m+k}(B, I^{m+k} \setminus C_{m+k,m}(B));\, k = 1, \ldots, k_0,\, B \in \mathscr{B}_{m+k,m}\}, \quad m \geqq 1 .$$

(7.16) Theorem. *Assume that the assumptions* (R), (KM), (LS) *and* (B-1) *are satisfied. Then there are* $\rho > 0$ *and* $c_0, c_1 \in (0, \infty)$ *such that*

$$c_0 \cdot N^{-n} \rho^n \leqq R_n \leqq c_1 \cdot N^{-n} \rho^n, \quad n \geqq 1 , \tag{7.17}$$

and

$$c_0 \cdot \rho^n \leqq \lambda_n \leqq c_1 \cdot \rho^n, \quad n \geqq 1 . \tag{7.18}$$

In particular, the assumption (B-2) *holds.*

Proof. By Lemma 7.14 and the assumptions (KM) and (LS), we see that there is a constant $C > 0$ such that $\lambda_n \leqq C \cdot N^n R_n$, $n \geqq 1$. Then combining this with Theorem 2.1 and Proposition 4.1, we have our assertion.

By combining all results in this paper, we have the following.

(7.19) Theorem. *Assume that the assumptions* (R), (KM), (LS), (GB) *and* (B-1) *are satisfied. Then there is a regular local Dirichlet form* $(\mathscr{E}, \mathscr{D}om(\mathscr{E}))$ *satisfying the following.*

(1) $\mathscr{D}om(\mathscr{E}) = \mathscr{D}_0 \subset C(E; \mathbb{R})$.

(2) $\mathscr{E}(f, g) = \rho N^{-1} \cdot \sum_{i \in I} \mathscr{E}(f \circ \psi_i, g \circ \psi_i)$ *for any* $f, g \in \mathscr{D}om(\mathscr{E})$.

(3) *Let L be the associated generator with the Dirichlet form* $(\mathscr{E}, \mathscr{D}om(\mathscr{E}))$. *Then for any connected open sets* G_1, G_2 *in E with* $\overline{G_1} \subset G_2$, *there is a* $\delta > 0$ *such that*

$$\delta \cdot \max_{x \in G_1} f(x) \leqq \min_{x \in G_1} f(x)$$

for any $f \in \mathscr{D}om(\mathscr{E})$ *with* $f|_{G_2} > 0$ *and* $Lf|_{G_2} \leqq 0$.

Proof. The assertions (1) and (2) follow from Theorems 6.9, 7.2 and 7.16. If $u_n \in C(I^n; [0, 1]))$, $n \geqq 1$, and $\sup_n \lambda_n N^{-n} \mathscr{E}_n(u_n, u_n) < \infty$, then by Proposition 5.2 and Theorem 5.4, we see that any cluster point of $\{\iota_n u_n\}_{n=1}^{\infty}$ in $L^2(E, d\nu)$ with respect to the weak topology belongs to $\mathscr{D}_0$. So we see that if K_0 and K_1 are connected compact sets in E with $K_0 \cap K_1 = \emptyset$, then there is an $f \in \mathscr{D}_0$ with $f|_{K_0} = 0$ and $f|_{K_1} = 1$. Therefore the assertion (3) is proved by a similar method to the proof of Lemma 7.8. Also, we have the regularity of the Dirichlet form.

This completes the proof.

8 Examples

Example 1 (Sierpinski carpet). Let $D = 2$, $N = 8$ and $\alpha = 3$. Let $x_i \in \mathbb{R}^2$, $i = 1, \ldots, 8$ be given by $z_1 = (0, 0)$, $z_1 = (0, \frac{1}{2})$, $z_3 = (0, 1)$, $z_4 = (\frac{1}{2}, 1)$, $z_5 = (1, 1)$, $z_6 = (1, \frac{1}{2})$, $z_7 = (1, 0)$, and $z_8 = (\frac{1}{2}, 0)$. Let ψ_i, $i = 1, \ldots, 8$, be given by $\psi_i(z) = \frac{1}{3} \cdot (z - z_i) + z_i$, $z \in \mathbb{R}^2$. Then the associated set E is called Sierpinski carpet.

Let $\gamma_0 = 1$. Then $q_{xy}^{(n)}$, $x, y \in I^n$, $n \geqq 1$, are determined. It is easy to check that the assumptions (A-1)–(A-4), (GB) and (LS) are satisfied. By Barlow and Bass [1], we also see that the assumptions (KM) is satisfied. Since $d_f = (\log 3)^{-1}(\log 8) < 2$, the assumption (R) is satisfied.

Now we will show that the assumption (B-1) is satisfied. This is essentially proved by Barlow and Bass [1] and [2] relying on a Harnack inequality. But the proof here relies only on a symmetry and it works for a lot of fractals.

(8.1) Proposition. *There is a* $C > 0$ *such that* $\sigma_n \leqq C \cdot \lambda_{n+2}^{(D)}$, $n \geqq 1$, *for Example* 1. *So the assumption* (B-1) *holds.*

Proof. Let $B = 1 \cdot I^n$ and $B' = 2 \cdot I^n$. Then $B, B' \in \mathscr{B}_{n+1,n}$ and $B_n \underset{1}{\tilde{+}} B'$. Note that $\sigma_n = \sigma_{n+1,n}(B, B')$. So there is a $u \in C(B \cup B'; \mathbb{R})$ such that $\mathscr{E}_{n+1, B \cup B'}(u, u) = 1$ and $(\langle u \rangle_B - \langle u \rangle_{B'})^2 = N^{-n}\sigma_n$.

Let $T_i: \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2$, be given by $T_1(\xi_1, \xi_2) = (\frac{1}{3} - \xi_1, \xi_2)$ and $T_2(\xi_1, \xi_2) = (\xi_1, \frac{2}{3} - \xi_2)$, $(\xi_1, \xi_2) \in \mathbb{R}^2$. Then it is easy to see that there are $S_i: B \cup B' \to B \cup B'$, $i = 1, 2$, such that $\psi_{S_i(x)}(E) = T_i(\psi_x(E))$, $i = 1, 2$. We may assume that $u(S_1(x)) = u(x)$ and $u(S_2(x)) = -u(x)$, $x \in B \cup B'$. Also, we may assume that $u(x) \geqq 0$, $x \in B'$.

Let $v_0 \in C(I^n \to \mathbb{R})$ be given by $v_0(x) = u(2 \cdot x)$, $x \in I^n$. Then we see that $\mathscr{E}_n(v_0, v_0) \leqq \mathscr{E}_{n+1, B \cup B'}(u \vee 0, u \vee 0) \leqq 1$ and $\langle v_0 \rangle_{I^n}^2 = \frac{1}{4} \cdot N^{-n}\sigma_n$. Let $T_i: \mathbb{R}^2 \to \mathbb{R}^2$, $i = 3, 4$, be given by $T_3(\xi_1, \xi_2) = (\xi_2, \xi_1)$, and $T_4(\xi_1, \xi_2) = (1 - \xi_2, \xi_1)$, $(\xi_1, \xi_2) \in \mathbb{R}^2$. Then there are $S_i: I^n \to I^n$, $i = 3, 4$, such that $\psi_{S_i(x)}(E) = T_i(\psi_x(E))$, $x \in I^n$. Now let $v_i \in C(I^n; \mathbb{R})$, $i = 1, 2$, be given by

$$v_1(x) = \begin{cases} v_0(S_3(x)) & \text{if } \psi_x(E) \subset \{(\xi_1, \xi_2) \in \mathbb{R}^2; \xi_1 + \xi_2 \geqq 1\} \\ v_0(x) & \text{otherwise}, \end{cases}$$

and

$$v_2(x) = \begin{cases} v_0(S_4(x)) & \text{if } \psi_x(E) \subset \{(\xi_1, \xi_2) \in \mathbb{R}^2; \xi_1 \leqq \xi_2\} \\ v_0(x) & \text{otherwise}. \end{cases}$$

Then one can see that $v_i \geqq 0$ and $\mathscr{E}_n(v_i, v_i) \leqq 2 \cdot \mathscr{E}_n(v_0, v_0)$, $i = 1, 2$. Now let $v \in C(I^{n+2}; \mathbb{R})$ be given by $v(8 \cdot 3 \cdot x) = v_1(x)$, $v(8 \cdot 4 \cdot x) = v_0(x)$, $v(8 \cdot 5 \cdot x) = v_2(x)$, and $v(y \cdot x) = 0$, $y \in I^2 \setminus \{(8, 3), (8, 4), (8, 5)\}$. Then we see that $v|_{\partial I^n} = 0$, $\langle v \rangle_{I^{n+2}}^2 \geqq N^{-2}\langle v_0 \rangle_{I^n}^2 = \frac{1}{4} \cdot N^{-n-2}\sigma_n$, and $\mathscr{E}_{n+2}(v, v) \leqq 5 \cdot \mathscr{E}_{n+1, B \cup B'}, (u \vee 0, u \vee 0) \leqq 5$.

This implies our assertion.

So by Theorem 7.19, we see that there is a self-similar regular local Dirichlet form in $L^2(E, \mathrm{d}\nu)$ and its domain is contained in $C(E, \mathbb{R})$. So there is a good self-similar diffusion process on E.

As far as we take γ_0 in $(0, d_f)$, the relation $\underset{n}{\sim}$ is the same, and so conclusions are the same. However, if we let $\gamma_0 = 0$, then the relation $\underset{n}{\sim}$ has changed and the assumption (GB) fails in this case. This shows that the assumption (GB) is rather unstable. Probably we have to replace this assumption by more stable assumption to handle more general fractals.

Example 2 Let $D \geqq 3$, $I = \{0, 1, 2\}^D \setminus \{(1, 1, 1)\}$ and $\alpha = 3$. Let $\psi_i: \mathbb{R}^D \to \mathbb{R}^D$, $i \in I$, be given by $\psi_i(z) = \frac{1}{3}(z - \frac{1}{2} \cdot i) + \frac{1}{2} \cdot i$. Here we regard I as a subset of $\mathbb{R}^D$. In this case $d_f = (\log 3)^{-1}(\log(3^D - 1))$. Let $\gamma_0 = D - 1$. Then the assumptions (A-1)–(A-4), (GB) and (LS) are satisfied. We believe that the assumption (KM) can be verified by Barlow Bass' "Knight Moves" argument and that the assumption (B-1) can be shown by the similar argument to the proof of Proposition 8.1. However, the assumption (R) fails in this case. We do not know how to check the assumption (B-2) and the regularity of the Dirichlet form in Theorem 4.5.

Example 3 (Carpet with holes). Let $D \geqq 3$. Let $\ell, m \geqq 1$ and $I = \{i \in \{0, \ldots, (2\ell + m - 1)\}^D;\ \#(\{k = 1, \ldots, D; \ell \leqq i_k \leqq \ell + m - 1\}) \leqq 1\}$. Let $\psi_i\colon \mathbb{R}^D \to \mathbb{R}^D$, $i \in I$, be given by $\psi_i(z) = (2\ell + m)^{-1}(z - (2\ell + m - 1)^{-1} \cdot i) + (2\ell + m - 1)^{-1} \cdot i$, $z \in R^D$. Then $d_f = (\log(2\ell + m))^{-1}(\log((2\ell)^D + D \cdot (2\ell)^{D-1} m))$. Let $\gamma_0 = (\log(2\ell + m))^{-1}(\log((2\ell)^{D-1} + (D-1) \cdot (2\ell)^{D-2} m))$. Then the assumptions (A-1)–(A-4), (GB) and (LS) are satisfied. By using comparison argument for resistance, one can see that the assumption (R) is satisfied if

$$m(2\ell)^{-(D-1)} + (2\ell)\{(2\ell)^{D-1} + (D-1)(2\ell)^{D-2} m\}^{-1} > 1\,.$$

So we see that there is a good self-similar diffusion process on E in this case, if we check the assumptions (KM) and (B-1). But we believe that one can verify them by using Barlow-Bass' idea and the proof of Proposition 8.1.

Example 4 If a nested fractal satisfies the assumptions (A-3), then it is easy to check the assumptions (A-1)–(A-4), (B-1), (KM), (GB) and (LS). The assumption (R) holds in general. But we do not know how one can check it without using Lindstrøm's result in general. In the case that it is embedded in 2-dimensional Euclidean space, we can check it easily. So our approach gives a new proof for the existence of self-similar diffusion processes on certain nested fractals like a snow-flake fractal, although Lindstrøm's proof in [5] is much more elegant.

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