Probab. Theory Relat. Fields 93, 137–152 (1992)

Probability Theory and Related Fields

Exit probability estimates for martingales in geodesic balls, using curvature

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Received July 7, 1991; in revised form January 6, 1992

Summary. Kallenberg and Sztencel have recently discovered exponential upper bounds, independent of dimension, on the probability that a vector martingale will exit from a ball in Euclidean space by time t. This article extends their results to martingales on Riemannian manifolds, including Brownian motion, and shows how exit probabilities depend on curvature. Using comparison with rotationally symmetric manifolds, these estimates are easily computable, and are sharp up to a constant factor in certain cases.

Mathematics Subject Classification (1980): 58G32, 60G48, 60G46, 60J65

0 Introduction

The following is a paraphrase of part of the introduction to Kallenberg and Stzencel (1991). "A familiar result (see Rogers and Williams (1987, p. 77)) states that

$$P(\sup\{|M_t|: t \geqq 0\} > r) \leqq 2\exp\{-r^2/2\} \tag{0.1}$$

where M is a continuous real-valued martingale, with $M_0 = 0$ and quadratic variation $[M, M]_t \leqq 1$. Applying (0.1) componentwise to a martingale $M = (M^1, \ldots, M^m)$ in $\mathbb{R}^m$ with $[M, M]_t = [M^1, M^1]_t + \cdots + [M^m, M^m]_t \leqq 1$ gives the bound

$$P(\sup\{|M_t|: t \geqq 0\} > r) \leqq 2m\exp\{-r^2/2m\} \tag{0.2}$$

which turns out to be of completely the wrong order, since (0.1) is in fact true in arbitrary (even infinite) dimension, possibly apart from a numerical factor outside the exponential." Kallenberg and Stzencel (1991) go on to derive such dimension-free estimates, valid even for discontinuous martingales. (Note that other dimension-free estimates for continuous martingales are found in Jacka and Yor (1990).)

The present paper attempts to isolate the geometric aspect of Kallenberg and Stzencel's striking results for continuous martingales in Euclidean and Hilbert spaces, in such a way as to extend them to martingales in a Riemannian manifold (V, g) (see Emery (1989) for basic stochastic calculus on manifolds, and Kendall

(1990) for some applications of these martingales). The main result (Theorem 2.1) is of the following form: if X is a martingale on (V, g) started at p, with Riemannian quadratic variation (explained below) $[X, X]_t \leqq t$, and if $\tau(a)$ is the first time X leaves the geodesic ball of radius a about p, then

$$P(\tau(a) \leqq t) \leqq \frac{4P(Z \geqq f(a)/\sqrt{t})}{P(Z \geqq 1)} \leqq \gamma \frac{\sqrt{t}}{f(a)} \exp\{-f(a)^2/2t\} \tag{0.3}$$

where Z is a Normal(0, 1) random variable, $\gamma = 4/P(Z \geqq 1)\sqrt{2\pi}$, and f is a "persistence function" determined by the geometry of (V, g); Kallenberg and Stzencel (1991) studied the Euclidean case where $f(a) = a$. Using comparison techniques similar to those used by Ichihara (1984) and others, this function f can be computed explicitly (Theorem 2.3). For example, if the sectional curvature in planes including a radial tangent vector is bounded below by $-c^2$, then (0.3) holds with

$$f(a)^2 = 2c^{-2} \log\cosh(ac) . \tag{0.4}$$

In Sect. 3 we consider the special case of Brownian motion on a Riemannian manifold. The resulting estimates (Theorems 3.1, 3.2) do not appear to be contained in the extensive literature on the heat kernel on a Riemannian manifold (see e.g. Chavel (1984), Li and Yau (1986), and Davies (1989)); it is conjectured, however, that they are not the best possible estimates for small t. Note that for Brownian motion, mean exit times from geodesic balls in a Riemannian manifold were calculated by Gray and Pinsky (1983), exponential estimates for $P(\tau(a) \leqq t)$ were given by Hsu and March (1985) and by Hsu (1989), and asymptotics for $P(\tau(a) > t)$ as $a \to 0$ were given by Karp and Pinsky (1987).

1 Geometric preliminaries

Suppose (V, g) is a smooth Riemannian manifold, possibly with boundary, modelled on m-dimensional Euclidean space $\mathbb{R}^m$, and let $p \in V$. If $\mathscr{B}[0, r]$ denotes the closed ball in T_pV of radius r about 0, and $\mathscr{B}[0, \infty]$ is interpreted as the whole of T_pV, it is well known that there exists $0 < b \leqq \infty$ such that the exponential map $\exp_p: T_pV \to V$ is a diffeomorphism from $\mathscr{B}[0, b]$ onto its image $\mathscr{B}' \subseteqq V$. By deleting the parts of V outside $\mathscr{B}'$ if necessary, and identifying T_pV with $\mathbb{R}^m$, we suppose henceforward that the polar co-ordinate map

$$\psi: I_0 \times S^{m-1} \to V - \{p\} \tag{1.1}$$

defined by $\psi(r, v) \equiv \exp_p(rv)$, is a diffeomorphism, where $I_0 = (0, b]$ (or $(0, \infty)$ if $b = \infty$).

Definition 1.1 We shall say that (V, g, p, b) is a regular Riemannian ball if the situation described in (1.1) holds.

According to the Gauss Lemma (see Gallot et al. (1990, p. 89)), the metric g may be expressed in geodesic polar coordinates by

$$g = \mathrm{d}r \otimes \mathrm{d}r + \tilde{g}_{(r,v)} \tag{1.2}$$

where $\tilde{g}_{(r,v)}$ is the metric induced by g at $\psi(r, v)$ on $\psi(\{r\} \times S^{m-1})$, and r also stands for the function on V such that $r(\psi(s, v)) = s$. Let ∂ denote the "radial" vector field on $V - \{p\}$, which is the push-forward under $r \to \psi(r, v)$ of the vector field $\partial/\partial r$. The *radial curvature* means the restriction of the curvature function to all the planes in $T_x V$ containing $\partial(x)$. See Greene and Wu (1979) for more details. For any smooth function f on V, the *second covariant derivative*, or Hessian, of f at x is the symmetric form

$$\nabla \mathrm{d}f(x)(u, w) = U(\langle \mathrm{d}f, W\rangle_x) - \langle \mathrm{d}f, \nabla_U W\rangle_x \tag{1.3}$$

where U and W are vector fields whose values at $x \in V$ are u and w respectively, $\langle .,. \rangle$ denotes the duality between 1-forms and vector fields, and ∇ stands for covariant differentiation with respect to the Riemannian connection.

Definition 1.2 If $I = [0, b]$, or $[0, \infty)$ if $b = \infty$, we say that $f: I \to [0, \infty)$ is a *persistence function* for (V, g, p, b) if the following conditions hold:
(i) $f(0) = 0$, f is C^2 on I, $f'(0) \leqq 1$, $f'(x) > 0$ and $f''(x) \leqq 0$ for $x > 0$.
(ii) If $t > 0$, $r(x) = t$, $w \in T_x V$, and $g(w, \partial) = 0$, then

$$\frac{\nabla \mathrm{d}r(x)(w, w)}{g(w, w)} \leqq \frac{1}{f(t) f'(t)} . \tag{1.4}$$

The reason for the term "persistence function" will become clear in Theorem 2.1. Examples will be given in Sect. 4. The relationship of persistence functions to radial curvature is clarified by the following result. As before $I = [0, b]$, or $[0, \infty)$ if $b = \infty$.

Proposition 1.3 (Comparison Theorem) *Let (V_1, g, p_1, b) and (V_2, h, p_2, b) be regular Riemannian balls, and assume that for all geodesics $\gamma_1: I \to V_1$ and $\gamma_2: I \to V_2$, parametrized by arclength, with $\gamma_i(0) = p_i$, and for all $t \in I$,*

$$\textit{each radial curvature at } \gamma_1(t) \leqq \textit{each radial curvature at } \gamma_2(t) . \tag{1.5}$$

If $f: I \to [0, \infty)$ is a persistence function for (V_1, g, p_1, b), then it is also a persistence function for (V_2, h, p_2, b).

Proof. As in (1.2), express the metrics g and h as

$$g = \mathrm{d}r \otimes \mathrm{d}r + \tilde{g}_{(r,u)}, \quad h = \mathrm{d}s \otimes \mathrm{d}s + \tilde{h}_{(r,v)}$$

where r and s denote the radial distance functions on V_1 and V_2 respectively. We shall abusively use the same symbols ψ, ∂ and ∇ as above on both V_1 and V_2. Let $t \in [0, b]$; according to the Hessian Comparison Theorem of Greene and Wu (1979, p. 19), if $w_1 \in T_{\psi(t,u)} V_1$ such that $g(w_1, \partial) = 0$ and $\tilde{g}_{(t,u)}(w_1, w_1) = 1$, and $w_2 \in T_{\psi(t,v)} V_2$ such that $h(w_2, \partial) = 0$ and $\tilde{h}_{(t,v)}(w_2, w_2) = 1$, then

$$0 \leqq \nabla \mathrm{d}s(w_2, w_2) \leqq \nabla \mathrm{d}r(w_1, w_1) .$$

Since f is a persistence function for (V_1, g, p_1),

$$f(t) f'(t) \nabla \mathrm{d}r(w_1, w_1) \leqq \tilde{g}_{(t,v)}(w_1, w_1) = 1 .$$

Since $f(r) \geqq 0$ and $f'(r) \geqq 0$

$$f(t) f'(t) \nabla \mathrm{d}s(w_2, w_2) \leqq f(t) f'(t) \nabla \mathrm{d}r(w_1, w_1) = 1 = \tilde{h}_{(t,v)}(w_2, w_2) ,$$

and the result clearly extends by linearity to tangent vectors w_2 which are not of unit length. □

The following lemma will be useful for the study of martingales on (V, g).

Lemma 1.4 *For all* $x \in V = \{p\}$, *and every vector field* U *on* V,

$$\nabla \mathrm{d}r(x)(\partial, U) = 0 \,. \tag{1.6}$$

Proof. There is a direct proof in local coordinates, but the author is indebted to M. Emery for the following concise proof, which is equally valid in infinite dimensions. First note that the vector field grad(r), denoted ∇r, is the same as ∂. Take $f = r$ and $W = \partial = \nabla r$ in (1.3), to obtain

$$\begin{aligned} \nabla \mathrm{d}r(x)(U, \partial) &= U(\langle \mathrm{d}r, \nabla r\rangle_x) - \langle \mathrm{d}r, \nabla_U(\nabla r)\rangle_x \\ &= U(g(\nabla r, \nabla r))_x - \tfrac{1}{2} U(g(\nabla r, \nabla r))_x = 0 \end{aligned} \tag{1.7}$$

since $g(\nabla r, Y) = \langle \mathrm{d}r, Y\rangle$ for any vector field Y, and $g(\nabla r, \nabla r) \equiv 1$. □

2 Passage time estimates for martingales

Let (V, g, p, b) be a regular Riemannian ball. Let $(\Omega, \mathscr{F}, P, (\mathscr{F}_t)_{t \geqq 0})$ be a filtered probability space satisfying the usual conditions. Our reference for manifold-valued processes is Emery (1989), from which we recall the following definitions. A continuous V-valued process X is called a *semimartingale* if $f \circ X$ is a real semimartingale for all smooth $f \colon V \to \mathbb{R}$. For a semimartingale X on V, there exists a unique linear mapping, denoted

$$\beta \to \textstyle\int \beta(\mathrm{d}X, \mathrm{d}X) \tag{2.1}$$

from the space of all bilinear forms on M to the space of real continuous processes with finite variation, such that for all smooth functions f and g on V,

$$\textstyle\int (f\beta)(\mathrm{d}X, \mathrm{d}X) = \int (f \circ X)\,\mathrm{d}(\int \beta(\mathrm{d}X, \mathrm{d}X)) \tag{2.2}$$

$$\textstyle\int (\mathrm{d}f \otimes \mathrm{d}g)(\mathrm{d}X, \mathrm{d}X) = [f \circ X, g \circ X] \tag{2.3}$$

where $[Y, Z]$ denotes the usual joint quadratic variation of continuous semimartingales Y and Z. The process $\int \beta(\mathrm{d}X, \mathrm{d}X)$ is called the *β-quadratic variation* of X. When β is chosen to equal g, the metric tensor, we call the process the *Riemannian quadratic variation*, denoted

$$[X, X] \equiv \textstyle\int g(X)(\mathrm{d}X, \mathrm{d}X) \tag{2.4}$$

whose expression in local coordinates is simply $\int g_{ij}(X)\,\mathrm{d}[X^i, X^j]$.

A V-valued semimartingale is called a *martingale on* (V, g) (or more strictly, a Γ-martingale, with respect to the Riemannian connection Γ) if for all smooth $f \colon V \to \mathbb{R}$,

$$f \circ X - f \circ X_0 - \tfrac{1}{2} \textstyle\int \nabla \mathrm{d}f(X)(\mathrm{d}X, \mathrm{d}X) \in \mathscr{M}^c_{\mathrm{loc}} \tag{2.5}$$

where $\mathscr{M}^c_{\mathrm{loc}}$ denotes the set of continuous local martingales, and $\nabla \mathrm{d}f$ is as in (1.3).

Henceforward in this section we shall study a martingale X on (V, g) by projecting X onto $I \times S^{m-1}$ using the inverse ψ^{-1} of the map ψ mentioned in (1.1). We shall write X in terms of its *radial* and *angular* parts as follows:

$$(R_t, \Theta_t) \equiv \psi^{-1}(X_t) \quad \text{on} \quad \{X_t \neq p\} \tag{2.6}$$

and $R_t \equiv 0$ on $\{X_t = p\}$; it is not necessary to define Θ_t on $\{X_t = p\}$. In other words, R_t is the Riemannian distance from X_t to p. Some more notation:

σ denotes the stopping-time $\inf\{t : R_t = b\}$;

Y_t^* denotes $\sup\{|Y_s| : 0 \leqq s \leqq t\}$, for any real stochastic process Y;

$\gamma = 4/P(Z \geqq 1)\sqrt{2\pi}$, where Z is a Normal(0, 1) random variable.

Theorem 2.1 (Generalized Kallenberg–Stzencel estimate) *Let X be a martingale on (V, g), with $X_0 = p$ and $[X, X]_t = t \wedge \sigma$ (see (2.4)), and suppose that f is a persistence function on (V, g, p, b). If B denotes a Brownian motion on $\mathbb{R}$ started at 0, then the radial part of X satisfies*

$$P(R_t^* \geqq a) \leqq \frac{P(B_t^* \geqq f(a))}{P(B_1 \geqq 1)} \leqq \gamma \min\left\{\frac{\sqrt{t}}{f(a)}, \sqrt{\frac{\pi}{2}}\right\} \exp\{-f(a)^2/2t\}, \quad a \in (0, b)\,. \tag{2.7}$$

Remarks. (a) Now it should be clear why f is called a persistence function; the greater f is, the smaller is the bound on the right side of (2.7), and so the more likely the martingale is to "persist" in the vicinity of where it started. Loosely speaking, the tendency to persist increases as sectional curvatures increase, as the next corollary shows.

(b) Cranston and Hsu point out that the restriction of this result to geodesic balls is unnatural, because existence of a cut locus can only reduce, not increase, the probability on the left side of (2.7). Unfortunately the present proof in Sect. 5 seems to depend on existence of $\nabla\, \mathrm{d}r(x)$ for $x \neq p$, and so the generalization of Theorem 2.1 to an arbitrary Riemannian manifold must await a different proof.

(c) For applications to Brownian motion on manifolds, see Sect. 3.

(d) To see how sharp this estimate is for various kinds of martingales, consider the situation where (V, g) is flat Euclidean space $\mathbb{R}^m$, and so $f(a) = a$ (see Example 4.3(a) below). (This remark relates to a question of Sznitman (Zürich).)

Case (i). Let $\bar{X}$ be one-dimensional Brownian motion run along one of the coordinate axes, and so $P(\bar{R}_t^* \geqq a) = P(B_t^* \geqq a)$; this shows that (2.7) is sharp up to a constant factor.

Case (ii). Let $\tilde{X}_t \equiv W_{t/m}$, where W is Brownian motion on $\mathbb{R}^m$; if $\tau(a)$ denotes the first exit time from the ball of radius a, so $P(\tau(a) \leqq t) = P(\tilde{R}_t^* \geqq a)$, then the mean exit time is

$$\mathbb{E}[\tau(a)] = a^2 = \int_0^\infty P(\tilde{R}_t^* < a)\,\mathrm{d}t$$

which is the same as in Case (i). It is easy to see, however, that the exit time $\tau(a)$ has a lower variance in Case (ii). It follows that, for fixed a, there exist $0 < t_1 \leqq a^2 \leqq t_2 < \infty$ such that

$$P(\bar{R}_t^* \geqq a) > P(\tilde{R}_t^* \geqq a) \quad \text{for } t < t_1$$

$$P(\bar{R}_t^* \geqq a) < P(\tilde{R}_t^* \geqq a) \quad \text{for } t > t_2\,.$$

Since the upper estimate for $P(\bar{R}_t^* \geqq a)$ is sharp for all t (up to a constant factor), it follows from the second inequality that the upper estimate for $P(\tilde{R}_t^* \geqq a)$ is sharp for large t; however (2.7) is not very informative when $t > a^2$ (or $f(a)^2$), because in that case the right side is approximately 1. This suggests that Theorem 2.1 is likely to be most useful in the case where the martingale is quite dissimilar to Brownian motion, in the sense of having anisotropic local characteristics. See also Problem 6.1 below.

Corollary 2.2 *Suppose (V_1, g, p_1, b) and (V_2, h, p_2, b) are Riemannian balls satisfying the conditions of Proposition* 1.3, *i.e. the radial curvatures of V_2 are bounded below by those of V_1. If $f: I \to [0, \infty)$ is a persistence function for (V_1, g, p_1, b), and if X is a martingale on (V_2, h), with $X_0 = p_2$ and $[X, X]_t = t \wedge \sigma$, then the radial part of X satisfies* (2.7).

In the following theorem and corollaries, we assume that X is a martingale on (V, g), with $X_0 = p$ and $[X, X]_t = t \wedge \sigma$.

Theorem 2.3 *Suppose (V, g, p, b) is a Riemannian ball such that every radial curvature at $x \in V - \{p\}$ is bounded below by $K(r(x))$, where $K: I \to (-\infty, \infty)$ is the radial curvature of some rotationally symmetric Riemannian manifold, with the property that:*
(i) *If $u: I \to [0, \infty)$ satisfies $u(0) = 0$ and $u' = 1 + Ku^2$, then u' is positive and nonincreasing.*
Then (2.7) *holds for the function $f(x) = \sqrt{2 \int_0^x u(r)\, dr}$.*

Remark. Note that (i) holds if K is nonpositive, nonincreasing, and piecewise constant; for if $K(r) = -c^2$ for $r \in (a_1, a_2)$, then $u'(r) = \operatorname{sech}^2(l + cr)$ and $u(r) = c^{-1} \tanh(l + cr)$ on (a_1, a_2), for some constant l; thus u' is positive and nonincreasing on (a_1, a_2). Since K is nonincreasing, $u' (= 1 + Ku^2)$ must also be nonincreasing across the junction of two intervals on which K is piecewise constant.

Corollary 2.4 *If the radial curvatures of (V, g, p, b) are bounded below by $-c^2 < 0$, then the radial part of X satisfies*

$$P(R_t^* \geqq a) \leqq \frac{P(B_t^* \geqq c^{-1}\sqrt{2 \log(\cosh(ac))})}{P(B_1 \geqq 1)}, \quad a \in (0, b),$$

$$\leqq \frac{P(B_t^* \geqq \min\{0.688a, \sqrt{3a/2c}\})}{P(B_1 \geqq 1)}, \quad a \in (0, b). \tag{2.8}$$

Remark. The usefulness of this kind of result is as follows: if the martingale X appears as the solution of some stochastic differential equation on V, then (2.8) can be used to prove nonexplosion, i.e. that the solution exists on V for all time, a.s.

Corollary 2.5 *If the radial curvatures of (V, g, p, b) are nonnegative, then, for γ as in* (2.8)

$$P(R_t^* \geqq a) \leqq \frac{P(B_t^* \geqq a)}{P(B_1 \geqq 1)} \leqq \gamma \min\left\{\frac{\sqrt{t}}{a}, \sqrt{\frac{\pi}{2}}\right\} \exp(-a^2/2t), \quad a \in (0, b).$$

Remark. The first inequality is best possible, up to a multiplicative constant. To see this let $\alpha: (-b, b) \to V$ be a geodesic parametrized by arc-length, with $\alpha(0) = p$, let

B' be a Brownian motion on $\mathbb{R}$ started at 0, and let X be the martingale $\alpha \circ B'$, which satisfies $[X, X]_t = t$; see Emery (1989, p. 40). Clearly the right side of (2.7) is exactly $P(B_1 \geqq 1)^{-1}$ times bigger than the left side for all t and a.

Theorem 2.6 (Law of the iterated logarithm) *Let X be a martingale on (V, g), with $X_0 = p$ and $[X, X]_t = t \wedge \sigma$, and suppose that f is a persistence function on (V, g, p, b). Then as $t \to 0$,*

$$\limsup \frac{f(R_t)}{\sqrt{2t \log|\log t|}} \leqq 1 \quad \text{a.s.} \tag{2.9}$$

and the same holds as $t \to \infty$, provided $\sigma = \infty$ a.s. Moreover this bound is sharp.

Remarks. (a) Theorem (2.6) is almost a restatement of part of Theorem 4.6 of Kallenberg and Stzencel (1991).

(b) For the hyperbolic space of constant negative curvature $-c^2 < 0$, $f(r) \sim \sqrt{2r/c}$ as $r \to \infty$ (see (4.6)), so the Law of the Iterated Logarithm implies

$$\limsup \frac{R_t}{t \log|\log t|} \leqq c \quad \text{a.s. as } t \to \infty\ . \tag{2.10}$$

3 Application to Brownian motion on manifolds

If W is a Brownian motion on (V, g), started at p, then $[W, W]_t = mt$, where m is the dimension of V (see Emery (1989, p. 64)). It follows that the results of Sect. 3 apply to $X_t \equiv W_{t/m}$, since it is a martingale on (V, g) with $[X, X]_t = t$. However it follows from the properties of Brownian motion (see Emery (1989, p. 62)) that, for this X,

$$\nabla \mathrm{d}r(X)(\mathrm{d}X, \mathrm{d}X) = \frac{1}{m} \Delta r(X)$$

and careful study of the proofs in Sect. 5 shows the function f appearing in Theorem 2.1 need not be a persistence function, but merely a "*Δ-persistence function*" which is the same as in Definition 1.2, except that inequality (1.4) is replaced by

$$\frac{\Delta r(x)}{m} \leqq \frac{1}{f(r(x)) f'(r(x))}, \quad x \neq p\ . \tag{3.1}$$

Greene and Wu's (1979) Hessian Comparison Theorem may be replaced by their Laplacian Comparison Theorem (p. 26) in Proposition 1.3, and consequently radial curvatures by Ricci curvatures of the form $\mathrm{Ric}(\partial, \partial)$ in condition (1.5). Similar methods were used by Debiard et al. (1976) and Ichihara (1984) in solving other problems about Brownian motion on a manifold. Using these geometric ideas, and the obvious Brownian rescaling, we may weaken the conditions of Theorems 2.1 and 2.3 as follows. Let $\tau(a)$ denote the first exit time of W from the ball of radius a around p, and let ∂ be the radial vector field as before. Note that an estimate of the following kind also appears in Hsu (1989, p. 1252).

Theorem 3.1 *Let W be a Brownian motion on (V, g) started at p, and suppose f is a Δ-persistence function on (V, g, p, b). If B denotes Brownian motion on $\mathbb{R}$ started at* 0, *and γ is as in Theorem* 2.1, *then for $a \in (0, b)$*,

$$P(\tau(a) \leqq t) \leqq \frac{P(B_t^* \geqq f(a)/\sqrt{m})}{P(B_1 \geqq 1)} \leqq \gamma \min\left\{\frac{\sqrt{tm}}{f(a)}, \sqrt{\frac{\pi}{2}}\right\} \exp\{-f(a)^2/2tm\} \tag{3.2}$$

Theorem 3.2 *Suppose (V, g, p, b) is a Riemannian ball whose Ricci curvatures satisfy*,

$$\frac{\operatorname{Ricci}(\partial, \partial)(x)}{(m-1)g(\partial, \partial)} \geqq K(r(x)) \tag{3.3}$$

for some function $K: I \to (-\infty, \infty)$, with the property:
(i) *If $u: I \to [0, \infty)$ satisfies $u(0) = 0$ and $u' = 1 + Ku^2$, then u' is positive and nonincreasing.*
Then (3.2) *holds for the function $f(x) = \sqrt{2 \int_0^x u(r)\, dr}$.*

Remarks. (a) See the remark after Theorem 2.3 for instances when (i) holds.

(b) When t is fixed and $a \to 0$, Pinsky has pointed out that, by results of Karp and Pinsky (1987), there exist constants $c_1 < 0 < c_0$, depending only on m, such that

$$P(\tau(a) > t) \sim \exp\left\{t\left(-\frac{c_0}{a^2} + c_1\sigma + O(a^2)\right)\right\}, \quad \text{as } a \to 0\,,$$

where σ is the scalar curvature at p. This is an improvement on (3.2) for small a and fixed t. On the other hand, for fixed a and small t, the discussion in Remark (c) following Theorem 2.1 suggests that (3.2) is not the best possible estimate; see Problem 6.2 below.

(c) When $K(r(x)) \equiv 0$ in (3.3) (i.e. radial Ricci curvatures are nonnegative), then we obtain an estimate analogous to Corollary 2.5. Likewise when $K(r(x)) \equiv -c^2 < 0$ in (3.3), we obtain as in (2.8) the estimate

$$P(\tau(a) \leqq t) \leqq \gamma \sqrt{\frac{\pi}{2}} \exp(-\min\{0.47a^2, 3a/2c\}/2tm), \quad a \in (0, b)\,. \tag{3.4}$$

Note that the well-known result of Yau (1978), that no explosion for Brownian motion is possible if the Ricci curvature is bounded below by a constant, follows easily from (3.4); the sum over $n \geqq 1$ of $P(\tau(n^2) \leqq n)$ is seen to be finite, so $\tau(n^2) > n$ for all but finitely many n a.s., by the first Borel–Cantelli Lemma, which implies nonexplosion.

4 Examples of persistence functions

To calculate a persistence function for an arbitrary Riemannian manifold could be difficult. Fortunately Proposition 1.3 and Corollary 2.2 allow us to use a persistence function calculated on a suitably chosen "comparison manifold", such as one with rotational symmetry.

We shall say that (V, g, p, b) is a *model* if every linear isometry $\iota: T_pV \to T_pV$ is realized as the differential of an isometry $\Psi: V \to V$, i.e. $\Psi(p) = p$ and $T_p\Psi = \iota$. In this case (see Greene and Wu (1979)), the metric in (1.2) takes the form

$$g = \mathrm{d}r \otimes \mathrm{d}r + \tilde{g}_{(r,v)} = \mathrm{d}r \otimes \mathrm{d}r + \varphi(r)^2 \hat{g}_v \,. \tag{4.1}$$

It is well known (see Stoker (1969, pp. 179–183)) that $\varphi(0) = 0$, $\varphi(r) > 0$ for $r > 0$, and $\varphi'(0) = 1$. Let $K(r)$ denote the sectional curvature of any two-dimensional subspace of $T_{\iota(r,v)}V$ containing ∂, the radial vector field; this satisfies the Jacobi equation $\varphi''(r) = -K(r)\varphi(r)$. Define f to be the nonnegative function such that

$$f(t)^2 \equiv \int_0^t \frac{2\varphi(r)\,\mathrm{d}r}{\varphi'(r)} \,. \tag{4.2}$$

We shall see that this f satisfies (1.4) with equality; the main question is whether $f''(x)$ is nonpositive.

Proposition 4.1 *If $K(r)$ is non-positive, the function f defined in* (4.2) *is a persistence function if and only if $u^2 \geqq 2u' \int u$, where $u \equiv \varphi/\varphi'$ and $\int u$ is short for $\int_0^x u(r)\,\mathrm{d}r$. In particular, if u' is positive and nonincreasing then f is a persistence function.*

Corollary 4.2 *Suppose that whenever $u(r)$ satisfies $u' = 1 + Ku^2$ and $u(0) = 0$, then u' is positive and nonincreasing, where K denotes the radial curvature function $K(r)$. Then the nonnegative function f such that $f(x)^2 = \int_0^x 2u(r)\,\mathrm{d}r$ is a persistence function.*

Proof. First we check condition (i) of Definition 1.2. The Jacobi equation implies $\varphi'' \geqq 0$, and so $\varphi' > 0$; (4.2) implies $f(r)f'(r) = \varphi(r)/\varphi'(r)$, so $f'(r) > 0$ when $r > 0$. Clearly $f(0) = 0$ by (4.2), and $f(r) > 0$ for $r > 0$. Since $\varphi(t) = t + \mathrm{O}(t^2)$ as $t \to 0$,

$$f(r)^2 = \int_0^r \frac{2t + \mathrm{O}(t^2)}{1 + \mathrm{O}(t)}\,\mathrm{d}t = r^2 + \mathrm{O}(r^3), \quad \text{as } r \to 0 \,,$$

and so $f(r) = r + \mathrm{o}(r)$ as $r \to 0$, so $f'(0) = 1$. It only remains to prove that $f''(r) \leqq 0$ for $r > 0$. Differentiating the identity $f(r)f'(r)\varphi'(r) = \varphi(r)$ gives:

$$f''(r)f(r) = 1 - f'(r)^2 - \varphi''(r)\varphi(r)/\varphi'(r)^2 \,,$$

so it suffices to prove that $(f')^2 \geqq 1 - \varphi''\varphi/(\varphi')^2$. Since $(f')^2 = (\varphi/\varphi')^2/\int 2\varphi/\varphi'$, this is equivalent to

$$\varphi^2 \geqq [(\varphi')^2 - \varphi\varphi''] \int 2\varphi/\varphi'$$

$$\Leftrightarrow u^2 \geqq 2\frac{(\varphi')^2 - \varphi\varphi''}{(\varphi')^2} \int u = 2u' \int u \,.$$

Observe also that u' positive and nonincreasing implies

$$u(r)u'(r) \geqq u(r)u'(x) \quad \text{for } 0 \leqq r \leqq x \,,$$

$$\Rightarrow u^2(x)/2 = \int_0^x uu' \geqq u'(x) \int_0^x u$$

$$\Rightarrow f''(x) \leqq 0 \quad \text{for } x > 0 \,.$$

Condition (ii). For this f, Greene and Wu (1979, p. 30) show that if $r(x) = t > 0$, $w \in T_x V$, $w \neq 0$, and $g(w, \partial) = 0$, then

$$\frac{\nabla \, dr(x)(w, w)}{g(w, w)} = \frac{\varphi'(t)}{\varphi(t)} = \frac{1}{f(t) f'(t)}$$

which verifies (1.4). □

Proof of the corollary. Since $\varphi'' = -K\varphi$ and $u = \varphi/\varphi'$,

$$u' = \frac{\varphi'}{\varphi'} - \frac{\varphi\varphi''}{(\varphi')^2} = 1 - u^2 \varphi''/\varphi = 1 + Ku^2 .$$

Thus $u(r)$ satisfies $u' = 1 + Ku^2$ and $u(0) = 0$, and so $u'' \leqq 0$. Hence the conclusion follows from the last sentence of the proposition. □

Examples 4.3 (a) *Euclidean space.* When (V, g) is $\mathbb{R}^m$ with the Euclidean metric, with a pole at 0, then $\varphi(r) = r$, $u(r) = r$, $u' = 1$, and $f(r) = r$ is a persistence function. Theorem 2.1 is now exactly the same as Theorem 4.5 of Kallenberg and Sztencel (1991), namely

$$P(R_t^* \geqq a) \leqq \frac{P(B_t^* \geqq a)}{P(B_1 \geqq 1)}, \quad a \in (0, \infty) . \tag{4.3}$$

(b) *Spaces of constant negative curvature (e.g. hyperbolic spaces).* Note that, by the theorem of Hadamard–Cartan (see Gallot et al. (1987)), any simply-connected manifold with non-positive sectional curvatures has a global geodesic polar coordinate system. If all sectional curvatures equal $-c^2$, for some $c > 0$, then the metric g can be expressed in geodesic polar coordinates (see Spivak (1979, vol. 2, p. 327)) as

$$g = dr \otimes dr + \frac{1}{c^2} \sinh^2(cr)(d\vartheta)^2$$

where $(d\vartheta)^2$ denotes the induced Riemannian metric on the geodesic unit sphere. Here $\varphi(r) = \sinh(cr)/c$, $u(r) = \tanh(cr)/c$, $u'(r) = \mathrm{sech}^2(cr)$ which is positive and decreasing in r on $[0, \infty)$, and consequently

$$f(r) = c^{-1} \sqrt{2 \log(\cosh(cr))} \tag{4.4}$$

is a persistence function. Here Theorem 2.1 gives

$$P(R_t^* \geqq a) \leqq \frac{P(B_t^* \geqq c^{-1} \sqrt{2 \log(\cosh(ac))})}{P(B_1 \geqq 1)}, \quad a \in (0, \infty) . \tag{4.5}$$

Note that $f(r) \sim \sqrt{2r/c}$ as $r \to \infty$, and $f(r) \sim r$ as $r \to 0$. By the concavity of log, and the inequalities $\cosh x \geqq 1 + x^2/2$, $\cosh x \geqq e^x/2$, applied on the intervals $(0, \log 16]$ and $(\log 16, \infty)$ respectively, we deduce that

$$f(r) \geqq \min\left\{0.688r, \sqrt{\frac{3r}{2c}}\right\} . \tag{4.6}$$

(c) *Spaces of constant positive curvature.* A manifold V where all sectional curvatures equal $c^2 > 0$ does not possess global geodesic polar coordinates. However if $p \in V$, then the geodesic ball about p of radius $a < \pi/(2c)$ excludes the cut locus of p. The metric g can be expressed in geodesic polar coordinates (see Spivak (1979, vol. 2, p. 327)) as

$$g = \mathrm{d}r \otimes \mathrm{d}r + \frac{1}{c^2}\sin^2(rc)(\mathrm{d}\vartheta)^2$$

where $(\mathrm{d}\vartheta)^2$ denotes the induced Riemannian metric on the geodesic unit sphere. Here $\varphi(r) = \sin(cr)/c$, $u(r) = \tan(cr)/c$, $u'(r) = \sec^2(cr)$, but formula (4.2) no longer yields a persistence function, because u' is an increasing function and $f''(x)$ fails to be nonpositive. The linear function $f(r) = r$ is a persistence function by Proposition 1.3 (Hessian Comparison Theorem), and the author is not aware of any other persistence function which exceeds it.

5 Proofs

Before proving the theorem and corollaries, we shall need some preliminary results. The following result is similar to a theorem of Kendall (1987).

Lemma 5.1 *There exists $\xi \in \mathcal{M}^c_{\mathrm{loc}}$ such that*

$$1_{\{R \neq 0\}}\,\mathrm{d}R = \mathrm{d}\xi + \tfrac{1}{2}1_{\{R \neq 0\}}\nabla\,\mathrm{d}r(X)(\mathrm{d}\Theta, \mathrm{d}\Theta)\,. \tag{5.1}$$

Remark. In local coordinates, writing X_t as $(R_t, \Theta_t^2, \ldots, \Theta_t^m)$, formula (5.1) becomes

$$\mathrm{d}R_t = \mathrm{d}\xi_t + \tfrac{1}{4}\partial g_{jk}/\partial r(X_t)\mathrm{d}[\Theta^j, \Theta^k]_t \quad \text{on} \quad \{R_t \neq 0\}\,. \tag{5.1'}$$

Proof. The function $x \to r(x)^2$ is smooth on V, so R^2 is a semimartingale, and by (2.5)

$$R^2 - R_0^2 - \tfrac{1}{2}\int \nabla\mathrm{d}(r^2)(X)(\mathrm{d}X, \mathrm{d}X) = \zeta \in \mathcal{M}^c_{\mathrm{loc}}\,. \tag{5.2}$$

It follows from (1.3) that $\nabla\mathrm{d}(r^2)(x) = 2r\nabla\,\mathrm{d}r(x) + 2(\mathrm{d}r \otimes \mathrm{d}r)(x)$ on $\{x \neq 0\}$, and by writing $\nabla\,\mathrm{d}r(X)(\mathrm{d}X, \mathrm{d}X)$ in local coordinates, Lemma 1.4 shows that $\nabla\,\mathrm{d}r(X)(\mathrm{d}X, \mathrm{d}X) = \nabla\,\mathrm{d}r(X)(\mathrm{d}\Theta, \mathrm{d}\Theta)$. Since $\int 1_{\{R=0\}}\,\mathrm{d}[R, R] = 0$, we obtain from (5.2):

$$R^2 = R_0^2 + \zeta + [R, R] + \int 1_{\{R \neq 0\}}R\nabla\,\mathrm{d}r(X)(\mathrm{d}\Theta, \mathrm{d}\Theta)\,.$$

Given $\varepsilon > 0$, take a smooth function $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(x) = \sqrt{x}$ on $[\varepsilon, \infty)$, and apply Itô's formula to $h(R)$; this yields

$$\begin{aligned} 1_{\{R \geqq \varepsilon\}}\mathrm{d}R &= \tfrac{1}{2}1_{\{R \geqq \varepsilon\}}\{R^{-1}\mathrm{d}\zeta + R^{-1}\mathrm{d}[R, R] + \nabla\,\mathrm{d}r(X)(\mathrm{d}\Theta, \mathrm{d}\Theta) \\ &\quad - \tfrac{1}{4}R^{-3}\mathrm{d}[R^2, R^2]\} \\ &= \tfrac{1}{2}1_{\{R \geqq \varepsilon\}}\{R^{-1}\mathrm{d}\zeta + \nabla\,\mathrm{d}r(X)(\mathrm{d}\Theta, \mathrm{d}\Theta)\}\,. \end{aligned}$$

Since this holds for all $\varepsilon > 0$, setting $\xi \equiv \frac{1}{2}\int 1_{\{R \neq 0\}}R^{-1}\mathrm{d}\zeta$ gives the desired formula. $\square$

Let us now restrict to martingales X on (V, g) with the time scaled such that the Riemannian quadratic variation process is equal to $t \wedge \sigma$. By the Gauss Lemma, referred to in (1.2),

$$[X, X] = [R, R] + [\Theta, \Theta] \tag{5.3}$$

where $[\Theta, \Theta] = \int \tilde{g}(X)(\mathrm{d}\Theta, \mathrm{d}\Theta)$. The expression $[R, R]$ makes sense, as the preceding Lemma shows. The following result and part of the proof are modelled on Theorem 4.4 of Kallenberg and Sztencel (1991).

Proposition 5.2 In the situation of Theorem 2.1, define $A_t \equiv \int_0^t 1_{\{R \neq 0\}} \mathrm{d}[R, R]$. Then there exists a Brownian motion B such that

$$f(R_t^*) \leqq B^* \circ A_t + \sqrt{t - A_t} \tag{5.4}$$

$$f(R_t^*) \leqq \sup\{|B_s| + \sqrt{t - s} : s \leqq A_t\}\ . \tag{5.5}$$

Proof. Step I. For simplicity, assume $\sigma(\equiv \inf\{t : R_t = b\}) = \infty$ a.s.; the general case involves notational changes only. Let f be a persistence function for (V, g, p, b), and let

$$H_t \equiv f(R_t) - \sqrt{t - A_t}\ . \tag{5.6}$$

We may define a predictable process (α_t), with $0 \leqq \alpha_t \leqq 1$, by

$$(\alpha_t)^2 \mathrm{d}t = 1_{\{R \neq 0\}} \mathrm{d}[R, R]_t = \mathrm{d}A_t\ . \tag{5.7}$$

Observe that, since $f(0) = 0$ and $f(x) > 0$ on $(0, b]$, it follows that $R_t > 0$ on the set $\{H_t > 0\}$. Let $\mathrm{d}H_t^+$ denote the positive part of H, and define an increasing process J, which increases only on $\{H = 0\}$, by

$$J_s \equiv H_s^+ - \int_0^s 1_{\{H(t) > 0\}} \left\{ f'(R_t)\mathrm{d}R_t + \frac{1}{2} f''(R_t)\alpha_t^2\, \mathrm{d}t - \frac{1}{2}\frac{1 - \alpha_t^2}{\sqrt{t - A_t}} \mathrm{d}t \right\}.$$

The third integrand on the right is understood to be 0 if $A_t = t$. So by (5.1), if ξ is the local martingale part of R, we may write in stochastic differential notation

$$\begin{aligned} \mathrm{d}H_t^+ &= 1_{\{H(t) > 0\}} f'(R_t)\, \mathrm{d}\xi - \\ \frac{1}{2} 1_{\{H(t) > 0\}} &\left\{ \frac{1 - \alpha_t^2}{\sqrt{t - A_t}} \mathrm{d}t - f'(R_t) \nabla \mathrm{d}r(X_t)(\mathrm{d}\Theta, \mathrm{d}\Theta)_t - f''(R_t)\alpha_t^2\, \mathrm{d}t \right\} + \mathrm{d}J_t \\ &= \mathrm{d}Y_t - \mathrm{d}V_t + \mathrm{d}J_t \end{aligned} \tag{5.8}$$

where $Y \in \mathscr{M}_{\mathrm{loc}}^c$, and the quadratic variation of Y satisfies

$$[Y, Y]_t = \int_0^t 1_{\{H(s) > 0\}} \alpha_s^2 f'(R_s)^2\, \mathrm{d}s \tag{5.9}$$

Step II. Next we will prove that V, appearing in (5.8), is an increasing process. Using the Definition 1.2 of persistence function, we see that, on $\{R_t > 0\}$,

$$f'(R_t) \nabla \mathrm{d}r(X_t)(\mathrm{d}\Theta, \mathrm{d}\Theta)_t \leqq f(R_t)^{-1} \mathrm{d}[\Theta, \Theta]_t = (1 - \alpha_t^2) f(R_t)^{-1}\, \mathrm{d}t \tag{5.10}$$

where the last equality follows from (5.3), (5.7), and the assumption that $[X, X]_t = t$. Using the fact that $f(R_t) > \sqrt{t - A_t}$ on the set $\{H_t > 0\}$, we obtain

$$2\mathrm{d}V_t = 1_{\{H(t)>0\}}\left\{\frac{1-\alpha_t^2}{\sqrt{t-A_t}}\mathrm{d}t - f'(R_t)\nabla \mathrm{d}r(X_t)(\mathrm{d}\Theta, \mathrm{d}\Theta)_t - f''(R_t)\alpha_t^2\,\mathrm{d}t\right\}$$

$$\geqq 1_{\{H(t)>0\}}\left\{\frac{1-\alpha_t^2}{\sqrt{t-A_t}}\mathrm{d}t - (1-\alpha_t^2)f(R_t)^{-1}\,\mathrm{d}t - f''(R_t)\alpha_t^2\,\mathrm{d}t\right\}$$

$$\geqq -1_{\{H(t)>0\}}f''(R_t)\alpha_t^2\,\mathrm{d}t \geqq 0$$

where the last inequality follows from Definition 1.2(i).

Step III. Since $R_0 = 0$, we may assume that the processes Y, V and J appearing in (5.8) satisfy $Y_0 = V_0 = J_0 = 0$. By Skorohod's lemma (see Rogers and Williams (1987, p. 117)),

$$J_t = \sup\{-(0 \wedge (Y_s - V_s)) : 0 \leqq s \leqq t\}$$
$$= -\inf\{(Y_s - V_s) : 0 \leqq s \leqq t\}$$

since already $Y_0 - V_0 = 0$. So

$$H_t^+ = Y_t - V_t - \inf\{(Y_s - V_s) : 0 \leqq s \leqq t\}$$
$$= \sup\{Y_t - V_t - Y_s + V_s) : 0 \leqq s \leqq t\}$$
$$\leqq \sup\{Y_t - Y_s : 0 \leqq s \leqq t\} + \sup\{V_s - V_t : 0 \leqq s \leqq t\}$$
$$= Y_t - \inf\{Y_s : 0 \leqq s \leqq t\}$$

using the fact that $\sup\{V_s - V_t : 0 \leqq s \leqq t\} = 0$, since V is increasing by Step II. Since $Y \in \mathcal{M}^c_{\mathrm{loc}}$, $Y_t = B' \circ [Y, Y]_t$ for some Brownian motion B', and by Lévy's theorem, there exists another Brownian motion B such that

$$B'_t - \inf\{B'_s : 0 \leqq s \leqq t\} = |B_t| \,. \tag{5.11}$$

Hence

$$H_t^+ \leqq B' \circ [Y, Y]_t - \inf\{B'_s : 0 \leqq s \leqq [Y, Y]_t\} = |B \circ [Y, Y]_t| \,. \tag{5.12}$$

By Definition 1.2, $f''(x) \leqq 0$, and consequently $0 < f'(x) \leqq f'(0) \leqq 1$ for $x \in (0, \infty)$. It follows from (5.9) that $[Y, Y]_t \leqq A_t$. Combining (5.6) and (5.12) gives: for $0 \leqq s \leqq t$,

$$f(R_s) \leqq H_s^+ + \sqrt{s - A_s}$$
$$\leqq |B \circ [Y, Y]_s| + \sqrt{s - A_s} \,. \tag{5.13}$$

Since $f(x)$ is increasing in x, and $\sup\{|B \circ [Y, Y]_s| : s \leqq t\} \leqq \sup\{|B \circ A_s| : s \leqq t\}$, this implies

$$f(R_t^*) \leqq B^* \circ A_t + \sqrt{t - A_t} \,.$$

Moreover since $(s - A_s)$ is increasing in s, (5.13) also implies

$$f(R_t^*) \leqq \sup\{|B_s| + \sqrt{t - s} : s \leqq A_t\} \,. \qquad \square$$

Proof of Theorem 2.1 (taken directly from Theorem 4.5 of Kallenberg and Sztencel (1991)). By (5.5), since $A_t \leqq t$,

$$f(R_t^*) \leqq \sup\{|B_s| + \sqrt{t-s} : s \leqq t\} . \tag{5.14}$$

Fix $a \in (0, b)$ and a time $t > 0$, and define

$$\tau \equiv \inf\{s \in [0, t] : B_s + \sqrt{t-s} \geqq a\}$$

where $\inf\{\emptyset\} = \infty$. Since f is monotone increasing,

$$\begin{aligned} P(R_t^* \geqq f^{-1}(a)) &= P(f(R_t^*) \geqq a) \\ &\leqq P(\sup\{|B_s| + \sqrt{t-s} : s \leqq t\} \geqq a) \\ &\leqq P(\sup\{B_s + \sqrt{t-s} : s \leqq t\} \geqq a) + P(\sup\{-B_s + \sqrt{t-s} : s \leqq t\} \geqq a) \\ &= 2P(\tau \leqq t) . \end{aligned}$$

By Brownian motion scaling, and the Strong Markov Property of B at τ,

$$\begin{aligned} P(B_t^* \geqq a) &\geqq 2P(B_t \geqq a) \\ &\geqq 2P(B_t - B_\tau \geqq \sqrt{t-\tau}, B_\tau + \sqrt{t-\tau} \geqq a) \\ &= 2\mathbb{E}[P(B_t - B_\tau \geqq \sqrt{t-\tau} \mid \mathscr{F}_\tau); B_\tau + \sqrt{t-\tau} \geqq a] \\ &= 2P(B_1 \geqq 1)P(\tau \leqq t) \\ &\geqq P(B_1 \geqq 1)P(R_t^* \geqslant f^{-1}(a)) \end{aligned}$$

which proves the first inequality in (2.7). By the Reflection Principle and Brownian motion scaling,

$$P(B_t^* \geqq f(a)) \leqq 4P(B_t \geqq f(a)) = 4P(B_1 \geqq f(a)/\sqrt{t}) .$$

Now use the standard estimates

$$\frac{1}{\sqrt{2\pi}} \int_s^\infty \exp\{-x^2/2\}\,\mathrm{d}x \leqq \frac{1}{s\sqrt{2\pi}} \int_s^\infty x \exp\{-x^2/2\} = \frac{1}{s\sqrt{2\pi}} \exp\{-s^2/2\}$$

and

$$\left(\frac{1}{\sqrt{2\pi}} \int_s^\infty \exp\{-x^2/2\}\,\mathrm{d}x\right)^2 \leqq \frac{1}{4} \int_{s\sqrt{2}}^\infty r \exp\{-r^2/2\}\,\mathrm{d}r = \frac{1}{4}\exp\{-s^2\}$$

$$\Rightarrow P(B_t^* \geqq f(a)) \leqq \frac{4}{\sqrt{2\pi}} \min\left\{\frac{\sqrt{t}}{f(a)}, \sqrt{\frac{\pi}{2}}\right\} \exp\{-f(a)^2/2t\}$$

which yields the second inequality of (2.7). □

Proofs of 2.2 to 2.5 Corollary 2.2 is immediate from Proposition 1.3 and Theorem 2.1. Theorem 2.3 follows from Corollary 2.2 and Corollary 4.2. Corollaries 2.4 and 2.5 follow from Theorem 2.3 and the fact, proved in Examples 4.3(a) and (b), that $f(r) = c^{-1}\sqrt{2\log(\cosh(cr))}$ and $f(r) = r$ are persistence functions for the hyperbolic space of constant curvature $-c^2$ and for Euclidean space, respectively. The second inequality of Corollary 2.4 follows from (4.6). □

Proof of Theorem 2.6 Write

$$L(t) = \sqrt{2t|\log|\log t||}, \quad t > 0$$

and recall from the law of the iterated logarithm for one-dimensional Brownian motion that, as $t \to 0$ or ∞,

$$\limsup \frac{B_t^*}{L(t)} = 1 \text{ a.s.}$$

Now (2.9) follows from (5.5) in Proposition 5.2, and the bound is attained if the process $\alpha \equiv 1$ (i.e. X is Brownian motion run along a unit speed geodesic through p, started at p). □

6 Unsolved problems

Problem 6.1
A continuous martingale with fastest rate of escape. Let χ denote the set of continuous local martingales X in $\mathbb{R}^m$ with $X_0 = 0$ and $[X, X]_t = t$. If R denotes the radial component of X, then define

$$h_X(t, a) = P(R_t^* \geqq a), \, a, t \in (0, \infty) .$$

Does there exist $Y \in \chi$ such that $h_Y(t, a) = \sup\{h_X(t, a): X \in \chi\}$ for all $a, t \in (0, \infty)$, in other words a "fastest escaping martingale"? Remark (c) following Theorem 2.1 suggests that a possible candidate is of the following type. Let B and W denote independent Brownian motions in $\mathbb{R}$ and $\mathbb{R}^m$ respectively, let $\tilde{\tau}(a) = \inf\{t: |B_t| = a\}$, and fix $c > 0$; now take

$$Y(t) = \begin{cases} (B_t, 0, \ldots, 0), & \text{for } 0 \leqq t \leqq \tilde{\tau}(c) \\ Y(\tilde{\tau}(c)) + W(t/m) - W(\tilde{\tau}(c)/m), & \text{for } t > \tilde{\tau}(c) . \end{cases}$$

The conjecture is that this Y solves the problem for a suitable choice of c (approximately a^2). If so, this may generalize to martingales on manifolds.

Problem 6.2
Best estimates for exit probabilities for Brownian motion on a manifold, for small t. It seems likely that the estimates for Brownian motion in Sect. 3 can be improved in something like the following way. If radial Ricci curvatures are non-negative (i.e. $K(r(x)) \equiv 0$ in (3.3)), then for each $\varepsilon > 0$ there may exist a constant A, depending on ε, a and m, such that

$$P(\tau(a) \leqq t) \leqq A\sqrt{t}\exp[-a^2/2(t + \varepsilon)], \quad \text{for } t > 0 .$$

Likewise if $K(r(x)) \equiv -c^2 < 0$ in (3.3), then for each $\varepsilon > 0$ there may exist a constant A, depending on ε, a and m, such that

$$P(\tau(a) \leqq t) \leqq A\{\exp[-a^2/2(t + \varepsilon)] + \exp[-a/c(t + \varepsilon)]\}, \quad \text{for } t > 0 .$$

Acknowledgement. The author thanks Olav Kallenberg for providing a preliminary version of Kallenberg and Sztencel (1991), and for pointing out an improvement. The comments and suggestions of D. Stroock and of some participants in the 1991 Haifa Workshop on Probabilistic Methods in D.E. (M. Cranston, M. Emery, E. Hsu, M. Pinsky, and A. Sznitman) are gratefully acknowledged. An anonymous referee helped to correct several significant technical errors.

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