Probab. Th. Rel. Fields 85, 307–364 (1990)

Probability Theory and Related Fields

Symmetric exclusion on random sets and a related problem for random walks in random environment

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Received June 16, 1988

Summary. We study symmetric exclusion on a random set, where the underlying kernel $p(x, y)$ is strictly positive. The random set is generated by Bernoulli experiments with success probability q.

We prove that for certain values of the involved parameters the transport of particles through the system is drastically different from the classical situation on $\mathbb{Z}$. In dimension one and $r := \lim_{|x| \to \infty} |(|x|^{-1} \log p(0, x))| > |\log(1-q)|$ the transport of particles occurs on a nonclassical scale and is (on a macroscopic scale) *not* governed by the heat equation as in the case: $r < |\log(1-q)|$ on a random set, or in the classical situation on $\mathbb{Z}$.

The reason for this behaviour is, that a random walk with jump rates $p(x, y)$ restricted to the random set, converges to Brownian motion in the usual scaling if $r < |\log(1-q)|$ but yields nontrivial limit behaviour only in the scaling $x \to u^{-1}x$, $t \to u^{1+\alpha}t$ $(\alpha > 1)$ if $+\infty > r > |\log(1-q)|$. We calculate α and study the limiting processes for the various scalings for fixed random sets. We shortly discuss the case $r = +\infty$, here in general a great variety of scales yields nontrivial limits.

Finally we discuss the case of a "stationary" random set.

A. Motivation and main results

0. Introduction

In the last years the theory of particle systems with spatially inhomogeneous evolution mechanism has attracted attention for its significance in application and for new interesting phenomena occurring. The interest has focused so far on various types of Branching processes (Dawson and Fleischman [1], Greven [5, 6]) or the contact process (Bramson, Durrett, and Schonmann; Liggett). Here we focus on a different type of question: we are interested in the transport properties of particle systems evolving in an inhomogeneous medium, that is with spatially varying conductivity. This will lead to a problem for a random walk in random environment, which is of interest in itself. We have to extend work of Kawazu-Kesten [8] to more general situations, that is non-nearest neighbour models. Compared with [8] or Sinai's work [15] also some new phenomena occur.

Consider the following evolution on a subset of $\mathbb{Z}^d$: Given is a random set A of accessible sites. We define a Markov process evolving on $\{0,1\}^A$ according to the following rule: Particles move from x to y at an exponential rate given below provided the system is in the state $\eta \in \{0,1\}^A$:

$$(0.1) \qquad (1_A(x)p(x,y)1_A(y))\eta(x)(1-\eta(y)).$$

(Here $\eta(y)=1$ or 0 depending wether site x is occupied or empty.)

The set A is generated by a random mechanism but is then fixed throughout all time. We are always interested in the evolution for given random environment.

In principle there are two main problems of interest:

(A) Suppose the process starts in an inhomogeneous situation for example: one half space occupied, the other one empty. Now analyse the flux of particles on a macroscopic scale, that is: study in a first and main step the rescaled function $u(t,x)=E(\eta_t(x)|A)$.

(B) Take the process in equilibrium, tag a particle and analyse its motion on a macroscopic scale.

We start in this paper the analysis of question (A) for the case where:

$$(0.2) \quad p(x,y)=p(y,x), \quad p(x,y)=p(0,y-x), \quad p(x,y)>0 \quad \forall (x,y)\in \mathbb{Z}^d\times\mathbb{Z}^d$$
$$\sum_y p(0,y)y^2<\infty.$$

In the classical situation where $A=\mathbb{Z}^d$ we have the situation that the path of the tagged particle tends in the usual scaling to Brownian motion and the density of particles $u(t,x)=E(\eta_t(x))$ fulfills in the scaling $x\to\varepsilon^{-1}x$, $t\to\varepsilon^{-2}t$ in the limit $\varepsilon\to 0$ the heat equation:

$$(0.3) \qquad \frac{\partial}{\partial t}\hat{u}(t,x) = -\frac{1}{2}\sigma^2\left(\frac{\partial}{\partial x}\right)^2 \hat{u}(t,x).$$

We shall show in this paper that in dimension $d=1$ and for random sets A generated by Bernoulli experiments with success probability q, we have the classical picture as far as problem (A) is concerned for almost all realizations of the random set if:

$$(0.4) \qquad \overline{\lim_{|x|\to\infty}}\, ((|x|)^{-1}|\log p(0,x)|) < |\log(1-q)|$$

but something drastically different in the case where the relation above is violated. Then the particles move at a slower speed and the motion depends also in the macroscopic picture on the random environment ω defined by A.

The basic tool in studying this problem is the fact that problem (A) leads to a problem for a *random walk on a random set*. To see this make the following observations:

The density $\hat{E}(\eta_t(x))$ of particles at x at time t is given by: $(\hat{E}:=E(\cdot|A))$

$$(0.5) \qquad \hat{E}(\eta_t(x)) = \sum_{y\in A} P_t(x,y)\hat{E}(\eta_0(y)),$$

P_t has generator $1_A(x)(p(x,y)-\delta(x,y))1_A(y)$.

Therefor our problem (A) leads us in the symmetric situation to the study of a random walk on a random set. (For proofs of (0.5) see Liggett [11].) On the other hand due to the duality relations, it suffices in the symmetric case to treat the n-particle problem. The n-particle problem however is straightforward once the one particle problem is solved. We formulate for that reasons our results as results on random walks in random environments. But note that this reduction is valid only in the case of *symmetric* exclusion processes.

Problem (B) is more difficult especially in the subdiffusive situation where quite some additional work has to be done. We focus in this paper solely on problem (A) even though our techniques will be useful for problem (B) too.

The organization of the paper is as follows: In Sect. 1 we formulate and explain our results on a continuous time random walk on a random set which appears in problem (A). In Sect. 2 we show how to reduce our problems to problems for *discrete Markov chains on* $\mathbb{Z}$. Section 3 prepares the important tools of our analysis: we construct certain random harmonic and subharmonic functions $h(x)$, $f(x)$ for our chain and analyse their asymptotic behaviour for $|x|\to\infty$. In Sect. 4 we apply these results to the diffusive case and finally in Sect. 5, 6 to the subdiffusive case. In both cases it is the main point to get control over the behaviour of the Markov chain introduced in Sect. 2. Finally in Sect. 7 we put everything together to prove our theorems. We exploit a point of view on the ergodic theory of Markov chains developed by the author in [4].

The main work has to be done to deal with the fact that the restricted chain can not be reduced to a model with nearest neighbour jumps or a model where jumprates are assigned to the bonds. Models of the last mentioned type have been treated in the literature, see for example Kawazu and Kesten [8] for a rigorous treatment and an extensive list of references especially to the physics literature.

1a. The model

We start by introducing the ingredients we need to define our random walk precisely.

(i) Suppose $\{Z(x)\}_{x\in\mathbb{Z}^d}$ are i.i.d. Bernoulli-variables with:

$$\text{Prob}\,(Z(x)=+1)=q. \tag{1.1}$$

A realization of $(Z(x))_{x\in\mathbb{Z}^d}$ we shall denote with ω.

The random set A on which our process will move is defined as:

$$A=\{x|Z(x)=+1\}. \tag{1.2}$$

(ii) Furthermore we have a Markov transition kernel $p(x,y)$ on $\mathbb{Z}^d\times\mathbb{Z}^d$ with the properties stated in (0.2) (i.e. homogeneous, symmetric, strictly positive transition matrix, finite variance).

Now we are ready to define the process $(X(t))_{t\in\mathbb{R}^+}$ we are interested in as follows:

Definition 1. $(X(t))_{t\in\mathbb{R}^+}$ is a continuous time random walk on $\mathbb{Z}^d$ with generator L defined on $L^\infty(\mathbb{Z}^d)$ as follows:

$$(Lf)_{(x)}=\sum_y (p(x,y)-\delta(x,y))1_{A\times A}(x,y)(f(y)), \quad \text{choose } X(0)=x\in A. \tag{1.3}$$

Our main interest is focused on the behaviour of $X(t)$ as $t\to\infty$ and the question whether we can rescale time and space so that we obtain a limiting process.

Furthermore in which cases do we get Brownian motion as the limiting process? What is the structure of the limiting process in the subdiffusive case? *We organize the results in a such a way that we start with results carrying over to more general situations and then proceed to results (Theor. 3) which depend on the very special form of the model.* In this paper we are concerned with the one dimensional situation. Throughout the paper we assume (0.2) and $d=1$.

1b. The results

The behaviour of the process $X(t)$ can be described very well provided the tails of $p(0,\cdot)$ behave fairly regular. The important requirement for a *detailed* analysis is that the following limit exists:

$$(1.4) \qquad r := \lim_{|x|\to\infty} \left|\left(\frac{1}{|x|}\log p(0,x)\right)\right| \quad (+\infty \text{included}).$$

We denote by $Y_\sigma(t)$ the Brownian motion with variance σ, with $\mathscr{L}(X)$ the law of X.

Theorem 1.

Case 1. $r<|\log(1-q)|$

$$(1.5) \qquad \mathscr{L}\left(\left(\frac{1}{n}X(n^2t)\right)_{t\in\mathbb{R}^+}|\omega\right) \underset{n\to\infty}{\Longrightarrow} \mathscr{L}((Y_\sigma(t))_{t\in\mathbb{R}^+}) \quad \omega\text{-a.s.}$$

with σ positive and independent of ω.

Case 2. $r>|\log(1-q)|$ or: $r=|\log(1-q)|$ *and* $p(0,x)\sim ce^{-r|x|}$

$$(1.6) \qquad \mathscr{L}\left(\left(\frac{1}{n}X(n^2t)\right)_{t\in\mathbb{R}^+}|\omega\right) \underset{n\to\infty}{\Longrightarrow} \delta_{\{Y_t\equiv 0\}} \quad \omega\text{-a.s.} \quad \square$$

Remark. Case 1 and 2 can be distinguished for general $p(x,y)$ fulfilling (0.2) by the criterion: $\sum_{n=1}^{\infty}(1-q)^n(p(0,n))^{-1}<\infty$ or $+\infty$. Studying the second case further, requires *necessarily* regularity assumptions for transparent results. Therefore we focused here already on cases were (1.4) holds.

Remark. For σ one can give a representation (see Coroll. 6 in Sect. 4) but no simple formula.

The Theorem 1 raises of course immediately the question: what is the right rescaling in the second case where the motion is slowed down too much by the gaps in the thinned out random set.

The results do depend very much on the form of the tails of $p(0,\cdot)$ if we are in the *subdiffusive* case. If we want to see something in a deterministic rescaling we have to have subexponential tails of $p(0,\cdot)$. Our techniques work for quite general $p(0,\cdot)$ we focus however here on assumptions which allow transparent results. Very concise results can be given if we assume that the tails behave asymptotically exponential. For convenience only we shall assume in the sequel even:

$$(1.7) \qquad p(0,x)=c\exp(-r|x|), \qquad c=\left(\sum_{x\in Z}p(0,x)\right)^{-1}.$$

A remark on terminology and notation. We call a measure trivial if it is concentrated on the points 0, respectively the process $\equiv 0$. With weak convergence of a sequence we mean convergence as processes if we write the sequence as $\mathscr{L}((X_t)_{t\in\mathbb{R}})$ and weak convergence of the marginals if we write $\mathscr{L}(X_t)$. (Compare [2], Chap. 3.)

Theorem 2. *Assume that* (1.7) *holds.*

a) *For* $\infty > r > |\log(1-q)|$ *we have with* $\alpha = r|(\log(1-q))^{-1}|$:

(1.8) $$\left\{\mathscr{L}\left(\left(\frac{1}{n}X(n^{1+\alpha}t)\right)_{t\in\mathbb{R}^+}\right)\right\}_{n\in\mathbb{N}^+}$$ *is tight and all weak limit points are nontrivial.*

For $r = |\log(1-q)|$ *we have*:

(1.9) $$\left\{\mathscr{L}\left(\left(\frac{1}{n}X((n^2\ln n)t)\right)_{t\in\mathbb{R}^+}\right)\right\}_{t\in\mathbb{N}}$$ *is tight and all weak limit points are nontrivial.*

b) *We obtain in* (1.8) *and* (1.9) *convergent subsequences if n runs through any subsequence* $(n_i(k))_{k\in\mathbb{N}}$ *with the property: (*$[x]$ *denotes the largest integer smaller than x.)*

(1.10) $$(((|\log(1-q)|)^{-1}\log n_i(k)) - [(|\log(1-q)|)^{-1}\log(n_i(k))]) \underset{k\to\infty}{\longrightarrow} i \in [0\ 1).$$

For subsequences with different i we obtain in (1.8) *different limits.* □

Remark. If we replace $n^{1+\alpha}$ by a function $f(n) \ll n^{1+\alpha}$ ($\gg n^{1+\alpha}$) we obtain δ_0 as a limit (respectively the sequence is not tight).

The theorem above tells us roughly how fast the particle can move, but we would like to have some information what happens for *fixed* ω, at least if we exclude ω in a set with small probability. This point of view is analogues to the procedure in Sinai's work [15]. Furthermore we want to know what is the structure of the limit processes. Both questions require a fairly complex analysis and we need to introduce various quantities:

The subdiffusive behaviour is due to the fact that the process needs too much time to cross large gaps (= consecutive points $\notin A$) when r is too big. It is therefore important to store the information about *location and size of the large gaps.* In our situation it turns out that large gaps at the level n of rescaling are gaps of size:

(1.11) $$[(|\log(1-q)|)^{-1}\log(n)].$$

since the expected waiting time for a jump across such a gap is of order n^α and on the other hand the occurrence of larger gaps in $[-n, n]$ has probability tending to 0 as $n\to\infty$.

We define therefore first the following sequence $(\hat{W}^\delta_{n,\omega}(x))_{x\in\mathbb{R}}$ of processes. These processes store all information about large gaps, namely their location and size.

Here is the definition: Denote by $((x^\delta_{i,n}))_{i\in\mathbb{Z}}$ the location of points such that a block not belonging to A follows to the right which has length:

(1.12) $$[(|\log(1-q)|)^{-1}\log n] - a^\delta_i \quad \text{with} \quad -\infty < a^\delta_i \leqq \frac{|\log\delta|}{r} \quad \text{and} \quad \delta \leqq 1.$$

Define ($\alpha = |\log(1-q)|^{-1} r$ as before, $c = (1+e^{-r})(1-e^{-r})^{-1}$ as in (1.7)):

$$(1.13)\quad \hat{W}^{\delta}_{n,\omega}(x) := \begin{cases} n^{\alpha} \sum_{i \in I(x)} c e^{r a_i^{\delta}} & I(x) = \{i \mid x^{\delta}_{i,n} \in [0,x) \cap A\}, \quad x>0 \\ -n^{\alpha} \sum_{i \in I(x)} c e^{r a_i^{\delta}} & I(x) = \{i \mid x^{\delta}_{i,n} \in [x,0) \cap A\}, \quad x \leqq 0. \end{cases}$$

Different from a walk with jumps to nearest neighbours only we have also to take into account that the behaviour of our process will be different in the following two situations:

$$\underset{\text{large gap}}{\cdots (x\,x\,x\,x) \cdots \uparrow} \qquad\qquad \underset{\text{large gap}}{\cdot x \cdot (x\,x\,x\,x) \cdot x \cdot \uparrow}$$

In words: The *local structure* of *A close to the large gap* influences the time the process needs to cross this gap.

We shall show that we can find a jump process $(\tilde{c}_y)_{y \in \mathbb{R}}$ with jumppoints $x_i \subseteq \mathbb{Z}$ and values $\tilde{c}_y \in [\underline{c}, \bar{c}]$ $\underline{c} > 0$, $\bar{c} < \infty$ which describe this effect. In our special situation where $p(0,x) = c e^{-r|x|}$ we can give the following explicit formula:

$$(1.14)\quad \tilde{c}_y = ((1+FG)F)^{-1}, \qquad F = \left(1 + \sum_{k=-1}^{-\infty} e^{rk} y_i\right) \frac{1+e^{-r}}{1-e^{-r}}, \qquad G = \sum_{k=1}^{\infty} e^{-rk} y_i,$$

$$y_k = \begin{cases} 1_{\{k \in A - [y]\}}, & k<0 \\ 1_{\{k \in A - z\}, z = \inf(x \in A \mid x > y)}, & k>0. \end{cases}$$

Remark. A general device to find this process $(\tilde{c}_y)$ is to construct a harmonic function for the jumpchain of our process (which can be uniquely determined by some additional requirements: it will be constructed in Sect. 3) and setting:

$$(1.14')\quad \tilde{c}_y = (h(z) - h(y)) p(y,z) \left(\sum_{x \in A} p(y,x)\right)^{-1} \quad \text{for } y = x_i, \quad \text{where}$$

$$z = \inf(x \in A \mid x > x_i), \qquad \{x_i\}_{i \in \mathbb{Z}} : \{x \in \mathbb{Z} \mid x \in A,\ x+1 \notin A\}.$$

The information *about large gaps and the local structure around them*, is condensed in the following processes:

$$(1.15)\qquad \bar{W}^{\delta}_{n,\omega}(x) = \int_0^x \tilde{c}_y \, d\hat{W}^{\delta}_{n,\omega}(y), \qquad x \in \mathbb{R}.$$

Since in the end we want to study the properly rescaled process we introduce

$$(1.16)\qquad W^{\delta}_{n,\omega}(x) = n^{-\alpha} \bar{W}^{\delta}_{n,\omega}(nx) \quad \alpha = |\log(1-q)|^{-1} r.$$

Now we re ready to introduce the crucial process $Z^{\delta}_n(t,\omega)$ which gives on level n a good approximation for the behaviour of $X(t)$ for given ω and n large (provided ω isn't in a certain exceptional set with small probability and δ is small enough):

$$(1.17)\qquad Z^{\delta}_n(t,\omega) := (W^{\delta}_{n,\omega})^{-1}(Y(V^{-1}_{n,\omega,\delta}(t))),$$

here

$Y(t)$: Brownian motion with variance σ (σ is chosen later).
$V_{n,\omega,\delta}(t) := \int L(t, W^{\delta}_{n,\omega}(x))dx$, $L(t,x)$ is the local time of $Y(t)$.
For h nondecreasing we set $h^{-1}(u) = \inf(t|h(t) > u)$.

Three questions have to be answered: does $Z^{\delta}_n(t,\omega)$ approximate $n^{-1}X(n^{1+\alpha}t)$, does $Z^{\delta}_n(t,\omega)$ converge for $n\to\infty$, $\delta\to 0$ and finally what is the structure of that limit? The answers are given in a), b), c) of the theorem below. The theorem has a form analogues to Sinai's result for a walk in a random potential: (A nicer looking though less informative consequence is formulated in a Corollary below.)

Theorem 3. *Assume that* $p(0,x) = c\exp(-r|x|)$ *and* $\alpha = r(|\log(1-q)|)^{-1} > 1$, *and set* $\hat{E} = E(\cdot|\omega)$.

a) *There exists* $\sigma \in (0,\infty)$ *such that* $\forall \eta > 0$, $\varepsilon > 0$ *and* $f \in \mathscr{C}_0(\mathbb{R})$ *exist* n_0, $\delta_0 > 0$ *such that*:

$$\text{Prob}\left(\omega \left| \left| \hat{E}f\left(\frac{1}{n}X(n^{1+\alpha}t)\right) - \hat{E}f(Z^{\delta}_n(t,\omega))\right| \leqq \varepsilon \right.\right) \geqq 1-\eta \qquad \forall n \geqq n_0, \quad \delta \leqq \delta_0. \tag{1.18}$$

b) *Suppose* $(n_i(k))_{k\in\mathbb{N}}$ *is a sequence such that: ([x] = largest integer smaller x)*

$$((\log(1-q))^{-1}\log n_i(k)) - [(\log(1-q))^{-1}\log n_i(k)] \underset{k\to\infty}{\rightarrow} i \in [0,1). \tag{1.19}$$

Then the approximating processes $(Z^{\delta}_{n_i(k)}(t,\omega))_{t\in\mathbb{R}^+}$ *have the following asymptotics for* $k\to\infty$, $\delta\to 0$:

$$\mathscr{L}((Z^{\delta}_{n_i(k)}(t,\omega))_{t\in\mathbb{R}^+}) \underset{k\to\infty}{\Rightarrow} \mathscr{L}((Z^{\delta}_i(t))_{t\in\mathbb{R}^+}) \quad \forall i \in [0,1), \tag{1.20}$$

$$\mathscr{L}((Z^{\delta}_i(t))_{t\in\mathbb{R}^+}) \underset{\delta\to 0}{\Rightarrow} \mathscr{L}((Z_i(t))_{t\in\mathbb{R}^+}) \quad \forall i \in [0,1). \tag{1.21}$$

c) $(Z_i(t))$, $(Z^{\delta}_i(t))$ *with* $i \in [0,1)$ *have the following structure:*
(α) $Z_i(t)$ *is given by*

$$Z_i(t) = W_i^{-1}(Y(V_i^{-1}(t))), \tag{1.22}$$

where the ingredients are defined as follows:

(i) $Y(t)$ *is Brownian motion with variance* σ *(the same as in a),*
(ii) $V_i(t) = \int L(t, W_i(x))dx$ *with* $L(t,x)$ *the local time of* $Y(t)$.
(iii) $W_i(t) = \int_0^t c_y dY^{\alpha}_i(y)$.

Here the $(Y^{\alpha}_i(y))_{y\in\mathbb{R}}$ *are stable processes with positive increments and with index* α^{-1} *uniquely determined by:*

$$\mathscr{L}\left[\left(cn^{-\alpha}\sum_{x=0}^{[ny]}(p(x,y(x)))^{-1}e(x)1_A(x)\right)_{y\in\mathbb{R}^+}\right] \underset{\substack{n\to\infty \\ m\subseteq N_i}}{\Rightarrow} \mathscr{L}(Y^{\alpha}_i(y))$$

with:

$$y(x) = \inf(z|z > x, z\in A), \qquad e(x) = \sum_{y\in A} p(x,y), \qquad c = \frac{1+e^{-r}}{1-e^{-r}}$$

The $(c_y)_{y\in\mathbb{R}}$ form a jump process with values in $[\underline{c}, \bar{c}]$ and common jump points with $(Y_i^\alpha(y))_{y\in\mathbb{R}}$. If $(y_k)_{k\in I}$ denotes the jumppoints of Y_i^α then $\mathscr{L}((c_{y_k})_{k\in I}|(y_k)_{k\in I})$ is i.i.d.:

$$\mathscr{L}(c_0)=\mathscr{L}\left(\left(1+\left(1+\sum_{k=1}^{\infty} e^{-rk}B_k\right)\frac{1}{c}\left(\sum_{k=-1}^{-\infty} e^{+rk}B_k\right)\right)^{-1}\left(1+\sum_{k=1}^{\infty} e^{-rk}B_k\right)^{-1}\right)$$

with

$$\mathscr{L}((B_i)_{i\in Z})=\bigotimes_{\mathbb{Z}}\mathscr{B}(1,q), \qquad c=(1-e^{-r})^{-1}(1+e^{-r}).$$

(β) *To obtain Z_i^δ replace in* (1.22) *simply Y_i^α by $Y_i^{\alpha,\delta}$ with:*

$$Y_i^{\alpha,\delta}(t)-Y^{\alpha,\delta}(t_-)=Y_i^\alpha(t)-Y_i^\alpha(t_-)1_{\{Y_i^\alpha(t)-Y_i^\alpha(t_-)\geqq\delta\}}. \quad \square$$

Another, though less informative, way to phrase our result above is:

Corollary. *With the assumptions and notations as in Theorem* 3 *we have:*

$$\text{(1.18')} \quad \mathscr{L}\left(\left(\frac{1}{n_i(k)}X(n_i^{1+\alpha}(k)t)\right)_{t\in\mathbb{R}^+}\right)\underset{k\to\infty}{\Rightarrow}\mathscr{L}((Z_i(t))_{t\in\mathbb{R}^+}), \qquad \forall i\in[0,1)$$

$Z_i(t)$ is selfsimilar with index $(1+\alpha)^{-1}$ for all i and has continuous path.

The second line follows from (1.22) with Lemma 3, 4 from [8].

Remark. We can consider $(Z_i(t))_{t\in\mathbb{R}^+}$ again as a process in a random medium $\tilde{\omega}$ defined through a realization of $((Y_i^\alpha(t))_{t\in\mathbb{R}}, (c_y)_{y\in\mathbb{R}})$. This process can be viewed in a sense as an Ornstein-Uhlenbeck process in the random potential

$$U(y)=\int_0^y c_x dY_i^\alpha(x), \qquad \text{that is as solution } X_t \text{ of } dX_t=dY_t-U'(X_t)dt.$$

Then $\mathscr{L}(t^{-1/1+\alpha}X_t)=\mathscr{L}(X_1)$, that is on the macroscopic scale diffusion occurs at speed $t^{1/1+\alpha}$.

Remark. The (c_y) appearing in (1.22) represent the effects of the unboundedness of the range of the jump rate of the walk restricted to A. Otherwise the form of the limit process is similar to the one found in [8].

Of course one could ask now what happens in the case when $r=+\infty$. The behaviour in this case is very complex in the sense that there exist typically *various* scales in which we obtain nontrivial behaviour even for fixed ω but typically these scales are not comparable. We also have *localization.*

As an example take $p(0,x)=c\exp(-x^2)$. Consider the scales (corresponding to gaps $\approx(|\log(1-q)|^{-1}\log n-\beta)$):

$$\text{(1.23)} \qquad f_\beta(n):=n^{b^2\ln n}\cdot n^{-2\beta b}, \quad b=(|\log(1-q)|)^{-1}, \quad \beta\in\mathbb{N}$$

and define $T_n=\inf(t|X_t\notin[-n,n])$. Then we have:

Theorem 4.

$$\text{(1.24)} \qquad \mathscr{L}\left(\frac{1}{n}X(n^{\tilde{\beta}\log n}t)|\omega\right)\underset{n\to\infty}{\Rightarrow}\delta_0 \quad \omega\text{-a.s.}$$

for all $\tilde{\beta}<(|\log(1-q)|)^{-2}=b^2$

(1.25) *For every* $\beta \in \mathbb{N}$:

$$\left\{\mathscr{L}\left(\frac{1}{n} X(n f_\beta(n) t)\right)\right\}_{n \in \mathbb{N}} \text{ is weakly compact with nontrivial weak limit points.}$$

(1.26) *For every* $\beta \in \mathbb{N}$:

$$\left[\sup_{T \in \mathbb{R}^+} \left(\overline{\lim_{n \to \infty}} \left(\sup_{t \leq T} \left|\frac{1}{n} X(n f_\beta(n) t)\right|\right)\right)\right] < \infty \quad \omega\text{-a.s.}$$

(1.27)
$$\mathscr{L}\left(\frac{1}{\log n} \log T_n\right) \underset{n \to \infty}{\Rightarrow} \delta_{b^2}. \quad \square$$

Remark. The effect in (1.25), (1.26) above is simply that the walk is trapped between gaps of size $(|\log(1-q)|)^{-1} \log n - \beta^i$ $(i=1,2)$ with $\beta^i < \beta$.

Remark. This result says especially that it is in the case $r = +\infty$ not possible to find a scale independent of the realization of the medium, which still gives a reasonable picture of the motion of the particle. The motion is on different scales in different parts of space, and the various parts have random extensions. So we have in fact three regions in parameterspace with qualitative very different behaviour of the random walk: $r < |\log(1-q)|$, $\infty > r \geqq |\log(1-q)|$, $r = +\infty$.

1c. Outlook

We conclude with some remarks on a more general form of the random medium. Consider a stationary ergodic process of the form $(T_i^1, T_i^2)_{i \in \mathbb{Z}}$ with T_i^1, T_i^2,$\mathbb{N}$-valued random variables. With T_i^1 we describe the length of strings of points belonging to A and with T_i^2 the length of strings not belonging to A. Assume that at 0 a string belonging to A starts. If we have $E_\omega((p(0, T_0^2+1))^{-1}) < \infty$, then the random walk on A behaves diffusive ω-a.s. and there is no difficulty to adapt our methods of proof. The behaviour in the subdiffusive situation can be described via a ω-independent rescaling essentially only in the case where the $(T_i^2)_{i \in \mathbb{Z}}$ are independent and the Prob $(T_0^{(2)} \geqq n)$ behaves fairly regular. We don't have the space to go into detail, however the methods we develop in this paper suffice to treat the general model and the reader having a special situation in mind won't have any problem in working out this case along the line of our arguments. Note however that the limiting processes in the subdiffusive case *do* depend on the special form of $p(0, \cdot)$ and $\mathscr{L}(T_0^{(2)})$! We shall work out some cases in a forthcoming article.

The case of higher dimensions shows different features and our methods have to be refined *substantially*, we cannot discuss details here.

B. Basic tools

Let us shortly summarize the idea behind our approach. In order to study $X(t)$ we should investigate the following random times: Consider an interval I and define $T_x(I) =$ exit time from I, when the process $X(t)$ starts in x. Controlling the behaviour of these random variables should be the core of the problem. In order to study $T_x(I)$ we need to know at least three things: What are the probabilities to leave the interval

I to the left or the right, how often do we visit a point $y \in I$ befor leaving I and how much time do we spend in the average in a point.

The first two properties are properties of the jump chain only. Our first idea is that we can *separate* the analysis of the jump chain from the question how long it takes to make n-jumps as $n \to \infty$ (Sect. 2a). Our second idea is to control the number of points visited befor $T_x(I)$ and the exit probabilities to the right, left with (random) *subharmonic* respectively *harmonic functions* and this way turn our problem into a potential theoretic one, namely to estimate these (sub)harmonic functions (Sect. 2b, 3). The chapter B turns these ideas into mathematics.

2. *The embedded jump chain and the resistance between points*

This section especially part b) is basic for the rest of the paper, it introduces the important discrete time chain $(\hat{X}_n)_{n \in \mathbb{N}}$ on $\mathbb{Z}$. The first purpose of this section is to prove that it suffices to study a discrete time Markov chain on A, the jumpchain. Next we relabel points in A by $\mathbb{Z}$ in order to obtain a new random Markov kernel $\hat{p}(x,y)$ on $\mathbb{Z}$. For this kernel we define the notion of resistance and derive basic properties of that quantity which is crucial in the study of $(\hat{X}_n)_{n \in \mathbb{N}}$, as $n \to \infty$.

a) Reduction to a discret time problem: the embedded jump chain. The first step towards proving our theorems consists in reducing our problem to a discrete time problem.

Definition 2. $(X_n)_{n \in \mathbb{N}}$ is the jump chain belonging to $(X(t))_{t \in \mathbb{R}^+}$. This chain has transition kernel $\bar{p}(x,y)$: (on $A \times A$)

$$(2.1) \qquad \bar{p}(x,y) := (p(x,y) 1_{A \times A}(x,y))(e(x))^{-1}, \qquad e(x) := \sum_{y \in A} p(x,y).$$

We define a measure $\Pi(\cdot)$ with support on A by setting:

$$(2.2) \qquad \Pi(B) = \sum_{x \in B} e(x) \quad \forall B \subseteq A, \qquad \Pi(z) = 0 \quad \text{for } z \notin A. \quad \square$$

The following proposition tells us in (2.4) that it suffices for our purposes to study $(X_n)_{n \in \mathbb{N}}$ instead of $(X(t))_{t \in \mathbb{R}^+}$ and that the chain $(X_n)_{n \in \mathbb{N}}$ is reversible with respect to Π.

Proposition 1. *For every kernel* $p(x,y)$ *on* $\mathbb{Z}^d$ *which is symmetric and strictly positive for* $x \neq y$ *the following holds for the jump chain of the walk restricted to A (as defined in* (2.1)*):*

a) Π *is a reversible invariant measure of* $\bar{p}$ *for every A.*

b)

(i) *For every realization of* A:

$$(2.3) \qquad \mathcal{L}(X(t)) = \mathcal{L}(X_{N(t)}); \qquad N(t) = \inf\left(n \,\Big|\, \sum_{1}^{n+1} T_i \geqq t\right),$$

where conditioned on $(X_i)_{i \in \mathbb{N}}$ *the* $(T_i)_{i \in \mathbb{N}}$ *are independent with* $\mathcal{L}(T_{i+1}) = \exp(e(X_i))$.

(ii) *Consider now for given* ω *the normed sum of the jump times* T_i *belonging to* $(X(t))_{t \in \mathbb{R}^+}$:

$$(2.4) \qquad n^{-1}\left(\sum_{i=1}^{n} T_i\right) \underset{n \to \infty}{\overset{\text{a.s}}{\longrightarrow}} (E_\omega(e(0)))^{-1} \quad \omega\text{-a.s.}$$

Proof of Proposition 1. a) For each $x, y \in A$ we check the detailed balance conditions as follows:

$$e(x)\bar{p}(x,y) = e(x)p(x,y)(e(x))^{-1} = p(x,y) = p(y,x) = e(y)p(y,x)(e(y))^{-1}$$
$$= e(y)\bar{p}(y,x).$$

b) The part (i) is of course an immediate consequence of the Definition of $e(x)$. More interesting is part (ii). Here we shall adapt an idea of Papanicolaou and Varadhan in [14] for our purposes, that is: study the medium as seen from the walker, use for this process Birkhoffs ergodic theorem and then derive the consequences for the problem in question. The details of that program require some work:

Step 1. Note first that conditioned on ω and $(X_i)_{i \in N}$, the $(T_i)_{i \in N}$ are independent and exponentially distributed with $E(T_{i+1}|\omega, (X_i)_{i \in N}) = (e(X_i))^{-1}$.

Therefore we shall study first the quantity $\sum_{i=1}^{n} (e(X_i))^{-1}$ and prove that:

$$\frac{1}{n} \sum_{i=1}^{n} (e(X_i))^{-1} \underset{n \to \infty}{\longrightarrow} (E_\omega(e(0)))^{-1} \quad \text{a.s.} \tag{2.5}$$

For that purpose we consider the Markov process χ_k on $\{0,1\}^{\mathbb{Z}^d}$, which is defined on the joint probability space generated by $\omega = \{Z(x)\}_{x \in \mathbb{Z}^d}$ and $(X_n)_{n \in \mathbb{N}}$, through the following relation: (χ_k is the medium seen from the walker at step k)

$$\chi_n := \begin{cases} +1 & \text{on } A - X_n \\ 0 & \text{elsewhere} \end{cases} \tag{2.6}$$

$$a(j, \varphi) := \left(\sum_k p(j,k) 1_{\{\varphi(k) = +1\}} \right)^{-1}; \quad j \in \mathbb{Z}^d, \quad \varphi \in \{0,1\}^{\mathbb{Z}^d}.$$

With this definition we can write: (using the homogeneity of $p(x,y)$)

$$\frac{1}{n} \sum_{k=1}^{n} (e(X_k))^{-1} = \frac{1}{n} \sum_{k=1}^{n} a(0, \chi_k). \tag{2.7}$$

Suppose now that we could find a measure μ on $\{0,1\}^{\mathbb{Z}^d}$ such that: (Such a μ is automatically unique!)

(2.8) μ is invariant measure of the Markov process χ.

μ is equivalent to $\bigotimes_{x \in \mathbb{Z}^d} \mathscr{B}_q$ and shift-ergodic.

$(\mathscr{B}_q := q\delta_1 + (1-q)\delta_0)$.

In this situation the ergodic theorem for stationary processes would imply that $(a(0, \cdot) \geqq 0!)$

$$\frac{1}{n} \sum_{k=0}^{n} a(0, \chi_k) \underset{n \to \infty}{\longrightarrow} E^\mu(a(0, \cdot)) \quad \left(\bigotimes_{x \in \mathbb{Z}^d} \mathscr{B}_q \right)\text{-a.s.} \tag{2.9}$$

Assuming (2.8) we could conclude the proof as follows:

We can assume without loss of generality that for the jumprates $p(x,y)$ we have: $p(x,x) \geqq \delta > 0$! Note that given $(e(X_i))$ the $T_i(X_i)$ are independent and exponentially distributed, therefore by conditioning on $\{e(X_i)\}_{i=1,\dots,n}$ a straightforward calculation using an extended law of large numbers [3] page 243, shows that (2.9), (2.7) imply

$$(2.10) \qquad \frac{1}{n}\sum_{i=1}^{n} T_i \underset{n\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} E^{\mu}(a(0,\cdot)) \text{ for fixed } \omega, \qquad \omega\text{-a.s.}$$

Step 2. Now it remains to construct μ and to evaluate $E^{\mu}(a(0,\cdot))$.

The measure μ will be obtained as a weak limit of a sequence μ_N of probability measures with support on periodic continuations of elements in $\{0,1\}^{E_N}$ with $E_N=(-N,N]^d$. In order to define μ_N we shall need the kernels $\bar{p}_N(x,y)$ obtained from the walk induced by $\bar{p}(x,y)$ on $A\cap E_N$ as follows: Denote by $A_N=\{y|y \bmod (2N)\in A\cap E_N\}$ and define $\bar{p}_N(x,y)$ as the walk $\bar{p}(x,y)$ restricted to A_N as in (2.1) and then projected on the restclasses mod $(2N)$.

$$(2.11) \qquad e_N(x):=\sum_{y\in A_N-x}\bar{p}(x,y)1_A(x), \qquad S_N:=\sum_{x\in E_N\cap A} e_N(x),$$

$$(2.12) \qquad \Pi_N(B):=\left(\sum_{x\in B} e_N(x)\right)(S_N)^{-1}; \qquad B\subseteq A\cap E_N.$$

Note that Π_N is a reversible invariant probability measure for $\bar{p}_N$.

Let $(\bar{X}_k^{(N)})_{k\in\mathbb{N}}$ be the stationary Markov process on $E_N\cap A$ with marginal Π_N and transition kernel $\bar{p}_N(\cdot,\cdot)$. Now define μ_N as follows: (Denote by τ_x the shift by $x\in E_N$ and by $\omega_N=\{Z(x)\}_{x\in E_N}$, respectively by $\tilde{\omega}_N$ the periodic continuation of ω_N to a $\{0,1\}$-valued function on $\mathbb{Z}^d$).

$$(2.13) \qquad \mu_N:=\mathscr{L}(\tau_{\bar{X}_k^{(N)}}\omega_N)\ \forall k\in\mathbb{N}, \qquad \tilde{\mu}_N(\omega):=\mu_N((\omega|_{E_N}))1_{\{\omega=(\tilde{\omega}|_{E_N})\}}.$$

The next observation is now that (the details of the proof are straightforward and left to the reader):

$$(2.14) \qquad e_n(x)\underset{n\to\infty}{\longrightarrow} e(x)\ \omega\text{-a.s.}; \qquad \sup_{|x|\leqq an}|e(x)-e_n(x)|\underset{n\to\infty}{\longrightarrow} 0\ \omega\text{-a.s. } \forall a<1$$

$$S_n(2n)^{-d}\underset{n\to\infty}{\longrightarrow} E_\omega(e(0))\ \omega\text{-a.s.}$$

Let $b(\cdot)$ be a bounded and local functional of χ. Then we calculate using (2.12) and (2.14):

$$(2.15) \qquad E^{\tilde{\mu}_n}(b)=E_\omega\sum_{x\in E_n}\frac{e_n(x)}{S_n}b(\tau_x\tilde{\omega}_n)=E_\omega\left(\frac{S_n}{n^d}\right)^{-1}\left(\frac{1}{n^d}\sum_{x\in E_n}e_n(x)b(\tau_x\tilde{\omega}_n)\right)$$

$$\sim E_\omega\left(\frac{S_n}{n^d}\right)^{-1}\left(\frac{1}{n^d}\sum_{x\in E_n}e(x)b(\tau_x\omega)\right).$$

Suppose we can show that then the first factor is uniformly integrable which will be done in step 4. A minute's thought shows that then the relation above implies with the second part of (2.14) and the L^1-ergodic theorem for stationary fields indexed by $\mathbb{Z}^d$, that we can define μ as follows:

$$(2.16) \qquad \tilde{\mu}_n\underset{n\to\infty}{\Longrightarrow}\mu, \qquad \mu \text{ is translation invariant, } \mu \text{ is } \chi\text{-invariant}.$$

Since (according to (2.14)) $S_n((2n)^d E_\omega(e(0)))^{-1}\underset{n\to\infty}{\longrightarrow} 1$ ω-a.s., we have especially:

$$(2.17) \qquad E^{\mu}(a(0,\cdot))=(E_\omega(e(0)))^{-1}.$$

Step 3. It remains to show that μ is shift-ergodic and equivalent to $\bigotimes_{x\in\mathbb{Z}^d}\mathscr{B}_q$, this measure we abbreviate by ν. The relation $\mu\triangleq\nu$ implies of course that μ is shift ergodic.

We show first that $\mu\ll\nu$. First observe the following facts: $\bar{p}(n,x,\cdot)$, the n-th power of $\bar{p}$, is equivalent to counting measure, $e_N(x)\leqq 1$, and $S_n((2n)^d E_\omega(e(0)))^{-1}\to 1$. Therefore looking at (2.12) shows us that, we have the following estimate for the density Φ_N of Π_N with respect to the normalised counting measure on E_N:

Define $c_n(\omega)=S_n(2n)^{-d}$, then:

$$\Phi_N^{(\omega)}\leqq\frac{1}{c_N(\omega)},\qquad c_N(\omega)\underset{N\to\infty}{\longrightarrow}E_\omega(e(0))\quad\bigotimes_{x\in\mathbb{Z}^d}\mathscr{B}_q\text{-a.s.}\tag{2.18}$$

We shall prove in step 4 that

$$\int\frac{1}{c_N(\omega)}\,d\nu\underset{N\to\infty}{\longrightarrow}\frac{1}{E_\omega(e(0))}\,\nu.\tag{2.19}$$

This implies with (2.13) and the second part of (2.18) that: ($\tilde{}$ as i (2.13))

$$\int\left|\frac{d\tilde{\mu}_N}{d\tilde{\nu}_N}\right|\wedge m\,d\tilde{\nu}_N\underset{N\to\infty}{\longrightarrow}1\quad\text{for some } m\in\mathbb{R}^+,\qquad \nu_N=\bigotimes_{x\in E_N}\mathscr{B}_q.\tag{2.20}$$

Since a ν-nullset can be approximated by a set M with $\nu(M)\leqq\varepsilon$ and such that M depends only on a finite number of sites and furthermore $\tilde{\mu}_N(M)\to\mu(M)$, we can conclude from (2.20):

$$\mu\ll\nu.\tag{2.21}$$

In order to obtain the relation $\nu\ll\mu$ we use the fact that $\bar{p}^N(n,x,\cdot)\triangleq$counting measure on E_N for every $n\in\mathbb{N}$. The argument then is word by word the same as in Papanicolaou, Varadhan [14], p. 551, we refer the reader to that paper for details.

Step 4. Now we prove (2.19). Our chain is derived as jumpchain of a continuous time process, we are therefore allowed to assume w.l.o.g. that: $p(x,x)\geqq\delta>0$. This implies especially that $\bar{p}^N(x,x)\geqq\delta$ for all $N\in\mathbb{N}$. Therefore we have the estimate

$$c_N(\omega)(2N)^d=\sum_{x\in E_N}e_N(x)\geqq\delta|A\cap E_N|.\tag{2.22}$$

Note that we are only interested in those ω where $|A\cap E_N|>0$. Observe furthermore that for those ω:

$$\frac{1}{c_N(\omega)}\leqq(2N)^d\delta^{-1}:=\tilde{N}\tag{2.23}$$

Now consider the following two events: ($\varepsilon>0,\varepsilon<q$)

$$\begin{aligned}C_1&:=\{\omega|\quad |A\cap E_N|\geqq\varepsilon\cdot(2N)^d\}\\ C_2&:=\{\omega|0<|A\cap E_N|<\varepsilon\cdot(2N)^d\}\end{aligned}\tag{2.24}$$

and estimate with (2.22) and (2.23) as follows:

$$E_\omega\left(\frac{1}{c_N(\omega)}\right)\leqq E_\omega(\tilde{N}1_{C_2})+E_\omega\left(\frac{1}{c_N(\omega)}\,1_{C_1}\right).\tag{2.25}$$

Since we have by (2.18) and (2.22):

$$(2.26)\quad E_\omega\left(\frac{1}{c_N(\omega)}1_{C_1}\right)\underset{N\to\infty}{\to}\left(E_\omega(e(0))\right)^{-1}, \qquad (\mathrm{Prob}(C_1)\underset{N\to\infty}{\to} 1 \quad \text{for} \quad \varepsilon<q!)$$

and by the large deviation principle for Bernoulli-variables:

$$(2.27)\quad \mathrm{Prob}_\omega(|A\cap E_N|<\varepsilon\cdot(2N)^d)\leqq e^{-\delta(\varepsilon)N^d} \quad \text{with} \quad \delta(\varepsilon)>0 \quad \text{for} \quad \varepsilon<q$$

we obtain from (2.25) immediately the assertion (2.19) with the Lemma of Fatou (by observing that we can make ε arbitrarily small, so especially smaller than q).

b) The induced chain on $\mathbb{Z}$. In order to study the chain (X_n) it is convenient to relabel the points in A by $\mathbb{Z}^d$, so that we obtain a new chain $(\hat{X}_n)$ on $\mathbb{Z}^d$. This chain will be the main object for our analysis in the next chapters. We focus on $d=1$ from now on.

Definition 3. $(\hat{X}_n)$ is a Markov chain on $\mathbb{Z}$ with transition kernel $\hat{p}(x,y)$ with

$$(2.28)\qquad \hat{p}(x,y):=\bar{p}(j,k),$$

where

$$\begin{aligned} j&:=\inf(l\,|\,\#\{m\in A, 0\leqq m\leqq l\}\geqq x)\\ k&:=\inf(l\,|\,\#\{m\in A, 0\leqq m\leqq l\}\geqq y)\end{aligned}\qquad \text{for } x,y\geqq 0$$

similar for $x,y\leqq 0$. □

The qualitative behaviour of $(\hat{X}_n)_{n\in\mathbb{N}}$ is, in the case of dimension 1, determined to a large extent by a quantity $c_{x,y}$ which we call (stressing the metaphor somewhat) the resistance between x and y.

$$(2.29)\qquad c_{x,y}:=\sum_{x\leqq Z\leqq y-1}\frac{1}{\hat{p}(Z,Z+1)}; \qquad y>x.$$

In the case $y<x$ we replace $\hat{p}(z,z+1)$ by $\hat{p}(z,z-1)$.

In the next step we investigate the behaviour of the resistance $c_{x,y}$ for $|y-x|\to\infty$.

Proposition 2. *Suppose* $d=1$ *and* $p(x,y)$ *fulfills* (0.2).

a) Case 1. $E_\omega\left(\frac{1}{\hat{p}(0,1)}\right)<+\infty$.

Then:

$$(2.30)\qquad \frac{1}{2|y|}c_{-y,y}\underset{y\to\infty}{\to}E_\omega\left(\frac{1}{\hat{p}(0,1)}\right) \qquad \omega\text{-a.s.}$$

Case 2. If $E_\omega\left(\frac{1}{\hat{p}(0,1)}\right)=+\infty$, *we consider the case where* $\lim_{x\to\infty}\left(\frac{1}{|x|}|\log p(0,x)|\right)$ $=r\in(|\log(1-q)|,\infty)$.

Define $\beta=|\log(1-q)|r^{-1}$. *Then we have*

$$(2.31)\qquad \overline{\lim_{|y|\to\infty}}\left(\frac{1}{(2|y|)^{1/\gamma}}c_{-y,y}\right)=\begin{cases}+\infty & \gamma>\beta\\ 0 & \gamma<\beta\end{cases}$$

b) *If we have* $p(0,x)\underset{|x|\to\infty}{\sim}ce^{-r|x|}$ *for some* r *with* $r>|\log(1-q)|$ *then:*

$$(2.32)\qquad \left\{\mathscr{L}\left(\frac{1}{(2|y|)^{1/\beta}}c_{-y,y}\right)\right\}_{y\in N}$$ *is weakly relative compact. The weak limit points are stable laws on* $\mathbb{R}^+$ *with index* β.

Furthermore we have as criterion for convergence along a subsequence:

(2.33)
$$\mathscr{L}\left(\frac{1}{|2y_k|^{1/\beta}}\, c_{-y_k,y_k}\right) \underset{k\to\infty}{\Rightarrow} \textit{stable law with index } \beta$$
$$\Leftrightarrow ((\log y_k) - [\log y_k]) \textit{ converges for } k\to\infty$$

For every possible limit of $(\log y_k - [\log y_k])_{k\in\mathbb{N}}$ *we obtain a different weak limit point for the law of the resistance along that subsequence. (They differ by a scalefactor only, of course).*

Proof. a) In case 1 we apply simply the strong law of large numbers for i.i.d. integrable random variables. Note that in this case a stationary ergodic random set would give the same result! We treat case 2 under point b).

b) The assumption on $p(0,x)$ implies the following relation for $\hat{p}(\cdot,\cdot)$:

(2.34)
$$\mathrm{Prob}_\omega\left(\frac{c}{\hat{p}(0,1)} \geqq e^{rn}\right) = (1-q)^{n-1} = e^{-n|\log(1-q)|}(1-q)^{-1}$$
$$= \frac{1}{(e^{rn})^\beta}\,(1-q)^{-1}$$

The assertion (2.31) follows now from Stout [17], p. 130–132 by straightforward arguments.

Now note that for x between n and $n+1$ the quantity $(e^{rx})^{-\beta}$ varies by a factor between 1 and $e^{r\beta}$. That means that the Laplacetransform $F(s)$ of $\mathscr{L}\left(\frac{1}{\hat{p}(0,1)}\right)$ obeys according to (2.34) above (see [3]):

(2.35)
$$0 < c \leqq \underset{s\to 0}{\underline{\lim}}\left(\frac{1-F(s)}{s^\beta}\right) \leqq \underset{s\to 0}{\overline{\lim}}\left(\frac{1-F(s)}{s^\beta}\right) \leqq \bar{c} < \infty$$

which immediately implies

(2.36)
$$0 < e^{-\frac{\lambda^\beta}{\bar{c}}} \leqq \underset{n\to\infty}{\underline{\lim}}\,(F(\lambda n^{-1/\beta}))^n \leqq \underset{n\to\infty}{\overline{\lim}}\,(F(\lambda n^{-1/\beta}))^n \leqq e^{-\frac{\lambda^\beta}{c}}.$$

The standard continuity theorem for Laplace transforms yields now that the sequence $\mathscr{L}\left(\frac{1}{2|y|^{1/\beta}}\, c_{-y,y}\right)$ is weakly compact with nontrivial weak limit points.

To proceed further note that (2.34) implies for a sequence $y_k \subseteq \mathbb{N}$ with: $[\log y_k] - \log y_k$ converges as $k\to\infty$, that:

(2.37)
$$\mathrm{Prob}_\omega\left(\frac{C}{\hat{p}(0,1)} \geqq \exp(r\log y_k)\right) \underset{k\to\infty}{\sim} \mathrm{const}\,(\exp(r\log y_k))^{-\beta}.$$

An explicit, but tedious calculation shows now that this implies for the Laplace transform $F(s)$ of $\mathscr{L}\left(\frac{1}{\hat{p}(0,1)}\right)$ that:

(2.38)
$$1 - F(s y_k^{-1/\beta}) \underset{k\to\infty}{\sim} \mathrm{const}\, s^\beta y_k^{-1}.$$

It is then a standard calculation to show that (compare [3]):

$$(2.39)\qquad \mathscr{L}\left(\frac{1}{|2y_k|^{1/\beta}}\, c_{-y_k,y_k}\right) \underset{k\to\infty}{\Longrightarrow} \text{stable law with index } \beta .$$

This finishes the proof of (2.33) and (2.32).

3. A harmonic function and an associated martingale

The basic idea of our approach to Theorem 1, 2 is to use martingale central limit theorems, and potential theoretic arguments in order to study $\mathscr{L}(n^{-\gamma}\hat{X}_n)$ and exit times from intervals as $n\to\infty$.

For that purpose we shall first construct a harmonic function h for the kernel $\hat{p}(x,y)$ (unbounded of course!), so that $Y_n := h(\hat{X}_n)$ is a martingale. The next step will be to analyse $h(x)$ as $|x|\to\infty$ in order to be able to use information about Y_n to conclude something about $\hat{X}_n$. Since the ideas behind these constructions work in more general situations, we state and proof first general results: in Sect. a) existence of a *suitable* h, in b) asymptotic properties of h and then we show in Sect. c) how they apply in our situation. Essential is the proper use of the potential theory for discrete time Markov chains.

a) Construction of an unbounded harmonic function for certain Markov chains. We start by proving a crucial fact about Markov chains on $\mathbb{Z}$ (our Prop. 3). Here we use many ideas from [4] and [12]. We shall use especially the fact that a recurrent chain with kernel P on $\mathbb{Z}$ has up to multiplicative constants a unique σ-finite positive invariant measure Π and the equation $(\cdot)(I-P)=\delta_x-\delta_y$ has a solution bounded by multiples of Π and all solutions of that kind differ only by a multiple of Π (see for example [4], Theor. 2). Furthermore we shall need the potential kernel $K(x,y)$ constructed in [12].

Consider a Markov kernel P on $\mathbb{Z}$ with the following four properties:

(i) P is recurrent and irreducible. Fix an positive σ-finite invariant measure Π.

(ii) Let η be a solution of the Poissonequation $(\cdot)(I-P)=\delta_x-\delta_y(x>y)$, which is bounded by a multiple of the invariant measure Π of the chain.

Now we require that for each pair x,y exist $a^+,a^-\in\mathbb{R}$ such that:

$$(3.1)\qquad \lim_{z\to+\infty}(\eta(z)-a^+\Pi(z))=0 \qquad \lim_{z\to-\infty}(\eta(z)-a^-\Pi(z))=0 .$$

We shall denote by $\hat{a}^+_{x,y}, \hat{a}^-_{x,y}$ the numbers defined by (3.1), when η is the minimal positive solution of $(\cdot)(I-P)=\delta_x-\delta_y$ and by $a^+_{x,y}, a^-_{x,y}$ the equivalent numbers if η is choosen as $(\delta_x-\delta_y)K$ where K is a potential kernel of P in the sense of [12]. Note that $a^+_{x,y}-a^-_{x,y}$ is independent of the choice of K and equal to $\hat{a}^+_{x,y}-\hat{a}^-_{x,y}$.

(iii) $\left(c_{x,y} \text{ denotes the resistance between } x \text{ and } y\text{: } \sum_{x\leqq Z\leqq y-1}\frac{1}{P(Z,Z+1)}\right)$

$$(3.2)\qquad \sum_y P(x,y)c_{x,y}<\infty \quad \forall x\in\mathbb{Z}, \qquad \sum_y P(x,y)|y|<\infty \quad \forall x\in\mathbb{Z}$$

(iv) The invariant measure Π of the chain P is bounded by a positive multiple of the counting measure from above and from below.

Remark. Note $a^{+;-}_{x,y}$ depend on the choice of Π!

Proposition 3. *If P is a Markov transition kernel on $\mathbb{Z}$ which fulfills* (i)–(iv) *above then there exists a (unique) function $h:\mathbb{Z}\to\mathbb{R}$ such that for a given choice of Π:*

(3.3) $(\alpha)\ P^*(h)=h$

$(\beta)\ h(0)=0$

$(\gamma)\ h(x+1)-h(x)=(a^+_{x+1,x}-a^-_{x+1,x})=(\hat{a}^+_{x+1,x}-\hat{a}^-_{x+1,x})$.

Remark. The point of this construction or choice of h is the representation (γ)! It is based on the "renewal property" (ii). We do not use (γ) as definition since the h we construct exist also in cases where (ii) does not hold, so that we have $\alpha)$ $\beta)$ but not $\gamma)$!

Remark. We have not yet excluded that $h\equiv 0$ that is $a^+_{x+1,x}-a^-_{x+1,x}=0$ for all $x\in\mathbb{Z}$. We will show in Proposition 4, 5 that for our application $h\neq 0$.

Our method of proof allows to show with minor modification the existence of certain subharmonic functions, which also allow a representation in terms of $a^{\pm}_{x,y}$. We shall only prove Proposition 3 in detail.

Proposition 3′. *Suppose* (i)–(iv) *of Proposition 3 are fulfilled. We can construct a function $f\geqq 0$ such that:*

(3.4) $(P^*-I)(f)=2\cdot 1_{\{0\}}$ (f *is subharmonic*)

$f(0)=0$,

(3.5) $f(x)=(\hat{a}^+_{x,0}+\hat{a}^-_{x,0})c \quad f(x)=c(a^+_{x,0}+a^-_{x,0}-2(K(x,0)-K(0,0)))$

$f(x+1)-f(x)=c(a^+_{x+1,x}+a^-_{x+1,x}-2(K(x+1,0)-K(x,0))c^{-1})$

$(f(x+1)-f(x)) \underset{x\to+\infty}{\sim} c(a^+_{x+1,x}-a^-_{x+1,x})=(\hat{a}^+_{x+1,x}-\hat{a}^-_{x+1,x})c$

$f(x-1)-f(x) \underset{x\to-\infty}{\sim} c(a^-_{x-1,x}-a^+_{x-1,x})=(\hat{a}^-_{x-1,x}-\hat{a}^+_{x-1,x})c$

$c=\Pi(0)$.

Furthermore we can write f in the form:

(3.6) $f=f^+ + f^- \quad f^{+,-}\geqq 0$

$(P^*-I)(f^{+-})=1_{\{0\}}$

$f^+(x)=\hat{a}^+_{x,0}c \quad f^-(x)=\hat{a}^-_{x,0}c$.

Proof of Proposition 3. First we shall define a sequence of functions $(h_n)_{n\in\mathbb{N}}$, harmonic except at the two points $-n, n$. And later we shall show that this sequence has a limit h which has the desired properties. Step 1 introduces h_n, step 2 derives properties, step 3 yields h and step 4 finally shows that h is harmonic.

Step 1. The sequence h_n is uniquely defined through the following properties:

(3.7) $P^*(h_n)-h_n=\bar{\bar{1}}_{\{-n\}}-\bar{\bar{1}}_{\{n\}} \quad \bar{\bar{1}}_{\{a\}}=1_{\{a\}}\cdot(\Pi(a))^{-1}$

$h_n(0)=0$

$h_n\in L^\infty(\mathbb{Z})$.

To see this define a Markov transition kernel $\bar{P}(y,x)=P(x,y)\,\frac{\Pi(x)}{\Pi(y)}$, where Π is a σ-finite positive invariant measure of P (which exists due to the fact that $P(\cdot,\cdot)$ is recurrent, see for example [4]).

This kernel $\bar{P}$ is recurrent (easily checked using (iv) and the Lemma of Harris quoted in [4]) and the same measure Π is an invariant measure. Define for given function f a new function $\bar{f}(x)=f(x)\,(\Pi(x))$. Then we can rewrite (3.7) as follows: (use (iv)!)

$$(\bar{h}_n)\bar{P}-\bar{h}_n=\delta_{\{-n\}}-\delta_{\{n\}} \tag{3.8}$$
$$\bar{h}_n(0)=0$$
$$\bar{h}_n d\Pi \leqq C\cdot\text{counting measure}.$$

Now standard theorems about solutions of the Poisson equation yield the assertion that h_n is determined by (3.7). (See for example Greven [4], Neveu [10]).

Remark. In the same fashion we can turn a Poissonequation with P for a measure $\eta:(\eta)(I-P)=\delta_x-\delta_y$ into one for functions and for $\bar{P}$:

$$(I-\bar{\mathrm{P}})(\bar{\bar{\eta}})=\bar{1}_{\{x\}}-\bar{1}_{\{y\}},\qquad \bar{\bar{f}}(\cdot)=f(\cdot)(\Pi(\cdot))^{-1},\qquad \bar{f}(\cdot)=f(\cdot)\Pi(\cdot). \tag{3.8'}$$

This fact will be used later frequently.

Step 2. Now we shall derive a bound on $\sup_n\,[|h_n(x)|]$ and a representation for $h_n(x)$.

For this purpose we introduce first some notation: Let η be the minimal positive solution of the equation $(\cdot)(I-P)=\delta_x-\delta_0$. There exists a number b_x such that: (see [4]).

$$\eta\leqq b_x\cdot\bar{\Pi} \tag{3.9}$$

where Π denotes the invariant measure normalized with $\bar{\Pi}(\{0\})=1$. We shall see below that this allows us to estimate $|h_n(x)|$ as follows:

$$|h_n(x)|\leqq 2b_x(\bar{\Pi}(n)\vee\bar{\Pi}(-n))\left(\inf_{x\in\mathbb{Z}}(\bar{\Pi}(x))^{-1}\right). \tag{3.10}$$

To prove this we calculate as follows: (The forth equality uses a basic property of potentials see [12], p. 109).

$$\begin{aligned}h_n(x)&=h_n(x)-h_n(0)=\langle h_n,\delta_x-\delta_0\rangle=\langle h_n,-\eta P+\eta\rangle\\&=-\langle P^*(h_n)-h_n,\eta\rangle=\langle\bar{\bar{1}}_{\{n\}}-\bar{\bar{1}}_{\{-n\}},\eta\rangle\\&=\eta(\{n\})(\Pi(n))^{-1}-\eta(\{-n\})(\Pi(-n))^{-1}.\end{aligned} \tag{3.11}$$

So that by (3.9) and by $\Pi(x)\geqq C^{-1}>0$ we can conclude from (3.11):

$$\begin{aligned}|h_n(x)|&\leqq C(\eta(\{n\})+\eta(\{-n\}))\leqq Cb_x(\Pi(n)+\Pi(-n))\\&\leqq 2b_xC(\Pi(n)\vee\Pi(-n)).\end{aligned} \tag{3.12}$$

Step 3. By our Assumption (iv) we can conclude from (3.12) that:

$$\sup_n\,[|h_n(x)|]\leqq C2b_x\sup_n(\Pi(n))<\infty. \tag{3.13}$$

This means that we have via assumption (ii) and (3.11):

(3.14) $$h_k(x) \underset{k\to\infty}{\longrightarrow} h(x) \quad \forall x \in \mathbb{Z}, \qquad |h(x)| \leqq b_x \bar{C} \quad \forall x \in \mathbb{Z},$$

(3.15) $$h(x) = (a^+_{x,0} - a^-_{x,0}) = (\hat{a}^+_{x,0} - \hat{a}^-_{x,0})$$
$$h(x+1) - h(x) = (a^+_{x+1,x} - a^-_{x+1,x}) = (\hat{a}^+_{x+1,x} - \hat{a}^-_{x+1,x}).$$

The last equality proves the assertion (γ).

Step 4. In order to prove the remaining point (α) we observe that due to $P^*(h_n) - h_n = \bar{1}_{\{-n\}} - \bar{1}_{\{+n\}}$ and (3.13) it suffices to show that: (apply dominated convergence theorem)

(3.16) $$\sum_y P(x,y)|h(y)| < \infty \ \forall x \in \mathbb{Z}$$
$$\sum_x P(0,x) b_x < \infty .$$

Note that the second inequality implies the first one.

Since η is the minimal positive solution of the Poissonequation we obtain $(\hat{a}^-_{y,y+1} \geqq 0!)$

(3.17) $$b_x \leqq \left(\inf_x \Pi(x)\right)^{-1} \sum_{0 \leqq y \leqq x} \sup_z [\eta_{y,y-1}(z)] \quad \forall x > 0 .$$

On the other hand we shall show below that the following estimate holds for the effect $\eta_{y,y-1}$ of the (δ_y, δ_{y-1}) – Filling scheme (= minimal positive solution of $(\cdot)(I-P) = \delta_y - \delta_{y-1}$) which amounts in this case (discrete state space) to saying $\eta_{x,x-1}(A) = E\#(\text{visits of } (X^{(x)}_k)_{k \in \mathbb{N}} \text{ to } A \text{ before first reaching } (x-1))$ ([4], Lemma A):

(3.18) $$\eta_{y,y-1}(\cdot) \leqq \left(\frac{1 - P(y,y-1)}{P(y,y-1)}\right) (\Pi(\{y\}))^{-1} \Pi(\cdot).$$

This implies due to assumption (iv) together with (3.17) that:

(3.19) $$b_x \leqq \left[c \cdot \sum_{0 \leqq y \leqq x} \frac{1}{P(y,y-1)}\right] \leqq \text{const} \cdot c_{x,0} .$$

Now our assumption (iii) gives immediately the assertion (3.16).

In order to prove (3.18) above, note first that: (Lemma of M. Kac quoted in [4])

(3.20) $$E(\#\text{visits of } X^{\{x\}}_k \text{ in } A \text{ before the first return to } x) = \bar{\Pi}(A)$$
$$\bar{\Pi}(\cdot) := (\Pi\{x\})^{-1} \Pi(\cdot)$$

and

(3.21) $$E(\#\text{visits of } X^{\{x\}}_k \text{ to } x \text{ before a jump from } x \text{ to } x-1 \text{ occurs})$$
$$= \frac{1 - P(x,x-1)}{P(x,x-1)}.$$

Both (3.20) and (3.21) together give immediately the desired inequality using that $\eta_{x,x-1}(A)$ is equal to the expected number of visits in A starting in x before first hitting $x-1$.

b) Asymptotic properties of harmonic functions. In this section we prove a result about the asymptotic behaviour ($|x|\to\infty$) of harmonic functions as constructed in Proposition 3. Here we consider a general class of random transition kernels with certain properties. That this class contains our kernels $\hat{p}(x,y)$ is nontrivial and shown in part c. The strategy in this section is to compare $h(x)$ with the resistance $c_{0,x}$ employing the representation of h given in (3.3) (γ). Via (3.5) line three this yields automatically information about the subharmonic functions f with $(P-I)(f)=1_{\{x\}}$ too.

In this section we consider an ergodic stationary process with values in $M_1(\mathbb{Z})$, which is denoted by $\{P(x,\cdot)\}_{x\in\mathbb{Z}}$. We assume that $\mathscr{L}(P(x,\cdot))=\mathscr{L}(P(x,-\cdot))$. Assume also that for almost all realizations the assumption (i)–(iv) of Proposition 3 are fulfilled, where (iv) holds uniformly in ω. Choose the σ-finite positive invariant measure which is only unique up to multiplicative constansts Π for each ω such that $\overline{\lim_{x\to\infty}}\,\Pi(x)=\overline{\lim_{x\to-\infty}}\,\Pi(x)=1$. Now it makes sense to talk about $E_\omega(a^+_{1,0})$, $E_\omega(a^-_{1,0})$, since we fixed Π and we fixed it such that $\{\Pi(x)\}_{x\in\mathbb{Z}}$ is stationary. Assume furthermore $E_\omega(a^+_{1,0})\neq E_\omega(a^-_{1,0})$.

Denote by h the harmonic function constructed in the last paragraph, respectively f the subharmonic function constructed in Proposition 3′ with $(P^*-I)=2\,1_{\{0\}}$, and $c_{x,y}$ is the resistance between x and y ((2.29)).

Proposition 4. *In the situation described above the following holds:*

a) *If* $E_\omega\left(\frac{1}{P(0,1)}\right)<\infty$ *then we find* $c\in\mathbb{R}^+$ *independent of* ω *such that for* $\tilde{h}=c\,h$ *we have*

$$\tilde{h}(x)/x \underset{|x|\to\infty}{\longrightarrow} 1 \quad \omega\text{-a.s.} \tag{3.22}$$

b) *If* $E_\omega\left(\frac{1}{P(0,1)}\right)=+\infty$ *we consider the case where the following additional assumptions hold*:

$$\{\mathscr{L}(c_{0,x}/|x|^\alpha)\}_{x\in\mathbb{N}} \text{ is weakly relative compact with only nontrivial limit points.} \tag{3.23}$$

$$P(x,y+z)\leqq P(x,y)\,P(y,y+z) \quad \omega\text{-a.s.}, \quad \text{for } y>x,\ z>0 \text{ or } y<x,\ z<0 \tag{3.24}$$

(3.25) *P is reversible with respect to the invariant measure* Π ω-a.s.

Then h respectively f have the property:

(3.26) $\{\mathscr{L}(h(x)/|x|^\alpha)\}_{x\in\mathbb{Z}}$
(3.27) $\{\mathscr{L}(f(x)/|x|^\alpha)\}_{x\in\mathbb{Z}}$
is relatively weakly compact and all weak limit points are nontrivial.

$$f(y)\neq 0 \quad \text{for} \quad y\neq 0 \quad \omega\text{-a.s.}$$

Note that in b) $h(x)x^{-\alpha}$ does not converge ω-a.s.!

Proof of Proposition 4. a) The general strategy is to use 3.3 (γ) to represent h in terms of the $a^{+;-}_{x+1,x}$ and then to apply the ergodic theorem. Here are the details:

Step 1. Due to a result of Neveu [12] we can construct a potential kernel $K(x, y)$ for P, that is we have especially

(3.28) $(K(x,\cdot)-K(y,\cdot))(I-P)=\delta_x-\delta_y \quad K(x,\cdot)\leqq\alpha\cdot\Pi$ for some $\alpha\in\mathbb{R}^+$.

Due to our assumption (ii) we have then automatically:

(3.29)
$$[(K(x,z)-K(y,z))-a^+_{x,y}\Pi(z)] \underset{z\to+\infty}{\longrightarrow} 0$$
$$[(K(x,z)-K(y,z))-a^-_{x,y}\Pi(z)] \underset{z\to-\infty}{\longrightarrow} 0.$$

With the representation for h derived earlier in (3.3) (γ), we know that whatever σ-finite invariant measure we select, h has the property:

(3.30) $$h(y+1)-h(y)=c(a^+_{y+1,y}-a^-_{y+1,y}) \quad \text{for some } c\in\mathbb{R}.$$

We can represent $h(y)$ therefore in the form:

(3.31) $$h(y)=c\sum_{0<x\leqq y}(a^+_{x,x-1}-a^-_{x,x-1})$$

Note that a^+, a^- depend on the choice of Π. The point now is that Π is choosen here in a translation invariant fashion. This is the case because we have choosen Π for each ω such that $\limsup_{x\to\infty}\Pi(x)=\limsup_{x\to-\infty}\Pi(x)=1$ and then we have that $(\Pi(x))_{x\in\mathbb{Z}}$ is stationary and ergodic (for the latter use (3.20)!). So having chosen Π:

(3.32) $$h(y)=\sum_{0<x\leqq y}(a^+_{x,x-1}-a^-_{x,x-1}), \quad \left(\frac{1}{\Pi(x)}\right)_{x\in\mathbb{Z}} \text{ stationary ergodic}.$$

The $(a^+_{x-1,x}-a^-_{x-1,x})_{x\in\mathbb{Z}}$ form a stationary $\mathbb{R}$-valued process, since the difference $a^+_{x-1,x}-a^-_{x-1,x}$ is independent of the choice of the special function in the construction of $K(\cdot,\cdot)$ and since $\{(\Pi(x))^{-1}\}_{x\in\mathbb{Z}}$ is stationary! (See [12] or [4], Theor. 2, for the fact that two admissible solutions of the Poissonequation differ by a multiple of Π only!)

The underlying process $\{P(x,\cdot)\}_{x\in\mathbb{Z}}$ is ergodic so that the sequence $(a^+_{x-1,x}-a^-_{x-1,x})$ is also ergodic, since the tail field of that sequence is contained in the tailfield of $\{P(x,\cdot)\}_{x\in\mathbb{Z}}$. The proof of that fact is based on an identity we shall prove later namely (3.53). The details are straightforward and omitted here. Note that by assumptions $E_\omega(a^+_{1,0})\neq E_\omega(a^-_{1,0})$.

Provided we can show that $E_\omega(|a^+_{1,0}-a^-_{1,0}|)<\infty$, the ergodic theorem for stationary processes and (3.32) tells us that:

(3.33) $$h(y)/y \underset{y\to+\infty}{\longrightarrow} E_\omega(a^+_{1,0}-a^-_{1,0})\neq 0 \quad \omega\text{-a.s.}, \quad \text{q.e.d.}$$

Step 2. In order to verify that the expectation above is finite we shall use the following fact, which can be found for example in [4], Theorem 2: If $\eta_{x,y}$ denotes the effect of the δ_x,δ_y-Fillingscheme (or equivalent the minimal positive solution of $(\cdot)(I-P)=\delta_x-\delta_y$) then we have:

(3.34) $$\eta_{x,y}+\eta_{y,x}=a_{x,y}\cdot\Pi, \quad a_{x,y}\in\mathbb{R}^+.$$

This implies with (3.1) that

(3.35) $\hat{a}^{+}_{y,x}+\hat{a}^{-}_{y,x}\leqq 2a_{x,y}$ (for notation see (3.1)).

With the same considerations as in (3.18)–(3.21) in the proof of Proposition 3 we derive the estimate:

(3.36) $a_{1,0}\leqq C\cdot\left(\frac{1}{P(1,0)}+\frac{1}{P(0,1)}\right)$ for some $C\in\mathbb{R}^{+}$ independent of ω.

Now (3.35) and (3.36) yield immediately that $E_{\omega}(|a^{+}_{1,0}-a^{-}_{1,0}|)<\infty$, since $a^{+}_{1,0}-a^{-}_{1,0}=\hat{a}^{+}_{1,0}-\hat{a}^{-}_{1,0}$ and by definition $\hat{a}^{+}_{1,0},\hat{a}^{-}_{1,0}\geqq 0$ and $E_{\omega}((P(0,1))^{-1})=E_{\omega}((P(1,0))^{-1})<\infty$ by assumption.

b) The starting point here is again the representation $h(y)=\sum_{0}^{y}(a^{+}_{x,x-1}-a^{-}_{x,x-1})$ obtained in (3.32). The task is to relate the sum on the right to the resistance between 0 and y (which was denoted by $c_{0,y}$) and then we can use Proposition 2b in order to obtain our assertion.

Step 1. We saw already that (see (3.35) and (3.36) above)

(3.37) $$\left|\sum_{1}^{y}(a^{+}_{x,x-1}-a^{-}_{x,x-1})\right|\leqq\tilde{C}\cdot\sum_{1}^{y}\left(\frac{1}{P(x,x-1)}+\frac{1}{P(x-1,x)}\right)$$
$$=\tilde{C}(c_{0,y}+c_{y,0}),\quad\text{with }\tilde{C}\text{ independent of }\omega.$$

Since on the otherhand the invariant measure Π is bounded below and above by the counting measure and P is assumed reversible with respect to Π in this part we have:

(3.38) $$\left|\sum_{0}^{y}(a^{+}_{x,x-1}-a^{-}_{x,x-1})\right|\leqq C\cdot\sum_{0}^{y}\frac{1}{P(x,x-1)}=C\cdot c_{0,y},$$
C independent of ω.

Step 2. It remains to obtain a corresponding estimate from below. It suffices to know something for y very large, since we want only to assert something about the behaviour of $h(y)/y^{\alpha}$ for $y\to\infty$.

We know from the ergodic theorem that for all $\varepsilon>0$ and $\alpha>1$ the following holds:

(3.39) $$\left(\sum_{x=0}^{y}\frac{1}{P(x,x-1)}1_{\{P(x,x-1)\geqq\varepsilon\}}\right)y^{-\alpha}\underset{y\to\infty}{\longrightarrow}0\quad\omega\text{-a.s.}$$

so that it suffices to prove for some $\varepsilon>0$ an estimate of the form:

(3.40) $$\sum_{x=0}^{y}(a^{+}_{x,x-1}-a^{-}_{x,x-1})\geqq\underline{c}\sum_{x=0}^{y}\frac{1}{P(x,x-1)}1_{\{P(x,x-1)\leqq\varepsilon\}},\qquad\underline{c}>0.$$

This will be done in the next Lemma below.

First we finish the proof of Proposition 4 assuming (3.40) to be true. Consider first part one of assertion (3.26) dealing with h. Having (3.38) and (3.39) combined with (3.40), the assertion (3.26) follows immediately with (3.31) from the assumptions about the behaviour of the resistance in (3.23). For the first part of assertion (3.27) about f we use (3.5) instead of (3.31).

The second part ($f(y)>0$ for $y \neq 0$) works as follows:

The relations $(P^*-I)(f)=2 1_{\{0\}}$, $f \geqq 0$ imply that either $1_{\{x \leqq 0\}} f \equiv 0$ or: $f(y)=0 \Rightarrow y=0$. $1_{\{x \leqq 0\}} f \equiv 0$ would however imply according to the relation (3.5) and (3.40) that $P(x, x+1) \geqq \delta > 0$ $\forall x \in \mathbb{Z}^-$ for some $\delta > 0$, where δ can be chosen (due to the ergodicity of $\{P(x, \cdot)\}_{x \in \mathbb{Z}}$) independent of ω. This in turn means that $E_\omega\left(\frac{1}{P(0,1)}\right) < \infty$ in contradiction to our assumption. Therefore (3.27) part two holds.

We turn back to proving (3.40). It suffices to prove:

Lemma 3.1. *Suppose that a transition kernel $P(x,y)$ fulfills the conditions* (i)–(iv) *of Proposition 3 together with the following conditions for some $c \in \mathbb{R}^+$:*

(3.41) $$\sum_{k=2}^{\infty} P(x, x \pm k) \leqq c P(x, x \pm 1) \quad \forall x \in \mathbb{Z},$$

(3.42) $$P(y, x-1)/P(y, x) \leqq c \cdot P(x, x-1) \quad \forall y > x$$
$$P(y, x)/P(y, x-1) \leqq c \cdot P(x-1, x) \quad \forall y < x.$$

Then we have the following estimate: (ε, $\underline{c}$ depending only on c and $\inf_x \Pi(x)$*)*

(3.43) $$\exists_{\substack{\underline{c}>0 \\ \varepsilon>0}}: \ (a^+_{x,x-1} - a^-_{x,x-1}) \geqq \underline{c} \cdot \frac{1}{P(x,x-1)} 1_{\{P(x,x-1) \leqq \varepsilon\}} 1_{\{P(x-1,x) \leqq \varepsilon\}}.$$

Remark. (i) In our case $P(x-1,x) \leqq CP(x,x-1)$ as mentioned in (3.37) before!

(ii) The assumption (3.24) of Proposition 4 implies of course (3.41), (3.42).

Proof. First remember that $a^+_{x,x-1} - a^-_{x,x-1} = \hat{a}^+_{x,x-1} - \hat{a}^-_{x,x-1}$ where the $\hat{\ }$-quantities are derived from the minimal positive solution of $(\cdot)(I-P) = \delta_x - \delta_{x-1}$. Denote this solution again with $\eta_{x,x-1}(\cdot)$. Then we know $\eta_{x,x-1}(A) = E\#\{\text{visits of } X^{\{x\}}_k \text{ to } A$ before reaching $(x-1)\}$ ([4] I, Lemma A). Therefore:

(3.44) $$\eta_{x,x-1}(x-1) = 0.$$

To prove (3.43) we shall proceed in two steps: first estimate $\eta_{x,x-1}(x)$ from below and then in the second step use the fact that $\eta_{x,x-1}(x-1)=0$ to obtain a result about $(\hat{a}^+_{x,x-1} - \hat{a}^-_{x,x-1})$ by passing to the dual chain and applying the optional stopping theorem for marginales.

Step 1. Let $(X_n)_{n=0,1,\ldots}$ be the Markov chain with transition kernel P. With $\text{Prob}_{\{x\}}(\cdot)$ we denote the probability measure for the corresponding process starting in point x.

In order to write down our estimate we shall need the following quantity:

(3.45) $$\beta(x) := \text{Prob}_{\{x\}} \text{ (The chain } X_n \text{ hits } (x-1) \text{ before } x \text{ and } X_1 \notin \{x-1, x\}).$$

Then with the same argument as in (3.19) to (3.21) we have:

(3.46) $$\eta_{x,x-1}(x) \geqq \frac{1-p(x,x-1)-\beta(x)}{\beta(x)+p(x,x-1)} = \frac{1-p(x,x-1)\gamma(x)}{p(x,x-1)} \frac{1}{\gamma(x)},$$
$$\text{with } \gamma(x) = (\beta(x)/p(x,x-1)) + 1$$

Suppose we can show that:

$$\gamma(x) \leqq K \quad \forall x \in \mathbb{Z} \tag{3.47}$$

then we can write for some $\varepsilon \in (0, K^{-1})$:

$$\eta_{x,x-1}(x) \geqq \frac{1-Kp(x,x-1)}{p(x,x-1)} \frac{1}{K} \geqq c \frac{1}{p(x,x-1)} 1_{\{p(x,x-1) \leqq \varepsilon\}}. \tag{3.48}$$

The estimate on $\gamma(x)$ we obtain as follows:

The fact that $\sum_{k=2}^{\infty} p(x,x-k) \leqq cp(x,x-1)$ leaves us with finding a bound from above on $\gamma'(x) := \beta'(x)/p(x,x-1)$ with: (denote with $B_x = \{X_k^{\{x\}} > x$ before it hits $x-1$ or $x\}$)

$$\beta'(x) := \mathrm{Prob}_{\{x\}}\,((\text{the chain } X_n \text{ hits } (x-1) \text{ before it hits } x) \cap B_x). \tag{3.49}$$

In order to get control over this quantity we introduce:

$$c_y = \mathrm{Prob}_{\{x\}}(B_x \cap (\text{the last point before } X_n \text{ hits } \{x-1, x\} \text{ is } y)). \tag{3.50}$$

From here we obtain our assertion (3.47) by applying to the relation above our assumption (3.42) to obtain:

$$\beta'(x) \leqq \sum_{y \geqq x+1} c_y \frac{P(y,x-1)}{P(y,x)} \leqq c' p(x,x-1). \tag{3.51}$$

Step 2. Having (3.48), we are left with the task to estimate $(\hat{a}^{+}_{x,x-1} - \hat{a}^{-}_{x,x-1})$by the quantity $\eta_{x,x-1}(x) - \eta_{x,x-1}(x-1) = \eta_{x,x-1}(x)$. Compare (3.44)).

For that purpose we define the quantities: $\left(\bar{X}_k\text{: process with kernel } \bar{P}(x,y) = P(y,x)\frac{\Pi(y)}{\Pi(x)}\right)$

$$\bar{H}_y(x,x-1) = \mathrm{Prob}_{\{y\}}(\bar{X}_k \text{ hits } \{x, x-1\} \text{ first in } x). \tag{3.52}$$

Since $\eta_{x,x-1}$ is the minimal positive solution for the Poissonequation $(\cdot)(I-P) = \delta_x - \delta_{x-1}$, we can obtain with the same manipulations as in (3.7), (3.8′) an Poissonequation in terms of $\bar{P}$. Then we have by the well known optional stopping theorem for martingales:

$$\begin{aligned}(\hat{a}^{+}_{x,x-1} - \hat{a}^{-}_{x,x-1}) = \lim_{y \to \infty} (\bar{H}_{+y}(x,x-1) \\ - \bar{H}_{-y}(x,x-1))\eta_{x,x-1}(x)(\Pi(x))^{-1}.\end{aligned} \tag{3.53}$$

If we plug (3.48) in (3.53) and use $\Pi(x) \leqq C$ we see that our assertion (3.43) is proved if we can show that:

$$\begin{aligned}(\bar{H}_{+\infty}(x,x-1) - \bar{H}_{-\infty}(x,x-1)) \geqq \delta > 0 \\ \text{provided: } P(x,x-1) + P(x-1,x) \leqq \varepsilon.\end{aligned} \tag{3.54}$$

But this is an immediate consequence of our two assumptions (3.41), (3.42) and assumption (iv) in Proposition 3 because they imply for sufficiently small ε (depending on c in (3.41), (3.42)) that for some $\delta > 0$: $\bar{H}_{-\infty}(x,x-1)) \leqq 1/2 - \delta$,

$\bar{H}_{+\infty}(x, x-1) \geqq (1/2)+\delta$. The straightforward details are left to the reader (compare technique leading to (3.51)).

c) The Application to the Case of a Walk Restricted to a Random Set. In this section we shall investigate the question whether our random kernel $\hat{p}(x, y)$ introduced in paragraph 2b) fulfills the assumptions which were needed to construct the harmonic function h (of Prop. 3) and whether Proposition 4 on the asymptotic form of these functions is applicable.

Proposition 5. *Consider our original kernel $p(x, y)$. Assume that we have in addition that:*

$$\lim_{|x| \to \infty} \left(\frac{1}{|x|} \log p(0, x) \right) > -\infty . \tag{3.55}$$

Denote by $\bar{x}$ the point which is mapped onto x relabelling A with $\mathbb{Z}$.

Then we have:

The $\{\hat{p}(x, \cdot)\}_{x \in \mathbb{Z}}$ form a stationary and shift ergodic process with the following properties:

(1) *$\hat{p}(x, y)$ satisfies the assumptions* (i)–(iv) *of Proposition 3 for almost all realizations of the random set A.*

(2) *We can choose a reversible invariant measure with weights $\hat{e}(x) = \left(\sum_{y \in A} p(\bar{x}, y) \right)$ and with that choice the process $\{\hat{e}(x)\}_{x \in \mathbb{Z}}$ is stationary ergodic with $E_\omega(a^+_{1,0}) \neq E_\omega(a^-_{1,0})$ and $\hat{p}(x, z) \leqq \hat{p}(x, y)\hat{p}(y, z)$ for $z > y > x$. Therefor is Prop. 4 applies to $\hat{p}(\cdot, \cdot)$.*

Corollary 5. *Assume that either $E_\omega\left(\frac{1}{\hat{p}(0, 1)}\right) < +\infty$ or $p(0, x) = ce^{-r|x|}$ for some r. Then for $\hat{p}(x, y)$ exists for almost all ω an harmonic function $h_\omega(\cdot)$ with $h_\omega(0) = 0$. It is determined uniquely by requiring* (3.3) (γ) *and choosing $\Pi(x) = \hat{e}(x)$. It has the property that Proposition 4 holds. Furthermore $\{x \mid |h_\omega(x)| \leqq \lambda\}$ is compact for every $\lambda \in \mathbb{R}$.*

In the sequel we shall assume that we have choosen this harmonic function h for each ω and in the notation we suppress the dependence on ω for convenience.

Proof of Proposition 5. The fact that $\{\hat{p}(x, \cdot)\}_{x \in \mathbb{Z}}$ is stationary follows from $p(x, y) = p(0, y-x)$ and $\mathscr{L}(A-x) = \mathscr{L}(A)$, ($A$ = random set). The ergodicity follows from the ergodicity of $\{1_A(x)_{x \in \mathbb{Z}}\}$ and $p(0, y) \underset{|y| \to \infty}{\longrightarrow} 0$.

We start proving assertion (1). (i) Since $p(x, y) > 0$ it is clear that $\bar{p}(x, y) > 0$ on $A \times A$ and therefore $\hat{p}(x, y) > 0$ on $\mathbb{Z} \times \mathbb{Z}$ which implies of course that $\hat{p}(\cdot, \cdot)$ is irreducible. In order to show that $\hat{p}(\cdot, \cdot)$ is recurrent we observe first that this is the same as showing that $\hat{p}(\cdot, \cdot)$ is recurrent. Since both $p(\cdot, \cdot)$ and $\bar{p}(\cdot, \cdot)$ are reversible we can apply the Nash-Williams recurrence criterion, ([11]), to show that $\bar{p}(\cdot, \cdot)$ is recurrent.

Define

$$\begin{aligned} \alpha(x, y) &= 1 \cdot p(x, y) \\ \bar{\alpha}(x, y) &= e(x)\bar{p}(x, y) = e(x) 1_A(x) p(x, y) 1_A(y) \end{aligned} \tag{3.56}$$

and note that $\bar{\alpha} \leqq \alpha$.

Then the Nash-Williams criterion tells us that the recurrence of $p(\cdot,\cdot)$ implies the recurrence of $\bar{p}(\cdot,\cdot)$. (Compare [11].)

(iv) Note that the assumption $p(x,x)\geqq\delta>0$ is no lost of generality here since p is kernel of the jumpchain of a continuous time process. We have then the estimate:

$$(3.57)\quad 0<\delta=p(x,x)\leqq e(x)\leqq 1\quad \forall x\in A,\qquad \overline{\lim_{x\to-\infty}}\ e(x)=\overline{\lim_{x\to+\infty}}\ e(x)=1 \text{ a.s.}$$

which proves immediately the assertion.

(iii) As long as $E_\omega\left(\frac{1}{\hat{p}(0,1)}\right)<\infty$ we know that for almost all ω, we have for ω fixed:

$$(3.58)\qquad c_{0,x}\leqq \text{const}\cdot|x|$$

and the assertion is therefore implied by our assumption $\sum_y p(x,y)y^2<\infty$.

In the case where $E_\omega\left(\frac{1}{p(0,1)}\right)=+\infty$, we know from our additional assumption (3.55) that:

$$(3.59)\qquad \hat{p}(0,x)\leqq c_\delta\frac{1}{|x|^\delta}\quad \forall\delta<\infty.$$

So that in the cases where $r<\infty$, Proposition 2 tells us that the assertion holds. (Since then $c_{0,x}=0(x^\delta)$ for some $\delta>0$, compare Prop. 2, (2.31).)

(ii) We fix ω throughout this section. We observe first that by the same manipulation as in (3.7)–(3.8′), we can (by introducing the dual chain $\bar{P}$ again) transform assertions about solutions of $(\cdot)(I-P)=\delta_x-\delta_y$ into assertions about $(\bar{P}^*-I)(f)=g$.

We shall need the following quantities for the dual chain $(\bar{X}_k)$ (compare (3.52) for the definition):

$$(3.60)\qquad \bar{H}_y(B,z):=\text{Prob}\,(\bar{X}_k^{\{y\}} \text{ hits } B \text{ first in point } z).$$

We see (using the translation mechanism mentioned above) that the convergence of $(K(x,z)-K(y,z))(\Pi(z))^{-1}$ for $z\to+\infty$, $-\infty$ (here $K(\cdot,\cdot)$ is the potential kernel for P) is implied by showing that the bounded solutions f of $(\bar{P}^*-I)(\cdot)=\bar{1}_{\{x\}}-\bar{1}_{\{y\}}$ $(\bar{g}(x):=g(x)(\Pi(x)))$ have the property that $f(z)$ converges as $z\to+\infty$, $-\infty$. This in turn is equivalent to showing (by the martingale optional stopping theorem, see (3.53))

$$(3.61)\qquad \bar{H}_y(B,z) \text{ converges for } y\to+\infty,\quad y\to-\infty \text{ for finite sets } B.$$

This last fact is implied by a Coupling-result. Suppose we can define $(\bar{X}_k^{\{y-x\}})_{k\in\mathbb{Z}}$ and $(\bar{X}_k^{\{y\}})_{k\in\mathbb{Z}}$ on a joint probability space such that with S, T denoting the hitting times of B for the two processes:

$$(3.62)\qquad \text{Prob}\,(\bar{X}_S^{\{y\}}=\bar{X}_T^{\{y-x\}})\underset{y\to-\infty}{\longrightarrow}1\quad \text{uniformly in } z\leqq 0$$

then (3.61) would hold.

It is known that (3.62) above holds for classical random walks ([4] Theorem 1 and Corollary or [13]). In order to get it for our random walk simply note that we can find, for almost all realizations of A for every $n, m \in \mathbb{N}$ points z_1, z_2 such that:

(3.63) $\mathbb{Z} \cap (z_1 - n, z_1) \subseteq A, \quad \mathbb{Z} \cap (z_2, z_2 + n) \subseteq A, \quad z_1 \leqq -m$ and $z_2 \geqq m$.

So that we can deduce (3.62) from the result for classical random walks with standard analysis.

Next we prove assertion (2) of Proposition 5. The first part of assertion (2) of Proposition 5 is trivial due to Proposition 1a, so is the subexponentiality of $\hat{p}$. The proof of the second part of assertion (2) about $E_\omega(a_{1,0}^+ - a_{1,0}^-)$ is somewhat more involved, even though the fact itself is intuitively obvious. We start for a warm up and for showing the spirit of our approach by showing that

(3.64) $$\mathrm{Prob}_\omega(\hat{a}_{1,0}^+ > 0) > 0 \quad \text{(for notation see Prop. 3')}.$$

For that it suffices to consider ω such that $[-n, n] \subseteq A$, where we shall choose n in a minute.

For a random walk on $\mathbb{Z}$, which is recurrent and fulfills $\sum_y p(x,y)(y)^2 < \infty$ we have due to a result of Ornstein [13] that $\tilde{\hat{a}}_{1,0}^+ > 0 = \tilde{\hat{a}}_{1,0}^-$. The $\sim$ indicates that we talk about the unrestricted walk $p(x,y)$ here rather than about $\hat{p}(x,y)$. It is now of course standard analysis to show that for an $\varepsilon > 0$ we can choose n such, that (η is the minimal positive solution of $(\cdot)(I-P) = \delta_1 - \delta_0$ again).

(3.65) $$\overline{\lim_{x \to +\infty}} \, (\eta(x)(\Pi(x))^{-1}) \geqq \hat{a}_{1,0}^+ - \varepsilon \quad \text{for } \omega \text{ with } [-n, n] \subseteq A.$$

This proves (3.64) by choosing $\varepsilon = 1/2\hat{a}_{1,0}^+$. Now note that (3.64) implies according to Proposition 3' that f^+, f^- constructed there are not identically 0, for ω with $[-n, n] \subseteq A$.

Now we start with proving $E_\omega(a_{1,0}^+) \neq E_\omega(a_{1,0}^-)$. We do this by showing $E_\omega(a_{1,0}^+) = E_\omega(a_{1,0}^-)$ leads to a contradiction.

Step 1. We know $E_\omega(a_{1,0}^+) = E_\omega(a_{1,0}^-)$ implies ((3.33)!) that $h(x) = o(|x|)$ ω-a.s., with h defined through (3.3). Therefore with f defined by (3.5) in Prop. 3'

(3.66) $$E_\omega(a_{1,0}^+) = E_\omega(a_{1,0}^-) \quad \text{implies: } f(x) = o(|x|) \quad \omega\text{-a.s.}$$

since the representation formula (3.32) for h implies together with (3.5) that $f(x) \sim c|h(x)|$ as $|x| \to \infty$, with $1 \geqq c \geqq \delta > 0$ uniformly in ω.

We shall show now that for $p(x,y) = p(y,x)$ and $\sum_y p(x,y) y^2 < \infty$ (note $d = 1$ was already used in (3.64) above) the kernel $p(x,y)$ has the property that the subharmonic function f from (3.5) fulfills:

(3.67) $$f(x) \geqq c(\omega)|x|, \qquad \mathrm{Prob}\,(c(\omega) > 0) > 0.$$

Of course (3.67) and (3.66) together show that $E_\omega(a_{1,0}^+) = E_\omega(a_{1,0}^-)$ is impossible in our model.

Step 2. It remains therefore to show that (3.67) above is true. For that purpose observe first that for given $n \in N$ the points x such that $[x-n, x+n] \cap \mathbb{Z} \subseteq A$ have positive density.

Since $\sum_y \hat{p}(x,y)y^2 \leqq c \sum p(x,y)y^2 < \infty$, we can choose for ε fixed $m(n)$ such that for $n \geqq n_0$ and some $a \in (0,1)$:

$$(3.68) \qquad \sum_{k>0} \hat{p}(0, m(n)+k) \leqq \frac{\varepsilon}{n^2} \quad \text{and } m(n) \leqq an$$

$\left(\text{use Chebychev: } P(X \geqq n) \leqq n^{-2} \int_n^\infty x^2 P(dx) = o(n^{-2})\right)$.

The idea is now to show that $f(\cdot)$ grows at least linear in $[x-n+m, x+n-m]$, which would immediately give (3.67). Note first that for $\delta > 0$ small enough and $m(n)$ choosen such that (3.68) holds

$$(3.69) \qquad \overline{\lim_{n\to\infty}} \operatorname{Prob}[\hat{X}_k^{\{x\}} \text{ makes a jump } \geqq m(n) \text{ before time } \delta n^2] < 1.$$

Now fix n and therefore $m(n)$ such that the probabilities above are less than $1-\varepsilon'$ and (3.68) holds. We write m for $m(n)$ now.

If $(\hat{X}_k^{\{x\}})$ doesn't make jumps $\geqq m$ during a time period it looks like a symmetric random walk till it leaves $[x-n+m, x+n-m]$. It therefor has the property that $(I := [x-n+m, x+n-m])$

$$(3.70) \qquad E(T_I) \geqq \text{const}\, n^2 \quad \text{where} \quad T_I = \inf(k \mid X_k^{\{x\}} \notin I).$$

This implies that $\hat{X}_k^{\{x\}}$ needs in the average a timespan bigger than $c(\omega)K^2$ to leave $[-K, K]$, where $c(\omega)$ depends on the density of points such that $[x-n+m, x+n-m] \subseteq A$.

We shall show that this implies that:

$$(3.71) \qquad \hat{a}_{x,0}^+ + \hat{a}_{x,0}^- \geqq \text{const}\, c(\omega)|x| \quad \text{or} \quad \hat{a}_{-x,0}^+ + \hat{a}_{-x,0}^- \geqq \text{const}\, c(\omega)|x|$$

which finishes the proof of (3.67) using the representation in Proposition 3′.

Step 3. The last implication can be seen as follows: The chain starting in 0 will have the property that (by (3.70)) it visits points in $(-K, K)$ in the mean at least $c(\omega)K^2$ times before leaving this intervall. Denote by ξ the minimal positive solution of:

$$(3.72) \qquad (\cdot)(I-\hat{P}) = 2\delta_0 - (\delta_{-K} + \delta_K).$$

Note that ξ is bigger than the minimal positive solution of $(\cdot)(I-\hat{P}) \geqq 2\delta_0 - \infty 1_{\{\mathscr{C}(-K,K)\}}$, so that due to the remark above we have:

$$(3.73) \qquad \xi((-K, K)) \geqq c(\omega)K^2.$$

Due to the fact $0 < \underline{c} \leqq \Pi \leqq \bar{c}$ and $\xi(\cdot)(\Pi(\cdot))^{-1}$ is maximal at 0, we can conclude:

$$(3.74) \qquad \xi(0) \geqq \text{const} \cdot c(\omega)K.$$

Now consider ξ^1, ξ^2 the minimal positive solution of the equations $(\cdot)(I-\hat{P}) = \delta_0 - \delta_K(\delta_0 - \delta_{-K})$. Since $\xi^1 + \xi^2$ is a positive solution of (3.72) we conclude from (3.74) that:

$$(3.75) \qquad \xi^1(0) + \xi^2(0) \geqq \text{const}\; c(\omega)K.$$

Similar to the procedure in (3.52)–(3.54) we obtain then again with the martingale optional stopping theorem:

$$(3.76)\qquad (\hat{a}^{+}_{K,0}+\hat{a}^{-}_{K,0})+(\hat{a}^{+}_{-K,0}+\hat{a}^{-}_{-K,0})\geqq \text{const}\, c(\omega)K. \quad \text{q.e.d.}$$

We don't repeat the details here. $\left(\text{Use} \lim_{K\to+\infty}\lim_{y\to+\infty} \bar{H}_y(\{K,0\},0)=1.\right)$

C. Proof of Theorem 1 to 4

The Sects. 4, 5, 6 apply our results from Sect. 3 to the chain $(\hat{X}_k)_{k\in\mathbb{N}}$. They contain the important results we shall need in Sect. 7 to prove Theorem 1 to 4. Section 4 aims at Theorem 1, Sect. 5 at Theorem 2 and Sect. 6 at Theorem 3. The proof of Theorem 4 is a byproduct of the results we have by then.

4. Asymptotic behaviour of the associated martingale: diffusive case

In this chapter we consider the case $E(\hat{p}(0,1))^{-1}<\infty$ and we apply first the martingale central limit theorem to study $h(\hat{X}_k)$, the associated martingale of our random walk $(\hat{X}_k)$ with h as introduced in Sect. 3 (Coroll. 5), and then we derive the implications for the jumpchain of our original walk $X(t)$.

Notation. By $B^2_\sigma(t)$ we denote (in this section only!) Brownian motion with diffusion constant σ^2. We shall write $Y(s)$ for $Y_{[S]}$, in case we regard a discrete chain as a random variable with values in $D(\mathbb{R})$, the space of right continuous functions with left limits. With $\hat{e}(x)$ we denote again $\sum_y p(\bar{x},y)1_A(y)$, where $\bar{x}$ corresponds to x when relabelling A with $\mathbb{Z}$.

Proposition 6. *Let h denote the harmonic function for $\hat{p}(x,y)$ constructed in Sect. 3. Suppose that $E_\omega\left(\frac{1}{\hat{p}(0,1)}\right)<\infty$. Define $Y_n=h(\hat{X}_n)$ and $\sigma^2=E_\omega\left(\sum_y h^2(y)\hat{p}(0,y)\hat{e}(0)\right)$. Then:*

$$(4.1)\qquad \frac{1}{2k}\left(\sum_{x=-k}^{+k}\sum_y (h(x+y)-h(x))^2\alpha(x,y)\right)\underset{k\to\infty}{\longrightarrow}\sigma^2<\infty \text{ a.s.},$$

$$\alpha(x,y):=\hat{e}(x)\hat{p}(x,y).$$

$$(4.2)\qquad \mathcal{L}\left(\frac{Y_n}{\sigma\sqrt{n}}\,|\omega\right)\underset{n\to\infty}{\Rightarrow}\mathcal{N}(0,1)\qquad \omega\text{-a.s.}$$

$$(4.3)\qquad \mathcal{L}\left(\left(\frac{1}{n}\,Y(n^2\sigma^2 t)\right)_{t\in\mathbb{R}^+}|\omega\right)\underset{n\to\infty}{\Rightarrow}\mathcal{L}((B_1(t))_{t\in\mathbb{R}^+})\qquad \omega\text{-a.s.}$$

Corollary 6. *Assume $E_\omega\left(\frac{1}{\hat{p}(0,1)}\right)<\infty$ then we have for $\tilde{\sigma}=(E_\omega(|a^{+}_{1,0}-a^{-}_{1,0}|))^{-1/2}\sigma<\infty$:*

$$(4.4)\qquad \mathcal{L}\left(\left(\frac{1}{n}\,\hat{X}(n^2 t)\right)_{t\in\mathbb{R}^+}|\omega\right)\underset{n\to\infty}{\Rightarrow}\mathcal{L}((B_{\tilde{\sigma}^2}(t))_{t\in\mathbb{R}^+})\qquad \omega\text{-a.s.}$$

and for the jumpchain $(X_k)_{k\in\mathbb{N}}$ *of our original Markov process,* (4.4) *holds with* $\bar{\sigma}^2$ *where*

$$\bar{\sigma}=q^{-1}\tilde{\sigma} \quad (\sigma=\sigma(q,p(0,\cdot))!)\,. \tag{4.5}$$

The first assertion of the Corollary is seen by combining Proposition 4 ((3.22)), 5 and 6. The second assertion follows from the observation that we have the following relation between $\hat{X}_k$ and X_k: $\hat{X}_k=n\Leftrightarrow X_k$ equals the n-th point in A which is bigger 0, which yields the assertion using the fact that the gaps between points in A are geometrically distributed (parameter $1-q$).

Proof of Proposition 6. The process $(Y_n)_{n\in\mathbb{N}}$ is for every ω a centered martingale with respect to the σ-fields $\mathscr{A}((\hat{X}_i)_{i\leqq k})$, $k=0,1,\ldots$ To prove the Proposition we shall use the standard central limit theorems for martingales.

Step 1. Therefor an important quantity is the conditional variance and quadratic variation:

$$E((Y_{n+1}-Y_n)^2|\hat{X}_n=x,\omega)=\sum_y (h(x+y)-h(x))^2\hat{p}(x,x+y)\,. \tag{4.6}$$

Note that this functional of the medium has a stationary (in x) distribution (Prop. 4a) which proves (4.1) via the ergodic theorem, if we can show it is integrable (see step 2).

We can write the conditional quadratic variation of the rescaled martingale in the form:

$$\frac{1}{n}\sum_1^n E((Y_{k+1}-Y_k)^2|\hat{X}_k,\omega)=\frac{1}{n}\sum_1^n\sum_y (h(\hat{X}_k+y)-h(\hat{X}_k))^2\hat{p}(\hat{X}_k,\hat{X}_k+y)\,. \tag{4.7}$$

The right hand side of the equation above is now treated according to the scheme of the proof of Proposition 1, namely considering it as a mean along a path of the canonical Markov process describing the environment as seen from the random walker. We don't repeat the details here and refer the reader to Sect. 2a. We obtain for almost all ω provided the right hand side is finite:

$$\begin{aligned}&\frac{1}{n}\sum_1^n\sum_y (h(\hat{X}_k+y)-h(\hat{X}_k))^2\hat{p}(\hat{X}_k,\hat{X}_k+y)\\ &\underset{n\to\infty}{\longrightarrow} E_\omega\left(\sum_y (h(x+y)-h(x))^2\alpha(x,y)\right)\\ &\qquad\text{a.s. and in}\quad L^1(\mu,\mathscr{L}(\hat{X}_n))\ (\text{see }(2.16))\,.\end{aligned} \tag{4.8}$$

If we can show that: $E_\omega\left(\frac{1}{\hat{p}(0,1)}\right)<\infty$ implies that $E_\omega\left(\sum_y (h(x+y)-h(x))^2\alpha(x,y)\right)$ $<\infty$, then we have by (4.8) the a.s. and L^1-convergence of the conditional quadratic variation. The L^1-convergence gives us a Lindeberg-condition, ω-a.s. The invariance principle for martingales ([7]), Theor. 4.4, p. 100) gives us now (4.2) and (4.3).

Step 2. In order to show that: $E_\omega\left(\frac{1}{\hat{p}(0,1)}\right)<\infty$ implies finite expected quadratic variation, we distinguish two cases: (r as defined in Theorem 1 and $\bar{y}\Leftrightarrow y$ relabelling A with $\mathbb{Z}$.).

Case 1. $r=0$. Then $E_\omega\left(\left(\frac{1}{\hat{p}(0,1)}\right)^2\right)<\infty$ and we can estimate:

$$\sum_y (h(y))^2 \hat{p}(0,y)\hat{e}(0) \leqq \sum_y h^2(y) p(0,\bar{y}), \tag{4.9}$$

and therefor by combining (3.32) and (3.38): (C independent of ω)

$$E_\omega\left(\sum_y h^2(y)\hat{p}(0,y)\hat{e}(0)\right) \leqq \left(E_\omega\left(\frac{C}{\hat{p}(0,1)}\right)^2\right) \sum_y y^2 p(0,y) < \infty. \tag{4.10}$$

Case 2. $r>0$.
First assume that: $p(0,x) = \text{const}\, e^{-r|x|}$.

Next we use the property that

$$\hat{p}(0,y) \leqq \prod_0^{y-1} \hat{p}(x,x+1). \tag{4.11}$$

Now we can estimate with (3.32) together with (3.38) as follows (C independent of ω).

$$\begin{aligned} \sum_y h^2(y)\hat{p}(0,y) &\leqq C\left(\sum_y \left(\sum_1^y \frac{1}{\hat{p}(x,x+1)}\right)^2 \hat{p}(0,y)\right) \\ &\leqq C\left(\sum_y \left(\sum_1^y \frac{1}{\hat{p}(x,x+1)}\right)\left(\sum_1^y \prod_{x=1,x\neq k}^{y} \hat{p}(x,x+1)\right)\right) \\ &\leqq C\left(\sum_y \left(\sum_1^y \frac{1}{\hat{p}(x,x+1)}\right)(y a^{|y|-1})\right); \end{aligned} \tag{4.12}$$

$a<1$ and a independent of ω.

Taking expectation over ω in (4.12) yields now immediately the assertion for $p(0,x)=e^{-r|x|}$.

We see immediately that what we need for the argument is $p(0,x) \leqq Ce^{-r|x|}$ for some C, r. This is however assured by (1.4), so that we have proved the assertion in general.

5. *Asymptotic behaviour of the associated martingale: subdiffusive case*

In the case of infinite expected resistance we examine the behaviour of $(\hat{X}_k)$ when we average over ω and especially we determine the appropriate scale for our process. In the next section we shall discuss the more complicated question to determine the behaviour for fixed ω. In this section we derive again a more general result for random transition kernels generating a chain X_k and later we will use Proposition 5 to specialize the results to our case.

The approach we take here is to use the submartingales $f(X_k)$ (with f as in Prop. 3′) to estimate after what time the process will leave an interval $[-n,n]$ for n large. This random time determines the rate at which we have to rescale time, if we scale space by n^{-1}.

Therefor we define the following stopping times $T(n)$ for a Markov chain (X_k) on $\mathbb{Z}$:

(5.1) $$T(n)=\inf(k|X_k\in\mathscr{C}(-n,n)).$$

The essential property of these stopping times is given by:

Proposition 7. *Suppose that $P(x,y)$ is a random transition kernel which fulfills the conditions of Proposition 4b (see (3.22) to (3.27)). Then we have:*

(5.2) $$\{\mathscr{L}(T(n)n^{-(1+\alpha)})\}_{n\in\mathbb{N}} \quad \textit{is weakly relative compact.}$$

The weak limit points are different from δ_0.

Proof. The crucial step in the proof is to study the behaviour of:

(5.3) $$L(x,n):=\sum_{i=1}^{T(n)} 1_{\{X_i=x\}},$$

$$L^\delta(x,n)=L(x,n)1_{A_n^\delta},$$

$$\text{with}\quad A_n^\delta=\left\{\omega|\sum_{-n}^{n}\frac{1}{P(x,x+1)}\leqq\delta n^\alpha\right\}\cap\left\{\omega|\sum_{|y|>n} b^{|y-n|}\frac{1}{P(y,y+1)}\leqq\delta\right\},$$

$$b:=\sup_x P(x,x+1)<1.$$

We shall show later in this section that:

Lemma 5.1. *Suppose the kernel for (X_k) fulfills the assumptions of Proposition 7. Then:*

(5.4) $$\left\{\mathscr{L}\left(\frac{L^\delta(x,n)}{n^\alpha}\right)\right\}_{n\in\mathbb{N}} \quad \textit{is weakly relative compact with nontrivial weak limit points.}$$

(5.5) $$\left(\frac{L^\delta(x,n)}{n^\alpha}\right)_{n\in\mathbb{N},x\in[-n,n]} \quad \textit{are uniformly integrable over } \omega \textit{ and the process.}$$

(5.6) $$0<\inf_n\inf_{|x|\leqq an} E\left(\frac{L^\delta(x,n)}{n^\alpha}\right)\leqq\sup_n\sup_{|x|<n} E\left(\frac{L^\delta(x,n)}{n^\alpha}\right)=\delta C<\infty$$

with $a<1$.

To proceed further note that:

(5.7) $$T(n)=1+\sum_{x=-n}^{+n} L(x,n);$$

$$T^\delta(n):=T(n)1_{A_n^\delta}=1+\sum_{x=-n}^{+n} L^\delta(x,n).$$

Since according to Proposition 4b: $\liminf_{n\to\infty}\text{Prob}_\omega(A_n^\delta)$ goes to 1 as $\delta\uparrow+\infty$, it suffices in order to show (5.2), to prove this result for $T^\delta(n)n^{-(1+\alpha)}$ for arbitrary $\delta\in\mathbb{R}^+$.

The relations (5.6) combined with (5.7) above implies first of all that (expectation over ω and process)

$$\overline{\lim_{n\to\infty}}\,[E(T^\delta(n)n^{-(1+\alpha)})]<\infty\,,\qquad \liminf_{n\to\infty}[E(T^\delta(n)n^{-(1+\alpha)})]>0 \tag{5.8}$$

This implies the assertion (5.2) for $T^\delta(n)n^{-(1+\alpha)}$ by combining (5.7) with the relation (5.5). (Note in (5.5) we have the uniform integrability in n *and* $x\in[-n,n]$ so that $(n^{-(1+\alpha)}T^\delta(n))_{n\in\mathbb{N}}$ is uniformly integrable!) This proves according to our remark above our Proposition 7 by letting $\delta\uparrow+\infty$ and it remains to show Lemma 5.1.

Proof of Lemma (5.1). To prove our Lemma we use the subharmonic function f from Proposition 3'. We start by observing the following general fact about subharmonic functions:

Step 1. Suppose that S is a stopping time of the chain $(X_k^\nu)_{k\in\mathbb{N}}$ and f is a positive subharmonic function. Define $g=(P^*-I)(f)$ and $\mu=\mathscr{L}(X_S^\nu)$.

Then the following holds (see [4], part II, Lemma 1a):

$$\langle\mu,f\rangle=\langle\nu,f\rangle+\langle\eta,g\rangle-\lim_{n\to\infty}\langle\nu_n,f\rangle \tag{5.9}$$

$$\eta(A)=E\left(\sum_{k=0}^{S-1}1_{\{X_k^\nu\in A\}}\right),\qquad \nu_n(A)=\mathrm{Prob}\,(X_n^\nu\in A,n<S)\,.$$

If we apply this to our situation, that is $S=T_n$, $\nu=\delta_y$ and $f=f$ as constructed in Proposition 3', $g=21_{\{x\}}$, we easily obtain for some C_1,C in $\mathbb{R}^+$ that:

$$\begin{aligned}
&\langle\mu,f\rangle=f(y)+2E(\#\text{visits of } X_j^{\{y\}} \text{ to } x \text{ before reaching } \mathscr{C}((-n,n))\\
&\langle\mu,f\rangle\geqq(f(n)\wedge f(-n))C_1\qquad C_1>0, C_1 \text{ independent of } \omega \text{ and } n\\
&\langle\mu,f\rangle\leqq(f(n)\vee f(-n))+C_2(\omega)\,,\qquad C_2(\omega)<\infty \text{ a.s., independent of } x\\
&C_2(\omega)=C\sum_{|y|\geqq n}b^{|y-n|}(P(y,y+1))^{-1}\,,\qquad b=\sup_x(P(x,x+1))<1\,.
\end{aligned} \tag{5.10}$$

Remark. The last two lines follow from the fact that by assumption: $P(x,y+z)\leqq P(x,y)P(y,y+z)$, P is reversible with respect to Π and Π bounded by the counting measure from above and below. Furthermore $|f(x+y)-f(x)|$ can be bounded by the resistance between x and $x+y$. We leave the straightforward details to the reader. (Compare (4.12)! and Prop. 4a, case 2).

Step 2. We start by showing the first inequality of (5.6). (The second one is implied by (5.5)). We shall study $\{\mathscr{L}(E(L^\delta(x,n)|\omega)n^{-\alpha})\}_{n\in\mathbb{N}}$, in the case where $x=x(n)\leqq\sigma n$ for some $\sigma<1$. Due to stationarity and ergodicity of $\left\{\dfrac{1}{P(x,x+1)}\right\}_{x\in\mathbb{Z}}$ and due to the assumption that $\{L(c_{0,n}n^{-\alpha})\}_{n\in\mathbb{N}}$ has nontrivial weak limit points, we conclude that for some $\varepsilon>0$: $\lim_{n\to\infty}\mathrm{Prob}\,(\omega|c_{-[\sigma n],-n}\geqq\varepsilon n^\alpha, c_{[\sigma n],n}\geqq\varepsilon n^\alpha)>0$. This proves with (5.10) second line and with (3.5) in connection with (3.37) that uniformly for $x=x(n)\leqq\sigma n$:

$$\liminf_{n\to\infty}[\mathrm{Prob}\,(\omega|E(L(x,n)|\omega)\geqq\varepsilon' n^\alpha)]>0\,,\qquad \varepsilon'>0\,. \tag{5.11'}$$

From here the first inequality in (5.6) follows now immediately. It remains to show (5.4) and (5.5).

Step 3. In order to show (5.4) and (5.5) we shall define in (5.13) below a suitable sequence $a_n^{\{x\}}(\omega)$ and write $L^\delta(x,n)n^{-\alpha}=(L^\delta(x,n)(a_n^{\{x\}}(\omega))^{-1}(a_n^{\{x\}}(\omega)n^{-\alpha})$. Then (5.14) in the Lemma 5.2 below tells us that the first factor converges weakly to exp (1) and the expectation to 1. Next (5.18), (5.19) in Lemma 5.3 below tell us that the second factor is bounded above on A_n^δ by $C\delta$ for $|x|\leqq n$. This proves in connection with (5.11′) immediately our assertions (5.4), (5.5).

Lemma 5.2. *Suppose* $(X_j)_{j\in\mathbb{N}}$ *is a Markov chain on* $\mathbb{Z}$ *which is recurrent. Define:*

(5.12) $$q_n^{\{x\}}=\operatorname{Prob}(\{X_j^{\{x\}} \text{ returns to } \{x\} \text{ before reaching } \mathscr{C}[-n,n]\})$$

(5.13) $$a_n^{\{x\}}:=(1-q_n^{\{x\}})^{-1}.$$

If $a_n^{\{x\}}\underset{n\to\infty}{\longrightarrow}+\infty$ *then* ($\mathscr{L}, E$ *with respect to* $(X_k)_{k\in\mathbb{N}}$)

(5.14) $$\mathscr{L}\left(\frac{L(x,n)}{a_n^{\{x\}}}\right)\underset{n\to\infty}{\Rightarrow}\exp(1),\qquad E\left(\frac{L(x,n)}{a_n^{\{x\}}}\right)\underset{n\to\infty}{\longrightarrow}1$$

Proof. Remember

(5.15) $$E(e^{sY})=\frac{1}{s+1}\quad\text{for}\quad \mathscr{L}(Y)=\exp(1).$$

We have:

(5.16) $$E(\exp(-sL(x,n)))=\frac{1-q_n}{1-e^{-S}q_n}$$

and therefor for $a_n\to+\infty$ we can calculate as follows:

(5.17) $$E\left(\exp\left(-\frac{sL(x,n)}{a_n}\right)\right)=\frac{1-q_n}{1-q_ne^{-s/a_n}}\underset{n\to\infty}{\longrightarrow}\frac{1}{(s+1)}.$$

This proves the first part of (5.14), the second part is obtained again by explicit calculation.

Lemma 5.3. *For our chain* $(X_k^{\{x\}})_{k\in\mathbb{N}}$ *we have the relations:*

Denote by f_x *is the solution of* $P^*f_x-f_x=2 1_{\{x\}}, f_x(x)=0$ *constructed in Proposition 3′* ($f_x(y)>0$ *for* $y\neq x$, *see* (3.27)!). *Then for* $C_2(\omega)$ *as defined in* (5.10):

(5.18) $$(a_n^{\{x\}})\leqq C_2(\omega)+\max(f_x(-n),\ f_x(n)),$$

(5.19) $$\sup_{|x|\leqq n}\max(f_x(n), f_x(-n))\leqq\sum_{-n}^{+n}\frac{C}{P(y,y+1)}.$$

Proof. Since f_x is subharmonic we obtain by applying (5.9) and (5.10) third line with elementary calculations:

(5.20) $$(1-q_n(x))\geqq((\max f_x(-n),\ f_x(n))+C_2(\omega))^{-1},$$

which yields with the definition of $a_n^{\{x\}}$ in (5.13) the assertion (5.18).

The relation (5.19) is an immediate consequence of (3.5) and (3.35), (3.36). q.e.d.

As a Corollary of the proof of Proposition 7 we observe that:

Corollary 7. *Under the assumptions of Proposition 7 we have for every* $c \in \mathbb{R}^+$:

(5.21) $$\mathscr{L}(T(cn)n^{-2}) \underset{n\to\infty}{\longrightarrow} \delta_\infty \quad \omega\text{-a.s.}$$

We can replace here assumption (3.23) *by* $\mathscr{L}(c_{0,x/|x|}) \Rightarrow \delta_\infty$, *which is equivalent to* $E_\omega\left(\frac{1}{P(0,1)}\right) = \infty$.

Proof. The observation to make here is, that due to (3.5) and (3.40) the solution to $(P^* - I)(\cdot) = 2 1_{\{x\}}$ we constructed in Proposition 3′ has the property:

(5.22) $$f(n)n^{-1} \underset{n\to\infty}{\Rightarrow} +\infty \quad \omega\text{-a.s.}$$

With this relation plugged into the estimates of this paragraph we arrive at (5.22). We leave the straightforward but tedious details to the reader.

6. The behaviour of $\hat{X}_k$ *for fixed* ω *in the subdiffusive case*

The dynamics in the case $E_\omega\left(\frac{1}{p(0,1)}\right) = +\infty$ and $p(0,x) = ce^{-r|x|}$ looks roughly as follows: The dynamics of the process is determined by the fact that $\mathscr{C}A$ contains intervals of the size $|\log(1-q)|^{-1}\log n$. To cross these gaps the random walk needs a certain amount of macroscopic time (namely $n^{1+\alpha}, \alpha > 1$). However in between these large gaps the walk moves on a faster scale. The difficulty of the analysis is now rooted in the fact that in order to cross gaps of size $[|\log(1-q)|^{-1}\log(n) - a]$ we need also a macroscopic amount of time namely $\varepsilon n^{1+\alpha}$ steps with $\varepsilon = e^{-ra}$. Since a can be arbitrarily large we obtain "in the limit" a set of gaps which are dense in the macroscopic space. Furthermore is the structure of the set A close to these large gaps of importance for the ability of the walk to cross a large gap.

In order to overcome this difficulty we consider first *auxiliary processes where all gaps are of size* $|\log(1-q)|^{-1}\log(n) - a$ with $a \leqq c < \infty$. We analyse these auxiliary processes first and then in the second step we send c to $+\infty$. Finally we incorporate the local structure close to the gaps into our picture.

If we have only large gaps we describe for $n \to \infty$ the motion of the walker by a Markov chain governing the transition from one of the intervals between large gaps to the other. Inside such an interval the position of the walk should be uniformly distributed. (Since it looks here like (reflected) Brownian motion provided we use the scaling $x \to \frac{1}{n}x, t \to n^2 t$.) Taking $c \uparrow +\infty$ means our subdivision of macroscopic space into intervals becomes finer and finer resulting in the picture given in the theorem.

We proceed now as follows: First we consider auxiliary processes namely a walker on a set derived from $\mathbb{Z}$ by locating gaps at points $[x_i n]$ of size $[|\log(1-q)|^{-1}\log n] - a_i, a_i \leqq c$ and let n tend to infinity, later we let c tend to infinity. This is in Sect. 6a. In a second Sect. 6b we incorporate the local structure of A close to large gaps. These results are then connected with our original problem in a third Sect. 6c. In the fourth Sect. 6d we collect the proofs of the results related to the potential theory of the involved Markov chains.

a) A sequence: $(\bar{X}_k^{\varepsilon,n})_{k\in\mathbb{N}}$ of auxiliary walks on a set with large gaps only. The auxiliary processes we shall need are random walks with transition kernel $p(x,y)$ restricted to a set $A^{\varepsilon,n}$ containing only large gaps. In order to define these sets $A^{\varepsilon,n}$ we shall make use of the following ingredients:

Given are sequences $(x_i^\varepsilon)_{i\in\mathbb{Z}}$, $(a_i^\varepsilon)_{i\in\mathbb{Z}}$ with the following properties:

$$(6.1)\qquad (x_i^\varepsilon\in\mathbb{R}, \quad x_{i+1}^\varepsilon > x_i^\varepsilon \quad \forall i\in\mathbb{Z}, \quad |\{i|x_i^\varepsilon\in[a,b]\}|<\infty \quad \forall(a,b)\in\mathbb{R}^2$$

$$(6.2)\quad a_i^\varepsilon\in\mathbb{Z}\,; \quad a_i^\varepsilon\leqq\frac{|\log\varepsilon|}{r}, \quad \sum_k \exp(-r|k|-ra_{i+k}^\varepsilon)<\infty \quad \forall i\in\mathbb{Z}, \quad \varepsilon\in(0,1).$$

The sets $A^{\varepsilon,n}$ are now defined as follows: (Set $p=|\log(1-q)|^{-1}$)

$$(6.3)\qquad A^{\varepsilon,n}=\mathbb{Z}\Big\backslash\Big(\bigcup_{i\in\mathbb{Z}}([nx_i^\varepsilon],\ [nx_i^\varepsilon+(p\log(n)-a_i^\varepsilon)^+])\Big).$$

This means that we place gaps of a fixed size at the locations given by the x_i^ε and then we blow up the picture in a proper fashion. Note that the possible overlap between the intervals we remove from $\mathbb{Z}$, becomes empty for any set of the form $[-an, an]$ if n is large enough, that is we can forget this effect since we want to study the case $n\to\infty$.

We denote by $(\bar{X}_k^{\varepsilon,n})_{k\in\mathbb{N}}$ the jump chain of the random walk restricted to $A^{\varepsilon,n}$ that is this random walk has the following transition matrix:

$$(6.4)\qquad (1_{A^{\varepsilon,n}}(x)p(x,y)1_{A^{\varepsilon,n}}(y))\left(\sum_{y\in A^{\varepsilon,n}} p(x,y)\right)^{-1}.$$

We make use of the following abbreviation ($[y]$: largest integer below y for $y\in\mathbb{R}$).

$$(6.5)\qquad I_{i+1}^{\varepsilon,n}=([nx_i^\varepsilon],[nx_{i+1}^\varepsilon]] \qquad I_i^\varepsilon=(x_i^\varepsilon,x_{i+1}^\varepsilon].$$

Our aim is to describe $\text{Prob}(\bar{X}_{[kn^{1+\alpha}]}^{\varepsilon,n}\in I_i^{\varepsilon,n})$ for $n\to\infty$. The description of these limiting probabilities will be given via a Markov procen $(Y_t^\varepsilon)_{t\in\mathbb{R}^+}$ on $(x_i^\varepsilon)_{i\in\mathbb{Z}}$, which describes the transition from one interval I_i^ε to the other. The process Y_t^ε will be defined with the help of the function introduced below.

Define $\hat{a}_i^\varepsilon=\exp(-ra_i^\varepsilon)$ and then set:

$$(6.6)\qquad h^\varepsilon(x)=\begin{cases}\displaystyle\sum_{i=0}^{k(x)}\hat{a}_i^\varepsilon & k(x)=\sup(j|x_j^\varepsilon<x) \quad \text{for} \quad x>0\\ \displaystyle-\sum_{i=0}^{k'(x)}\hat{a}_i^\varepsilon & k'(x)=\inf(j|x_j^\varepsilon>x) \quad \text{for} \quad x<0.\end{cases}$$

With this function $h^\varepsilon(\cdot)$ we can define a transition kernel $q^\varepsilon(i,j)$ on $\mathbb{Z}\times\mathbb{Z}$ as follows:

$$(6.7)\qquad q^\varepsilon(j,j-1)=\frac{h^\varepsilon(x_{j+1}^\varepsilon)-h^\varepsilon(x_j^\varepsilon)}{h^\varepsilon(x_{j+1}^\varepsilon)-h^\varepsilon(x_{j-1}^\varepsilon)}$$

$$q^\varepsilon(j,j+1)=\frac{h^\varepsilon(x_j^\varepsilon)-h^\varepsilon(x_{j-1}^\varepsilon)}{h^\varepsilon(x_{j+1}^\varepsilon)-h^\varepsilon(x_{j-1}^\varepsilon)}$$

First we define a process on $\mathbb{Z}$ and then use this process to define the one on the intervals where we identify the intervals with the right endpoint.

Definition 4. The process $(\tilde{Y}_t^\varepsilon)_{t\in\mathbb{R}^+}$ is a Markov process on $\mathbb{Z}$ with the following transitions and jumprates: (The constant c appearing below will be specified later on)

(6.8) transitions: according to $q^\varepsilon(i,j)$

$$\text{jumprate in } i\in\mathbb{Z}: (\hat{a}_i^{-1}+\hat{a}_{i+1}^{-1})(|x_{i+1}^\varepsilon - x_i^\varepsilon|)^{-1}c, \qquad c\in\mathbb{R}^+.$$

The process Y_t^ε is now simply defined as $Y_t^\varepsilon = x_{\tilde{Y}_t^\varepsilon}^\varepsilon$. □

Define the following sets of subsequences of $\mathbb{N}$: $(n_j)\in N_k$ if $[\log n_j]-\log n_j$ converges to $-k$.

Proposition 8. *Assume that* $p(0,x)=\text{const}\exp(-r|x|)$ *and* $\alpha^{-1}=\frac{|\log(1-q)|}{r}<1$.

Then the following holds: (We suppres the lenghty explicit form for c below). For every $k\in[0,1)$ *there exists* $c\in(0,\infty)$ *such that with choosing that c in* (6.8) *we have:*

$$\text{(6.9)}\qquad \operatorname{Prob}\left(\frac{1}{n}\bar{X}_{[tn^{1+\alpha}]}^{\varepsilon,n}\in(x_i^\varepsilon, x_{i+1}^\varepsilon]\right)\underset{\substack{n\to\infty\\ n\subseteq N_k}}{\longrightarrow}\operatorname{Prob}(\tilde{Y}_t^\varepsilon=i+1)=\operatorname{Prob}(Y_t^\varepsilon=x_{i+1}^\varepsilon)$$

If $l(\cdot)$ *denotes the Lebesgue measure and B a Borelset contained in* $(x_i^\varepsilon, x_{i+1}^\varepsilon]$ *and* $s(n)=o(n^{1+\alpha}); s(n)\gg n^2$, *then:*

$$\text{(6.10)}\qquad \operatorname{Prob}\left(\frac{1}{n}\bar{X}_{[tn^{1+\alpha}+s(n)]}^{\varepsilon,n}\in B\,\Big|\,\frac{1}{n}\bar{X}_{[tn^{1+\alpha}]}^{\varepsilon,n}\in(x_i^\varepsilon, x_{i+1}^\varepsilon]\right)\underset{\substack{n\to\infty\\ n\subseteq N_k}}{\longrightarrow} l(B)/_{|x_{i+1}^\varepsilon - x_i^\varepsilon|}.$$

To get an idea how to prove this observe: The limiting process is a birth and death Markov process. Approximately the process counting the index of the interval where $\bar{X}_k^{\varepsilon,n}$ sits moves to nearest neighbours only, at least in the scale we use (Lemma 6.3 below). So we need to know the time our process spends in an interval and the probabilities to leave it to the left or the right. For this purpose we use of course our harmonic and subharmonic functions from Proposition 3, 3′. The assumptions made there are easily checked following the arguments in the proof of Proposition 5, assertion 1, we leave this straightforward modification to the reader.

So in order to prove our Proposition 8, a minutes thought shows that it suffices ([2] chap. 4, Theor. 2.6)) to show the following four Lemmata: ((6.1) proves (6.10) while (6.2) and (6.3) proves (6.9)).

Lemma 6.1. *Denote by* (Y_t) *Brownian motion in* $[x_i^\varepsilon, x_{i+1}^\varepsilon]$ *with reflection at the boundary and diffusion constant* σ^2.

Then for $\sigma=\left(\sum_y p(0,y)y^2\right)^{1/2}$ *we have:*

$$\text{(6.11)}\qquad \mathcal{L}\left(\left(\frac{1}{n}\bar{X}_{s+tn^2}^{\varepsilon,n}\right)_{t\in\mathbb{R}^+}\Big|\bar{X}_s^{\varepsilon,n}=[yn]\in I_i^{\varepsilon,n}\right)\Rightarrow\mathcal{L}((Y_t^{(y)})_{t\in\mathbb{R}^+}),\qquad \forall i\in\mathbb{Z}$$

Lemma 6.2. *Denote by $T_j^{\varepsilon,n}(x)$ the exit time of $\bar{X}_k^{\varepsilon,n}$ from $I_j^{\varepsilon,n}$ that is:*

(6.12) $$T_j^{\varepsilon,n}(x)=\inf(k|\bar{X}_k^{\varepsilon,n}\in\mathscr{C}I_j^{\varepsilon,n}) \quad \textit{for } \bar{X}_0^{\varepsilon,n}=x\in I_j^{\varepsilon,n}.$$

Consider now subsequences of $\mathbb{N}$ *contained in* N_i.

Then for each $i\in[0,1)$ there exist $c\in(0,\infty)$ independent of j such that uniform in $x\in I_j^{\varepsilon,n}$:

(6.13) $$\mathscr{L}\left(\frac{T_j^{\varepsilon,n}(x)}{n^{1+\alpha}}\right)\underset{\substack{n\to\infty\\ n\subseteq N_i}}{\Rightarrow}\exp(c|x_{j+1}^\varepsilon-x_j^\varepsilon|^{-1}(\hat{a}_{j+1}^{-1}+\hat{a}_j^{-1})).$$

Lemma 6.3.

(6.14) $$\sum_{j:|j-l|>1}\ \sum_{k\leqq tn^{1+\alpha}}\operatorname{Prob}(\bar{X}_{k+1}^{\varepsilon,n}\in I_j^\varepsilon|\bar{X}_k^{\varepsilon,n}\in I_l^\varepsilon)\underset{n\to\infty}{\longrightarrow}0 \quad \forall t\in\mathbb{R}^+$$

where $I_k^\varepsilon=(x_k^\varepsilon,x_{k+1}^\varepsilon]$.

Proof. Obvious which the help of (5.12), (5.13) and (5.19) combined with the fact that in any given interval I on $\mathbb{R}: x_{i+1}^\varepsilon-x_i^\varepsilon\geqq\delta(I)>0$.

Lemma 6.4. *We construct, according to the device given in Proposition 3* (3.3), *harmonic functions $h^{\varepsilon,n}(\cdot)$ for $(\bar{X}_k^{\varepsilon,n})_{k\in\mathbb{N}}$.*

Then we can achieve after multiplying $h^{\varepsilon,n}(\cdot)$ by a factor depending possibly on n that: (Call the resulting function $\tilde{h}^{\varepsilon,n}$)

(6.15) $$n^{-\alpha}\tilde{h}^{\varepsilon,n}(nx)\underset{n\to\infty}{\longrightarrow}h^\varepsilon(x) \textit{ in the Skorohod metric}.$$

Denote by $\nu^{\varepsilon,n}=\mathscr{L}(n^{-1}X_{T_j^{\varepsilon,n}(x)})$, then (6.15) *can be strenghened to:*

(6.15′) $$\int|n^{-\alpha}\tilde{h}^{\varepsilon,n}(ny)-h^\varepsilon(y)|d\nu^{\varepsilon,n}(y)\underset{n\to\infty}{\longrightarrow}0.$$

The proofs of 6.1, 6.2 and 6.4 need no fundamentally new ideas beyond the techniques from Chaps. 3 and 5, we defer these proofs, potential theoretic in spirit, to the last Sect. 6d of this chapter.

The next step is now to study what happens with Y_t^ε if we let ε tend to 0, which corresponds to refine the subdivision (x_i^ε). Consider therefor the following situation: given is $\{(x_i^\varepsilon)_{i\in\mathbb{Z}},(a_i^\varepsilon)_{i\in\mathbb{Z}}\}_{\varepsilon\in(0,1]}$ such that (in addition to (6.1) and (6.2)) the conditions (6.16) to (6.19) below hold. Define $I(x)=\{i|x_i^\varepsilon\in[-x,x]\}$. Here are the conditions:

(6.16) $$\{x_i^{\varepsilon'}|i\in\mathbb{Z}\}\subseteq\{x_i^\varepsilon|i\in\mathbb{Z}\} \quad \text{for } \varepsilon'<\varepsilon,$$

(6.17) $$\sup_{i\in I(x)}(x_{i+1}^\varepsilon-x_i^\varepsilon)\underset{\varepsilon\to0}{\longrightarrow}0 \quad \text{for all } x\in\mathbb{R}^+;$$

(6.18) $$\left(\bigcup_{\varepsilon>0}\{x_i^\varepsilon|i\in\mathbb{Z}\}\right) \quad \text{is countable},$$

(6.19) $$\sup_{\varepsilon>0}\left(\sum_{i\in I(x)}\hat{a}_i^\varepsilon\right)<\infty, \qquad \text{where again } \hat{a}_i^\varepsilon=e^{-ra_i^\varepsilon}.$$

Then we can define

(6.20) $$h(x)=\lim_{\varepsilon\to0}\left(\sum_{i:0\leqq x_i^\varepsilon<x}\hat{a}_i^\varepsilon\right) \quad \text{for } x>0, \quad (\text{analogues for } x\leqq0).$$

Let $Y(t)$ be Brownian motion with diffusion constant c (as appears in (6.8)) and $L(t,x)$ it's local time. Denote by $V(t)$ the time transformation $V(t)=\int L(t,h(x))dx$. If we replace h by h^ε in these formulas we can write $\mathscr{L}(Y_t^\varepsilon)=\mathscr{L}((h^\varepsilon)^{-1}Y((V^\varepsilon)^{-1}(t)))$ as an elementary calculation shows (compare [8]). It is now no surprise that for $\varepsilon\to 0$:

Proposition 9. *Assume that* $(x_i^\varepsilon)_{i\in\mathbb{Z}}, (a_i^\varepsilon)_{i\in\mathbb{Z}}$ *fulfill* (6.16)–(6.19). *We have for* $Y_t^\varepsilon := x^\varepsilon_{\hat{Y}_t^\varepsilon}$:

$$\mathscr{L}((Y_t^\varepsilon)_{t\in\mathbb{R}^+}) \underset{\varepsilon\to 0}{\longrightarrow} \mathscr{L}((h^{-1}(Y(V^{-1}(t))))_{t\in\mathbb{R}^+}). \tag{6.21}$$

Proof. This is a consequence of a limit theorem for birth and death processes of C. Stone. The details of this reduction are (besides notation) the same as in Kawazu-Kesten [8] on page 565–567, furthermore the result is very intuitive we therefor refer the reader to that paper.

b. A refined carricature $(\hat{X}_k^{\varepsilon,n})$ *of a walk on a random set.* The last two Propositions allow us to control a walk passing through a medium with large gaps, which are macroscopically well separated. That means that so far we have not accounted for the fact that the medium may look as follows around the large gap:

$$\begin{array}{c} \downarrow \qquad\qquad\qquad\qquad \downarrow \\ x \cdot x\,x\,x\,x\,(\cdot\cdot\cdot\cdot\cdot\cdot)\;x\,x \cdot x\,x\,. \\ x \text{ large gap } x+1 \end{array}$$

The two one point gaps marked have of course quite an influence on the behaviour of the random walker and his ability to cross the large gap. We incorporate this effect into a new carricature $(\hat{X}_k^{\varepsilon,n})_{k\in\mathbb{N}}$ of our process $\hat{X}_k$. Loosely speaking we consider a set with large gaps and possibly small gaps close to these large gaps: we go back to our kernel $p(x,y)$ on the random set A and we obtain a carricature by *filling the little gaps far away from large ones and making only nearest neighbour steps.*

Precisely: The information about the above mentioned small trouble spots is hidden in the harmonic function we constructed in Proposition 3, 5. We define therefor: (h denotes the harmonic function for $\hat{p}(x,y)$ based on the choice $\Pi(x) = e(\tilde{x})$, (here $x\leftrightarrow\tilde{x}$ relabelling A with $\mathbb{Z}$), constructed in Prop. 3)

$$\tilde{c}_x = \frac{h(x+1)-h(x)}{(\hat{p}(x,x+1))^{-1}}. \tag{6.22}$$

Definition 5. We denote by $(\hat{X}_k^{\varepsilon,n})$ the Markov chain on $\mathbb{Z}$ defined by $\hat{h}^{\varepsilon,n}$ according to the Eq. (6.7) where $\hat{h}^{\varepsilon,n}(x)$:

$$\hat{h}^{\varepsilon,n}(y) := 2y + \int_0^y \tilde{c}_x d(h^{\varepsilon,n}(x)), \tag{6.23}$$

$$\text{with: } h^{\varepsilon,n}(x) = \begin{cases} \sum_0^{x-1} \dfrac{1}{\hat{p}(y,y+1)}\, 1_{\{\hat{p}(y,y+1)\le(\varepsilon n)^{-\alpha}\}}, & x>0 \\ 0, x=0 \\ -\sum_x^{-1} \dfrac{1}{\hat{p}(y,y+1)}\, 1_{\{\hat{p}(y,y+1)\le(\varepsilon n)^{-\alpha}\}}, & x<0. \end{cases}$$

We know by (3.37), (3.40) that for $n \geqq n_0$, $\varepsilon \leqq \varepsilon_0$:

(6.24) $$0 < \underline{c} < \tilde{c}_x \leqq \bar{c} < \infty \quad \text{for } x \text{ with: } h^{\varepsilon,n}(x+1) \neq h^{\varepsilon,n}(x).$$

Note that the process $\{\tilde{c}_x(\omega)\}_{x\in\mathbb{Z}}$ is not i.i.d. (it is stationary of course) and it's appearance is due to the fact that the restricted chain is not a nearest neighbour chain after the relabelling.

It is easy to check here by explicit calculation (nearest neighbour steps!) that again the assumptions of Proposition 3 are fulfilled and if we apply the construction of Proposition 3 to $\hat{X}_k^{\varepsilon,n}$ we recover as our harmonic function h the functions $\hat{h}^{\varepsilon,n}$. (This is checked immediately by explicit calculation using (3.3)γ.) The first task is now to use the ideas of part a) of this section to control the behaviour of $\left(\frac{1}{n}\hat{X}_{tn^{1+\alpha}}^{\varepsilon,n}\right)$, since it is the process we hope to be a good approximation for n large, ε small to the rescaled process $\frac{1}{n}\hat{X}_{[tn^{1\alpha}]}$, the one we are interested in finally (that is the one with kernel $\hat{p}(x, y)$).

In the next Proposition we determine the behaviour of $\hat{X}_k^{\varepsilon,n}$) for $n\to\infty$, $\varepsilon\to 0$ in terms of a *time transformed Brownian motion* $Z_t^{\varepsilon,n}$, of the type which occurred already in Proposition 9. Knowing Proposition 9 it is no surprise that we shall need the following ingredients:

(6.25) $$Z_t^{\varepsilon,n} = (\bar{h}^{\varepsilon,n})^{-1}(Y((\bar{V}^{\varepsilon,n})^{-1}(t))) \quad (f^{-1}(u) = \inf(t|f(t)=u)),$$

where the quantities on the right are defined as follows:

(6.26) $$\bar{h}^{\varepsilon,n}(x) = n^{-\alpha} \sum_{0}^{[n,x]-1} \tilde{c}_y(\hat{p}(y, y+1))^{-1} 1_{\{\hat{p}(y,y+1)\leqq(\varepsilon n)^{-\alpha}\}},$$

$$x>0, \quad (\tilde{c}_y \text{ as in } (6.22)).$$

For $x \leqq 0$ sum from $-[nx]$ to -1 and multiply by -1.

$$\bar{V}^{\varepsilon,n}(t) = \int L(t, \bar{h}^{\varepsilon,n}(x))dx,$$

here Y_t = Brownian motion with variance $\sigma^2 = 1$ and $L(t, x)$ it's local time.

Comparing $(\hat{X}_k^{\varepsilon,n})$ with $(Z_t^{\varepsilon,n})$ makes of course only sense if we known more about the behaviour of $(Z_t^{\varepsilon,n})$ for $n\to\infty$, $\varepsilon\to 0$. This is the case since it turns out for this purpose we need only information about $\bar{h}^{(\varepsilon,n)}(\cdot)$ which is closely related to the wellstudied resistance. This information will be provided later on in the Proposition 11. Note that $Z_t^{\varepsilon,n}$ moves on the subsequence of points x with the property that the resistance between x and $x+1 = c_{x,x+1} \geqq \varepsilon n^\alpha$.

Proposition 10. *Under the assumptions of Theorem 3 we have* ($\hat{E} := E(\cdot|\omega)$) *for n running through a sequence in some N_i:*

For all $\eta > 0$, $f \in \mathscr{C}_0(\mathbb{R})$:

(6.27) $$\lim_{\varepsilon\to 0}\left(\overline{\lim_{n\to\infty}} \operatorname{Prob}\left(\omega \left| \hat{E}f\left(\frac{1}{n}\hat{X}_{[tn^{1+\alpha}]}^{\varepsilon,n}\right) - \hat{E}f(Z_t^{\varepsilon,n})\right| \geqq \eta\right)\right) = 0.$$

The convergence for $\varepsilon\to 0$ is uniform in n.

Proof. First we use the scheme of Proposition 8 developed in (6.7)–(6.8) and try to approximate for fixed ε and n large the process $(\hat{X}_k^{\varepsilon,n})_{k\in\mathbb{N}}$ by a process $(\hat{Y}_t^{\varepsilon,n})_{t\in\mathbb{R}^+}$

(a particular version of $Z_t^{\varepsilon,n}$) which lives on the right endpoints of the rescaled intervals $(x_i^{\varepsilon,n}, x_{i+1}^{\varepsilon,n}]$, where $(x_i^{\varepsilon,n})_{i\in\mathbb{Z}}$ counts the points $x\in\mathbb{Z}$ with $\hat{p}(x,x+1)\leq \varepsilon n)^{-\alpha}$. The additional technical problem we have here compared with Proposition 8 is that $(x_i^{\varepsilon,n})_{i\in\mathbb{Z}}$ do depend on n now. The main purpose of the following is to show how to handle this problem.

Step 1. To introduce $\hat{Y}_t^{\varepsilon,n}$ consider now the rescaled positions of the gaps that is introduce: $\bar{x}_i^{\varepsilon,n}=n^{-1}x_i^{\varepsilon,n}$ and the intervals $(\bar{x}_i^{\varepsilon,n}, \bar{x}_i^{\varepsilon,n}+1]$. Furthermore define $\hat{a}_i^{\varepsilon,n} = \tilde{c}_y a_i^{\varepsilon,n}$ where $y=x_i^{\varepsilon,n}$ and $a_i^{\varepsilon,n}=n^{-\alpha}(\hat{p}(x_i^{\varepsilon,n}, x_i^{\varepsilon,n}+1))^{-1}$. Now define $(\hat{Y}_t^{\varepsilon,n})$ by (6.6)–(6.8), in the latter formula $c=1$.

Step 2. Here we construct a process Y_t^{ε}, in a sense the limit of $Y_t^{\varepsilon,n}$, in order to get rid of the n-dependence of the $(\bar{x}_i^{\varepsilon,n})_{i\in\mathbb{Z}}$: The Proposition 2b tells us that $\mathscr{L}((\bar{x}_i^{\varepsilon,n}, a_i^{\varepsilon,n})_{i\in\mathbb{Z}})$ converges for $n\to\infty$, $n\subseteq N_\theta$. In fact we use here a stronger version, which says that $\mathscr{L}((n^{-\alpha}c_{0,[nx]})_{x\in\mathbb{R}})$ converges for $n\to\infty$, n running through a subsequence in N_θ to a stable process with index α^{-1} (this is of course with (2.32) a classical result, we just quote here). We shall prove later on in a Corollary to Proposition 11 (6.31) that this implies the convergence of $\mathscr{L}((\bar{x}_i^{\varepsilon,n}, \hat{a}_i^{\varepsilon,n})_{i\in\mathbb{Z}})$. We denote the jump points and jump heights of the limiting object by $(\bar{x}_i^{\varepsilon}, \hat{a}_i^{\varepsilon})_{i\in\mathbb{Z}}$ here. As a consequence ([8], p. 567) we can construct $((\bar{x}_i^{\varepsilon,n}, \hat{a}_i^{\varepsilon,n})_{i\in\mathbb{Z}})_{n\in\mathbb{N}}$, $(x_i^{\varepsilon}, \hat{a}_i^{\varepsilon})_{i\in\mathbb{Z}}$ on a common probability space such that the distribution for fixed n is the given one and such that in addition:

$$(\bar{x}_i^{\varepsilon,n}, \hat{a}_i^{\varepsilon,n}) \underset{n\to\infty}{\longrightarrow} (\bar{x}_i^{\varepsilon}, \hat{a}_i^{\varepsilon}) \text{ a.s.}, \qquad n\subseteq N_\theta. \tag{6.28}$$

Denote this big probability space by $\tilde{\Omega}$ and an element by $\tilde{\omega}$, that is $\tilde{\omega}=[((\bar{x}_i^{\varepsilon,n})_{i\in\mathbb{Z}})_{n\in\mathbb{N}}, (\bar{x}_i^{\varepsilon})_{i\in\mathbb{Z}}], [(\hat{a}_i^{\varepsilon})_{i\in\mathbb{Z}}, ((\hat{a}_i^{\varepsilon,n})_{i\in\mathbb{Z}})_{n\in\mathbb{N}}]$. These objects above define (see (6.7), (6.8)) for every n the scale and speedmeasure of $(\hat{Y}_t^{\varepsilon,n})_{t\in\mathbb{R}^+}$ respectively of a process $({}^{\theta}\hat{Y}_t^{\varepsilon})$ and therefor define these processes uniquely, if we choose in (6.8) the constant $c=1$.

Now we can of course conclude with standard arguments hat $\mathscr{L}((\hat{Y}_t^{\varepsilon,n})_{t\in\mathbb{R}^+}) \underset{n\to\infty}{\Rightarrow} \mathscr{L}(({}^{\theta}\hat{Y}_t^{\varepsilon})_{t\in\mathbb{R}^+})$ (if $n\subseteq N_\theta$). (Process moves to nearest neighbours only!)

It is easy to see with an explicit calculation that in fact $\mathscr{L}(\hat{Y}_t^{\varepsilon,n})=\mathscr{L}(Z_t^{\varepsilon,n})$ (compare [8], p. 566). Therefor we can conclude that: $(\tilde{E}=E(\cdot|\tilde{\omega}))$

$$\overline{\lim_{n\to\infty}} \left|\tilde{E}f\left(\frac{1}{n}\hat{X}_{tn^{1+\alpha}}^{\varepsilon,n}\right) - \tilde{E}f(Z_t^{\varepsilon,n})\right| \leq \overline{\lim_{n\to\infty}} \left(\left|\tilde{E}f\left(\frac{1}{n}\hat{X}_{tn^{1+\alpha}}^{\varepsilon,n}\right) - \tilde{E}f({}^{\theta}Y_t^{\varepsilon})\right|\right), \qquad n\subseteq N_\theta.$$

Step 3. In order to prove our assertion it remains to check first that for $\varepsilon\to 0$ the claims of Proposition 8 holds, if we replace $(\bar{X}_t^{\varepsilon,n}, Y_t^{\varepsilon})$, by $(\hat{X}_t^{\varepsilon,n}, \hat{Y}_t^{\varepsilon})$. As before we have to show that the modified versions of Lemma 6.2 and 6.4 hold. For (6.4) we just use (6.28) above together with Proposition 2b. This relation (6.28) replaces also (6.69) in the proof of (6.2) and the rest of the proof carries over. We leave the straightforward details to the reader. Having done this we can conclude now that with

$I_\varepsilon(f)=\{i|\bar{x}_i^{\varepsilon}\in\sup(f)\}$, we have for $f\in\mathscr{C}_0(\mathbb{R})$:

$$\overline{\lim_{n\to\infty}} \left|\tilde{E}f\left(\frac{1}{n}\hat{X}_{tn^{1+\alpha}}^{\varepsilon,n}\right) - \tilde{E}f({}^{\theta}Y_t^{\varepsilon})\right| \leq \sup_{i\in I_\varepsilon(f)} \left(\sup_{\bar{x}_i^{\varepsilon}\leq x\leq \bar{x}_{i+1}^{\varepsilon}} |f(\bar{x}_{i+1}^{\varepsilon})-f(x)|\right)$$

with $n\to\infty$ through N_θ.

But we know from Proposition 2b, especially the characterization of the limit points of $\mathscr{L}((n^{-\alpha}c_{0,[ny]})_{y\in\mathbb{R}})$ that:

$$\operatorname{Prob}\left(\omega \,\Big|\, \sup_{i\in I_\varepsilon(f)} (|\bar{x}_{i+1}^\varepsilon - \bar{x}_i^\varepsilon|) \leqq \delta\right) \underset{\varepsilon\to\infty}{\longrightarrow} 0 \quad \forall \delta > 0.$$

This proves of course our assertion (6.27), if we combine this result with Step 2.

Step 4. In order to see the for $\varepsilon \to 0$ uniform convergence in n, we simply observe that the $\hat{a}_i^\varepsilon$ can be realized by realizing an α^{-1}-stable process and looking for the jumps which are bigger than ε, similar $a_i^{\varepsilon,n}$ arises as the values of $(\hat{p}(x,x+1))^{-1}$ for $x = x_i^{\varepsilon,n}$ where $\hat{p}(x,x+1) \leqq (\varepsilon n)^{-\alpha}$. Now write all the processes in the form given in (6.25): Let $Y(t)$ be a Brownian motion with diffusion constant σ^2 and define

$$\hat{h}^\varepsilon(x) = \operatorname{sign}(x) \sum_{i\in I(x)} \hat{a}_i^\varepsilon \quad (I(x) = \{i \in \mathbb{N} | x >_{(<)} x_i^\varepsilon >_{(<)} 0),$$

$$\hat{V}^\varepsilon(t) = \int L(t, \hat{h}^\varepsilon(x))\, dx.$$

Then as in the end of Step 2

$$\mathscr{L}({}^\theta\hat{Y}_t^\varepsilon) = \mathscr{L}((\hat{h}^\varepsilon)^{-1} Y((\hat{V}^\varepsilon)^{-1}(t)).$$

Using $\hat{h}^{\varepsilon,n}$ instead of $\hat{h}^\varepsilon$, we can represent $\hat{X}_k^{\varepsilon,n}$ in the form given above. Observe that the expressions $\left(\sum_{i\in I(x)} \hat{a}_i^{\varepsilon,n}\right)$ are monotone in ε if we consider $\mathbb{R}^+$, $\mathbb{R}^-$ separately (decreasing for $x>0$, increasing for $x<0$) therefor the same monotonicity properties holds for $\hat{h}^{\varepsilon,n}$, in (6.23). This proves immediately the assertion of Proposition 10.

The next step is to study $Z_t^{\varepsilon,n}$ for $n\to\infty$, $\varepsilon\to 0$. A look at (6.25) shows that it is essential now to construct a limiting object of the harmonic functions $\bar{h}^{\varepsilon,n}(x)$ (compare (6.26)) which define the approximating Markov chains to our real process. Observe according to (6.23) and (6.26): $\bar{h}^{\varepsilon,n}(x) = n^{-\alpha} \int_0^x \tilde{c}_y d(h^{\varepsilon,n}(y))$ and we know by the Proposition 2 that $\mathscr{L}\{(n^{-\alpha}h^{\varepsilon,n}(ny))_{y\in\mathbb{R}}\}$ tends weakly to a stable process with index α^{-1} and where jumps smaller ε are omitted as $n\to\infty$ through a sequence in N_i. It remains therefor to study the behaviour of $(\tilde{c}_{[ny]})_y$ for $n\to\infty$ for given $h^{\varepsilon,n}(x)$, that is given location and size of large gaps. We are going to construct first a modified process $(c_x^{\delta,n})$ which takes into account that in the representation formula for $\bar{h}^{\varepsilon,n}(x)$ only those $\tilde{c}_x$ count where x is such that the resistance $c_{x,x+1}$ is of the order of magnitude of at least εn^α.

(6.28) $c_x^{\delta,n} := \tilde{c}_y$ y is the largest integer smaller or equal than x such that

$$c_{y,y+1} \geqq \delta n^\alpha.$$

It can be easily shown via (3.3) and [12] that $\tilde{c}_y$ is a measurable function of the medium. Below we give the asymptotic behaviour of these objects and in a corollary the consequences for our process $Z_t^{\varepsilon,n}$. We call a function stepfunction if it is of the form $\sum_{i\in\mathbb{Z}} a_i I_i$ where I_i are indicators of intervals with $UI_i = \mathbb{R}$. By $h^{\varepsilon,n}$ we denote the function introduced in (6.23) part two, which describes location and size of the large gaps. Denote by $I(g)$ the set of jumppoints of a stepfunction g on $\mathbb{R}$.

Proposition 11.

$$\mathscr{L}((c_{[nx]}^{\delta,n})_{x\in\mathbb{R}^+}|n^{-\alpha}h^{\delta,n}(n\cdot)=g) \underset{n\to\infty}{\Rightarrow} \bigotimes_{x\in I(g)} \mathscr{L}(c), \tag{6.29}$$

$\forall g := stepfunction\ on\ \mathbb{R}$,

$$\mathscr{L}(c)\ as\ in\ (1.14), \quad \operatorname{Supp}(\mathscr{L}(c)) \subseteq (0,\infty). \tag{6.30}$$

Corollary 11.

$$\mathscr{L}((\bar{h}^{\varepsilon,n}(n,x))_{x\in\mathbb{R}}) \underset{n\to\infty}{\overset{n\in N_i}{\Rightarrow}} \mathscr{L}\left(\left(\int_0^y c_x dY_{i,\varepsilon}^{\alpha}(x)\right)_{y\in\mathbb{R}^+}\right). \tag{6.31}$$

$Y_{i,\varepsilon}^{\alpha}$ and c_x are defined as follows: (recall $c_{x,y}$ denotes the resistance)

$$Y_{i,\varepsilon}^{\alpha}(y) = \sum_0^y (Y_i^{\alpha}(x) - Y_i^{\alpha}(x_-))1_{\{Y_i^{\alpha}(x)-Y_i^{\alpha}(x_-)\geqq\varepsilon\}}$$

with Y_i^{α} a stable process with index α^{-1} given as the limit of $\mathscr{L}((n^{-\alpha}c_{0,[ny]})_{y\in\mathbb{R}^+})$ for $n\to\infty$, n running through a subsequence in N_i.

$$(c_x)_{x\in\mathbb{R}^+}: \mathscr{L}(c_x|Y_i^{\alpha}) = \bigotimes_{x\in I(Y_i^{\alpha})} \mathscr{L}(c), \qquad \mathscr{L}(c)\ as\ in\ (1.14).$$

Furthermore we have for $\varepsilon\to 0$:

$$\mathscr{L}\left(\left(\int_0^y c_x dY_{i,\varepsilon}^{\alpha}(x)\right)_{y\in\mathbb{R}^+}\right) \underset{\varepsilon\to 0}{\Rightarrow} \mathscr{L}\left(\left(\int_0^y c_x dY_i^{\alpha}(x)\right)_{y\in\mathbb{R}^+}\right). \tag{6.32}$$

Proof. We consider here $\hat{p}(x,y)$ and its harmonic function h. Our aim is to analyse $\tilde{c}_x = (h(x+1)-h(x))\hat{p}(x,x+1)$ for x with $\hat{p}(x,x+1)\sim n^{-\alpha}$.

Step 1. First we recall a representation of $h(x+1)-h(x)$ suitable for our purposes. We start with $h(x+1)-h(x) = (a_{x+1,x}^+ - a_{x+1,x}^-) = (\hat{a}_{x+1,x}^+ - \hat{a}_{x+1,x}^-)$ $= -(\hat{a}_{x,x+1}^+ - \hat{a}_{x,x+1}^-)$ (this latter version is easier to handle notation wise) (see Prop. 3, (3.3) to recall definitions).

In order to analyse $(\hat{a}_{x,x+1}^+ - \hat{a}_{x,x+1}^-)$ we have to study $\zeta^{(x)}$ which denotes the minimal positive solution of $(\cdot)(I-\hat{p}) = \delta_x - \delta_{x+1}$. We start introducing some quantities we shall need to give a useful representation of $\zeta^{(x)}$.

$$v^{(x)} = 1_{[x+2,\infty)}(\cdot)\hat{p}(x,\cdot) \tag{6.33}$$

$$\mu^{(x)} = 1_{(-\infty,x]}(\cdot)\hat{p}(x,\cdot) \tag{6.34}$$

(6.35) $\eta^{(x)}, \xi^{(x)}$ are the minimal positive solution of:

$$(I-\hat{p})(\cdot) \geqq v^{(x)} - (\delta_x + \delta_{x+1})$$

$$(I-\hat{p})(\cdot) \geqq \mu^{(x)} - (\delta_x + \delta_{x+1})$$

$$\beta^{(x)}:\ -(\eta^{(x)}(I-\hat{p}) - v^{(x)}) = \beta^{(x)}\delta_{x+1} + \alpha^{(x)}\delta_x \tag{6.36}$$

$$\gamma(x):\ -(\xi^{(x)}(I-\hat{p}) - \mu^{(x)}) = \delta^{(x)}\delta_x + \gamma^{(x)}\delta_{x+1}.$$

We know that $\eta^{(x)}(A) = E([\#\text{ visits of } \hat{X}_k \text{ to } A \text{ before reaching } x \text{ or } x+1]\,1_{\{\hat{X}_1\geqq x+2\}}$ (see [4], part I, Lemma 2) similar $\xi^{(x)}$. From this we conclude immediately:

Lemma 6.5.

$$(6.37)\qquad \zeta^{(x)} = \sum_{j=0}^{\infty} (1-\hat{p}(x,x+1)-\beta^{(x)}-\gamma^{(x)})^j(\eta^{(x)}+\xi^{(x)})$$
$$= (\hat{p}(x,x+1)+\beta^{(x)}+\gamma^{(x)})^{-1}(\eta^{(x)}+\xi^{(x)}).$$

This last equation will now be analyzed, the first factor in Step 2, the second in Step 3.

Step 2. We are interested in the behaviour of $h(x+1)-h(x)$ for those x where $h(x+1)-h(x) \geqq \varepsilon n^{\alpha}$. For the analysis it is most convenient to pass to a new random set A_n, which is interpreted as medium on the event: at 0 starts a large gap of size $[(\log(1-q))^{-1}\log n]-a$. To be precise we introduce the following notation:

$$(6.38)\qquad \begin{aligned} &(\omega_i^-)_{i\in\mathbb{N}} \quad \text{i.i.d. Bernoulli with success probability } q\\ &(\omega_i^+)_{i\in\mathbb{N}} \quad \text{i.i.d. Bernoulli with success probability } q\\ &b\in\mathbb{R}^+ \quad b=ce^{-ra} \quad \text{with } a\in(-\infty,\log\varepsilon/r)\end{aligned}$$

The new medium is defined as follows:

$$\text{let}\quad p(0,x)=ce^{-r|x|},\ y(x)=x-\left(\left[\frac{1}{|\log(1-q)|}\log n\right]-a\right),\qquad \text{then}$$

$$(6.39)\qquad x\in A_n\begin{cases} x<0 & \text{and} \quad \omega_{|x|}^- = +1\\ x=0 & \text{or} \quad y(x)=0\\ y(x)>0 & \text{and} \quad \omega_{y(x)}^+ = +1.\end{cases}$$

Note that if we denote by $\hat{p}_n(x,y)$ the kernel on $\mathbb{Z}\times\mathbb{Z}$ induced by this new medium A_n (that is use definition (2.1), (2.28) and replace A by A_n) then we have:

$$(6.40)\qquad \hat{p}_n(0,1)=bn^{-\alpha} \quad \text{for } n\in N_\theta \quad \text{with } \theta=0.$$

For notational convenience we focus on $n\in N_0$ (that is $[|(\log(1-p))^{-1}|\log(n)] - |(\log(1-p))^{-1}|\log(n) \underset{n\in\mathbb{N}_0}{\longrightarrow} 0$), since $n\in N_i$ requires just another constant in (6.40)).

We shall write $\eta^{(n)}, \xi^{(n)}\dots$ if we talk about the quantities: $\eta^{(0)}, \xi^{(0)}$ in the medium A_n. The invariant measure for $\hat{p}_n$ (choosen according to the convention from Prop. 5) is denoted by $\Pi^{(n)}$. Then we can prove: (Sect. 6.d)).

Lemma 6.6. *(Assume always n runs through a sequence in N_0 here)*

$$(6.41)\qquad \gamma^{(n)}n^{\alpha} \underset{n\to\infty}{\longrightarrow} \gamma b$$

$$(6.42)\qquad \beta^{(n)}n^{\alpha} \underset{n\to\infty}{\longrightarrow} \beta b$$

$$(6.43)\qquad \|\cdot\|_\infty - \lim_{n\to\infty}\eta^{(n)} \equiv 0, \qquad \|\cdot\|_\infty - \lim_{n\to\infty}\xi^{(n)} = \xi, \qquad \|\cdot\|_\infty - \lim_{n\to\infty}\Pi^{(n)} = \Pi$$

$$(6.44)\qquad \beta = \left(\sum_{k=1}^{\infty} e^{-rk}\omega_k^+\right), \qquad \gamma = \left(\frac{1-e^{-r}}{1+e^{-r}}\left(1+\sum_{k=1}^{\infty} e^{-rk}\omega_k^-\right)\right)\beta.$$

If we combine Lemma 6.6 with Lemma 6.5 and the relation (6.40) we obtain for $n \subseteq N_0$: (a.s. with respect to $(\omega_i^-, \omega_i^+)_{i \in \mathbb{N}}$)

$$\hat{p}_n(0,1)\zeta^{(n)} \underset{n\to\infty}{\overset{\|\cdot\|_\infty}{\longrightarrow}} \frac{1}{1+\beta+\gamma}\xi \quad a.s. \tag{6.45}$$

The next task will be to study $\xi(y)/\Pi(y)$ for $y \to \pm\infty$.

Step 3. Since as $n \to \infty$, the point1 becomes a barrier for our process in the medium A_n, we expect ξ to be determined by the following transition kernel $\hat{q}(x, y)$:

$$q(y,z) := 1_{\mathscr{B}}(y)p(y,z)1_{\mathscr{B}}(z)\left(\sum_{z\in\mathscr{B}} p(y,z)\right)^{-1}, \tag{6.46}$$

$$\mathscr{B} = \{x \mid x<0, \omega_{|x|}^- = 1\} \cup \{0\}$$

$\hat{q}$ is derived from q according to (2.28)

(relabelling A_n by $\mathbb{Z}$).

Define $\bar{\Pi}(z) = \sum_{y\leq 0} p(\tilde{z}, y)\omega_y^-$ (again $z \leftrightarrow \tilde{z}$, relabelling A_n by $\mathbb{Z}$) and by $\bar{\xi}$ the minimal positive solution of $(I-\hat{q})(\cdot) = \mu^{(0)}\|\mu^{(0)}\|^{-1} - \delta_0$ (see (6.34)). We shall prove:

Lemma 6.7.

$$\lim_{y\to\infty}(\xi(y)/\Pi(y)) = 0, \qquad \lim_{y\to-\infty}(\xi(y)/\Pi(y)) = A, \tag{6.47}$$

$$A = \lim_{y\to-\infty}(\bar{\xi}(y)/\bar{\Pi}(y)) = \left(\sum_{k=0}^{\infty} ce^{-rk}\omega_k^-\right)^{-1}, \tag{6.48}$$

$$\left(\omega_0^- := 1,\ c := \frac{1-e^{-r}}{1+e^{-r}}\right).$$

As a consequence we obtain via (6.45) that for $n \subseteq N_0$:

$$\hat{p}_n(0,1)\left(\lim_{y\to+\infty}(\zeta^{(n)}(y)/\Pi^{(n)}(y)) - \lim_{y\to-\infty}(\zeta^{(n)}(y)/\Pi^{(n)}(y))\right) \tag{6.49}$$

$$\underset{n\to\infty}{\longrightarrow} \frac{1}{1+\beta+\gamma}A$$

$$\Pi^{(n)}(\tilde{z}) = \left(\sum_y p(z,y)1_{A_n}(z)1_{A_n}(y)\right), \qquad z \leftrightarrow \tilde{z} \text{ relabelling } A_n \text{ by } \mathbb{Z}.$$

Step 4. Now we are going to apply (6.49) to our original situation. Consider the sequence $\{x_i(n)\}_{i\in\mathbb{Z}}$ of sets, where for fixed n, $\{x_i(n)\}_{i\in\mathbb{Z}}$ labels the points where for some fixed $\varepsilon > 0$:

$$\hat{p}(x_i(n), x_i(n)+1) \leq (\varepsilon n)^{-\alpha}. \tag{6.50}$$

We denote by $b_{i,n}$ the quantity defined by the equation

$$\hat{p}(x_i(n), x_i(n)+1) = b_{i,n}n^{-\alpha} \tag{6.51}$$

and by $\beta_{i,n}$, $\gamma_{i,n}$ etc. the quantities $\beta^{(x_i(n))}, \ldots$, as defined in Step 1 (6.33) and (6.36).

Observe that due to Proposition 2b the distance $x_{i+1}(n)-x_i(n)$ is of the order n, that is $\mathscr{L}(n^{-1}(x_{i+1}(n)-x_i(n)))$ converges to a distribution with no atom at 0. With (6.44) and (6.48) we can therefor conclude that the quantities $(\beta_{i,n})_{i\in\mathbb{Z}}$, $(\gamma_{i,n})_{i\in\mathbb{Z}}$, $(A_{i,n})_{i\in\mathbb{Z}}$, are for different i asymptotically independent. Therefor we obtain for $n\subseteq N_0$:

$$\mathscr{L}\left(\left(\frac{1}{1+\beta_{i,n}+\gamma_{i,n}} A_{i,n}\right)_{i\in\mathbb{Z}} \Big| \{x_i(n)\}_{i\in\mathbb{Z}}\right) \underset{n\to\infty}{\Rightarrow} \bigotimes_{i\in\mathbb{Z}} \Gamma. \tag{6.52}$$

Γ is a probability measure on $\mathbb{R}^+$ with $\operatorname{supp}(\Gamma)\subseteq(0,\infty)$. $\mathscr{L}(\Gamma) = \mathscr{L}(((1+FG)F)^{-1})$ with

$$F=\left(1+\sum_{k=-1}^{-\infty} e^{rk} y_i\right)\frac{1+e^{-r}}{1-e^{-r}}, \qquad G=\sum_{k=1}^{\infty} e^{-rk} y_i, \qquad \mathscr{L}((y_i)_{i\in\mathbb{Z}})=\bigotimes_{\mathbb{Z}^1} \mathscr{B}(1,q).$$

The representation of h in terms of $\hat{a}^{+;-}_{x+1,x}$ and relation (6.45) together with the last relation in (6.43) and (6.52) allows now to go back to our original problem and to derive from the relation above:

$$\begin{aligned} &\mathscr{L}(((\hat{p}(x_i(n), x_i(n)+1)(h(x_i(n)+1)-h(x_i(n))))_{i\in\mathbb{Z}} | n^{-\alpha} h^{\delta,n}(n\cdot)=g) \\ &\underset{\substack{n\to\infty\\ n\subseteq N_0}}{\Rightarrow} \bigotimes_{i\in\mathbb{Z}} \Gamma, \qquad \forall g := \text{stepfunction on } \mathbb{R}. \end{aligned} \tag{6.53}$$

This proves of course our assertion in Proposition 11.

It remains to prove our Lemma 6.6 and 6.7. Since the arguments are closely related to the ones needed to prove 6.4, and potential are theoretic in spirit we shall prove them together in Sect. 6d.

c) Comparison between the carricature $\hat{X}^{\varepsilon,n}_k$ *and* $\hat{X}_k$. The last important step towards Theorem 3 consists in showing that the rescaled original chain $\left(\frac{1}{n}\hat{X}_{tn^{1+\alpha}}\right)$ and the auxiliary process $\left(\frac{1}{n}\hat{X}^{\varepsilon,n}_{\tilde{t}n^{1+\alpha}}\right)$ with $\tilde{t}=(E_\omega(\hat{e}(0)^{-1}))^{-1}t$ are for fixed t and very large n very close for most environments ω provided ε is sufficiently small. The transformation $t\to\tilde{t}$ takes into account the effect of the small gaps in the real medium which are *not* close to large gaps (remember $\hat{p}(x,x)>0$!). Precisely:

Proposition 12. *Under the assumptions of Theorem 3 the following holds* $(\hat{E}:=E(\cdot|\omega))$:

$$\lim_{\varepsilon\to\infty}\left[\overline{\lim_{n\to\infty}}\ \operatorname{Prob}\left(\omega\,\middle|\,\left|\hat{E}f\left(\frac{1}{n}\hat{X}_{tn^{1+\alpha}}\right)-\hat{E}f\left(\frac{1}{n}X^{\varepsilon,n}_{\tilde{t}n^{1+\alpha}}\right)\right|\geq\delta\right)\right]=0 \tag{6.54}$$

for all $f\in\mathscr{C}_0(\mathbb{R})$ *and* $\delta>0$.

The convergence for $\varepsilon\to 0$ *is uniform in* n.

The proof of this Proposition is based on the following two Lemmata which show that the space and time structure of both rescaled processes become very similar for large n and small ε.

Lemma 6.8. *Denote by d_I the Skorohod metric for functions on the interval $I \subseteq \mathbb{R}$. With $h(\cdot)$, $\hat{h}^{\varepsilon,n}(\cdot)$ we denote the harmonic functions belonging to $(\hat{X}_k)_{k\in\mathbb{N}}$, $(\hat{X}_k^{\varepsilon,n})_{k\in\mathbb{N}}$ which we constructed in Proposition 3. The explicit form of the latter is given in* (6.23). *We extend these functions on $\mathbb{Z}$ to functions on $\mathbb{R}$ by setting $g(x)=g([x])$. Then the following holds:*

$$(6.55)\qquad \lim_{\varepsilon\to\infty}\left(\overline{\lim_{n\to\infty}}\ \mathrm{Prob}\left(\omega \middle| d_I\left(\frac{1}{n^\alpha}h(nx), \frac{1}{n^\alpha}\hat{h}^{\varepsilon,n}(nx)\right)\geqq\eta\right)\right)=0 \quad \forall\eta>0$$

$h(nx)$, $\hat{h}^{\varepsilon,n}(nx)$ are of course considered functions of x for the Skorohod metric. The convergence for $\varepsilon\to 0$ is uniform in n.

Proof. Since we can bound increments of $h(x)$ by multiples of the resistance between the relevant points, compare (3.38), (3.40), the relation (6.55) follows from Proposition 2b and the explicit formula for $\hat{h}^{\varepsilon,n}$ given in (6.23).

Lemma 6.9. *Denote by $L^\varepsilon(x,n)$, $L(x,n)$ the number of visits to x before reaching $\mathscr{C}[-n,n]$ in the processes $(\hat{X}_k^{\varepsilon,n})$, $(\hat{X}_k)$. Then we have with the convention $\tilde{L}^\varepsilon(x,n)=(e(\tilde{x}))^{-1}L^\varepsilon(x,n)$, where $\tilde{x}\leftrightarrow x$ relabells the random set A with $\mathbb{Z}$:*

$$(6.56)\qquad \lim_{\varepsilon\to\infty}\left(\overline{\lim_{n\to\infty}}\ E_\omega \sup_{|x|\leqq n}\left(\left\|\mathscr{L}\left(\frac{\tilde{L}^\varepsilon(x,n)}{n^\alpha}\middle|\omega\right)-\mathscr{L}\left(\frac{L(x,n)}{n^\alpha}\middle|\omega\right)\right\|\right)\right)=0.$$

The convergence for $\varepsilon\to 0$ is uniform in n.

Proof. As in Lemma 6.8 we conclude that (6.55) still holds replacing h, $\hat{h}^{\varepsilon,n}$ by the subharmonic functions f_x, $(e(\tilde{x}))^{-1}\cdot\hat{f}_x^{\varepsilon,n}(\cdot)$ constructed in Proposition 3′ (the x refers to $(P-I)(f_x)=2 1_{\{x\}}$). Following the scheme of Sect. 5 we show then with the Eq. (5.9) that $\overline{\lim\limits_{n\to\infty}}\ \mathrm{Prob}_\omega(|E(\tilde{L}^\varepsilon(x,n)n^{-\alpha}|\omega)-E(L(x,n)n^{-\alpha}|\omega)|>\delta)$ tends to 0 as $\varepsilon\to 0$ uniformly in $|x|\leqq n$. Now Lemma 5.2 proves the assertion (6.56).

Proof of Proposition 12. The proof of this Proposition is nothing but making precise the following idea: In the macroscopic scale the probabilities to leave a fixed interval to the left (right) and the distribution of the number of steps to leave this interval become for both processes very close for most ω if n is large and ε is small enough (according to Lemma 6.8 and 6.9). This should imply that the processes are close for ε small and n large for most ω.

Step 1. In order to make this "for most ω" precise, fix first an $f\in\mathscr{C}_0(\mathbb{R})$ and define for a compact interval $I=[a,b]\subseteq\mathbb{R}$ containing the support of f the set $\Omega_{n,\varepsilon}^{\theta,\tau}(I)$ of media. To do this we need the following ingredients:

(6.57) $L_{\varepsilon,n}(x,m)$, $\hat{L}_{\varepsilon,n}(x,m)$ denotes for given ω the number of visits to x before leaving $[x-m,x+m]n$ for $(\hat{X}_k^{\varepsilon,n})_{k\in\mathbb{N}}$, $(\hat{X}_k)_{k\in\mathbb{N}}$.

(6.58) $(x_i^{\varepsilon,n})_{i\in\mathbb{N}}$ labels the points x with $\hat{p}(x,x+1)\leqq\varepsilon n^{-\alpha}$.

With these ingredients we define $\Omega_{n,\varepsilon}^{\theta,\tau}(I)$: (Functions on $\mathbb{Z}$ are extended to $\mathbb{R}$ by setting $q(x)=q([x])$ and abbreviate $I(n)=\{(x,m)\in\mathbb{Z}^2 \mid [x-m,x+m]\subseteq nI\}$).

$$\begin{aligned}(6.59)\qquad \Omega_{n,\varepsilon}^{\theta,\tau}(I)=&\left\{\omega \mid \sup_{x\in I}(n^{-\alpha}|\hat{h}^{\varepsilon,n}(nx)-h(xn))|)\leqq\theta\right\}\\ &\cap\left\{\omega \mid \max_{n^{-1}x_i^{\varepsilon,n}\in I}(n^{-1}|x_{i+1}^{\varepsilon,n}-x_i^{\varepsilon,n}|\leqq\theta)\right\}\\ &\cap\left\{\omega \mid \min_{n^{-1}x_i^{\varepsilon,n}\in I}(n^{-1}|x_{i+1}^{\varepsilon,n}-x_i^{\varepsilon,n}|\geqq\tau)\right\}\\ &\cap\left\{\omega \mid c_{0,a},c_{0,b}\geqq\varepsilon n^{\alpha}(b-a)\right\}\\ &\cap\left\{\omega \mid \sup_{(x,m)\in I(n)}(\|\mathscr{L}(n^{-\alpha}(\hat{e}(x))^{-1}L_{\varepsilon,n}(x,m)|\omega)\right.\\ &\left.-\mathscr{L}(n^{-\alpha}\hat{L}_{\varepsilon,n}(x,m)|\omega)\|)\leqq\theta\right\}\\ &\cap\left\{\omega \mid \sum_{x<[an]}e^{-r|x-[an]|}(\hat{p}(x,x-1)^{-1}\right.\\ &\left.+\sum_{x>[bn]}e^{-r|x-[bn]|}(\hat{p}(x,x+1))^{-1}\leqq\theta\right\}.\end{aligned}$$

We shall show later on that for our $f\in\mathscr{C}_0(\mathbb{R})$: for every $\delta>0$ we can find a $\theta_0(\delta)$ such that:

$$(6.60)\qquad \forall\theta\leqq\theta_0(\delta),\quad t\leqq t_0(I)\quad\text{and}\quad \hat{X}_0^{\varepsilon,n}=X_0\in I:$$

$$\overline{\lim_{n\to\infty}}\left(1_{\Omega_{\varepsilon,n}^{\theta,\tau}(I)}\left|E\left(f\left(\frac{1}{n}\hat{X}_{\tilde{t}n^{1+\alpha}}^{\varepsilon,n}\right)\Big|\omega\right)-E\left(f\left(\frac{1}{n}\hat{X}_{tn^{1+\alpha}}\right)\Big|\omega\right)\right|\right)\leqq\delta$$

$$t_0(I)\uparrow\infty\quad\text{as}\quad I\uparrow\mathbb{R},\qquad \tilde{t}=(E(\hat{e}(0)^{-1})^{-1}t.$$

With Lemma 6.8, Proposition 2b and Lemma 6.9 applied to (6.59) we can conclude on the other hand that:

$$(6.61)\qquad \left[\lim_{n\to\infty}\operatorname{Prob}(\omega|\omega\in\Omega_{\varepsilon,n}^{\theta,\tau}(I))\right]\uparrow_{\tau\to 0}\left[\lim_{n\to\infty}\operatorname{Prob}(\omega|\omega\in\Omega_{\varepsilon,n}^{\theta,0}(I))\right],\quad \forall\theta>0,\quad I\subseteq\mathbb{R},$$

$$(6.62)\qquad \underline{\lim}_{n\to\infty}\operatorname{Prob}(\omega|\omega\in\Omega_{\varepsilon,n}^{\theta,0}(I))\uparrow_{\varepsilon\to 0}1\quad\forall\theta>0,\quad I\subseteq\mathbb{R}.$$

Both statements (6.60) and (6.61) together give immediately the assertion.

Step 2. It remains now to prove (6.60). For that purpose we consider again $(x_i^{\delta,n})_{i\in\mathbb{Z}}$, which label for δ, n fixed the points x such that $p(x,x+1)\leqq(\delta n)^{-\alpha}$. In the next step approximate $\hat{X}_k$ by a process $(\tilde{X}_k^{\delta,n})$ by ommitting all jumps to points which cross more than one of the points $(x_i^{\delta,n})_{i\in\mathbb{Z}}$. We check immediately that for all $t>0$ the following holds for $\tau > 0$ and $T_n(I)$ the exit time of $\hat{X}_k$ from nI:

$$(6.63)\qquad \overline{\lim_{n\to\infty}}\left(\text{Prob}\left(\sup_{k\leq tn^{1+\alpha}}[1_{\{T_n(I)\leq tn^{1+\alpha}\}}|n^{-1}\tilde{X}_k^{\delta,n}-n^{-1}\hat{X}_k|)]>0|\omega\right)\right)=0 \quad \forall\omega\in\Omega_{n,\delta}^{\theta,\tau}(I).$$

This process $(\tilde{X}_k^{\delta,n})_{k\in\mathbb{Z}}$ (which is Markov for given ω) we approximate in turn by $(\tilde{\tilde{X}}_k^{\delta,n})_{k\in\mathbb{Z}}$:

$$(6.64)\qquad \tilde{\tilde{X}}_k^{\delta,n} := \inf_i (x_i^{\delta,n}|x_i^{\delta,n}\geq \tilde{X}_k^{\delta,n}).$$

If we choose $\delta=\varepsilon$, than we have obviously for all $\theta\leq\varepsilon$ that:

$$(6.65)\qquad |n^{-1}\tilde{\tilde{X}}_k^{\varepsilon,n}-n^{-1}\tilde{X}_k^{\varepsilon,n}|\leq\varepsilon \quad \text{for} \quad \omega\in\Omega_{\varepsilon,n}^{\theta,0}(I), \quad k\leq T_n(I).$$

Apply this last procedure to $\hat{X}_k^{\varepsilon,n}$ and call the result $\tilde{\hat{X}}_k^{\varepsilon,n}$. Taking the relations (6.63) and (6.65) together we see that in order to finish our proof we are left with comparing $(n^{-1}\tilde{\tilde{X}}_k^{\varepsilon,n})_{k\in\mathbb{N}}$ and $(n^{-1}\tilde{\hat{X}}_k^{\varepsilon,n})_{k\in\mathbb{N}}$. Both these processes live on the sequence $(n^{-1}x_k^{\varepsilon,n})_{k\in\mathbb{Z}}$ and have only transition to the neighboring points.

Step 3. To prove (6.60) we shall use the fact that $\omega\in\Omega_{\varepsilon,n}^{\theta,\tau}(I)$ to show, that for every $\delta>0$ we can find a $\theta_0(\delta)$ and $t_0(I)>0$ such that for all $\omega\in\Omega_{\varepsilon,n}^{\theta,\tau}(I)$ and $t\leq t_0(I)$, $\theta\leq\theta_0(\delta)$:

$$(6.66)\qquad \overline{\lim_{n\to\infty}}\,|E(f(n^{-1}\tilde{\hat{X}}_{tn^{1+\alpha}}^{\varepsilon,n})|\omega)-E(f(n^{-1}\tilde{\tilde{X}}_{tn^{1+\alpha}}^{\varepsilon,n})|\omega)|\leq\delta$$

Putting the three relations (6.63), (6.65) and (6.66) together prove of course the first assertion in (6.60). It is however easy to derive with line three in (6.59) and Lemma 5.2 that $t_0(I)\uparrow+\infty$ as $I\uparrow R$.

Step 4. It remains to show (6.66) above. Call the probabilities to leave the interval $I_i=(x_i^{\varepsilon,n},x_{i+1}^{\varepsilon,n}]$ to the right starting in x: $a_1^{\varepsilon,n}(i,x)$ for the process $\hat{X}_k^{\varepsilon,n}$ and $a_2^{\varepsilon,n}(i,x)$ for the process $\tilde{X}_k^{\varepsilon,n}$. Denote the respective exit times from I_i by $S_i^{\varepsilon,n}(x)$, $T_i^{\varepsilon,n}(x)$. Finally let $L(I)$ be the set of indices such that $x_i^{\varepsilon,n}\in I$. Note in I_i, $L(I)$ we repress the dependence on ε and n in the notation.

Now we use (6.63) and line 1 and 5 respectively line 4 in (6.59) with [17], page 250, to conclude for all $\theta\leq\theta_0(\delta)$, $t\leq t_0(I)$, $\omega\in\Omega_{\varepsilon,n}^{\theta,\tau}(I)$:

$$\overline{\lim_{n\to\infty}}\left(\sup_{i\in L(I)}\sup_{x\in I_i}|a_1^{\varepsilon,n}(i,x)-a_2^{\varepsilon,n}(i,x)|\right)\leq\delta$$

$$\overline{\lim_{n\to\infty}}\left(\sup_{i\in L(I)}\sup_{x,y\in I_i}\|\mathcal{L}(\tilde{S}_i^{\varepsilon,n}(x)-\mathcal{L}(\tilde{T}_i^{\varepsilon,n}(y))\|\right)\leq\delta,\qquad \tilde{T}_{(x)}^{\varepsilon,n}=T_{(x)}^{\varepsilon,n}n^{-(1+\alpha)}$$

This proves (6.66) for $\tau>0$. q.e.d.

d) Proof of Lemma 6.1, 6.2, 6.4, 6.6 and 6.7. In this section we shall give the proofs of Lemma 6.1, Lemma 6.2 and of the group of Lemmata 6.4, 6.6 and 6.7. The main tool is the exploitation of the asymptotics of the subharmonic function f of Proposition 3′ and the spirit of this section is potential theoretic. This is of course the point where we exploit the fact that $p(x,z)=p(x,y)p(y,z)$ for $x<y<z$. Otherwise we would get at this stage instead of convergence several limit points.

Proof of Lemma 6.1. Abbreviate $[x_i^\varepsilon, x_{i+1}^\varepsilon]$ by I. The first observation is that with the techniques from Sect. 5, namely Corollary 7 (5.21), we can show that for the exit times T_n of $(\bar{X}_k^{\varepsilon,n})_{k\in\mathbb{N}}$ from the interval nI, we have that $n^{-2}T_n$ converges to ∞ in probability. Therefor

(6.67) $$\overline{\lim_{n\to\infty}}\ \mathrm{Prob}\,(n^{-1}\bar{X}_k^{\varepsilon,n} \text{ leaves } I \text{ before time } n^2 t)=0 \quad \forall t\in\mathbb{R}^+.$$

Replace now the discrete time chain by a continuous time rate 1 process and omit all jumps leading outside nI. Denote this new process by $(\bar{\bar{X}}_S^{\varepsilon,n})_{S\in\mathbb{R}^+}$, it is for n, ε fixed a Markov process.

We consider the action of the generator G_n^ε of the process $(n^{-1}\bar{\bar{X}}_{Sn^2}^{\varepsilon,n})$ on the set $D_1=\{f|f$ is a restriction of a $C^3(\mathbb{R})$ function to I with $f|_{\partial I}=0\}$. Observe that D_1 is dense in the set D with respect to the norm $\|f\|=\|f1_I\|_\infty+\|f'1_I\|_\infty$, where $D=\{f\in C^2(I)|f|_{\partial I}=0, (x-y)^{-2}(f(y)-f(x))$ converges for $x\to y\in\partial I, x\in I\}$.

The set D is the domain of the generator $\frac{1}{2}\sigma^2\left(\frac{d}{dx}\right)^2$ of reflected Brownian motion in D, where $\left(\frac{d}{dx}\right)^2$ is defined at the boundary by $\lim_{|x-y|\to 0}[2(f(y)-f(x))\cdot(x-y)^{-2}]$. A straighforward calculation shows that:

(6.68) $$G_n^\varepsilon(f) \overset{\|\cdot\|_\infty}{\underset{n\to\infty}{\Rightarrow}} \frac{1}{2}\left(\frac{d}{dx}\right)^2(f) \quad \forall f\in D_1.$$

This implies the convergence of the semigroup of $(n^{-1}(\bar{\bar{X}}_{Sn^2}^{\varepsilon,n}))_{S\in\mathbb{R}^+}$ to the semigroup of reflected Brownian motion (Compare [2], p. 9 formula (1.16)). By a result from [2], p. 167 we have then: $\mathscr{L}((n^{-1}\bar{\bar{X}}_{Sn^2}^{\varepsilon,n})_{S\in\mathbb{R}})$ converges to reflected Brownian motion on I in the sense of processes.

Due to the relation (6.67) above and due to the law of large numbers for the jump times this implies that $\mathscr{L}((n^{-1}\bar{X}_{[Sn^2]}^{\varepsilon,n})_{S\in\mathbb{R}^+})$ converges to reflected Brownian motion. (Note one process is obtained from the other by transforming time $t\to T(t)$ with $T(nt)(nt)^{-1} \underset{n\to\infty}{\longrightarrow} 1, \forall t\in\mathbb{R}^+$ and furthermore the limit process has continuous path. We leave out the tedious details of this measure theoretic puzzle).

Proof of Lemma 6.2. Step 1. First construct a subharmonic function $f_k^{\varepsilon,n}$ for $(\bar{X}_k^{\varepsilon,n})_{k\in\mathbb{N}}$ according to Proposition 3′ with, among other properties: $f_x^{\varepsilon,n}(x)=0, P_{\varepsilon,n}(f_x^{\varepsilon,n})-f_x^{\varepsilon,n}=2 1_{\{x\}}$ where $P_{\varepsilon,n}$ stands for the transition kernel of $(\bar{X}_k^{\varepsilon,n})_{k\in\mathbb{N}}$. Crucial is now again the behaviour of $(n^{-\alpha}f_x^{\varepsilon,n})_{n\in\mathbb{N}}$. We have:

(6.69) $$n^{-\alpha}f_x^{\varepsilon,n}([ny]) \underset{n\to\infty}{\longrightarrow} c\sum_0^{k(y)}\hat{a}_i^\varepsilon, \quad k(y)=\sup(i|x_i^\varepsilon<y), \quad c\in(0,\infty).$$

Remark. $c=(1-e^{-2r})(1+e^{-r})(e^{-r}+e^{-2r}-e^{-3r})^{-1}$. The proof of 6.69 proceeds as follows:

Denote the quantities $a_{x,y}^+, a_{x,y}^-$ introduced in Proposition 3 in Sect. 3 for a Markov chain by $a_{x,y}^+(\varepsilon,n)$ once they are constructed for $(\bar{X}_k^{\varepsilon,n})$. It is easy to see that Lemma 6.1 implies that for, $x>y$ and $\exists i\ \ x_i\in(y,x)$, we have $a_{x,y}^-(\varepsilon,n) \underset{n\to\infty}{\longrightarrow} 0$.

Now the relation (3.3) (3.5) in Proposition 3.3′ tell us that it suffices to prove the analogues statement for $h^{\varepsilon,n}$, that is (6.15) of Lemma 6.4 and Lemma 6.6. For this reason we refer the reader to the proof of Lemma 6.4, 6.6 for these facts.

Step 2. As a next step we decompose the exit time $T_j^{\varepsilon,n}$ from $n\cdot(x_j^\varepsilon, x_{j+1}^\varepsilon]$. Note we repress the dependence on the starting point of $(\bar{X}_k^{\varepsilon,n})_{k\in\mathbb{N}}$ in our notation.

$$(6.70)\qquad T_j^{\varepsilon,n}=\sum_{x=[nx_j]+1}^{[nx_{j+1}]} L^\varepsilon(x,n), \qquad L^\varepsilon(x,n)=\sum_{k=0}^{T_j^{\varepsilon,n}} 1_{\{\bar{X}_k^{\varepsilon,n}=x\}},$$
$$\bar{X}_0^{\varepsilon,n}=[yn] \quad \text{with} \quad y\in(x_j^\varepsilon, x_{j+1}^\varepsilon).$$

We shall prove below in Step 4 that uniformly in $x\in nI$:

$$(6.71)\quad E(n^{-\alpha}L^\varepsilon(x,n)) \underset{n\to\infty}{\longrightarrow} 2c(\hat{a}_j^{-1}+\hat{a}_{j+1}^{-1})^{-1}\ \forall x\in nI, \qquad \text{with } c \text{ as in (6.69)}.$$

So that together with (6.70) we have as a consequence:

$$(6.72)\qquad E(T_j^{\varepsilon,n})n^{-(\alpha+1)} \underset{n\to\infty}{\longrightarrow} 2c(\hat{a}_j^{-1}+\hat{a}_{j+1}^{-1})^{-1}(x_{j+1}^\varepsilon-x_j^\varepsilon).$$

Next apply our Lemma 5.2, (5.12) to (5.14), in order to obtain the uniform integrability of $n^{-(\alpha+1)}T_j^{\varepsilon,n}$ and then conclude from (6.72) that:

(6.73) $\{\mathscr{L}(n^{-(1+\alpha)}T_j^{\varepsilon,n})\}_n$ is weakly relative compact with nontrivial limit points. A limit point has mean $c(\hat{a}_j^{-1}+\hat{a}_{j+1}^{-1})^{-1}(x_{j+1}^\varepsilon-x_j^\varepsilon)$.

We are done once we can show that such a limit point has to be an exponential distribution.

Step 3. To prove exponentiality we use of course a coupling argument. In the remainder we work out the details. Consider for some $y\in(x_j^\varepsilon, x_{j+1}^\varepsilon)$ the stopping time:

$$(6.74)\quad T_n^x=\begin{cases}\inf(k|\bar{X}_k^{\varepsilon,n}=[yn], \bar{X}_0^{\varepsilon,n}=x) & \text{if } \bar{X}_k^{\varepsilon,n} \text{ reaches } [yn] \text{ before } T_j^{\varepsilon,n}\\ T_j^{\varepsilon,n} \text{ elsewhere}\end{cases}.$$

Observe that in order to show the characteristic property of the exponential distribution:

$$(6.75)\qquad \mathrm{Prob}(T_j^{\varepsilon,n}\geqq(t+s)n^{1+\alpha}|T_j^{\varepsilon,n}\geqq tn^{1+\alpha}) \underset{n\to\infty}{\sim} \mathrm{Prob}(T_j^{\varepsilon,n}\geqq sn^{1+\alpha})$$

it suffices to show that

$$(6.76)\qquad n^{-(1+\alpha)} \sup_{x\in(x_j,x_{j+1})n} (E(T_n^x)) \underset{n\to\infty}{\longrightarrow} 0.$$

It remains therefor to prove (6.71) and (6.76) in order to established our Lemma.

Step 4. We start with proving (6.71). Rewrite (6.71) in the form

$$(6.77)\qquad E(L^\varepsilon(x,n)) \underset{n\to\infty}{\sim} 2c\frac{\hat{a}_j\hat{a}_{j+1}}{\hat{a}_j+\hat{a}_{j+1}}n^\alpha$$

To prove this note first that according to (6.15) and (6.15′) we have:

$$(6.78)\qquad \mathrm{Prob}(\bar{X}_k^{\varepsilon,n} \text{ exits } n(x_j,x_{j+1}] \text{ first to the right}) \underset{n\to\infty}{\sim} \frac{\hat{a}_j}{\hat{a}_j+\hat{a}_{j+1}}.$$

Next apply the formula from (5.9): $\langle\mu P_S, f\rangle=\langle\mu,f\rangle+\langle\eta,g\rangle$, $g=Pf-f$ to: $f=f_x^{\varepsilon,n}$, $S=T_j^{\varepsilon,n}$, $\mu=\delta_x$ and conclude then with (6.69) and (6.78) that (6.77) holds.

The uniformity in x is obtained by applying the estimates for $f_x^{\varepsilon,n}$ in terms of the resistance obtained from combining (3.5) with (3.35), (3.36).

Step 5. The relation (6.76) is shown as follows: Define a function k_n as

$$k_n := \sum_{z \in nI} f_z^{\varepsilon,n}. \tag{6.79}$$

This function has the property $P_{\varepsilon,n}(k_n) - k_n = 21_{nI}$ and therefor if we combine the formula from (5.9) again: $\langle \mu P_S, f \rangle = \langle \mu, f \rangle + \langle \eta, g \rangle$ with (5.10) we obtain now the estimate:

$$E(T_n^x) \leqq k_n(y) + \operatorname{Prob}(T_n^x = T_j^{\varepsilon,n}(x))\, Cn \cdot \left(\max_{x \in nI} [(f_x([x_j n]) \vee f_x([x_{j+1} n]+1))] + C_2 \right). \tag{6.80}$$

The second summand is bounded by $\bar{C}n^2$ (apply (5.9), (6.23) to $f_y^{\varepsilon,n}$ and T_n^x, to bound the probability term by $n^{1-\alpha}$).

Combine now (3.5) with (3.36) to conclude from (6.80) that

$$\sup_{x \in nI} (E(T_n^x)) \leqq C \left(\sum_{[x_j n]+1}^{[x_{j+1} n]-1} \frac{1}{P_{\varepsilon,n}(x, x+1)} + \sum_{[x_j n]+2}^{[x_{j+1} n]} \frac{1}{P_{\varepsilon,n}(x, x-1)} \right) n + \bar{C}n^2 \tag{6.81}$$
$$\leqq \bar{\bar{C}} n^2. \quad \text{q.e.d.}$$

Proof of Lemma 6.4, 6.6 and 6.7. Observe first that 6.4 is included in 6.6 for the choice $\omega_k^+ = \omega_k^- = 1$ for all $k \in \mathbb{N}$, which makes the quantity $\left(\frac{1}{1+\gamma+\beta} A \right)$ to a constant depending only on $p(0,x)$ that is $r(p(0,x) = ce^{-r|x|})$. The proof of 6.6 and 6.7 proceeds in three steps. In step one we prove the assertions (6.41), (6.42) about β^n, γ^n and in Step 2 we study η^n, ξ^n (6.43) and in Step 3 the behaviour of $\xi(y)$ for $y \to \pm\infty$ (6.47). Recall the notation introduced in (6.33) to (6.36), and (6.39).

Step 1. Remember that we are in the situation were $p(0,x) = ce^{-r|x|}$ with

$$c = \left(\sum_{x \in Z} e^{-r|x|} \right)^{-1} \quad (\text{that is } c = (1-e^{-r})(1+e^{-r})^{-1}).$$

(i) We start with analyzing the behaviour of $\beta^{(n)}$. We first calculate the probability B_n, that the walk with transition kernel $\hat{p}_n(x,y)$ makes a jump from 0 to the right but not to 1:

$$B_n = bn^{-\alpha} \left(\sum_{k=1}^{\infty} e^{-rk} \omega_k^+ \right) \quad \text{for } n \in \mathbb{N}_0 \tag{6.82}$$

(see (1.10) for the definition of $\mathbb{N}_\theta$).

Observe that jumps from a point $y > 0$ into the left halfline have probability bounded by $C(n^{-\alpha} \hat{p}_n(y,1))$ for the kernel $\hat{p}_n(\cdot,\cdot)$ (since $\hat{p}_n(y,z) \leqq C\hat{p}_n(y,1)\hat{p}_n(1,0)\hat{p}_n(0,z)$) therefor we obtain from (6.82) above immediately (6.42).

(ii) In order to study γ^n we observe first that: the probability to jump at least twice from the left halfline to the right one or vice versa before hitting 0 or 1 is at most of the order $n^{-2\alpha} = 0(n^{-\alpha})$ and therefor negligible for our purpose.

Consider the chain $X^{(n)}$ with kernel $\hat{p}_n(\cdot,\cdot)$. Now introduce $d_y = \text{Prob}(X^{(n)}_{T-1} = y | X^{(n)}_0 = 0)$, T = hitting time of $\{y | y \geqq 0\}$. We have according to the remark above:

$$\gamma^{(n)} = \sum_{y<0} d_y \left(\frac{\sum_{k \geqq 1} \hat{p}_n(y,k)}{\hat{p}_n(y,0) + \sum_{k \geqq 1} \hat{p}_n(y,k)} \right) + 0(n^{-2\alpha}). \tag{6.83}$$

Now use that $p(0,x) = c\exp(-r|x|)$ and an explicit calculation yields (6.41) and (6.45). (Hint use $p(u,x) = p(u,v)p(v,w)p(w,x)$ with $u=y$, $v=0$, $w=1$, $x=k$.)

Step 2. (i) In order to prove (6.43) we introduce $\hat{\eta}^{(n)}$, the minimal positive solution of

$$(I - \hat{p}_n)(\cdot) = v^{(n)}(\|v^{(n)}\|)^{-1} - \delta_1. \tag{6.84}$$

We have by construction of $\eta^{(n)}$, $\hat{\eta}^{(n)}$:

$$\eta^{(n)} \leqq \hat{\eta}^{(n)}(\|v^{(n)}\|) \leqq \hat{\eta}^{(n)}(n^{-\alpha}C). \tag{6.85}$$

Observe that if we can show that $\hat{\eta}^{(n)} \leqq C\Pi \leqq \tilde{C}\cdot$(counting measure) for all n, then we have proved (6.43) part one, since (6.85) tells us that then $\|\eta^{(n)}\|_\infty = 0(n^{-\alpha})$. (We use the notation $\|\eta\|_\infty = \sup_x |\eta(x)|$.) With the techniques from Sect. 3, (compare (3.34) to (3.38)) we obtain:

$$\|\hat{\eta}^{(n)}\|_\infty \leqq C\left(\sum_{x>1} \bar{v}^n(x) \left(\sum_{y=2}^{x} (\tilde{p}_n(y,y-1))^{-1} \right)\right), \qquad \bar{v}^n = v^n \|v^n\|^{-1}. \tag{6.86}$$

Introduce next the medium generated by the set $\{0,1\} \cup \{-i | \omega_i^- = 1, i \in \mathbb{N}\} \cup \{i | \omega_{i-1}^+ = 1, i \geqq 1\}$. This medium induces a walk $\tilde{p}$ on $\mathbb{Z}$ by restricting $p(x,y)$ and then relabelling the points with $\mathbb{Z}$. Now note that for $y \geqq 2$: $\hat{p}_n(y,y-1) \geqq \tilde{p}(y,y-1)$ for the medium A_n for all n and furthermore $\hat{p}_n(0,y) \leqq Ca^{|y|}\|v^n\|$ with $a<1$. Therefor we estimate starting from (6.86) above as follows:

$$\|\hat{\eta}^{(n)}\|_\infty \leqq \tilde{C}\left(\sum_{x>1} a^x \cdot \left(\sum_{y=2}^{x} (\tilde{p}(y,y-1))^{-1} \right)\right) = \tilde{\tilde{C}} < \infty \tag{6.87}$$

which concludes the proof of (6.43) part one. (The fact $\tilde{\tilde{C}} < \infty$ follows from Prop. 1b.)

(ii) Note that with the same idea as above we obtain also:

$$\|\xi^{(n)}\|_\infty \leqq C < \infty \quad \forall n \in \mathbb{N} \tag{6.88}$$

(C depends of course on (ω^-, ω^+)!).

As a consequence we can select a pointwise convergent subsequence $\xi^{(n_k)}$. Denote it's limit with ξ. The estimate (6.88) above allows us to conclude from the fact: $\xi^{(n)}$ is (the unique) minimal positive solution of $(I-\hat{p}_n)(\cdot) = \mu^{(n)} - (\delta^{(n)}\delta_0 + \gamma^{(n)}\delta_1)$, that:

(6.89) ξ is minimal positive solution of: $(I-\hat{q})(\cdot) = \mu - \delta_0$, $\mu = q(0,\cdot)$

$\hat{q}(x,y)$ is obtained from $q(x,y)$ by relabelling $\mathscr{B}$ with $\mathbb{Z}^-$

$$q(x,y) := 1_{\mathscr{B}}(x)p(x,y)1_{\mathscr{B}}(y)\left(\sum_{y\in\mathscr{B}} p(x,y)\right)^{-1},$$

$$\mathscr{B} = \{x | x<0, \omega_{|x|}^- = 1\} \cup \{0\}.$$

This implies that ξ is independent of the choice of (n_k) and therefor $\xi^{(n)}$ converges pointwise to the unique ([4]) measure ξ which solves (6.89). It remains now to prove that this convergence takes place as $\|\cdot\|_\infty$-convergence. The techniques of proof are similar to arguments we worked out in detail in Sect. 3, here we only sketch the proof: For $\mu^{(n)}$, μ with bounded support this would be a consequence of the Balayage-principle or equivalent the optional stopping theorem for martingales for the dual chain (compare Sect. 3 (3.52) to (3.54)). Since we have unbounded support we have to estimate the influence of the tails of a measure μ, on the minimal solution of the Poissonequation for $\mu-\delta_0$ uniformly in the $\|\cdot\|_\infty$-norm. This is done in the fashion of (6.86), (6.87)!

By an explicit calculation one proves that $\Pi^{(n)}$ converges to some Π in the $\|\cdot\|_\infty$ norm.

Step 3. We conclude with the proof of Lemma 6.7. The relation (6.89) allows us to derive that $\xi=\bar{\xi}$ similar one has $\Pi=\bar{\Pi}$, so that especially (6.48) holds.

To proceed further recognise that $(\bar{\xi}+\delta_0)$ is $\hat{q}$-invariant and furthermore is the minimal positive solution of $(\hat{q}-I)(\cdot)=\sigma$ (=zero measure), which assigns measure one to the point 0. This implies:

$$\xi=\bar{\xi}=\bar{\Pi}(\bar{\Pi}(0))^{-1}-\delta_0 \tag{6.90}$$

and therefor

$$\bar{\xi}(y)/\bar{\Pi}(y) \underset{y\to-\infty}{\longrightarrow} (\bar{\Pi}(0))^{-1}. \tag{6.91}$$

Since according to our convention how to choose $\Pi^{(n)}$ we have: $\bar{\Pi}(0) = \sum_{k\leq 0} p(0,k)\omega_{|\bar{k}|}$. We have therefor proved Lemma 6.7.

7. *Proof of Theorem 1–4*

a) Theorem 1. We prove part a) of the theorem by combining Corollary 6 (4.4) with Proposition 1 (2.4) which gives immediately convergence of the finite dimensional distributions. In order to obtain convergence in the sense of processes for the continuous time process with Proposition 1, use simply Theorem 9.1 in [2], p. 142.

In order to apply Corollary 6 we have to show that $r<|\log(1-q)|$ implies that $E_\omega\left(\frac{1}{\hat{p}(0,1)}\right)<\infty$:

Observe that $r<|\log(1-q)|$ means that:

$$p(0,y)\geq ce^{-a|y|}; \quad c>0, \quad r<a<|\log(1-q)|. \tag{7.1}$$

Therefor

$$E_\omega\left(\frac{1}{\hat{p}(0,1)}\right)\leq 2c^{-1}\cdot\sum_{n=0}^{\infty} e^{-|\log(1-q)|n}\cdot e^{an}<\infty. \tag{7.2}$$

In order to prove part b) of the theorem we assume first that $p(0,x)\sim ce^{-r|x|}$ for $|x|\to\infty$. Then we combine Corollary 7 ((5.21)) with Proposition 1 and obtain the assertion for the case $r\geq|\log(1-q)|$ under the restriction made above on the tails of $p(0,x)$.

Next we relax the condition $p(0,x)\sim ce^{-r|x|}$ in the case $r>|\log(1-q)|$ by getting the analogues estimate to (7.1) above (that is $p(0,y)\leq c\exp(-ay)$ with

$a \in (|\log(1-q)|, r))$ and then check that this suffices in the estimates in the proof of Proposition 7.

b) Theorem 2. We consider first the case where $r < |\log(1-q)|$. By combining Proposition 7 (5.21) and Proposition 1 (2.4), one obtains with the definition

$$(7.3) \qquad \bar{T}(n) := \inf(t | X(t) \in \mathscr{C}[-n, n])$$

that

(7.4) $\{\mathscr{L}(\bar{T}(n) n^{-(1+\alpha)})\}_{n \in \mathbb{N}}$ is relatively weakly compact and the weak limit points are nontrivial.

To proceed further we use the stationary and independence structure of the distribution of the medium. We explain below how we can get out of the relation (7.4) that:

$$(7.5) \qquad \lim_{c \to \infty} \overline{\lim_{n \to \infty}} \left[\mathrm{Prob}(\exists t \leqq s n^{1+\alpha} : X(t) \in \mathscr{C}[-cn, cn])\right] = 0 \quad \forall s \in \mathbb{R}^+$$

$$(7.6) \qquad \exists t_i,\ b \in \mathbb{R}^+,\ \forall t \neq t_i \in \mathbb{R}^+ : \left[\overline{\lim_{n \to \infty}} \mathrm{Prob}(X(n^{1+\alpha} t) \in [-bn, bn])\right] < 1,$$

$$t_{i+1} - t_i \geqq a > 0.$$

(Note Prob$(\cdot)$ refers here to the measure on the product of medium and process.)

The relation (7.5) is obtained as follows: In order to reach $\mathscr{C}[-cn, cn]$ from point 0 we have to cross $[c]$-times an intervall of length n or we have to have at least one jump of size $\geqq n$. The variables $1_{\{x \in A\}}$ are independent for x belonging to different intervalls $[(k-1)n, kn]$. Furthermore the probability of a jump of size $\geqq \varepsilon n$ tends to 0 exponentially fast as $n \to \infty$. It is now straightforward analysis to derive (7.5) from (7.4).

In order to get (7.6) note that for every $c > 0$ with positive probability we leave $(1+b)[-n, n]$ before $cn^{1+\alpha}$-time units (uniformly in n), compare Sect. 5 Lemma 5.2 and 5.3 and Proposition 2b. In order to be back in $[-bn, bn]$ at time $n^{1+\alpha}$ we have to cross again an intervall of length n, which according to (7.4) we can accomplish in less than $(t-c)n^{1+\alpha}$-time units only with probability smaller than 1 for c large enough (uniformly in n). This means the values of t where the limit in question is 1 are isolated q.e.d. Note that with a little bit more work we get through with assuming only ergodicity (instead of independence) for the medium.

Both relations (7.5) and (7.6) together imply that

(7.7) $\left\{\mathscr{L}\left(\frac{1}{n} X(n^{1+\alpha} t)\right)\right\}_{n \in \mathbb{N}}$ is relatively weakly compact for every t and weak limit points are nontrivial, except for at most $t = t_i$ with a sequence t_i with $t_{i+1} - t_i \geqq a > 0$.

In order to prove (1.8) it remains now to show tightness of the sequence $\left\{\mathscr{L}\left(\left(\frac{1}{n} X(n^{1-\alpha} t)\right)_{t \in \mathbb{R}^+}\right)\right\}_{n \in \mathbb{N}}$, as measures on the space of right continuous functions with limits from the left. This is a fact we can derive best from Theorem 9.1 in [2], p. 142. Use (7.5) and (7.7) to verify the conditions needed there.

The proof of (1.9) follows the same lines, we simply have to replace $n^{1+\alpha}$ by $n\log n$ and prove the respective versions of Propositions 2b and 7. All we have to do to accomplish this, is to replace (2.32) and (2.33), which refer to classical results, compare [8] or [3]. We leave the straightforward details to the reader.

The remark following the Theorem 2 is consequence of Proposition 7 and the fact already used above that any interval can be left with positive probability before time $cn^{1+\alpha}$.

c) Theorem 3. 3a): The first observation is again that Proposition 1 (2.4) and the calculation following (4.4) imply that for our purposes we are allowed to replace $n^{-1}X(n^{1+\alpha}t)$ by $\frac{a}{n}\hat{X}_{[bn^{1+\alpha}t]}$ with $a=q$ and $b=(E_\omega(e(0)))$. Assume first $n\subseteq N_i$ for some i. We start by connecting the approximating process $Z_n^{\varepsilon,n}(t,\omega)$ from the theorem with the one more suitable for technical purposes used in Sect. 6 namely $Z_t^{\varepsilon,n}$. These processes are defined in (1.15) and (6.25) via $W_{n,\omega}^{\varepsilon}$, $V_{n,\omega,\varepsilon}$ respectively $\bar{h}^{\varepsilon,n}$, $\bar{V}^{\varepsilon,n}$. Proposition 11 (6.29) and Proposition 2b tell us that these quantities converge in distribution to the same limit as $n\to\infty$, $n\subseteq N_i$. From this we can conclude that the $\mathscr{L}(Z_t^{\varepsilon,n})$ and $\mathscr{L}(Z_t^{\varepsilon,n}(t,\omega))$ have for $n\to\infty$, $n\subseteq N_i$ the same weak limit point. To see this we use the fact that both processes are of the form $g_n^{-1}Y((\int L(0,g_n(x))dx)^{-1}(t))$ with g_n converging weakly to a nondecreasing jumpprocess with isolated jump points. Then the assertion is checked by an elementary calculation. If we then combine the result above, Proposition 10 (6.27) and Proposition 12 (6.54) we obtain the assertion (1.18) of Theorem 3 for $n\to\infty$ in N_i. To generalize this consider: If the relation wouldn't hold for a subsequence n_k it would be violated for a subsequence $(n_k')\subseteq(n_k)$ with $(n_k')\subseteq N_i$ for some i. Therefor (1.18) holds for $n\in\mathbb{N}$.

b) and c): Denote by c the number $\left(\sum_x e^{-r|x|}\right)^{-1}$.

First observe that in our notation $c(\hat{p}(y,y+1))^{-1}=n^{\alpha}\exp(ra_i^{\delta})$ for $y=x_i^{\delta}$ and that therefor we conclude with Proposition 11: the laws of $W_{n,\omega}^{\delta}$ and our function $\bar{h}^{\delta,n}$ constructed in Sect. 6b (6.26) have for $n\to\infty$ the same weak limit points. For the last object we derived a limit theorem, namely Corollary 11 from Sect. 6b, ((6.31)). It allows us to conclude that:

$$\mathscr{L}((W_{n,\omega}^{\delta}(x))_{x\in\mathbb{R}}) \underset{\substack{n\to\infty\\ n\subseteq\mathbb{N}_i}}{\Rightarrow} \mathscr{L}\left(\left(\int_0^x c_y dY_{i,\delta}^{\alpha}(y)\right)_{x\in\mathbb{R}}\right), \tag{7.8}$$

$$\mathscr{L}\left(\left(\int_0^x c_y dY_{i,\delta}^{\alpha}(y)\right)_{x\in\mathbb{R}}\right) \underset{\delta\to 0}{\Rightarrow} \mathscr{L}\left(\left(\int_0^x c_y dY_i^{\alpha}(y)\right)_{x\in\mathbb{R}}\right). \tag{7.9}$$

In order to prove (1.20)–(1.22) it remains now to show (in view of the two relations above) that the convergence of the $\mathscr{L}(W_{n,\omega}^{\delta})$ implies the convergence of $\mathscr{L}((Z_n^{\delta}(t,\omega))_{t\in\mathbb{R}})$ to $\mathscr{L}((Z_i^{\delta}(t))_{t\in\mathbb{R}})$ as $n\to\infty$, $n\subseteq N_i$ and of $\mathscr{L}((Z_i^{\delta}(t))_{t\in\mathbb{R}})$ to $\mathscr{L}((Z_i(t))_{t\in\mathbb{R}})$ as $\delta\to 0$, where the latter is defined via (1.22). For the last assertion we can follow word by word (besides the notation) the arguments of Kawazu-Kesten in [8], there they solve in Lemma 2, Proposition 1 on p. 567–569 this problem. We don't repeat these arguments here and refer the reader to that paper. For the first assertion we pointed out in 3a how to proceed. The uniformity of the convergence

$\delta \to 0$ in n follows immediately from $Y^{\alpha}_{i,\delta}(x) Y^{\alpha}_{i}(x)$, $W^{\delta}_{n,\omega}(x) W_{n,\omega}(x)$ as $\delta \to 0$ for $x \geqq 0$. (For $x<0$ we have decreasing sequences instead.)

d) Proof of Theorem 4. We only indicate how to modify our earlier arguments for the case of exponential tails:

Consider again the sequence of sets $\{x^{n,\alpha}_i\}_{i \in \mathbb{Z}}$, where for n and α fixed $\{x^{n,\alpha}_i\}_{i \in \mathbb{Z}}$ labels the points which have strings of forbidden points to the right of length at least $|\log(1-q)|^{-1} \log n - a$ with $a<\alpha$. The length of the string is written as $[|\log(1-q)|^{-1} \log n] - a^{n,\alpha}_i$. We saw already in Sect. 6 that $(n^{-1} x^{n,\alpha}_i, a^{n,\alpha}_i)_{i \in \mathbb{Z}}$ converges weakly for $n \to \infty$, $n \subseteq N_\theta$. So realize again the whole sequence on a common probability space such that $(n^{-1} x^{n,\alpha}_i, a^{n,\alpha}_i) \underset{n\to\infty}{\longrightarrow} (x^{\alpha}_i, a^{\alpha}_i)$. Denote by $x^{\alpha}_j = \inf(x^{\alpha}_i | x^{\alpha}_i > 0)$, so that $0 \in [x^{\alpha}_{j-1}, x^{\alpha}_j]$. Furthermore note that $a^{\alpha}_{j-1}, a^{\alpha}_j \neq a$ with probability one.

We shall show that in the scale $f_\alpha(n)$ the process doesn't leave the interval $[x^{\alpha}_{j-1}, x^{\alpha}_j]$ in the limit $n \to \infty$. For that purpose consider again the subharmonic function f from Proposition 3′ with $P^*(f) - f = 1_{\{0\}}$, $f(0)=0$ among other properties. Again we can estimate f in terms of the resistance (compare Sect. 3) and obtain:

$$f_\alpha^{-1}(n) f([n,x]) \to \begin{cases} +\infty & x < x^{\alpha}_{j-1} \\ 0 & x \in [x^{\alpha}_{j-1}, x^{\alpha}_j] \\ +\infty & x > x^{\alpha}_j . \end{cases} \tag{7.10))}$$

With the same arguments as in Sect. 5 we derive from the relation above that the following holds for $T_n = \inf(k | X_k \in \mathscr{C}[x^{n,\alpha}_j])$:

$$n^{-1} f_\alpha^{-1}(n) T_n \underset{n\to\infty}{\longrightarrow} \infty \quad \text{in probability}. \tag{7.11}$$

Now look at the exit times T'_n from an interval $[y', y''] \subseteq (x^{\alpha}_{j-1}, x^{\alpha}_j)$. Looking at (7.10) above again, we conclude with the scheme from Sect. 5 that:

$$n^{-1} f_\alpha^{-1}(n) T'_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in probability}. \tag{7.12}$$

The last two relations prove Theorem 4. We omit any further details.

Acknowledgements. We thank H. Kesten and H. Rost for useful discussions and K. Oelschläger for hints on existing literature. The work on this paper began, while the author stayed at the IMA in Minneapolis in spring 1986

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