The law of the iterated logarithm for local time of a Lévy process

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Summary. Let $\{X_t\}$ be a one-dimensional Lévy process with local time $L(t, x)$ and $L^*(t) = \sup\{L(t, x): x \in \mathbb{R}\}$. Under an assumption which is more general than being a symmetric stable process with index $\alpha > 1$, we obtain a LIL for $L^*(t)$. Also with an additional condition of symmetry, a LIL for range is proved.

1 Introduction

Let $\{X_t\}$, $t \geq 0$ be a one-dimensional Lévy process. Its characteristic function can be represented as follows;

$$E \exp(iuX_t) = \exp(t\psi(u))$$

where

$$\psi(u) = ibu + \int (e^{iux} - 1 - iux(1 + x^2)^{-1})v(dx).$$

Here v is a measure on $\mathbb{R} - \{0\}$ satisfying $\int (1 \wedge x^2)v(dx) < \infty$. Note that we don't include the Gaussian component in $\psi(u)$ since the behavior of B.M. is well-known. For $x > 0$, define

$$G(x) = \int_{|y| > x} v(dy),$$

$$K(x) = x^{-2} \int_{|y| \leq x} y^2 v(dy).$$

We assume that

$$\limsup \frac{G(x)}{K(x)} < 1 \tag{1.1}$$

as x tends both to 0 and ∞, and

$$EX_1 = 0. \tag{1.2}$$

For α-stable processes, $\lim G(x)/K(x) = (2 - \alpha)/\alpha$ as $x \to 0$ and $x \to \infty$, so that our assumption is more general than being a symmetric α-stable process with $\alpha > 1$. We

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also note that (1.1) implies that $E|X_1| < \infty$. Assumption (1.2) is necessary to guarantee recurrence of the process. In fact, under (1.1) and (1.2), X_t is point-recurrent since the following criteria for point-recurrence are satisfied; for any $\lambda > 0$,

$$\int_{-\infty}^{\infty} \text{Re} \frac{1}{\lambda - \psi(u)} \lesssim \infty \ ,$$

$$\int_{|u<1|} \text{Re}\left(-\frac{1}{\psi(u)}\right) du = \infty \ .$$

This work is mainly concerned with the asymptotic growth rate of local time of X. Under (1.1) and (1.2), Kesten and Bretagnolle's conditions [4, 10] for existence of a continuous version of $L(t, x)$ as a function of t are satisfied: they are as follows;

$$\int_{-\infty}^{\infty} \text{Re} \frac{1}{1 - \psi(u)} du < \infty \ , \tag{1.3}$$

$$\int(1 \wedge |x|)\nu(dx) = \infty \ . \tag{1.4}$$

Moreover, it is not hard to show that a jointly continuous version of $L(t, x)$ exists by checking the results obtained by Barlow and Hawkes [2] and Barlow [1]. They improved the result of Getoor and Kesten [5], and proved that under (1.3) and (1.4),

$$\int_{|x|<1/e} \frac{\bar{\varphi}(x)}{x(\log 1/x)^{1/2}} dx < \infty \ , \tag{1.5}$$

is necessary and sufficient for the existence of jointly continuous version of local time where

$$\varphi^2(y) = \int(1 - \cos uy)\text{Re} \frac{1}{1 - \psi(u)} du \ ,$$

and $\bar{\varphi}$ denotes the monotone rearrangement of φ. For a comprehensive list of results related to local times, the reader may consult [1]. When a jointly continuous $L(t, x)$ exists, it is clear that

$$L(t, x) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \chi_{(x-\varepsilon, x+\varepsilon)}(X(s)) ds \ .$$

This work is motivated by a result of Griffin [7] which described the asymptotic growth of the local time of a symmetric α-stable process with $\alpha > 1$. To describe his results, let

$$L^*(t) = \sup\{L(t, x): x \in \mathbb{R}\}$$

$$R(t) = \{X(s): 0 \leq s \leq t\}$$

and denote the Lebesgue measure of $R(t)$ by $m(R(t))$. Griffin showed that for some $c, C \in (0, \infty)$,

$$\limsup_{t \to \infty} t^{-1/\alpha}(\text{ll} t)^{-(1-1/\alpha)} m(R(t)) = c \quad \text{a.s.} \tag{1.6}$$

$$\liminf_{t \to \infty} t^{-(1-1/\alpha)}(\text{ll} t)^{1-1/\alpha} L^*(t) = C \quad \text{a.s.} \tag{1.7}$$

where $\mathrm{ll}t$ denotes $\log \log t$. In the case of Brownian Motion, the law of iterated logarithm implies that the result of type of (1.6) holds even with $\sup_{s \leq t} |X_s|$ replacing $m(R(t))$, and the result of type of (1.7) was obtained by Kesten [9]. For symmetric α-stable processes, it is well-known that (1.6) is no more valid with $\sup_{s \leq t} |X_s|$ instead of $m(R(t))$. We will now be concerned with the same question for the class of Lévy processes satisfying (1.1) and (1.2). For $x > 0$, define

$$f(x) = G(x) + K(x) \,.$$

It is easy to see that $f^{-1}(1/y)$ is well-defined for y large since f is continuous and strictly decreasing once it reaches the support of v. In this work, we prove that assuming (1.1) and (1.2), for some $C \in (0, \infty)$,

$$\liminf_{t \to \infty} L^*(t) f^{-1}(\mathrm{ll}t/t) \mathrm{ll}t/t = C \quad \text{a.s.}$$

Furthermore, under the extra condition that X_t is symmetric, it will be shown that

$$\limsup_{t \to \infty} \frac{m(R(t))}{f^{-1}(\mathrm{ll}t/t)\mathrm{ll}t} = C \quad \text{a.s.}$$

There are very important estimates used to derive the necessary probability estimates, which assert that for some constants c, C and any positive integer m,

$$E(L(t, x)) \leq \frac{ct}{f^{-1}(1/t)} \tag{1.8}$$

$$E(L(t, x) - L(t, x + y))^{2m} \leq (2m)! \left(\frac{Ct}{f^{-1}(1/t)} \right)^m \left(\frac{1}{|y|K(|y|)} \right)^m . \tag{1.9}$$

The proof involves a method employed previously by Jain and Pruitt [8] for the case of an integer-valued random walk, and it turned out to be very useful in our situation. The analogous estimate to (1.9) for Brownian local time was obtained by Borodin [4] using the Ray's formula [12] which is not valid for general Lévy processes.

In Sect. 2, we will briefly review the basic facts and obtain the important estimates on the local time. We will state and prove the main theorem about the local time in Sect. 3. In Sect. 4, the law of iterated logarithm for range will be presented. We will denote a finite positive constant by C. Its value may be different from line to line; whenever necessary, we will number the constants.

2 Basic estimates

In this section, we will assume that (1.1) and (1.2) hold. Also we will assume that $v(\mathbb{R}) = \infty$. Otherwise X_t is a compound Poisson process with a drift term. We start with some useful facts about $f(x)$. First not that $x^2 f(x)$ is continuous and strictly increasing. Also it will be frequently used that from (1.1), there exist $0 < \varepsilon_0 < 1$, a_0, A_0 such that for $x \in (0, a_0] \cup [A_0, \infty)$, $G(x) \leq (1 - \varepsilon_0)K(x)$, hence there exists $1 < \delta_0 < 2$ such that

$$\begin{aligned} &x^{\delta_0} f(x) \text{ is strictly decreasing on } (0, a_0] \cup [A_0, \infty), \\ &y^{1/\delta_0} f^{-1}(y) \text{ is strictly increasing on } (0, f(A_0)] \cup [f(a_0), \infty) \end{aligned} \tag{2.1}$$

(See Lemma 4.2 of [13]). Using this, we observe that for $M > 1$, small and large values of x and y,

$$M^{-2} f(x) \leqq f(Mx) \leqq M^{-\delta_0} f(x) \tag{2.2}$$

$$M^{-1/\delta_0} f^{-1}(y) \leqq f^{-1}(My) \leqq M^{-1/2} f^{-1}(y) . \tag{2.3}$$

Also we note that for any x, $G(x) \leqq CK(x)$ for some $C > 0$, which implies that there exists δ_1 such that $0 < \delta_1 \leqq \delta_0$ and $x^{\delta_1} f(x)$ is strictly decreasing for any x. Therefore we obtain the analogous inequalities to (2.2) and (2.3) with δ_1 replacing δ_0 which hold for any x and y. We will use the fact that

$$\limsup_{x \to 0} G(x)/K(x) < 1$$

implies that $\{X_t\}$ has density, since

$$\int_{|u| \geqq 1/a_0} |\exp(t\psi(u))| du = 2 \int_{1/a_0}^{\infty} \exp(t \int (\cos ux - 1) v(dx)) du$$

$$\leqq 2 \int_{1/a_0}^{\infty} \exp\left(t \int_{|ux| \leqq 1} (\cos ux - 1) v(dx) \right) du$$

$$\leqq 2 \int_{1/a_0}^{\infty} \exp\left(-Ct \int_{|ux| \leqq 1} u^2 x^2 v(dx) \right) du$$

$$= 2 \int_{1/a_0}^{\infty} \exp(-CtK(1/u)) du$$

$$= 2 \int_{0}^{a_0} \exp(-CtK(v)) v^{-2} dv$$

$$\leqq 2 \int_{0}^{a_0} \exp(-Ctv^{-\delta_0} f(a_0)) v^{-2} dv$$

$$< \infty$$

where $v^{\delta_0} K(v) \geqq 2^{-1} v^{\delta_0} f(v) \geqq 2^{-1} a_0^{\delta_0} f(a_0)$ for $v \leqq a_0$ is used. It is now convenient to introduce more notation though it will not be used so frequently as G and K. Define for $x > 0$,

$$\alpha(x) = \int_{|y| > x} y(1 + y^2)^{-1} v(dy) ,$$

$$\beta(x) = \int_{|y| \leqq x} y^3 (1 + y^2)^{-1} v(dy) .$$

For any $a > 0$, we may write X_t as the sum of two independent Lévy processes $X_t^1(a)$, $X_t^2(a)$ where

$$E \exp(iuX_t^1(a)) = \exp\left(itu(b - \alpha(a)) + t \int_{|x| \leqq a} (\exp(iux) - 1 - iux(1 + x^2)^{-1}) v(dx) \right),$$

$$E \exp(iuX_t^2(a)) = \exp\left(t \int_{|x| > a} (\exp(iux) - 1) v(dx) \right).$$

$X_t^2(a)$ is a compound Poisson process of parameter $tG(a)$ with all jumps of size greater than a up to time t. That is,

$$X_t^2(a) = \sum_{s \leq t} (X_s - X_{s-}) \mathscr{X}_{[-a,\, a]^c}(X_s - X_{s-}) \,.$$

By differentiating, it is easy to see that

$$EX_t^1(a) = t(b + \beta(a) - \alpha(a))$$

$$\mathrm{Var}\, X_t^1(a) = ta^2 K(a) \,.$$

Now we will derive two basic estimates (1.8) and (1.9) on the local time which are interesting themselves. The techniques used here heavily rely on the method used by Jain and Pruitt [8] in the case of an integer-valued random walk under analogous assumptions to ours. The argument starts from the well-known inversion formula to obtain the necessary probability estimate. First we prove a series of lemmas.

Lemma 2.1 *There exists C_1 such that for any t,*

$$\int |e^{t\psi(u)}| \mathrm{d}u \leq C_1/f^{-1}(1/t) \,.$$

Proof. We start with the estimate on $|e^{t\psi(u)}|$. For $u > 0$, we have

$$|e^{t\psi(u)}| = \exp\left(t \int (\cos ux - 1) v(\mathrm{d}x) \right)$$

$$\leq \exp\left(t \int_{|x| \leq 1/u} (\cos ux - 1) v(\mathrm{d}x) \right)$$

$$\leq \exp(- CtK(1/u))$$

$$\leq \exp(- Ctf(1/u)) \,. \tag{2.4}$$

Let $a_t = f^{-1}(1/t)$, and using (2.4), observe that

$$\int |e^{t\psi(u)}| \mathrm{d}u \leq 2/a_t + 2 \int_{1/a_t}^{\infty} \exp(- Ctf(1/u)) \mathrm{d}u$$

$$\leq 2/a_t + 2 \int_{1/a_t}^{\infty} \exp(- Ctu^{\delta_1} a_t^{\delta_1} f(a_t)) \mathrm{d}u$$

$$\leq 2/a_t + 2 \int_{1/a_t}^{\infty} \exp(- Cu^{\delta_1} a_t^{\delta_1}) \mathrm{d}u$$

$$\leq 2/a_t + 2 \int_{0}^{\infty} \exp(- Cv^{\delta_1}) \mathrm{d}v/a_t$$

$$= C_1/a_t \,. \quad \square$$

Lemma 2.2 *There exists C_2 such that for any $x, \varepsilon > 0$, $t > 0$,*

$$P(|X_t - x| < \varepsilon) \leq C_2 \varepsilon/f^{-1}(1/t) \,.$$

Proof. Essentially, we follow the proof of Theorem 3.6 of [6]. We start with the inversion formula,

$$\frac{1}{2\lambda}\int P(|X_t - (x - y)| \leq \lambda)\frac{1 - \cos ry}{\pi r y^2}\,dy$$

$$= \frac{1}{2\pi}\int_{|u| \leq r} \exp(-iux + t\psi(u))\frac{\sin \lambda u}{\lambda u}\left(1 - \frac{|u|}{r}\right)^+ du\;. \tag{2.5}$$

Note that for $0 < \eta < \lambda$,

$$\frac{1}{2\lambda}\int P(|X_t - (x - y)| \leq \lambda)\frac{1 - \cos ry}{\pi r y^2}\,dy$$

$$\geq \frac{1}{2\lambda} P(|X_t - x| \leq \lambda - \eta)\int_{|z| \leq r\eta} \frac{1 - \cos z}{\pi z^2}\,dz$$

$$\geq \frac{C}{\lambda} P(|X_t - x| \leq \lambda - \eta)$$

if we let $r\eta = 1$. Using the inversion formula (2.5) with Lemma 2.1, we have

$$P(|X_t - x| \leq \lambda - \eta) \leq C\lambda/f^{-1}(1/t)\;.$$

Now the assertion follows easily if we let $\eta = \varepsilon,\ \lambda = 2\varepsilon$.   □

Lemma 2.3 *There exists C_3 such that for any $x, \varepsilon > 0$ and sufficiently large t,*

$$\int_0^t P(|X_s - x| < \varepsilon)ds \leq C_3 \varepsilon t/f^{-1}(1/t)\;,$$

hence

$$EL(t, x) \leq C_3 t/f^{-1}(1/t)\;.$$

Proof. Using Lemma 2.2, we write

$$\int_0^t P(|X_s - x| < \varepsilon)ds \leq \int_0^t \frac{C_2\varepsilon}{f^{-1}(1/s)}\,ds\;.$$

Using (2.1), observe that for any $A_0 < f^{-1}(1/t)$,

$$\int_0^{1/f(a_0)} C_2\varepsilon/f^{-1}(1/s)ds + \int_{1/f(A_0)}^t C_2\varepsilon/f^{-1}(1/s)ds$$

$$\leq \frac{C_2\varepsilon}{a_0 f(a_0)^{1/\delta_0}}\int_0^{1/f(a_0)} s^{-1/\delta_0}ds + \frac{C_2\varepsilon t^{1/\delta_0}}{f^{-1}(1/t)}\int_{1/f(A_0)}^t s^{-1/\delta_0}ds$$

$$\leq C\varepsilon/(a_0 f(a_0)) + C\varepsilon t/f^{-1}(1/t)\;.$$

This completes the proof since obviously

$$\int_{1/f(a_0)}^{1/f(A_0)} \frac{C_2\varepsilon}{f^{-1}(1/s)}\,ds < C\varepsilon$$

and $t/f^{-1}(1/t) \to \infty$ as $t \to \infty$ □.

Lemma 2.4 *There exists C_4 such that for any x, y and $\varepsilon > 0$,*

$$\int_0^\infty |P(|X_s - x| < \varepsilon) - P(|X_s - (x+y)| < \varepsilon)| ds \leqq \frac{C_4 \varepsilon}{|y| K(|y|)}.$$

Proof. The proof runs similarly to Lemma 7 of [8]. □

Lemma 2.5 *There exists C_5 such that for any positive integer m, and t large enough,*

$$E(L(t, x) - L(t, x + y))^{2m} \leqq (2m)! \, (C_5 t / f^{-1}(1/t))^m (|y| K(|y|))^{-m}.$$

Proof. Fix x and y, and let

$$I_{z, \varepsilon} = (z - \varepsilon, z + \varepsilon), \qquad \Phi_\varepsilon = \mathscr{X}_{I_{x, \varepsilon}} - \mathscr{X}_{I_{x+y, \varepsilon}}.$$

It suffices to show that

$$E\left(\int_0^t \Phi_\varepsilon(X_s) ds \right)^{2m} \leqq (2m)! \left(\frac{C_5 \varepsilon t}{f^{-1}(1/t)} \right)^m \left(\frac{\varepsilon}{|y| K(|y|)} \right)^m.$$

Recall from Sect. 2 that X_t has density. Furthermore it suffices to consider the case when X_t has continuous density. Otherwise, we choose a symmetric stable process $\{Y_t\}$ of index α, independent of $\{X_t\}$ such that

$$(2 - \alpha)/\alpha < 1 - \limsup G(x)/K(x),$$

as x tends to 0 and ∞. For $\delta > 0$, set $Z_{\delta, t} = X_t + Y_{\delta t}$. Then the Lévy measure of $Z_{\delta, t}$ is given by

$$\mu_\delta(dx) = \nu(dx) + \delta |x|^{-1-\alpha} dx,$$

which implies that as $x \to 0$ and $x \to \infty$,

$$\limsup G_{\mu_\delta}(x)/K_{\mu_\delta}(x) < 1$$

and

$$f_{\mu_\delta}(x) = f(x) + 4\delta\alpha^{-1}(2 - \alpha)^{-1} x^{-\alpha},$$

where G_{μ_δ}, K_{μ_δ} and f_{μ_δ} denote the corresponding functions with μ_δ replacing ν in the definition of G, K and f respectively. Since $Z_{\delta, t}$ has continuous density, the general case follows easily by letting $\delta \to 0$ from the assertion for $Z_{\delta, t}$. Denote the density of X_t by $p(t, x)$ and note that Lemma 2.3 and 2.4 imply that for t large,

$$\int_0^t p(s, x) ds \leqq C_3 t / f^{-1}(1/t)$$

$$\int_0^\infty |p(s, x) - p(s, x + y)| ds \leqq C_4/(|y| K(|y|)). \tag{2.6}$$

Write, for $N > 2m$,

$$E\left(\int_0^t \Phi_\varepsilon(X_s) ds \right)^{2m} = E(S_1 + S_2 + \cdots + S_N)^{2m}$$

$$= \sum_Q E(S_1^{i_1} S_2^{i_2} \cdots S_N^{i_N})$$

where

$$J_k = [(k-1)t/N, kt/N),$$

$$S_k = \int_{J_k} \Phi_\varepsilon(X_s)\mathrm{d}s,$$

$$Q = \{(i_1, i_2, \ldots\ldots, i_N); 0 \leqq i_1, i_2, \ldots, i_N \leqq 2m, i_1 + i_2 + \cdots + i_N = 2m\}.$$

Let

$$Q_1 = \{(i_1, i_2, \ldots, i_N) \in Q; i_k = 0 \text{ or } 1 \text{ for all } k\}$$

$$Q_2 = Q \cap Q_1^c.$$

Calculating the cardinal number of Q_2, we observe that

$$\sum_{Q_2} E|S_1^{i_1} S_2^{i_2} \cdots S_N^{i_N}| \leqq \left(\frac{2t}{N}\right)^{2m}\left(\binom{N-1+2m}{2m} - \binom{N}{2m}\right)$$

which can be made sufficiently small if N is sufficiently large. Therefore,

$$E\left(\int_0^t \Phi_\varepsilon(X_s)\mathrm{d}s\right)^{2m} \sim \sum_{Q_1} E(S_1^{i_1} S_2^{i_2} \cdots S_N^{i_N})$$

$$= (2m)! \sum_{T_{2m}(1,N)} E(S_{i_1} S_{i_2} \cdots S_{i_{2m}})$$

where $T_k(i,j) = \{(i_1, i_2, \ldots, i_k): i \leqq i_1 < i_2 < \cdots i_k \leqq j\}$ and $\sim$ denotes that the ratio tends to 1 as $N \to \infty$. Now we show that by induction on m,

$$\left|\sum_{T_{2m}(1,N)} E(S_{i_1} S_{i_2} \cdots S_{i_{2m}})\right| \leqq \left(\frac{4C_3\varepsilon t}{f^{-1}(1/t)}\right)^m \left(\frac{C_4\varepsilon}{|y|K(|y|)}\right)^m. \qquad (2.7)$$

It is easy to see that (2.7) holds for $m = 1$. We write

$$\left|\sum_{T_{2m}(1,N)} E(S_{i_1} S_{i_2} \cdots S_{i_{2m}})\right| = \sum_{1 \leqq i_1 < i_2 \leqq N} \iint_{J_{i_1} \times J_{i_2}} E\{E(\Phi_\varepsilon(X_u)|X_v)\Phi_\varepsilon(X_v)$$

$$\times E_{X_v}\left(\sum_{T_{2m-2}(i_2+1,N)} S'_{i_3} S'_{i_4} \cdots S'_{i_{2m}}\right)\}\mathrm{d}u\,\mathrm{d}v \qquad (2.8)$$

where $S'_k = \int_{J_k} \Phi_\varepsilon(X_s - X_v)\mathrm{d}s$. Recall that X_t has density $p(t,x)$ and note that for $p(v,w) \neq 0$,

$$E(\Phi_\varepsilon(X_u)|X_v = w)$$

$$= p(v,w)^{-1}\left(\int_{I_{x,\varepsilon}} p(u,z)p(v-u, w-z)\mathrm{d}z - \int_{I_{x+y,\varepsilon}} p(u,z)p(v-u, w-z)\mathrm{d}z\right)$$

$$= p(v,w)^{-1}\left\{\int_{I_{x,\varepsilon}} (p(u,z) - p(u, z+y))p(v-u, w-z)\mathrm{d}z\right.$$

$$\left. + \int_{I_{x+y,\varepsilon}} (p(v-u, w+y-z) - p(v-u, w-z))p(u,z)\mathrm{d}z\right\}. \qquad (2.9)$$

Now we substitute (2.9) into (2.8) and take the absolute value of (2.8). Also applying the induction hypothesis on

$$E_{X_v}\left(\sum_{T_{2m-2}(i_2+1,N)} S'_{i_3} S'_{i_4} \cdots S'_{i_{2m}}\right)$$

and using (2.6), we obtain

$$\left| \sum_{T_{2m}(1,N)} E(S_{i_1} S_{i_2} \cdots S_{i_{2m}}) \right|$$

$$\leq \left(\frac{4C_3 \varepsilon t}{f^{-1}(1/t)} \right)^{m-1} \left(\frac{C_4 \varepsilon}{|y| K(|y|)} \right)^{m-1}$$

$$\times \sum_{1 \leq i_1 < i_2 \leq N} \left\{ \iint \left(\iint_{J_{i_1} \times J_{i_2}} R(u, v : z, w) \chi_{I_{x,\varepsilon}}(z) du\, dv \right) dz\, dw \right.$$

$$\left. + \iint \left(\iint_{J_{i_1} \times J_{i_2}} R(v - u, v : w - z, w) \chi_{I_{x+y,\varepsilon}}(z) du\, dv \right) dz\, dw \right\}$$

$$\leq \left(\frac{4C_3 \varepsilon t}{f^{-1}(1/t)} \right)^{m} \left(\frac{C_4 \varepsilon}{|y| K(|y|)} \right)^{m} ,$$

where

$$R(u, v : z, w) = |p(u, z) - p(u, z + y)| p(v - u, w - z) |\Phi_\varepsilon(w)| .$$

3 LIL for local time

In this section, under the assumptions (1.1) and (1.2), we will prove that

$$\liminf_{t \to \infty} L^*(t) f^{-1}(\mathrm{ll}t/t) \mathrm{ll}t/t > 0 \quad \text{a.s.} \tag{3.1}$$

$$\liminf_{t \to \infty} L^*(t) f^{-1}(\mathrm{ll}t/t) \mathrm{ll}t/t < \infty \quad \text{a.s.} \tag{3.2}$$

Throughout this section, we will assume that (1.1) and (1.2) hold. The necessary probability estimate to prove (3.1) is relatively easier to obtain. First we quote some results from [13]. Note that though the definition of f in this work is different from that in [13], it is not hard to obtain the following versions under our assumptions (1.1) and (1.2). Denote $A_t = \sup_{s \leq t} |X_s|$.

Lemma 3.1 (Theorem 4.6(2) of [13]) *There exists $0 < \theta_1 < 1$ such that for t sufficiently large*

$$P(X_t > 0) \geq \theta_1, \qquad P(X_t < 0) \geq \theta_1 .$$

Lemma 3.2 (Lemma 3.2(2) of [13]) *There exist C_6, C_7 such that for sufficiently large a, and t,*

$$P(A_t \leq a) \geq C_6 \exp(- C_7 t f(a)) .$$

Lemma 3.3 (Lemma 2.4 of [13]) (1) *For $tf(a) \leq \varepsilon(\delta_0 - 1) 2^{\delta_0 - 1}/\delta_0$,*

$$P(A_t \leq a) \geq C_8(\varepsilon) \exp(- 2^{-\delta_0} t f(a)) ,$$

where $C_8(\varepsilon) = 1 - 2(\delta_0 - 1)\varepsilon/(\delta_0(1 - \varepsilon)^2)$.

(2) *For* $tf(a) \leq \varepsilon(\delta_0 - 1)/\delta_0$,

$$P(A_t \geq a) \leq (1 - \varepsilon)^{-2} tf(a) .$$

Lemma 3.4 *For* $\zeta^{\delta_0} < (\delta_0 - 1)/(2\delta_0) \wedge 1$ *and* t *large*,

$$P(L^*(t) f^{-1}(1/t)/t \leq \zeta) \leq 4\zeta^{\delta_0} .$$

Proof. Using the fact that $t \leq L^*(t) A_t$ and Lemma 3.3(2), we observe that for t large

$$P(L^*(t) f^{-1}(1/t)/t \leq \zeta) \leq P(A_t \geq \zeta^{-1} f^{-1}(1/t))$$

$$\leq 4\zeta^{\delta_0}$$

since

$$tf(\zeta^{-1} f^{-1}(1/t)) \leq \zeta^{\delta_0} . \quad \square$$

Theorem 3.1

$$\liminf_{t \to \infty} L^*(t) f^{-1}(\mathrm{llt}/t) \mathrm{llt}/t > 0 \quad \text{a.s.}$$

Proof. Denote $h(t) = t/f^{-1}(1/t)$ and choose $\zeta < 1$ so that

$$\zeta^{\delta_0} < (\delta_0 - 1)/(2\delta_0) \wedge (2e)^{-2} .$$

By Lemma 3.4, we obtain for t large,

$$P(L^*(t) \leq 2^{-1} \zeta h(t/\mathrm{llt})) \leq \{P(L^*(t/\mathrm{llt}) \leq \zeta h(t/\mathrm{llt})\}^{\mathrm{llt}/2}$$

$$\leq (2\zeta^{\delta_0/2})^{\mathrm{llt}}$$

$$= e^{-\xi \mathrm{llt}} \qquad (3.3)$$

where $e^{-\xi} = 2\zeta^{\delta_0/2}$ and $\xi > 1$. Now setting $t_k = 2^k$, it suffices to show that

$$\sum P\{L^*(t_k) \leq 4^{-1} \zeta h(t_{k+1}/\mathrm{llt}_{k+1})\}$$

converges. Using that $h(t_{k+1}/\mathrm{llt}_{k+1})/h(t_k/\mathrm{llt}_k) \leq 2$ for k sufficiently large, and (3.3), we have

$$P\{L^*(t_k) \leq 4^{-1} \zeta h(t_{k+1}/\mathrm{llt}_{k+1})\} \leq \exp(-\xi \mathrm{llt}_k) ,$$

whose sum converges. $\square$

Next we prove the second half of the main result whose proof is much more involved. We need a lemma to estimate the tail of the distribution of $L^*(t)$ mainly based on the results in Sect. 2.

Lemma 3.5 *For* $\eta > 1$, $M > (4\delta_0\eta/(\delta_0 - 1))^{1/2} \vee 1$, *there exist* C_9, m_0 *such that for any integer* $m \geq m_0$ *and* t *large*

$$P(L^*(\eta t) f^{-1}(1/t)/t \geq 2M)$$

$$\leq C_3 \eta^{1/2}/M + (2m)! \, C_9(m)(\eta^{1/2}/M)^m + 16\eta/9M^2 ,$$

where $C_9(m)$ *depends only on* m.

Proof. For given $\eta > 1$, $M > (4\delta_0\eta/(\delta_0 - 1))^{1/2} \vee 1$, *let* $h(t) = t/f^{-1}(1/t)$, and $N = f^{-1}(1/M^2t)$. Then we have

$$P(L^*(\eta t) \geq 2Mh(t)) \leq P\left(\sup_{|x| \leq N} L(\eta t, x) \geq 2Mh(t)\right) + P(A_{\eta t} > N)$$

$$\leq P\left(\sup_{|x| \leq N} |L(\eta t, x) - L(\eta t, 0)| > Mh(t)\right)$$

$$+ P(L(\eta t, 0) \geq Mh(t)) + 16\eta/9M^2 . \tag{3.4}$$

where Lemma 3.3(2) is used with $\varepsilon = 1/4$. It is easier to handle the second term in (3.4) by using Lemma 2.3 which implies that

$$P(L(\eta t, 0) \geq Mh(t)) \leq \frac{EL(\eta t, 0)}{Mh(t)} \leq \frac{C_3\eta^{1/2}}{M} \tag{3.5}$$

since $f^{-1}(1/t)/f^{-1}(1/\eta t) < \eta^{-1/2}$. To obtain the upper bound for the first term in (3.4), let n be the integer such that $2^{n-1} < N \leq 2^n$ and $0 < \gamma < 1$ where γ will be chosen later. Observe that

$$P\left(\sup_{|x| \leq N} |L(\eta t, x) - L(\eta t, 0)| \geq Mh(t)\right)$$

$$\leq P\left(\sup_{|x| \leq 2^n} |L(\eta t, x) - L(\eta t, 0)| \geq Mh(t)\right)$$

$$\leq \sum_{k=0}^{n} \sum_{i=1}^{2^{n-k}} P\left(|L(\eta t, i2^k) - L(\eta t, (i - 1)2^k)| \geq \frac{M}{2}\gamma^{n-k}(1 - \gamma)h(t)\right)$$

$$+ \sum_{k=1}^{\infty} \sum_{i=1}^{2^{n+k}} P\left(\left|L\left(\eta t, \frac{i}{2^k}\right) - L\left(\eta t, \frac{i-1}{2^k}\right)\right| \geq \frac{M}{2}\gamma^k(1 - \gamma)h(t)\right). \tag{3.6}$$

Fix $\lambda > 0$ such that $1 + \lambda < \delta_0$ and let $m_0 = [(\delta_0 - 1 - \lambda)^{-1}] + 1$, $\gamma^2 = 2^{-\lambda}$ and $m \geq m_0$ where $[x]$ denotes the largest integer not exceeding x. We deal with the first and second term in (3.6) separately. To bound the first sum in (3.6), we let $A_1 = [\log_2 A_0] + 1$, and note that for $k \geq A_1$,

$$K(2^k) \geq f(2^k)/2 \geq (2^{n-k})^{\delta_0}f(2^n)/2 \sim C(2^{n-k})^{\delta_0}/(M^2t) \tag{3.7}$$

and for $0 \leq k < A_1$,

$$K(2^k) \geq Cf(2^k) \geq C2^{-2k}f(1) . \tag{3.8}$$

Now using the Markov Inequality, Lemma 2.5, (3.7) and (3.8), we have an upper bound for the first sum in (3.6),

$$\sum_{k=0}^{n} 2^{n-k}(2m)! \left(\frac{C_5\eta t}{2^k K(2^k)f^{-1}(1/\eta t)}\right)^m \left(\frac{2}{M\gamma^{n-k}(1 - \gamma)h(t)}\right)^{2m}$$

$$\leq (2m)! \left(\frac{4C_5\eta^{1/2}}{M^2(1 - \gamma)^2}\right)^m (f^{-1}(1/t)/t)^m 2^{n(1 + \lambda m)} \sum_{k=0}^{n} 2^{k(-\lambda m - m - 1)}K(2^k)^{-m}$$

$$\leq (2m)! \left(\frac{C\eta^{1/2}}{M^2(1-\gamma)^2} \right)^m 2^{n(1+\lambda m)} (f^{-1}(1/t))^m$$

$$\times \left\{ M^{2m} 2^{-n\delta_0 m} \sum_{A_1 \leq k \leq n} 2^{k(\delta_0 m - \lambda m - m - 1)} + (1/t)^m \sum_{0 \leq k < A_1} 2^{k(m-\lambda m - 1)} \right\}$$

$$\leq (2m)! \left(\frac{C\eta^{1/2}}{M^2(1-\gamma)^2} \right)^m (f^{-1}(1/t))^m \{ M^{2m} 2^{-nm} + 2^{n(1+\lambda m)} (1/t)^m \}$$

$$\leq (2m)! \left(\frac{C\eta^{1/2}}{M^2(1-\gamma)^2} \right)^m \{ M^m + M^{2(1+\lambda m)/\delta_0} f^{-1}(1/t)^{1+\lambda m + m} (1/t)^m \}$$

$$\leq (2m)! \left(\frac{C\eta^{1/2}}{M(1-\gamma)^2} \right)^m$$

$$= (2m)! \, C_9(m)(\eta^{1/2}/M)^m \tag{3.9}$$

since $f^{-1}(y)y^{1/\delta_0} \uparrow$ for y small implies that

$$f^{-1}(1/t)^{1+\lambda m + m}(1/t)^m = (f^{-1}(1/t)(1/t)^{1/\delta_0})^{1+\lambda m + m} t^{-m+(1+\lambda m + m)/\delta_0}$$

$$\leq Ct^{-m+(1+\lambda m + m)/\delta_0}$$

which converges to 0 as $t \to \infty$ since $m \geq m_0$. For the second term in (3.6), we let $a_1 = [\log_2 1/a_0] + 1$, and note that for $k \geq a_1$,

$$K(2^{-k}) \geq f(2^{-k})/2 \geq C2^{(k-a_1)\delta_0} f(2^{-a_1}) \tag{3.10}$$

and for $1 \leq k < a_1$,

$$K(2^{-k}) \geq Cf(2^{-k}) \geq C2^{k\delta_1} f(1) . \tag{3.11}$$

Again using the Markov Inequality, Lemma 2.5, (3.10) and (3.11), we have an upper bound for the second term in (3.6),

$$\sum_{k=1}^{\infty} 2^{n+k}(2m)! \left(\frac{C_5 \eta t}{2^{-k} K(2^{-k}) f^{-1}(1/\eta t)} \right)^m \left(\frac{2}{M\gamma^k (1-\gamma)h(t)} \right)^{2m}$$

$$\leq (2m)! \left(\frac{4C_5 \eta^{1/2}}{M^2(1-\gamma)^2} \right)^m (f^{-1}(1/t)/t)^m 2^n$$

$$\times \sum_{k=1}^{\infty} (2^{1+m+\lambda m})^k K(2^{-k})^{-m} .$$

Note that

$$\sum_{k=1}^{\infty} (2^{1+m+\lambda m})^k K(2^{-k})^{-m} \leq C^m \sum_{1 \leq k < a_1} (2^{1+m+\lambda m - \delta_1 m})^k$$

$$+ C^m \sum_{k \geq a_1} (2^{1+m+\lambda m - \delta_0 m})^k$$

which converges by the choice of λ and m_0. Therefore the second sum in (3.6) is bounded above by

$$(2m)! \, (C\eta^{1/2} M^{-2})^m 2^n (f^{-1}(1/t)/t)^m$$

$$\sim (2m)! \, (C\eta^{1/2} M^{-2})^m f^{-1}(1/M^2 t)(f^{-1}(1/t)/t)^m$$

$$\leq (2m)! \, (C\eta^{1/2} M^{-2})^m M^{2/\delta_0} (f^{-1}(1/t)(1/t)^{1/\delta_0})^{m+1}$$

$$\times (1/t)^{m-(m+1)/\delta_0} \tag{3.12}$$

which converges to 0 as $t \to \infty$. Combining (3.4), (3.5), (3.9) and (3.12) finish the proof. □

Remark. It was pointed out by the referee that Lemma 3.4 and 3.5 imply that $\{L^*(t)f^{-1}(1/t)/t, t \geq 1\}$ is stochastically compact, i.e. every sequence $\{L^*(t_n)f^{-1}(1/t_n)/t_n\}$ with $t_n \to \infty$ has a subsequence which converges to a non-degenerate law. It would be interesting to know what the subsequential limit laws are.

Finally we need following lemma to obtain the necessary probability estimate.

Lemma 3.6 *There exist $0 < \theta_2 < 1$, $0 < \rho_1 < 1$, $\rho_2 > 1$, such that for t sufficiently large,*

$$P(\rho_1 f^{-1}(1/t) \leq X_t \leq \rho_2 f^{-1}(1/t)) > \theta_2 .$$

Proof. We will determine ρ_1, and ρ_2 later. Let $a = f^{-1}(1/t)$. Lemma 3.3(1) implies that for ρ_2 large

$$P(|X_t| \leq \rho_2 a) \geq C(\rho_2)\exp(-2^{-\delta_0} t f(\rho_2 a))$$

$$\geq C(\rho_2)\exp(-1/(2^{\delta_0}\rho_2^{\delta_0})), \tag{3.13}$$

where $C(\rho_2) \to 1$ as $\rho_2 \to \infty$. Also Lemma 2.2 implies that

$$P(|X_t| \leq \rho_1 a) \leq C_2 \rho_1$$

$$< \theta_1/2, \tag{3.14}$$

by choosing ρ_1 small. Therefore using Lemma 3.1, (3.13), and (3.14), we obtain

$$P(\rho_1 a \leq X_t \leq \rho_2 a) \geq C(\rho_2)\exp(-1/(2^{\delta_0}\rho_2^{\delta_0})) + \theta_1/2 - 1$$

$$= \theta_2$$

which is positive if we choose ρ_2 large enough. □

Now we are ready to complete the main result.

Theorem 3.2

$$\liminf_{t \to \infty} L^*(t)f^{-1}(\mathrm{ll}t/t)\mathrm{ll}t/t < \infty \quad \text{a.s.}$$

Proof. Let $p(t) = (f^{-1}(\mathrm{ll}t/t)\mathrm{ll}t/t)^{-1}$, and $t_k = \exp(k^\lambda)$, $\lambda > 1$. We will use

$$\liminf_{t \to \infty} \frac{L^*(t)}{p(t)} \leq \limsup_{k \to \infty} \frac{L^*(t_k)}{p(t_{k+1})} + \liminf_{k \to \infty} \sup_x \frac{L(t_{k+1}, x) - L(t_k, x)}{p(t_{k+1})} . \tag{3.15}$$

To prove that the lim sup in (3.15) is finite, it suffices to show that $\sum P(L^*(t_k) \geqq Cp(t_{k+1}))$ converges. Let $\eta > 1$ be fixed and $r = [\text{llt}_k/\eta] + 1$. To apply Lemma 3.5, fix $m \geqq m_0$ and observe that

$$p(t_{k+1})/p(t_k) \geqq (t_{k+1}/t_k)^{1-1/\delta_0} . \tag{3.16}$$

By using Lemma 3.5, we have

$$P(L^*(t_k) \geqq Cp(t_{k+1})) \leqq rP(L^*(\eta t_k/\text{llt}_k) \geqq Cp(t_{k+1})/r)$$
$$\leqq Cr^2\eta^{1/2}p(t_k)/p(t_{k+1})$$
$$\leqq C\eta^{-3/2}(\text{llt}_k)^2(t_k/t_{k+1})^{1-1/\delta_0} \tag{3.17}$$

since (3.16) implies that in this setting, the first term of the upper bound obtained in Lemma 3.5 dominates the remaining terms. It is easy to see that (3.17) is summable for $\lambda > 1$. It remains to prove that

$$\sum_k P\left(\sup_x (L(t_{k+1}, x) - L(t_k, x)) < Cp(t_{k+1}) \right)$$

diverges. To obtain the necessary probability estimate, let $\gamma > 1$, $s = \gamma t/\text{llt}$, $a = f^{-1}(1/s)$, $\rho = \rho_1$ and $A = 2\rho_2\rho_1^{-1} - 2$ where ρ_1, and ρ_2 are the constants obtained in Lemma 3.6 and γ will be chosen later. Following Griffin's method [7], set

$$E_k = \left\{ \sup_x (L(ks, x) - L((k-1)s, x)) \leqq Mp(t) , \right.$$

$$\left. \sup_{0 \leqq u \leqq s} |X_{u+(k-1)s} - X_{(k-1)s}| \leqq \rho a, k\rho a \leqq X_{ks} \leqq (A+k)\rho a \right\}$$

and observe that for $r = [\text{llt}/\gamma] + 1$,

$$\bigcap_{k=1}^{r} E_k \subset \{L^*(t) \leqq AMp(t)\} .$$

Denote by $\mathscr{F}_t$ the smallest σ-field generated by $\{X_s, s \leqq t\}$, and note that

$$P\left(\bigcap_{k=1}^{r} E_k | \mathscr{F}_{(r-1)s} \right) = \prod_{k=1}^{r-1} \mathscr{X}_{E_k} P(E_r | \mathscr{F}_{(r-1)s})$$

$$= \prod_{k=1}^{r-1} \mathscr{X}_{E_k} P\left\{ \sup_x (L(rs, x) - L((r-1)s, x)) \leqq Mp(t), \right.$$

$$\left. \sup_{0 \leqq u \leqq s} |X_{u+(r-1)s} - X_{(r-1)s}| \leqq \rho a \right\}$$

$$\times P(r\rho a \leqq X_{rs} \leqq (A+r)\rho a | X_{(r-1)s}) \quad \text{a.s.}$$

Using Lemma 3.5 and Lemma 3.2 we have for fixed $m \geqq m_0$ and M large enough,

$$P\left(\sup_x (L(rs, x) - L((r-1)s, x)) \leqq Mp(t), \sup_{0 \leqq u \leqq s} |X_{u+(r-1)s} - X_{(r-1)s}| \leqq \rho a \right)$$

$$\geqq C_6 \exp(-C_7/\rho^2) - 2C_3\gamma^{1/2}/M - (2m)! \, C_9(m)(2\gamma^{1/2}/M)^m - 64\gamma/(9M^2)$$

$$= \theta_3 > 0 .$$

Lemma 3.6 implies that for $x \in [(r-1)\rho a, (r-1+A/2)\rho a]$,

$$P(r\rho a \leqq X_{rs} \leqq (A+r)\rho a | X_{(r-1)s} = x)$$

$$\geqq P(\rho a \leqq X_{rs} - X_{(r-1)s} \leqq (1+A/2)\rho a)$$

$$\geqq \theta_2 .$$

Similarly for $x \in [(r-1+A/2)\rho a, (A+r-1)\rho a]$,

$$P(r\rho a \leqq X_{rs} \leqq (A+r)\rho a | X_{(r-1)s} = x) \geqq \theta_2 .$$

By taking the iterated conditional expectations, we have

$$P(L^*(t) \leqq AMp(t)) \geqq (\theta_2\theta_3)^r$$

$$\geqq (\log t)^{-2\xi/\gamma}$$

where $e^{-\xi} = \theta_2\theta_3$. Therefore

$$P\left(\sup_x (L(t_{k+1}, x) - L(t_k, x)) \leqq AMp(t_{k+1}) \right) \geqq (k+1)^{-2\lambda\xi/\gamma}$$

whose sum diverges if $2\lambda\xi < \gamma$. $\quad\square$

4 LIL for range

In this section, assuming that X_t is symmetric, in addition to (1.1), we will prove that

$$\limsup_{t\to\infty} \frac{m(R(t))}{f^{-1}(\text{llt}/t)\text{llt}} = C \quad \text{a.s.} \tag{4.1}$$

Since (3.2) implies that

$$\limsup_{t\to\infty} \frac{m(R(t))}{f^{-1}(\text{llt}/t)\text{llt}} > 0 \quad \text{a.s.}$$

it suffices to prove that

$$\limsup_{t\to\infty} \frac{m(R(t))}{f^{-1}(\text{llt}/t)\text{llt}} < \infty \quad \text{a.s.}$$

We will modify Griffin's approach [7] to obtain the required probability estimates. Many calculation there are easier since a symmetric stable process has the scaling property. In fact, we have found that if we use the similar technique to [7], (4.1) holds under extra condition

$$\liminf_{x\to\infty} G(x)/K(x) > 0 ,$$

without assuming the symmetry of $\{X_t\}$. But for symmetric $\{X_t\}$, we can get the upper bound for Laplace transform of $m(R(t))$. It is interesting to compare the upper bounds for Laplace transform of $\sup_{s\leqq t} |X_s^1(a)|$ and $m(R(t))$ obtained in Lemma 4.1 and 4.2 respectively.

Define $J(t, a)$ to be the number of jumps of size greater than a up to time t. That is, recalling the definitions of X^1 and X^2 from Sect. 2, for $a > 0$

$$J(t, a) = \# \{s \le t : |X_s - X_{s-}| > a|\}$$

$$= \# \{s \le t : |X_s^2(a) - X_{s-}^2(a)| > a\} \ .$$

Define for $a > 0$,

$$\tau_0(a) = 0$$

$$\tau_n(a) = \inf \{s > \tau_{n-1}(a) : |X_s(a) - X_{s-}(a)| > a\}$$

$$= \inf \{s > \tau_{n-1}(a) : |X_s^2(a) - X_{s-}^2(a)| > a\}$$

and

$$Z_n(a) = \sup_{\tau_{n-1}(a) \le s < \tau_n(a)} |X_s - X_{\tau_{n-1}(a)}| \ .$$

It is clear that $Z_1(a), Z_2(a), \dots$, are i.i.d. and

$$Z_1(a) = \sup_{s < \tau_1(a)} |X_s| = \sup_{s \le \tau_1(a)} |X_s^1(a)| \ .$$

Furthermore $\{X_t^1(a)\}$ is independent of $\{\tau_n(a), n = 1, 2, \dots \}$ and $\tau_1(a)$ is exponentially distributed with parameter $G(a)$. As a consequence of these definitions, we have

$$m(R(t)) \le Z_1(a) + Z_2(a) + \cdots + Z_{J(t, a)} + Y_t(a) \tag{4.2}$$

where

$$Y_t(a) = \sup_{\tau_{J(t, a)} \le s < t} |X_s - X_{\tau_{J(t, a)}}| \ .$$

Now we prove two lemmas which yield the necessary probability estimate. Recall that (1.1) and symmetry of X_t are assumed throughout.

Lemma 4.1 *For any positive a, t, and u,*

$$E \exp \left(u \sup_{s \le t} |X_s^1(a)| \right) \le 4 \exp (t u^2 a^2 e^{ua} K(a)) \ .$$

Proof. Using the Lévy's Inequality, we have

$$P \left(\sup_{s \le t} |X_s^1(a)| > x \right) \le 2 P (|X_t^1(a)| > x) \ .$$

Also observe that

$$E \exp (u | X_t^1(a)|) \le E \exp (u X_t^1(a)) + E \exp (- u X_t^1(a))$$

$$\le 2 \exp \left(t \int_{|x| \le a} (e^{ux} - 1 - ux) v(dx) \right)$$

$$\le 2 \exp (t u^2 a^2 e^{ua} K(a)) \ .$$

Hence for $u > 0$,

$$E \exp\left(u \sup_{s \leq t} |X_s^1(a)|\right) = \int_0^\infty u e^{ux} P\left(\sup_{s \leq t} |X_s^1(a)| > x\right) dx$$

$$\leq 2 \int_0^\infty u e^{ux} P(|X_t^1(a)| > x) dx$$

$$= 2E \exp(u|X_t^1(a)|)$$

$$\leq 4 \exp(tu^2 a^2 e^{ua} K(a)). \quad \square$$

Lemma 4.2 *For any positive* $a, t, u,$

$$E \exp(um(R(t))) \leq 4 \exp(tu^2 a^2 e^{ua} K(a) + 9tG(a)) .$$

Proof. Let a be fixed and suppressed to simplify the notation in the following. We write, $s_1 < s_2$,

$$|X_{s_2}^1 - X_{s_1}^1|^* = \sup_{s_1 \leq t \leq s_2} |X_t^1 - X_{s_1}^1| .$$

Observe that by using Lemma 4.1,

$$E[\exp(u(Z_1 + Z_2 + \ldots + Z_{J(t)} + Y_t))| J(t) = N, \tau_1 = t_1, \tau_2 = t_2, \ldots \tau_N = t_N]$$

$$= E \exp[u(|X_{t_1}^1|^* + |X_{t_2}^1 - X_{t_1}^1|^* + \ldots + |X_{t_N}^1 - X_{t_{N-1}}^1|^* + |X_t^1 - X_{t_N}^1|^*)]$$

$$= E \exp(u|X_{t_1}^1|^*) E \exp(u|X_{t_2}^1 - X_{t_1}^1|^*) \ldots E \exp(u|X_t^1 - X_{t_N}^1|^*)$$

$$\leq 4^{N+1} \exp(tu^2 a^2 e^{ua} K(a))$$

$$\leq 4 e^{2N} \exp(tu^2 a^2 e^{ua} K(a)) .$$

Since $J(t, a)$ is a Poisson process with parameter $G(a)$,

$$E \exp(u(Z_1 + Z_2 + \ldots + Z_{J(t)} + Y_t) \leq 4 \exp(tG(a)(e^2 - 1) + tu^2 a^2 e^{ua} K(a))$$

$$\leq 4 \exp(9tG(a) + tu^2 a^2 e^{ua} K(a)) ,$$

from which the assertion follows by (4.2). $\quad \square$.

Theorem 4.1 *Suppose that* (1.1) *holds and* $\{X_t\}$ *is symmetric. Then*

$$\limsup_{t \to \infty} \frac{m(R(t))}{f^{-1}(\text{llt}/t)\text{llt}} = C \quad \text{a.s.}$$

Proof. Denote $k(t) = f^{-1}(\text{llt}/t)\text{llt}$. As we remarked earlier, it is enough to prove that

$$\limsup_{t \to \infty} \frac{m(R(t))}{k(t)} < \infty \quad \text{a.s.}$$

Let $a = f^{-1}(\text{llt}/t)$ and $ua = r$ where r will be chosen later. Then we have by using Lemma 4.2,

$$P(m(R(t)) \geq Ck(t)) \leq \exp(- Cu\text{allt}) E \exp(um(R(t)))$$

$$\leq \exp(- Cr\text{llt} + tr^2 e^r K(a) + 9tG(a))$$

$$\leq \exp(- (Cr - r^2 e^r - 9)\text{llt})$$

$$\leq (\log t)^{-2}$$

if we choose r and C so that $Cr - r^2 e^r > 11$. Hence Borel-Cantelli lemma implies that for $t_n = 2^n$,

$$\limsup_{n \to \infty} \frac{m(R(t_n))}{K(t_n)} \le C \quad \text{a.s.}$$

Note that by (2.3), for each n,

$$k(t_n)/k(t_{n-1}) \le 2^{1/\delta_0} \ .$$

Therefore the assertion follows since

$$\limsup_{t \to \infty} \frac{m(R(t))}{k(t)} \le \limsup_{n \to \infty} \frac{m(R(t_n))}{k(t_n)} \frac{k(t_n)}{k(t_{n-1})} \ . \quad \square$$

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