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Skorohod stochastic differential equations of diffusion type

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Summary. Let a, b be $C^2(R^1)$-functions with bounded derivatives of first and second order. We study stochastic differential equations

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}W_t + b(X_t)\,\mathrm{d}t\,, \quad 0 \leqq t \leqq 1\,,$$

whose initial value X_0 is a Fréchet differentiable random variable which may depend on the whole path of the driving Brownian motion (W_t). This anticipation requires to pass from the Itô-integral to the Skorohod-integral. We show that the equation has a unique local solution $\{X_t(\omega), 0 \leqq t \leqq t_0(\omega)\}$, for sufficiently small $t_0(\omega) > 0$, and we provide conditions for the existence of a global solution $\{X_t(\omega), 0 \leqq t \leqq 1\}$.

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1 Introduction

Let $(\Omega, \mathscr{F}, P)$ denote the Wiener space, $\Omega = C_0([0, 1])$, and (W_s) be the coordinate process. The theory of the stochastic integration of processes which are not necessarily adapted to (W_s) has been recently developed by several authors. In particular, Nualart and Pardoux [5] have developed an extended stochastic calculus for both the Skorohod and the generalized Stratonovich integrals. This theory allows to study stochastic differential equations (SDE) where the solution is a non-adapted process. We refer to [9] for a survey of different types of anticipating SDEs. A natural class of anticipating SDEs arises when we impose a random initial value of the solution depending on the whole path of (W_s). Equations of this type have been considered by Ocone and Pardoux [7, 8]. That means, one can consider SDEs of the form

$$\begin{aligned} \mathrm{d}X_s &= a(X_s)\circ \mathrm{d}W_s + b(X_s)\mathrm{d}s\,, \quad s \in [0, 1]\,, \\ X_0 &= G\,, \end{aligned} \tag{1.1}$$

where $\int_0^s a(X_r) \circ \mathrm{d}W_r$ denotes the Stratonovich integral, a and b are $C^2(R^1)$-functions with bounded derivatives of first and second orders, and G is a random variable over Ω. The special properties of the Stratonovich integral play a basic role in proving the existence and uniqueness of a solution of this equation.

The purpose of this paper is to study the SDE

$$\text{(1.2)} \qquad \mathrm{d}X_s = a(X_s)\mathrm{d}W_s + b(X_s)\mathrm{d}s\,, \quad s \in [0, t]\,,$$
$$X_0 = G\,,$$

where the stochastic integral $\int_0^s a(X_r)\mathrm{d}W_r$ is defined in the Skorohod sense. For the functions $a(x) = \alpha \cdot x$ and $b(x) = \beta \cdot x$ such an equation has been considered by Shiota in [10]. Under rather weak assumptions, which allow α and β to depend on the whole path of (W_s), this equation has been studied by Buckdahn in [1], using the Girsanov transformation. Here we will employ this in order to establish the uniqueness of a solution.

Setting $G = W_1$ and looking for a solution of the form $X_t = X_t(x)_{/X = W_1}$ one can rewrite (1.2) in the following form

$$\mathrm{d}X_s(x) + a'(X_s(x))\partial_x X_s(x)\mathrm{d}s = a(X_s(x)) \circ \mathrm{d}W_s + b(X_s(x))\,\mathrm{d}s\,, \quad s \in [0, t]\,,$$
$$X_0(x) = x\,,$$

which shows that in distinction to (1.1) one cannot expect the existence of a solution (X_s) of (1.2) on the whole Ω and for $t = 1$ in general. We show that for any bounded ball A in Ω and some $t = t_A > 0$ (depending on A) there is a unique solution (X_s) on A.

This requires considering the Skorohod integral on balls A and its domain in $L^1([0, t] \times A)$.

The paper is organized as follows: In Sect. 2 we introduce some elements of anticipating stochastic calculus, which will be needed later. In Sect. 3 we define the concept of a solution of a Skorohod SDE, give a short review on some results on linear SDEs and apply this for showing the uniqueness of a solution of (1.2). Finally, in Sect. 4 we present the main result concerning the existence of a solution X of (1.2) on $[0, t_A] \times A$, for any bounded ball $A \subset \Omega$ and some $t_A > 0$, and we provide conditions on the initial value G under which there is a solution X defined on $[0, 1] \times \Omega$. At the end of Sect. 4 we study additional conditions on G under which X_s has a density, for all $s \in (0, 1)$.

2 Basic notions. Skorohod integral

We will use $(\Omega, \mathscr{F}, P)$ to denote the Wiener space, i.e., $\Omega = C_0([0, 1])$ is the space of the continuous functions on $[0, 1]$ with initial value 0 and is equipped with the supremum norm $\|\cdot\|$, $\mathscr{F}$ is the Borel σ-field over Ω, and P is the standard Wiener measure on $(\Omega, \mathscr{F})$. Let (W_s) denote the coordinate process on Ω.

A random variable $F: \Omega \to R^1$ is said to be Fréchet differentiable if it belongs to $L^*(\Omega) = \bigcap_{p>1} L^p(\Omega)$ and if there is an $L^2([0, 1])$-valued random variable

$(D_sF) \in L^*(\Omega, L^2([0,1])) = \bigcap_{p>1} L^p(\Omega, L^2([0,1]))$ such that, for any $\omega \in \Omega$,

$$F\left(\omega + \int_0^\bullet h_s \mathrm{d}s\right) = F(\omega) + \int_0^1 D_sF(\omega)h_s \mathrm{d}s + o(|h|_{L^2([0,1])}),$$
$$\text{as } |h|_{L^2([0,1])} \to 0, \quad h \in L^2([0,1]) .$$

In particular, also all smooth Wiener functionals

$$F(\omega) = f(\omega_{t_1}, \ldots, \omega_{t_n}), \quad \omega \in \Omega ,$$
$$\text{with } f \in C_b^\infty(R^n), \quad 0 \leqq t_1 \leqq \cdots \leqq t_n \leqq 1, \quad n = 1, 2, 3, \ldots$$

belong to these Fréchet differentiable random variables. The set of all smooth Wiener functionals will be denoted by $\mathscr{S}$.

We introduce now some standard spaces. The Sobolev space $\mathbb{D}^{1,p}$, $p > 1$, is the completion of the set of all Fréchet differentiable random variables under the norm

$$\|F\|_{1,p} = \left(E[|F|^p] + E\left[\left(\int_0^1 |D_sF|^2 \mathrm{d}s\right)^{p/2}\right]\right)^{1/p},$$

and D also denotes the extension of the Fréchet derivative to $\mathbb{D}^{1,p}$. Then we set $\mathbb{D}^{1,*} = \bigcap_{p>1} \mathbb{D}^{1,p}$. The set $\mathscr{S}$ is dense in $(\mathbb{D}^{1,p}, \|\cdot\|_{1,p})$, $p > 1$, and in $\mathbb{D}^{1,*}$, too.

In generalization of the Fréchet differentiability, a random variable $F: \Omega \to R^1$ is called Malliavin differentiable if there exists an $L^2([0,1])$-valued random variable (D_sF) such that, for any $h \in L^2([0,1])$ and $\delta > 0$,

$$\lim_{\varepsilon \downarrow 0} P\left\{\omega : \left|\frac{1}{\varepsilon}\left(F\left(\omega + \varepsilon \int_0^\bullet h_s \mathrm{d}s\right) - F(\omega)\right) - \int_0^1 D_sF(\omega)h_s \mathrm{d}s\right| > \delta\right\} = 0 .$$

The set of the Malliavin differentiable random variables will be denoted by $\mathbb{D}^{1,0}$. Clearly,

$$\mathbb{D}^{1,*} \subset \mathbb{D}^{1,p} \subset \mathbb{D}^{1,0}, \quad p > 1 .$$

Lemma 2.1 *For any $h \in L^2([0,1])$ and $\theta_h \omega = \omega + \int_0^\bullet h_s \mathrm{d}s$, $\omega \in \Omega$, the spaces $\mathbb{D}^{1,0}$ and $\mathbb{D}^{1,*}$ are invariant relative to the transformation θ_h, i.e. for any $F \in \mathbb{D}^{1,0}(\mathbb{D}^{1,*})$ also $F(\theta_h)$ belongs to $\mathbb{D}^{1,0}(\mathbb{D}^{1,*})$ and*

$$D_s[F(\theta_h)] = (D_sF)(\theta_h), \quad \text{a.e.}$$

Lemma 2.2 (cf. [6]) *The random variables $M = \max_t W_t$, $m = \min_t W_t$ and $\|W\| = \max_t |W_t|$ are in $\mathbb{D}^{1,*}$, and with the notations $\tau_1 = \min\{t \in [0,1] : W_t = M\}$ and $\tau_2 = \min\{t \in [0,1] : W_t = m\}$ it holds*

$$D_sM = I_{[0,\tau_1]}(s) , \quad D_sm = I_{[0,\tau_2]}(s) , \quad \textit{and}$$

$$D_s[\|W\|] = I\{M > -m\} I_{[0,\tau_1]}(s) - I\{M < -m\} I_{[0,\tau_2]}(s) , \quad \text{a.e.}$$

In particular, (D_sM), (D_sm) and $(D_s[\|W\|])$ are bounded by 1.

By virtue of the Lemmata 2.1 and 2.2 the random variable $\|W(\theta_h)\|$ belongs to $\mathbb{D}^{1,*}$ for any $h \in L^2([0,1])$, and the derivative is bounded by 1.

Finally, we introduce the space $\mathbb{D}^{1,\infty}$ of all $F \in L^\infty(\Omega) \cap \mathbb{D}^{1,*}$ with $(D_sF) \in L^\infty([0,1] \times \Omega)$, and for any open ball $A = B_r(h)$ with radius $r > 0$ around $\int_0^\bullet h_s \mathrm{d}s$, $h \in L^2([0,1])$, the space $\mathbb{D}^{1,\infty}(A)$ of all $F \in \mathbb{D}^{1,\infty}$ whose support is in A and has a strictly positive distance from the boundary of A.

Lemma 2.3 *If, for any* $G \in L^1(A)$, *all* $F \in \mathbb{D}^{1,\infty}(A)$ *satisfy the relation*

$$\int_A FG\mathrm{d}P = 0\,, \tag{2.1}$$

then $G = 0$ a.e. *on* A.

Proof. Let $A = B_r(h)$ and (H_m) be a sequence of Fréchet differentiable random variables approximating sign $G \cdot I_A$ in $L^2(\Omega)$. Such a sequence exists, since $\mathscr{S}$ is dense in $L^2(\Omega)$. Moreover, suppose that, for any natural n, the function $\varphi_n \in C_0^\infty(R^1)$ is such that

$$\varphi_n(x) = 1, \quad \text{for } |x| \leqq r - \frac{1}{n}, \quad \varphi_n(x) = 0, \quad \text{for } |x| \geqq r - \frac{1}{2n}, \text{ and}$$

$$0 \leqq \varphi_n(x) \leqq 1, \quad \text{for all } x \in R^1.$$

Then, with the notations $\vee$, $\wedge$ for the maximum and the minimum, respectively, the sequence $\{((H_m \wedge 1) \vee (-1)) \cdot \varphi_n(\| W(\theta_h) \|), m = 1, 2, 3, \ldots\}$ of elements of $\mathbb{D}^{1,\infty}(A)$ is bounded in $L^\infty(\Omega)$ by 1 and approximates sign $G \cdot \varphi_n(\| W(\theta_h) \|)$ in $L^2(\Omega)$. Hence, (2.1) provides

$$\int_A |G| \varphi_n(\| W(\theta_h) \|)\mathrm{d}P = 0\,.$$

Since n is arbitrary, we see that G must vanish a.e. on A.

This lemma allows one to define the Skorohod integral $(\delta_A, \mathrm{Dom}\, \delta_A)$ as the adjoint of $(D, \mathbb{D}^{1,\infty}(A))$ for any ball $A = B_r(h), r > 0, h \in L^2([0, 1])$:

Definition. A process $(u_s) \in L^1([0, 1] \times A)$ is said to be *Skorohod integrable on A* if there exists a $\delta_A(u) \in L^1(A)$ such that

$$\int_A F\delta_A(u)\mathrm{d}P = \int_A \left(\int_0^1 D_sF \cdot u_s \mathrm{d}s \right) \mathrm{d}P, \text{ for all } F \in \mathbb{D}^{1,\infty}(A)\,. \tag{2.2}$$

In this case we write $(u_s) \in \mathrm{Dom}\, \delta_A$ and call the unique element $\delta_A(u) \in L^1(A)$ the Skorohod integral of (u_s) on A.

If $A = \Omega$, we denote the Skorohod integral on Ω by δ and its domain by Dom δ. Recall that the square integrable processes non-anticipating (W_s) as well as the elements of the space $L^2([0, 1], \mathbb{D}^{1,2})$ of all square integrable processes (u_s) with derivative $(D_r u_s)$ in $L^2([0, 1]^2 \times \Omega)$ are in Dom δ, and the Skorohod integral of a square-integrable non-anticipating process coincides with the Itô integral.

Immediately from the definition of $(\delta_A, \mathrm{Dom}\, \delta_A)$ we obtain the following characterization:

Lemma 2.4 *If A and B are any balls in* Ω *with* $A \subset B$, *then every process* $(u_s) \in$ Dom δ_B *considered as an element of* $L^1([0, 1] \times A)$ *belongs also to* Dom δ_A, *and*

$$\delta_A(u) = \delta_B(u)\,, \quad \text{a.e. } on\ A\,.$$

Obviously, Lemma 2.4 allows one to write $\int_0^t u_s \mathrm{d}W_s$ for $\delta_A(uI_{[0,t]})$ whenever the process $(u_s I_{[0,t]}(s))$ is in Dom δ_A for some $t \in [0, 1]$ and a ball $A \subset \Omega$. Moreover, Lemma 2.4 shows that, for any $(u_s) \in \mathrm{Dom}\, \delta_A$,

$$\delta_A(u) = 0 \quad \text{a.e on any ball } B \subset \left\{ \int_0^1 |u_s|^2 \mathrm{d}s = 0 \right\} (\subset A)\,.$$

This property is called the local property of δ. It allows one to define an integral $\delta_A^{\mathrm{loc}}(u)$ even if (u_s) does not belong to Dom δ_A, cf. [5]:

Definition. Let $A = B_r(h)$ be any ball in Ω. A process $(u_s) \in L^1([0,1] \times A)$ is called an *element of* (Dom $\delta_A)_{\mathrm{loc}}$ if there are a sequence $((u_s^n)) \subset \mathrm{Dom}\,\delta$ and a real sequence (r_n) monotonically increasing to r which are such that

$$\{u^n = u\} \supset B_{r_n}(h)\,, \quad n = 1, 2, 3, \ldots .$$

For such a $(u_s) \in (\mathrm{Dom}\,\delta_A)_{\mathrm{loc}}$ we set

$$\delta_A^{\mathrm{loc}}(u) = \delta(u_n) \text{ on } B_{r_n}\,, \quad n = 1, 2, 3, \ldots .$$

Under this definition we have:

Lemma 2.5 *Let* $(u_s) \in L^1([0,1] \times A)$. *Then,* $(u_s) \in \mathrm{Dom}\,\delta_A$ *if and only if* $(u_s) \in (\mathrm{Dom}\,\delta_A)_{\mathrm{loc}}$ *and* $\delta_A^{\mathrm{loc}}(u) \in L^1(A)$. *Moreover, if* $(u_s) \in \mathrm{Dom}\,\delta_A$, *then* $\delta_A^{\mathrm{loc}}(u) = \delta_A(u)$, a.e. *on* A.

Proof. Let $A = B_r(h)$ and $(u_s) \in \mathrm{Dom}\,\delta_A$. For any natural n and any $C_0^\infty(R^1)$-function φ_n with $\mathrm{supp}\,\varphi_n \subset (-r, r)$ and $\{\varphi_n = 1\} \supset \left(-r + \frac{1}{n}, r - \frac{1}{n}\right)$ we set $H_n = \varphi_n(\| W(\theta_h) \|)$. Clearly, $H_n \in \mathbb{D}^{1,\infty}(A)$. Let $u_s^n = H_n \cdot u_s$ on A and $u_s^n = 0$ outside A. Then

$$(u_s^n) \in L^1([0,1] \times \Omega)\,,$$

$$I(u^n) = H_n \delta_A(u) - \int_0^1 D_s H_n \cdot u_s \, ds \in L^1(\Omega), \text{ and}$$

$$E[F I(u^n)] = E\left[\int_0^1 D_s F \cdot u_s^n \, ds\right], \quad \text{for all } F \in \mathbb{D}^{1,\infty} .$$

Consequently, $(u_s^n) \in (\mathrm{Dom}\,\delta_A)$, and $\delta(u^n) = I(u^n)$. Thus, the local property of D

$$D_s H = 0 \quad \text{a.e. on } [0,1] \times \{H = 0\}, \text{ for all } H \in \mathbb{D}^{1,2} \text{ (cf. [5, Proposition 3.1])}$$

provides

$$\delta(u^n) = \delta_A(u) \quad \text{a.e. on } B_{r-(1/n)}(h)\,, \quad n = 1, 2, 3, \ldots .$$

Consequently $\delta_A^{\mathrm{loc}}(u) = \delta_A(u) \in L^1(A)$.

Conversely, suppose now that $(u_s) \in (\mathrm{Dom}\,\delta_A)_{\mathrm{loc}}$ and $\delta_A^{\mathrm{loc}}(u) \in L^1(A)$. Let $((u_s^n))$ be a sequence in Dom δ associated to (u_s) by the definition of $(\mathrm{Dom}\,\delta_A)_{\mathrm{loc}}$. Since, for any $F \in \mathbb{D}^{1,\infty}(A)$, there is a natural n with $\mathrm{supp}\,F \subset B_{r_n}(h)$, we have

$$\int_A F \delta_A^{\mathrm{loc}}(u) dP\left(= E[F\delta(u^n)] = E\left[\int_0^1 u_s^n \cdot D_s F \, ds\right]\right) = \int_A \left(\int_0^1 u_s \cdot D_s F \, ds\right) dP .$$

Hence, $u \in \mathrm{Dom}\,\delta_A$ and $\delta_A(u) = \delta_A^{\mathrm{loc}}(u)$ a.e. on A.

Finally, note that the notion of the Skorohod integral δ_A of processes $(u_s) \in L^1([0,1] \times A)$ on A developed here can be easily extended to any arbitrary open subset of Ω.

3 Skorohod SDE. Notion and uniqueness of the solution

Fix any $h \in L^2([0,1])$, $r > 0$ and $t \in [0,1]$, and set $A = B_r(h)$.

Definition. Let $X_0 \in L^1(A)$, $(a_s(x))$ and $(b_s(x))$ be in $L^1([0,t] \times A \times R^1)$. A process $(X_s) \in L^1([0,t] \times A)$ is called a *solution of the Skorohod SDE*

$$X_s = X_0 + \int_0^s a_r(X_r)\mathrm{d}W_r + \int_0^s b_r(X_r)\mathrm{d}r \quad \text{a.e. on } A, s \in [0,t] , \tag{3.1}$$

if

(i) $X_s \in L^1(A)$, $s \in [0,t]$, $(a_r(X_r))$ and $(b_r(X_r))$ are in $L^1([0,t] \times A)$,
(ii) $(a_r(X_r)I_{[0,s]}(r)) \in \mathrm{Dom}\, \delta_A$, for all $s \in [0,t]$, and
(iii) Eq. (3.1) holds, i.e., for any $F \in \mathbb{D}^{1,\infty}(A)$, $s \in [0,t]$, we have

$$\int_A \left(\int_0^s a_r(X_r) D_r F \mathrm{d}r \right) \mathrm{d}P = \int_A \left\{ X_s - X_0 - \int_0^s b_r(X_r)\mathrm{d}r \right\} F \mathrm{d}P.$$

For a statement on the uniqueness of the solution of (3.1) we first need some statements for the linear case of $a_s(\omega, x) = a_s(\omega) \cdot x$ and $b_s(\omega, x) = b_s(\omega) \cdot x$. For this, we review some results on linear Skorohod SDE of [1].

Proposition 3.1 *Let $(\sigma_s) \in L^\infty([0,t], \mathbb{D}^{1,\infty})$. Then, there are unique families $\{T_s, s \in [0,t]\}$ and $\{A_{r,s}, r \in [0,s], s \in [0,t]\}$, of transformations of Ω into itself with absolutely continuous image measures $P \circ [T_s]^{-1}$, $P \circ [A_{r,s}]^{-1}$ relative to P which satisfy the equations*

$$T_s\omega = \omega + \int_0^{s\wedge\cdot} \sigma_r(T_r\omega)\, \mathrm{d}r \quad \text{a.e.,} \quad s \in [0,t] ,$$

$$A_{r,s}\omega = \omega - \int_{r\wedge\cdot}^{s\wedge\cdot} \sigma_v(A_{v,s}\omega)\, \mathrm{d}v \quad \text{a.e.,} \quad r \in [0,s] ,$$

for any $s \in [0,t]$.

These transformations T_s, $A_{r,s}$ are invertible, the inverse A_s of T_s is given by

$$A_s = T_s^{-1} = A_{0,s} , \quad s \in [0,t] . \tag{3.2}$$

Moreover, the process $(\sigma_s(T_s))$ is in $L^\infty([0,t], \mathbb{D}^{1,\infty})$, and the densities $\mathscr{L}_s = \mathrm{d}P \circ [A_s]^{-1}/\mathrm{d}P$ and $L_s = \mathrm{d}P \circ [T_s]^{-1}/\mathrm{d}P$ have the following form:

$$\mathscr{L}_s = \exp\left\{ \int_0^s \sigma_r(T_r)\mathrm{d}W_r - \frac{1}{2}\int_0^s \sigma_r(T_r)^2 \mathrm{d}r - \int_0^s \int_0^r (D_v\sigma_r)(T_r) D_r[\sigma_v(T_v)]\mathrm{d}v\mathrm{d}r \right\} ,$$

$$L_s = \mathscr{L}_s(A_s)^{-1} , \quad s \in [0,t] .$$

This result is generalized in [12]. Proposition 3.1 allows us to state the following:

Proposition 3.2 *Assume that $(\sigma_s) \in L^\infty([0,t], \mathbb{D}^{1,\infty})$, $(b_s) \in L^\infty([0,t] \times \Omega)$ and $G \in L^\infty(\Omega)$. Then there exists a unique process $(X_s) \in L^1([0,t] \times \Omega)$ which solves the linear Skorohod* SDE

$$X_s = G + \int_0^s \sigma_r X_r \mathrm{d}W_r + \int_0^s b_r X_r \mathrm{d}r, \quad \text{a.e.,} \quad s \in [0,t] . \tag{3.3}$$

With the notations of Proposition 3.1, *this process is given by*

$$X_s = G(A_s)\exp\left\{\int_0^s b_r(A_{r,s})\mathrm{d}r\right\}L_s\,,\quad s\in[0,t]\,.$$

This global statement can be generalized to a local one.

Proposition 3.3 *Let* $(\tilde{\sigma}_s)$, $(\tilde{b}_s)$ *be in* $L^\infty([0,t]\times A)$ *and* $\tilde{G}\in L^\infty(A)$, *and suppose that there is a process* $(\sigma_s)\in L^\infty([0,t],\mathbb{D}^{1,\infty})$ *which coincides with* $(\tilde{\sigma}_s)$ a.e. *on* $A=B_r(h)$. *Then there exists a solution* $(\tilde{X}_s)\in L^1([0,t]\times A)$ *of the linear equation*

$$\tilde{X}_s=\tilde{G}+\int_0^s\tilde{\sigma}_r\tilde{X}_r\mathrm{d}W_r+\int_0^s\tilde{b}_r\tilde{X}_r\mathrm{d}r\quad\text{a.e. on } A,\ s\in[0,t]\,. \tag{3.4}$$

This solution $(\tilde{X}_s)$ *is unique in* $B_{r-3\varphi(t)}(h)$, *where*

$$\varphi(t)=\int_0^t\|\sigma_s\|_{L^\infty(\Omega)}\mathrm{d}s\,,$$

and if we set $G=\tilde{G}$ *on* A, $G=0$ *outside* A, $(b_s)=(\tilde{b}_s)$ *on* $[0,t]\times A$, *and* $(b_s)=0$ *outside* $[0,t]\times A$, *then* $(\tilde{X}_s)$ *is given on* $B_{r-3\varphi(t)}(h)$ *by the unique solution* (X_s) *of* (3.3).

Because the proof contains specifical technical arguments which are not needed later it is put into the appendix.

The uniqueness of the solution of a linear Skorohod SDE in a ball in Ω allows us to deduce the uniqueness of the solution also for a nonlinear Skorohod SDE. For this let $r>0$, $h\in L^2([0,1])$ and $A=B_r(h)$, and denote by $\mathscr{D}_t(A)$ the set of all $(u_s)\in L^1([0,t]\times A)$ which have an extension $(\tilde{u}_s)$ such that $\tilde{u}_s\in\mathbb{D}^{1,0}$, and $(D_r\tilde{u}_s)\in L^\infty([0,1]\times[0,t]\times\Omega)$ for any $s\in[0,t]$.

Theorem 3.4 *Let* $(a_s(x))$ *and* $(b_s(x))$ *be elements of* $L^1([0,t]\times A\times R^1)$ *which are such that* $a_s(\omega,.)$ *and* $b_s(\omega,.)$ *belong to* $C^2(R^1)$ a.e., *and both* $(\partial_x a_s(x))$ *and* $(\partial_x b_s(x))$ *are in* $L^\infty([0,t]\times A\times R^1)$. *Moreover, assume that there is a process* $(\gamma_s(x))\in L^\infty([0,t]\times R^1,\mathbb{D}^{1,\infty})$ *such that* $\gamma_s(\omega,.)\in C^1(R^1)$ a.e., $(\partial_x\gamma_s(x))\in L^\infty([0,t]\times\Omega\times R^1)$ *and* $(\gamma_s(x))=(\partial_x a_s(x))$ a.e. *on* A.

Set $\varphi(t)=\int_0^t\|\gamma_s\|_{L^\infty(\Omega\times R^1)}\mathrm{d}s$. *Then, for any* $X_0\in L^1(A)$, *there is at most one solution* $(X_s)\in\mathscr{D}_t(A)$ *of the Skorohod* SDE (3.1) *on* $B_{r-3\varphi(t)}(h)$.

Proof. Assume that we are given two solutions (X_s) and (Y_s) of SDE (3.1) which belong to $\mathbb{D}_t(A)$. For all $r\in[0,t]$, we set

$$\tilde{\sigma}_r=\int_0^1(\partial_x a_s)(\theta X_s+(1-\theta)Y_s)\mathrm{d}\theta\,,$$

$$\tilde{b}_r=\int_0^1(\partial_x b_s)(\theta X_s+(1-\theta)Y_s)\mathrm{d}\theta\,.$$

Clearly, $(\tilde{\sigma}_s)$ and $(\tilde{b}_s)$ are in $L^\infty([0,t]\times A)$. From the assumption $(X_s),(Y_s)\in\mathscr{D}_t(A)$ we know that there are extensions $(\tilde{X}_s),(\tilde{Y}_s)$ of (X_s) and (Y_s), respectively, with $(D_r\tilde{X}_s),(D_r\tilde{Y}_s)\in L^\infty([0,1]\times[0,t]\times\Omega)$. Hence, $(\sigma_s=\int_0^1\gamma_s(\theta\tilde{X}_s+(1-\theta)\tilde{Y}_s)\mathrm{d}\theta)$ is an extension of $(\tilde{\sigma}_s)$ and belongs to $L^\infty([0,t]\times\Omega)$; its derivative

$$D_r\sigma_s=\int_0^1(D_r\gamma_s)(\theta\tilde{X}_s+(1-\theta)\tilde{Y}_s)\mathrm{d}\theta+\int_0^1(\partial_x\gamma_s)(\theta\tilde{X}_s+(1-\theta)\tilde{Y}_s)$$
$$\times(\theta D_r\tilde{X}_s+(1-\theta)D_r\tilde{Y}_s)\mathrm{d}\theta$$

exists and is in $L^\infty([0,1]\times[0,t]\times\Omega)$. Therefore, $(\sigma_s)\in L^\infty([0,t],\mathbb{D}^{1,\infty})$. Set $Z_s = X_s - Y_s$, $s\in[0,1]$. Since $(X_s),(Y_s)\in L^1([0,t]\times A)$ are solutions of (3.1), we have

$$(Z_s)\in L^1([0,t]\times A),$$
$$(\tilde{\sigma}_r Z_r I_{[0,s]}(r) = a_r(X_r)I_{[0,s]}(r) - a_r(Y_r)I_{[0,s]}(r))\in \operatorname{Dom}\delta_A\,,\quad s\in[0,t]\,,$$

and

$$\begin{aligned} Z_s &= \int_0^s (a_r(X_r) - a_r(Y_r))\mathrm{d}W_r + \int_0^s (b_r(X_r) - b_r(Y_r))\mathrm{d}r \\ &= \int_0^s \tilde{\sigma}_r Z_r \mathrm{d}W_r + \int_0^s \tilde{b}_r Z_r \mathrm{d}r \quad \text{a.e. on } A\,,\quad s\in[0,t]\,. \end{aligned}$$

Consequently, all assumptions of Proposition 3.3 are satisfied so that $(Z_s)=0$ is the unique solution of this linear equation on $B_{r-3\varphi(t)}(h)$ with

$$\varphi(t) = \int_0^t \|\sigma_s\|_{L^\infty(\Omega)}\mathrm{d}s \leqq \int_0^t \|\gamma_s\|_{L^\infty(\Omega\times R^1)}\mathrm{d}s\,.$$

This completes the proof.

4 Skorohod SDE. Existence of a solution

Throughout this section we suppose that a and b are C^2-functions of R^1 into itself whose derivatives of first and second orders are bounded. The main aim of this section is to give a constructive proof of the existence of a solution of the Skorohod SDE (3.1) for

$$a_s(\omega,x) = a(x) \quad\text{and}\quad b_s(\omega,x) = b(x)\,.$$

Let us first introduce some notations and present the main results before coming to the fairly long and technical details and proofs.

For the construction of the solution (X_s) of the Skorohod SDE

$$X_s = G + \int_0^s a(X_r)\mathrm{d}W_r + \int_0^s b(X_r)\mathrm{d}r \quad \text{a.e. on } \Omega\,,\quad s\in[0,1]\,, \tag{4.1}$$

the associated Itô SDE

$$X_s(x) = x + \int_0^s a(X_r(x))\mathrm{d}W_r + \int_0^s b(X_r(x))\mathrm{d}r \quad \text{a.e. on } \Omega\,,\quad s\in[0,1]\,, \tag{4.2}$$

plays a basic role. The pathwise description of the solution $(X_s(x))$ introduced and discussed by Doss and Sussmann, e.g. [11], allows us to define the transformation

$$A_s(.,x):\Omega\to\Omega \tag{4.3}$$
$$A_s(\omega,x) = \omega - \int_0^{s\wedge .} a'(X_r(\omega,x))\mathrm{d}r\,,\quad \omega\in\Omega\,,$$

for any fixed $x\in R^1$.

Let $G : \Omega \to R^1$ be any Fréchet differentiable bounded random variable satisfying the following assumptions (G):

(G.1) For some real M it holds

$|D_s G(\omega)| \leqq M$, and

$|D_s G(\omega + \int_0^\bullet h_r \mathrm{d}r) - D_s G(\omega)| \leqq M |h|_{L^2([0,1])}$, for all $h \in L^2([0,1])$,

$(s, \omega) \in [0,1] \times \Omega$.

(G.2) There is a real $\delta > 0$ such that, with the notation $K = \|G\|_{L^\infty(\Omega)}$, we have

$1 + \int_0^s (D_r G)(A_s(\omega, x)) a''(X_r(\omega, x)) \partial_x X_r(\omega, x) \mathrm{d}r \geqq \delta$,

for all $(s, \omega, x) \in [0,1] \times \Omega \times [-K, K]$.

Then the implicit function theorem provides the following:

Lemma 4.1 *Under the assumption* (G) *there is a unique process* (U_s) *with values in* $[-K, K]$ *that satisfies the equation*

$$U_s(\omega) = G(A_s(\omega, U_s(\omega))) , \quad (s, \omega) \in [0,1] \times \Omega . \tag{4.4}$$

In particular, $U_0(\omega) = G(\omega)$. *The process* $(U_s(\omega))$ *is pathwise absolutely continuous with respect to the Lebesgue measure, and for any* $(s, \omega) \in [0,1] \times \Omega$, *there is a* $(D_r U_s(\omega)) \in L^2([0,1])$ *such that, for all* $h \in L^2([0,1])$, *it holds*

$$\int_0^1 D_r U_s(\omega) h_r \mathrm{d}r = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left[U_s\left(\omega + \varepsilon \int_0^\bullet h_r \mathrm{d}r \right) - U_s(\omega) \right] . \tag{4.5}$$

This will be the basis for the derivation of the main results:

Theorem 4.2 *Let* a *and* b *be* $C^2(R^1)$*-functions with bounded derivatives of first and second orders. Then, under condition* (G), *the process*

$$X_s(\omega) = X_s(\omega, U_s(\omega))$$

belongs to $\bigcap_{p>1} L^p([0,1], \mathbb{D}^{1,p})$, *and satisfies Eq.* (4.1),

$$X_s = G + \int_0^s a(X_r) \mathrm{d}W_r + \int_0^s b(X_r) \mathrm{d}r \quad \text{a.e.,} \quad s \in [0,1] .$$

Proposition 4.3 *Under assumption* (G) *the process* (X_s) *introduced in Theorem* 4.2 *has the following pathwise properties*:

For any $\omega \in \Omega$, *the mapping* $s \mapsto X_s(\omega)$ *is continuous, and there is a real constant* C *such that, with the notation*

$$\xi(\omega) = \int_0^1 \exp\{C|\omega_s|\} \, \mathrm{d}s ,$$

the random variables $X_s(\omega)$ *and* $D_r X_s(\omega)$ *are bounded by*

$$\exp\{C(\mathrm{e}^{C\xi(\omega)} + |\omega_s|)\} , \quad \textit{for all } s, r \in [0,1] , \quad \omega \in \Omega .$$

In particular, for any bounded ball A *in* Ω, *the process* (X_s) *belongs to* $\mathscr{D}_1(A)$.

Proposition 4.3 will allow us to derive the following statement about uniqueness.

Theorem 4.4 *Under condition* (G) *the process* (X_s) *defined in Theorem* 4.2 *is the unique solution of the Skorohod* SDE (4.1) *inside the class of all processes that, for any bounded ball* $A \subset \Omega$, *belong to* $\mathscr{D}_1(A)$.

Since assumption (G.2) is very restrictive, we should consider what happens if we omit (G.2) and impose only (G.1) on G. This will lead us to a "local" solution of (4.1).

Theorem 4.5 *Let* $a, b \in C^2(R^1)$ *such that the derivatives of first and second order are bounded, and assume* (G.1). *Then, for any bounded ball* $A = B_r(h)$, *there is a* $t > 0$ *such that the Skorohod* SDE (4.1) *has a unique solution* $(X_s) \in \mathscr{D}_t(A)$ *on* $[0, t] \times B_{r-3\varphi(t)}(h)$, *where* $\varphi(t) = t \cdot \sup_x |a'(x)|$.

After this presentation of the main results let us turn now to the details:

1 Review on the description of the solution of Eq. (4.1)

For any $x \in R^1$ let the continuous function $f: R^2 \to R^1$ be the solution of the equation

$$f(x, y) = x + \int_0^y a(f(x, z)dz\,, \quad y \in R^1\,. \tag{4.6}$$

Moreover, set

$$\Phi(x, y) = (\partial_x f(x, y))^{-1} \cdot (b - \tfrac{1}{2}\, a \cdot a')(f(x, y))\,,$$

$$\Psi(x, y) = (\partial_x f(x, y))^{-1} \cdot (b + \tfrac{1}{2}\, a \cdot a')(f(x, y))\,, \quad (x, y) \in R^2\,.$$

Obviously, there exists a real C_1 such that the functions $\Phi(x, y)$, $\partial_x \Phi(x, y)$, $\Psi(x, y)$ and $\partial_x \Psi(x, y)$ are bounded by $C_1(1 + |x|)\exp\{C_1|y|\}$, and $\partial_y \Phi(x, y)$ as well as $\partial_y \Psi(x, y)$ can be estimated by $C_2(1 + |x|^2)\exp\{C_2|y|\}$, for some real C_2. Hence, there are unique processes $(\varphi_s(x))$ and $(\psi_s(x))$ satisfying pathwise the equations

$$\varphi_s(\omega, x) = x + \int_0^s \Phi(\varphi_r(\omega, x), \omega_r)dr\,, \tag{4.7}$$

$$\psi_s(\omega, x) = x + \int_0^s \Psi(\psi_r(\omega, x), \omega_r)dr, \quad s \in [0, 1]\,,$$

for all $(\omega, x) \in \Omega \times R^1$.

Finally, we define

$$X_s(\omega, x) = f(\varphi_s(\omega, x), \omega_s), \tag{4.8}$$

$$Y_s(\omega, x) = f(\psi_s(\omega, x), \omega_s), \quad (s, \omega, x) \in [0, 1] \times \Omega \times R^1.$$

The such defined processes $(X_s(x))$ and $(Y_s(x))$ satisfy the Itô SDE

$$X_s(x) = x + \int_0^s a(X_r(x))dW_r + \int_0^s b(X_r(x))dr\,,$$

and

$$Y_s(x) = x + \int_0^s a(Y_r(x))\mathrm{d}W_r + \int_0^s (b + \tfrac{1}{2}aa')Y_r(x))\mathrm{d}r \quad \text{a.e.,} \quad s \in [0, 1] ,$$

respectively, which can be checked immediately by the non-anticipating Itô formula. Obviously, $X_s(\omega, .)$ and $Y_s(\omega, .)$ belong to $C^1(R^1)$, for all $(s, \omega) \in [0, 1] \times \Omega$, and $X_s(., x)$ as well as $Y_s(., x)$ are Fréchet differentiable, for all $(s, x) \in [0, 1] \times R^1$.

Lemma 4.6 *The processes $(X_s(\omega, x))$, $(Y_s(\omega, x))$ and their derivatives $(\partial_x X_s(\omega, x))$, $(\partial_x Y_s(\omega, x))$ are continuous in $[0, 1] \times \Omega \times R^1$. Moreover, there is a positive real C such that, with the notation*

$$\xi(\omega) = \int_0^1 \exp\{C|\omega_r|\}\, \mathrm{d}r , \quad \omega \in \Omega ,$$

these processes as well as their Fréchet derivatives $(D_r X_s(\omega, x))$ and $(D_r Y_s(\omega, x))$ are bounded by

$$\exp\{(1 + |x|)\mathrm{e}^{\xi(\omega)} + C|\omega_s|\}, \quad \textit{for all} \quad r, s \in [0, 1],\ \omega \in \Omega,\ x \in R^1 .$$

Proof. The continuity of $X_s(\omega, x)$, $Y_s(\omega, x)$ and their derivatives $\partial_x X_s(\omega, x)$, $\partial_x Y_s(\omega, x)$ follows immediately from (4.8). It remains to prove the estimates. For this recall that, for some real $C_2 > 0$, we have

$$|\Phi(x, y)| \leqq (1 + |x|)\exp\{C_2|y|\} ,$$

$$|\partial_x \Phi(x, y)| \leqq (1 + |x|)\exp\{C_2|y|\} ,$$

$$|\partial_y \Phi(x, y)| \leqq (1 + |x|^2)\exp\{C_2|y\} ,$$

so that we can derive from (4.7) and the relations

$$\partial_x \varphi_s(\omega, x) = \exp\left\{\int_0^s (\partial_x \Phi)(\varphi_r(\omega, x), \omega_r)\mathrm{d}r\right\} , \tag{4.9}$$

$$D_r \varphi_s(\omega, x) = \int_r^s (\partial_y \Phi)(\varphi_v(\omega, x), \omega_v)\, \frac{\partial_x \varphi_s(w, x)}{\partial_x \varphi_v(\omega, x)}\, \mathrm{d}v \cdot I_{\{r \leqq s\}}$$

that, for some real $C_3 > 0$, the variables $|\varphi_s(\omega, x)|$, $|\partial_x \varphi_s(\omega, x)|$ and $|D_r \varphi_s(\omega, x)|$ are less than

$$\exp\{(1 + |x|)\mathrm{e}^{C_3 \xi(\omega)}\} . \tag{4.10}$$

Thus, with regard to (4.8) and the following two relations

$$\partial_x X_s(\omega, x) = (\partial_x f)(\varphi_s(\omega, x), \omega_s)\partial_x \varphi_s(\omega, x),$$

$$D_r X_s(\omega, x) = (\partial_x f)(\varphi_s(\omega, x), \omega_s) D_r \varphi_s(\omega, x) + (\partial_y f)(\varphi_s(\omega, x), \omega_s) \cdot I_{\{r \leqq s\}}$$

we see that the estimations in the lemma are correct.

Analogously, we can deduce the estimations of $Y_s(\omega, x)$, $\partial_x Y_s(\omega, x)$ and $D_r Y_s(\omega, x)$.

Remark. From the proof of Lemma 4.6 it becomes clear that also the processes $(\varphi_s(\omega, x))$, $(\psi_s(\omega, x))$ and their derivatives are continuous with respect to (s, ω, x). The Fréchet derivatives $D_r\varphi_s(\omega, x)$ and $D_r\psi_s(\omega, x)$ are continuous with respect to (s, ω, x), uniformly relative to $r \in [0, 1]$.

In addition to the ω-wise estimation of $X_s(\omega, x)$ and $Y_s(\omega, x)$ we also need an a.e. estimation by a random variable of $L^*(\Omega)$.

Lemma 4.7 *There are a real $q > 0$ and a random variable $\zeta \in L^*(\Omega)$ which are such that* a.e. *the random variables*

$$X_s(x), \partial_x X_s(x), D_r X_s(x) \quad \text{and} \quad Y_s(x)$$

are bounded by $\zeta(1 + |x|^q)$, for all $s, r \in [0, 1]$, $x \in R^1$.

If, additionally, a has a bounded derivative of third order, then the same holds also for $\partial_x Y_s(x)$ and $D_r Y_s(x)$.

Proof. The Lemmata 2.1 and 2.2 of [8], including the proofs, provide the estimations we need.

Lemma 4.7 allows to deduce an a.e. estimate by a random variable of $L^*(\Omega)$ also for $\varphi_t(x)$, $\psi_t(x)$ and their derivatives.

Lemma 4.8 *There exist a real $q > 0$ and a random variable $\zeta \in L^*(\Omega)$ which are such that* a.e. *the random variables*

$$\varphi_s(x), \partial_x \varphi_s(x), D_r \varphi_s(x) \quad \text{and} \quad \psi_s(x)$$

are less than

$$\zeta(1 + |x|^q), \quad \text{for all} \quad r, s \in [0, 1], \; x \in R^1 .$$

If, additionally, a has a bounded derivative of third order, then the same is true for $\partial_x \psi_s(x)$ and $D_r \psi_s(x)$.

Proof. Without loss of generality we prove the statement only for $\varphi_s(x)$, $\partial_x \varphi_s(x)$ and $D_r \varphi_s(x)$. We first estimate $\varphi_s(x)$. Note that, by virtue of (4.7), there are reals $C_1, C_2 > 0$ such that

$$|\varphi_s(x)| \leqq |x| + \int_0^s e^{C_1|\omega_r|} \left| \left(b - \frac{1}{2} aa' \right)(X_r(x)) \right| dr$$

$$\leqq |x| + C_2 \int_0^s e^{C_1|\omega_1|}(1 + |X_r(x)|)\, dr .$$

Substituting now the estimate for $X_r(x)$ of Lemma 4.7 we see that the assertion for $\varphi_s(x)$ is true. The estimate for $\partial_x \varphi_s(x)$ we obtain from the relation

$$\partial_x \varphi_s(x) = (\partial_x f)(\varphi_s(x), \omega_s) \partial_x X_s(x)$$

and Lemma 4.7, the assertion for $D_r \varphi_s(x)$ can be proved by substituting the estimates of $\varphi_s(x)$ and $D_r X_s(x)$ in the relation

$$D_r \varphi_s(x) = (\partial_x f)(\varphi_s(x), \omega_s)^{-1} (D_r X_s(x) - (\partial_y f)(\varphi_s(x), \omega_s) I_{\{r \leqq s\}}) .$$

This completes the proof.

2 Transformations generated by $(X_s(x))$ and $(Y_s(x))$

For any $x \in R^1$, $s \in [0, 1]$, we define the following transformations of Ω into itself:

$$A_s(\omega, x) = \omega - \int_0^{s\wedge\cdot} a'(X_r(\omega, x))\mathrm{d}r ,$$

$$T_s(\omega, x) = \omega + \int_0^{s\wedge\cdot} a'(Y_r(\omega, x))\mathrm{d}r , \quad \omega \in \Omega .$$

Lemma 4.9 *For any $x \in R^1, s \in [0, 1]$, the transformations $A_s(x)$ and $T_s(x)$ are inverse to each other.*

Proof. Let $h \in C([0, 1])$. By virtue of (4.7) and (4.8), differentiation of $Y_s(\int_0^\bullet h_r \mathrm{d}r, x)$ and $X_s(\int_0^\bullet h_r \mathrm{d}r, x)$ relative to s provides

$$Y_s\left(\int_0^\bullet h_r \mathrm{d}r, x\right) = x + \int_0^s a\left(Y_v\left(\int_0^\bullet h_r \mathrm{d}r, x\right)\right)h_v \mathrm{d}v + \int_0^s \left(b + \frac{1}{2}aa'\right)\left(Y_v\left(\int_0^\bullet h_r \mathrm{d}r, x\right)\right)\mathrm{d}v ,$$

$$X_s\left(\int_0^\bullet h_r \mathrm{d}r, x\right) = x + \int_0^s a\left(X_v\left(\int_0^\bullet h_r \mathrm{d}r, x\right)\right)h_v \mathrm{d}v + \int_0^s \left(b - \frac{1}{2}aa'\right)\left(X_v\left(\int_0^\bullet h_r \mathrm{d}r, x\right)\right)\mathrm{d}v , \quad s \in [0, 1] .$$

Substitution of $T_1(\int_0^\bullet h_r \mathrm{d}r, x) \in C_0^1([0, 1])$ for $\int_0^\bullet h_r \mathrm{d}r$ in the second equation shows that $(X_s(T_1(\int_0^\bullet h_r \mathrm{d}r, x), x))$ is a solution of the first equation. Thus, the uniqueness of the solution of these differential equations implies that $Y_s(\int_0^\bullet h_r \mathrm{d}r, x)$ and $X_s(T_1(\int_0^\bullet h_r \mathrm{d}r, x), x)$ coincide.

In order to conclude the equality of $Y_s(\omega, x)$ and $X_s(T_1(\omega, x), x)$ for all $\omega \in \Omega$, we only have to recall that the functions $\omega \mapsto Y_s(\omega, x)$ and $\omega \mapsto X_s(\omega, x)$ are continuous, since this implies also the continuity of $\omega \mapsto X_s(T_1(\omega, x), x)$ $(= X_s(\omega + \int_0^\bullet a'(Y_r(\omega, x))\mathrm{d}r, x))$. Therefore, $Y_s(\omega, x) = X_s(T_1(\omega, x), x)$, $(s, \omega, x) \in [0, 1] \times \Omega \times R^1$. Since $(X_s(\omega, x))$ is non-anticipating, $X_s(T_1(\omega, x), x)$ and $X_s(T_t(\omega, x), x)$ coincide for all $s \leqq t$. Hence, for any $s \in [0, 1]$, we have

$$A_s(T_s(\omega, x), x) = T_s(\omega, x) - \int_0^{s\wedge\cdot} a'(X_r(T_s(\omega, x), x))\mathrm{d}r$$

$$= T_s(\omega, x) - \int_0^{s\wedge\cdot} a'(Y_r(\omega, x))\mathrm{d}r = \omega , \quad \omega \in \Omega .$$

Analogously, for any $s \in [0, 1]$, $x \in R^1$, we obtain

$$T_s(A_s(\omega, x), x) = \omega , \quad \omega \in \Omega .$$

This completes the proof.

3 The process (U_s)

Let $G:\Omega\to R^1$ be any Fréchet differentiable bounded random variable satisfying the assumptions (G).

We consider the equation $v. = G(A.(\bullet, v.))$. Its solution provides the process (U_s), which we have to substitute in $(X_s(x))$ in order to obtain a solution of (4.1). For this, recall the implicit function theorem, which we need in the folowing version:

Proposition 4.10 *Let* K *be a positive real. Assume that* $f:[0,1]\times[-K,K]\to[-K,K]$ *is a continuous function with bounded derivatives* $\partial_s f(s,x)$ *and* $\partial_x f(s,x)$ *for which it holds*

(i) *the mapping* $x\mapsto\partial_s f(s,x)$ *is continuous, uniformly relative to* $s\in[0,1]$.
(ii) $(s,x)\to\partial_x f(s,x)$ *is continuous, and there is a real* $\delta>0$ *such that*

$$1-\partial_x f(s,x)\geqq\delta,\quad \textit{for all } (s,x)\in[0,1]\times[-K,K]\,.$$

Then, for any $s\in[0,1]$, *there is a unique solution* $x=x_s\in[-K,K]$ *of the equation* $x=f(s,x)$, *and the function* $s\mapsto x_s$ *is absolutely continuous,*

$$\frac{\mathrm{d}}{\mathrm{d}s}x_s=\frac{\partial_s f(s,x_s)}{1-\partial_x f(s,x_s)}\,.$$

This proposition allows us to prove Lemma 4.1. We divide the proof into two parts and extend the statement.

Proof of Lemma 4.1

Step 1 Existence and uniqueness of the process (U_s) and its pathwise absolute continuity with derivative

$$\frac{\mathrm{d}}{\mathrm{d}s}U_s(\omega)=-\frac{a'(X_s(\omega,U_s(\omega)))(D_sG)(A_s(\omega,U_s(\omega)))}{1+\int_0^s(D_rG)(A_s(\omega,U_s(\omega)))a''(X_r(\omega,U_s(\omega)))(\partial_xX_r)(\omega,U_s(\omega))\,\mathrm{d}r}\,.$$

For any fixed $\omega\in\Omega$ we set

$$f(s,x)=G(A_s(\omega,x))\,.$$

Then, under (G), the chain rule shows that $\partial_x f(s,x)$ exists and has the form

$$\partial_x f(s,x)=-\int_0^s(D_rG)(A_s(\omega,x))a''(X_r(\omega,x))\partial_xX_r(\omega,x)\mathrm{d}r\,.$$

Consequently, $\partial_x f(s,x)$ satisfies the assumptions required in Proposition 4.10. For the proof of the existence of $\partial_s f(s,x)$ set

$$\kappa_\varepsilon(v)=0,\text{ for } v\leqq 0,\quad \kappa_\varepsilon(v)=\frac{1}{\varepsilon}v,\text{ for } v\in[0,\varepsilon],\text{ and}$$

$$\kappa_\varepsilon(v)=1,\text{ for } v\geqq\varepsilon\,,$$

where ε is any real greater than 0. Then, for any $h \in L^2([0, 1])$, the chain rule shows the existence of $(\mathrm{d}/\mathrm{d}s)G(\omega + \int_0^{\bullet} \kappa_\varepsilon(s-v)h_v \mathrm{d}v)$,

$$\frac{\mathrm{d}}{\mathrm{d}s} G\left(\omega + \int_0^{\bullet} \kappa_\varepsilon(s-v)h_v \mathrm{d}v\right) = \int_0^1 (D_r G)\left(\omega + \int_0^{\bullet} \kappa_\varepsilon(s-v)h_v \mathrm{d}v\right) \partial_s \kappa_\varepsilon(s-r)h_r \mathrm{d}r$$
$$= \frac{1}{\varepsilon} \int_{s-\varepsilon}^{s} (D_r G)\left(\omega + \int_0^{\bullet} \kappa_\varepsilon(s-v)h_v \mathrm{d}v\right) h_r \mathrm{d}r .$$

Thus,

$$(4.11) \qquad G\left(\omega + \int_0^{\bullet} \kappa_\varepsilon(t-v)h_v \mathrm{d}v\right)$$
$$= G(\omega) + \int_0^t \left\{\frac{1}{\varepsilon} \int_r^{r+\varepsilon} (D_r G)\left(\omega + \int_0^{\bullet} \kappa_\varepsilon(s-v)h_v \mathrm{d}v\right) \mathrm{d}s\right\} h_r \mathrm{d}r , \quad t \in [0, 1] .$$

Under (G.1) we have

$$\left|(D_r G)\left(\omega + \int_0^{\bullet} \kappa_\varepsilon(s-v)h_v \mathrm{d}v\right) - (D_r G)\left(\omega + \int_0^{r\wedge \cdot} h_v \mathrm{d}v\right)\right|$$
$$\leqq M\left(\int_{r-\varepsilon}^{r+\varepsilon} h_v^2 \mathrm{d}v\right)^{1/2}, \quad \text{for all } s \in [r, r+\varepsilon] .$$

Hence, the right-hand side of (4.11) tends to

$$G(\omega) + \int_0^t (D_r G)\left(\omega + \int_0^{r\wedge \cdot} h_v \mathrm{d}v\right) h_r \mathrm{d}r , \quad \text{as } \varepsilon \to 0 .$$

On the other hand, the convergence of the left-hand side of (4.11) to $G(\omega + \int_0^{t\wedge \cdot} h_v \mathrm{d}v)$ is obvious. Consequently, for any $h \in L^2([0, 1])$, the function $s \mapsto G(\omega + \int_0^{s\wedge \cdot} h_v \mathrm{d}v)$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}s} G\left(\omega + \int_0^{s\wedge \cdot} h_v \mathrm{d}v\right) = h_s (D_s G)\left(\omega + \int_0^{s\wedge \cdot} h_v \mathrm{d}v\right),$$

i.e., $\partial_s f(s, x)$ exists and

$$\partial_s f(s, x) = -a'(X_s(\omega, x))(D_s G)(A_s(\omega, x)) .$$

Clearly, $\partial_s f(s, x)$ is bounded, $a'(X_s(\omega, x))$ is continuous relative to (s, x) and, thus, from the estimate

$$|(D_s G)(A_s(\omega, x)) - (D_s G)(A_s(\omega, y))| \leqq M\left(\int_0^1 |a'(X_r(\omega, x)) - a'(X_r(\omega, y))|^2 \mathrm{d}r\right)^{1/2},$$
$$x, y \in [-K, K], (s, \omega) \in [0, 1] \times \Omega ,$$

we see that $x \mapsto \partial_s f(s, x)$ is continuous, uniformly with respect to $s \in [0, 1]$. Therefore, Proposition 4.10 can be applied. This yields the desired result.

Remark. In order to abbreviate the notations we introduce the transformations

$$A_s \omega = A_s(\omega, U_s(\omega)) , \quad T_s \omega = T_s(\omega, G(\omega)) , \quad \omega \in \Omega .$$

Since $T_s(x)$ and $A_s(x)$ are inverse to each other for all $x \in R^1$, Eq. (4.4) implies the same for T_s, A_s. Moreover, due to [2] the image measures $P \circ [T_s]^{-1}$, $P \circ [A_s]^{-1}$ are equivalent to the Wiener measure P.

Step 2 (of the proof of Lemma 4.1) Derivative $(D_r U_s(\omega))$ exists and is given by

$$D_r U_s(\omega) = \frac{(D_r G)(A_s\omega) - \int_r^s (D_v G)(A_s\omega) a''(X_v(\omega, U_s(\omega)))(D_r X_v)(\omega, U_s(\omega)) dv \cdot I_{\{r \leqq s\}}}{1 + \int_0^s (D_v G)(A_s\omega) a''(X_v(\omega, U_s(\omega)))(\partial_x X_v)(\omega, U_s(\omega)) dv} .$$

Fix any $(s, \omega) \in [0, 1] \times \Omega$ and any $h \in L^2([0, 1])$, and set

$$\begin{aligned} f(\varepsilon, x) &= G(A_s(\theta_{\varepsilon h}\omega, x)) \\ &= G\left(\theta_{\varepsilon h}\omega - \int_0^{s\wedge\cdot} a'(X_v(\theta_{\varepsilon h}\omega, x)) dv\right). \end{aligned}$$

For convenience we drop ω. Note that

$$\partial_\varepsilon [X_r(\theta_{\varepsilon h}, x)] = \int_0^r (D_v X_r)(\theta_{\varepsilon h}, x) h_v dv$$

so that the chain rule yields

$$\begin{aligned} \partial_\varepsilon f(\varepsilon, x) &= \int_0^1 (D_r G)(A_s(\theta_{\varepsilon h}, x)) \\ &\quad \times \left\{ h_r - I_{\{r \leqq s\}} a''(X_r(\theta_{\varepsilon h}, x)) \int_0^r (D_v X_r)(\theta_{\varepsilon h}, x) h_v dv \right\} dr . \end{aligned} \tag{4.12}$$

In analogy to the first step we can derive

$$\partial_x f(\varepsilon, x) = - \int_0^1 (D_r G)(A_s(\theta_{\varepsilon h}, x)) a''(X_r(\theta_{\varepsilon h}, x))(\partial_x X_r)(\theta_{\varepsilon h}, x) dr . \tag{4.13}$$

Obviously, in view of (4.12) and (4.13) the assumptions of Proposition 4.10 are satisfied here, too. Thus, the unique solution $(U_s(\theta_{\varepsilon h}))$ of the equation

$$\begin{aligned} U_s(\theta_{\varepsilon h}) &= G(A_s(\theta_{\varepsilon h}, U_s(\theta_{\varepsilon h}))) \\ &= f(\varepsilon, U_s(\theta_{\varepsilon h})) , \quad \varepsilon \in [0, 1] , \end{aligned}$$

is absolutely continuous with respect to ε, and

$$\frac{d}{d\varepsilon} U_s(\theta_{\varepsilon h}) = \int_0^1 h_r \left(\frac{(D_r G)(A_s) - \int_r^s (D_v G)(A_s) a''(X_v(U_s))(D_r X_v)(U_s) dv I_{\{r \leqq s\}}}{1 + \int_0^s (D_v G)(A_s) a''(X_v(U_s))(\partial_x X_v)(U_s) dv} \right) \circ \theta_{\varepsilon h} dr .$$

Setting $\varepsilon = 0$ we obtain the desired result. This completes the proof of Lemma 4.1.

Lemma 4.11 *Assume* (G). *Then, for any* $p > 1$, *the process* (U_s) *belongs to* $L^p([0, 1], \mathbb{D}^{1,p}) \cap L^\infty([0, 1] \times \Omega)$,

(i) *the mapping* $s \mapsto (D_r U_s) \in L^p([0, 1] \times \Omega)$ *is continuous, and*
(ii) $s \mapsto D_r U_s \in L^p(\Omega)$ *is continuous in* $[0, r]$, *uniformly with respect to* r.

Moreover, the process $((\mathrm{d}/\mathrm{d}s) U_s)$ *is bounded by some real constant,* $(D_r U_s)$ *is bounded by a random variable of* $L^*(\Omega)$, *and additionally, there is some real* $C > 0$ *such that, with the notation*

$$\xi(\omega) = \int_0^1 \exp\{C|\omega_s|\}\,\mathrm{d}s\,,$$

we have

$$|D_r U_s(\omega)| \leqq \exp\{\mathrm{e}^{C\xi(\omega)}\}, \quad \textit{for all} \quad s, r \in [0, 1],\ \omega \in \Omega\,. \tag{4.14}$$

Proof. Obviously, the process (U_s) as well as its derivative $((\mathrm{d}/\mathrm{d}s) U_s)$ are bounded by some real constant. By virtue of the second step of the proof of Lemma 4.1 there exists a real $C > 0$ such that

$$|D_r U_s(\omega)| \leqq C\left(1 + \int_r^s |(D_r X_v)(\omega, U_s(\omega))|\mathrm{d}v \cdot I_{\{r \leqq s\}}\right), \quad \omega \in \Omega\,.$$

Then, the Lemmata 4.6 and 4.7 imply that $(D_r U_s)$ is bounded by a random variable of $L^*(\Omega)$ and that there is a real C with (4.14).

Thus, it remains to show (i) and (ii). We first turn to (ii). Let $s \leqq r$. Then step 2 of the proof of Lemma 4.1 provides

$$D_r U_s = \frac{(D_r G)(A_s)}{1 + \int_0^s (D_v G)(A_s) a''(X_v(U_s))(\partial_x X_v)(U_s)\mathrm{d}v}\,. \tag{4.15}$$

Taking into account that

$$A_s \omega = \omega - \int_0^{s\wedge\cdot} a'(X_v(\omega, U_s(\omega)))\,\mathrm{d}v\,,$$

the Lipschitz condition (G.1) yields for all $r \in [0, 1]$ and all $s, t \in [0, 1]$ with $s \leqq t$ that

$$\begin{aligned} |(D_r G)(A_t) - (D_r G)(A_s)| &\leqq M\left(\int_0^s |a'(X_v(U_t)) - a'(X_v(U_s))|^2\,\mathrm{d}v\right)^{1/2} \\ &\quad + M\left(\int_s^t |a'(X_v(U_t))|^2\,\mathrm{d}v\right)^{1/2}\,. \end{aligned} \tag{4.16}$$

Since

$$\begin{aligned} &|a'(X_v(U_t)) - a'(X_v(U_s))| \\ &\leqq \sup_x |a''(x)| \cdot \sup_r \left|\frac{\mathrm{d}}{\mathrm{d}r} U_r\right| \int_0^1 |(\partial_x X_v)(\theta U_t + (1 - \theta) U_s)|\mathrm{d}\theta \cdot |t - s|\,, \end{aligned}$$

it follows from the estimate of $\partial_x X_v(x)$ in Lemma 4.7 that, for some $\zeta \in L^*(\Omega)$, we have

$$|(D_r G)(A_t) - (D_r G)(A_s)| \leqq \zeta|t - s|\,, \quad \text{for all} \quad r, s, t \in [0, 1]\,. \tag{4.17}$$

Consequently, for any $p > 1$, the mapping $s \mapsto (D_r G)(A_s) \in L^p(\Omega)$, $s \in [0, r]$, is continuous, uniformly relative to $r \in [0, 1]$. Since, on the other hand, by virtue of (G.2) and the Lemmata 4.1 and 4.6

$$s \mapsto 1 + \int_0^s (D_v G)(A_s) a''(X_v(U_s))(\partial_x X_v)(U_s) \mathrm{d}v \in L^p(\Omega) \tag{4.18}$$

is continuous and uniformly bounded, relation (4.15) now implies the correctness of statement (ii).

Finally, we prove (i). For this, note that by the same arguments as used above we can conclude that also the mapping

$$s \mapsto \int_r^s (D_v G)(A_s) a''(X_v(U_s))(D_r X_v)(U_s) \mathrm{d}v \cdot I_{\{r \leqq t\}} \in L^p(\Omega)$$

is continuous for all $r \in [0, 1]$, and from the Lemmata 4.7 and 4.11 we see

$$\sup_{r,s \in [0,1]} \left| \int_r^s (D_v G)(A_s) a''(X_v(U_s))(D_r X_v)(U_s) \mathrm{d}v \cdot I_{\{r \leqq s\}} \right| \in L^*(\Omega) .$$

This together with (4.17) and (4.18) gives (i).

The property (ii) of Lemma 4.11 allows to define

$$(D_- U)_s = L^2(\Omega) - \lim_{r \to s, r > s} D_s U_r , \quad s \in [0, 1] .$$

An immediate consequence of the Lemmata 4.1 and 4.11 is given by the following statement:

Lemma 4.12 *Under assumption* (G) *the process* $((D_- U)_s)$ *is bounded by some* $\zeta \in L^*(\Omega)$, *and*

$$\frac{\mathrm{d}}{\mathrm{d}s} U_s = -a'(X_s(U_s))(D_- U)_s , \quad \textit{for a.e.} \quad s \in [0, 1] .$$

4 *Proof of Theorems 4.2 and 4.4*

Note that by virtue of (4.8) the process (X_s) has the form

$$X_s = f(V_s, W_s) , \quad s \in [0, 1] , \tag{4.19}$$

where the process $V_s = \varphi_s(U_s)$ is absolutely continuous with respect to s: This is the reason why the Itô formula plays a key role in the proof of Theorem 4.2. Let us recall the anticipative Itô formula by Nualart and Pardoux (cf. 5, Theorem 6.1) for the special case we need.

Proposition 4.13 *Let f be a continuous function of R^2 into R^1 such that the derivatives $\partial_x f$, $\partial_y f$, $\partial^2_{y,x} f$ and $\partial^2_{y,y} f$ exist and are continuous. Moreover, let (V_s) be a continuous process with finite variation belonging to $L^2([0, 1], \mathbb{D}^{1,2})$ such that*

(i) $E\left[\int_0^1 \int_0^1 |D_r V_s|^4 \mathrm{d}r \mathrm{d}s\right] < +\infty$,

(ii) *the mapping* $s \mapsto DV_s \in L^4([0, 1] \times \Omega)$ *is continuous, and*

(iii) $s \mapsto D_r V_s \in L^4(\Omega)$ *is continuous in* $[0, r]$, *uniformly with respect to* $r \in [0, 1]$.

Then, for any $s \in [0, 1]$, *with the notation*

$$(D_- V)_s = L^2(\Omega) - \lim_{r \to s, r < s} D_s V_r ,$$

we have

$$f(V_s, W_s) = f(V_0, 0) + \int_0^s \partial_x f(V_r, W_r) \mathrm{d} V_r + \int_0^s \partial_y f(V_r, W_r) \mathrm{d} W_r$$

$$+ \frac{1}{2} \int_0^s \partial_{y,y}^2 f(V_r, W_r) \mathrm{d} r + \int_0^s \partial_{y,x}^2 f(V_r, W_r)(D_- V)_r \mathrm{d} r .$$

If $(\partial_y f(V_r, W_r) I_{[0,s]}(r)) \in \mathrm{Dom}\, \delta$, *then* $\int_0^s \partial_y f(V_r, W_r) \mathrm{d} W_r$ *denotes the Skorohod integral, otherwise it is the local Skorohod integral.*

Although, in Theorem 6.10 of [5] the Itô formula is established under the stronger assumption of the continuity of the process $s \mapsto D_r V_s \in L^4(\Omega)$, $s \in [0, 1]$, which is uniform with respect to $r \in [0, 1]$, it turns out that the proof given in [5] needs the weaker assumptions (ii) and (iii) of Proposition 4.13 only.

Proof of Theorem 4.2 We set $V_s = \varphi_s(U_s)$. Then, obviously, the process (V_s) is absolutely continuous,

$$\frac{\mathrm{d}}{\mathrm{d}s} V_s = \Phi(V_s, W_s) + (\partial_x \varphi_s)(U_s) \frac{\mathrm{d}}{\mathrm{d}s} U_s , \quad s \in [0, 1]\} , \tag{4.20}$$

and in view of

$$D_r V_s = (D_r \varphi_s)(U_s) + (\partial_x \varphi_s)(U_s) D_r U_s , \quad r, s \in [0, 1] , \tag{4.21}$$

we can derive from the remark to Lemma 4.6 and the Lemmata 4.8 and 4.11 that the process (V_s) satisfies the assumptions of Proposition 4.13. In particular, the process $((D_- V)_s)$ exists, and due to Lemma 4.12 we can deduce from (4.20) and (4.21) the relation

$$\frac{\mathrm{d}}{\mathrm{d}s} V_s = \Phi(V_s, W_s) - a'(X_s)(D_- V)_s , \quad s \in [0, 1] . \tag{4.22}$$

Since also the solution $f: R^2 \to R^1$ of Eq. (4.6) has the required properties, we can apply the anticipative Itô formula to $X_s = f(V_s, W_s)$: For this, we compute, on the basis of the Eqs. (4.6), (4.8), (4.4) and (4.19) defining f, $X_s(x)$, U_s and X_s, respectively, that

$$f(V_0, 0) = G,$$

$$\partial_y f(V_s, W_s) = a(X_s) ,$$

$$\partial_{yy}^2 f(V_s, W_s) = (a \cdot a')(X_s) ,$$

and taking into consideration Eq. (4.22) we see that

$$\partial_x f(V_s, W_s) \frac{\mathrm{d}}{\mathrm{d}s} V_s = \partial_x f(V_s, W_s)(\Phi(V_s, W_s) - a'(X_s)(D_- V)_s)$$

$$= (b - \tfrac{1}{2} a \cdot a')(X_s) - \partial_{y,x}^2 f(V_s, W_s)(D_- V)_s .$$

Hence, the anticipative Itô formula of Proposition 4.13 provides Eq. (4.1). From the estimations of $X_s(x)$, $\partial_x X_s(x)$, $D_r X_s(x)$, U_s and $D_r U_s$ in Lemma 4.7 and Lemma 4.11, respectively, we see that the process (X_s) belongs to $L^p([0, 1], \mathbb{D}^{1,p})$ for any $p > 1$. But, obviously we also have $(a(X_r)I_{[0,s]}(r)) \in \text{Dom}\,\delta$, for all $s \in [0, 1]$ then. Hence, (X_s) is a solution of the Skorohod SDE (4.1).

Remark. Note that by virtue of

$$X_s(\omega, x) = Y_s(A_s(\omega, x), x) \quad \text{and} \quad A_s\omega = A_s(\omega, U_s(\omega)) ,$$

it also holds

$$X_s(\omega) = Y_s(G) \circ A_s\omega , \quad (s, \omega) \in [0, 1] \times \Omega .$$

Now Theorem 4.2 allows us to prove Proposition 4.3 and then Theorem 4.4.

Proof of Proposition 4.3 The proof of the estimate follows immediately from the estimates of $X_s(\omega, x)$, $\partial_x X_s(\omega, x)$ and $D_r X_s(\omega, x)$ and those of $U_s(\omega)$ and $D_r U_s(\omega)$ in Lemma 4.6 and Lemma 4.11, respectively. Thus, for any bounded ball $A = B_r(h) \subset \Omega$ and any $H \in \mathbb{D}^{1,\infty}(B_{r+1}(h))$ with $\{H = 1\} \supset A$, the product (HX_s) is in $L^\infty([0, 1], \mathbb{D}^{1,\infty})$, and, since (HX_s) coincides with (X_s) on A, (X_s) is in $\mathscr{D}_1(A)$.

Proof of Theorem 4.4 Note that, for any bounded ball $A = B_r(h) \subset \Omega, r > 0$, $h \in L^2([0, 1])$, the solution (X_s) of (4.1) on Ω is also a solution of (4.1) on A. Since moreover, $(X_s) \in \mathscr{D}_1(A)$, we can apply Theorem 3.4 by setting $\gamma_s(\omega, x) = a'(x)$ and $\varphi(t) = \int_0^t \|\gamma_s\|_{L^\infty(\Omega \times R^1)} ds = t \sup_x |a'(x)|$. Hence, (X_s) is unique on $B_{r-3\varphi(t)}(h)$. This holds for any $r > 0$ and $h \in L^2([0, 1])$, and for Ω, too.

Remark. As we have shown, under assumption (G) there is a unique solution of the Skorohod SDE (4.1). The existence of such a solution on the whole probability space Ω is mainly ensured by assumption (G.2), whereas (G.1) is only some smoothness requirement for G which allows to apply the anticipative stochastic calculus. So we should pay a bit more attention to (G.2). Obviously, we can replace (G.2) by the stronger conditon (G.2′), namely

(G.2′) either (i) $D_s G(\omega) \geqq 0$, for all $(s, \omega) \in [0, 1] \times \Omega$, and $a''(x) \geqq 0$, for all $x \in R^1$, or (ii) $D_s G(\omega) \leqq 0$, for all $(s, \omega) \in [0, 1] \times \Omega$, and $a''(x) \leqq 0$, for all $x \in R^1$.

In particular, if $G = f(W_1), f \in C^2(R^1)$, then the requirements (i) and (ii) take the form $f'(x) \geqq 0$, $a''(x) \geqq 0$, for all $x \in R^1$, and $f'(x) \leqq 0$, $a''(x) \leqq 0$, for all $x \in R^1$, respectively.

This price of omitting (G.2) consists in the loss of the global existence of the solution of (4.1), assumption (G.1) guarantees only the existence of a local solution as described in Theorem 4.5.

5 *Proof of Theorem 4.5*

Proof. Let $\lambda \in C^\infty(R^1)$ with $\lambda(x) = |x|$, for $|x| \geqq 1$, and $|x| \leqq \lambda(x) \leqq 1$, for $|x| \leqq 1$. With regard to Lemma 4.6 there is some real $C > 0$ such that

$$\left(\int_0^1 |\partial_x X_s(\omega, x)|^2 \right)^{1/2} \leqq \exp\{C(1 + |x|)e^{\xi(\omega)}\} , \quad (\omega, x) \in \Omega \times R^1 ,$$

for $\xi(\omega) = \int_0^1 \exp\{C\lambda(\omega_s)\}\, ds$.

Fix any reals r_1, r_2 with $0 < r_1 < r_2$ and a $\kappa \in C_0^\infty(R^1)$ which is such that

$$\kappa(x) = 1\ , \quad \text{for } |x| \leqq r_1\ , \quad \kappa(x) = 0\ , \quad \text{for } |x| \geqq r_2\ , \quad \text{and}$$
$$0 \leqq \kappa(x) \leqq 1\ , \quad \text{for all } x \in R^1\ ,$$

and set $\tilde{G}(\omega) = G(\omega)\kappa(\xi(\omega))$.

Due to the assumption, G satisfies (G.1) and, obviously, so does $\tilde{G}$. In particular,

$$\tilde{K} = \|\tilde{G}\|_{L^\infty(\Omega)} \leqq \|G\|_{L^\infty(\Omega)}\ ,$$
$$\|D\tilde{G}\|_{L^\infty([0,1]\times\Omega)} \leqq \|DG\|_{L^\infty([0,1]\times\Omega)} + r_2 \sup_x |\kappa'(x)| \sup_x |\lambda'(x)| \cdot \|G\|_{L^\infty(\Omega)}\ ,$$

and in virtue of the relation

$$\xi(\omega) \leqq \exp\left\{C\left(1 + \sup_x |a'(x)|\right)\right\}\xi(A_s(\omega, x))$$

we have

$$\operatorname{supp}(D_r\tilde{G})(A_s(x)) \subset \{|\xi| \leqq r_2'\} \text{ with } r_2' = r_2 \exp\left\{C\left(1 + \sup_x |a'(x)|\right)\right\}.$$

Hence, there exists a real $C(r_2)$ with

$$\begin{aligned}
&\int_0^t |(D_r\tilde{G})(A_s(\omega, x))a''(X_r(\omega, x))(\partial_x X_r)(\omega, x)|\,\mathrm{d}r \\
&\leqslant C(r_2)\left(\int_0^1 |(\partial_x X_r)(\omega, x)|^2 \mathrm{d}r\right)^{1/2} t^{1/2} I\{|\xi| \leqq r_2'\} \\
&\leqq C(r_2)\exp\{C(1 + \tilde{K})\mathrm{e}^{r_2'}\}t^{1/2}, \\
&\qquad \text{for all } (s, \omega, x) \in [0, t] \times \Omega \times [-\tilde{K}, \tilde{K}]\ .
\end{aligned}$$

Let $\delta \in (0, 1)$. Then the expression in the last line is less than $1 - \delta$ if $t = t(r_2) > 0$ is small enough. In this case we have

$$1 + \int_0^s (D_r\tilde{G})(A_s(\omega, x))a''(X_r(\omega, x))\partial_x X_r(\omega, x)\mathrm{d}r \geqq \delta\ ,$$
$$\text{for all } (s, \omega, x) \in [0, t] \times \Omega \times [-\tilde{K}, \tilde{K}]\ ,$$

i.e., $\tilde{G}$ satisfies also (G.2) on the time interval $[0, t]$. Of course, this restriction of (G.2) to $[0, t]$ does not matter if we consider the Skorohod SDE (4.1) only for this time interval. So Theorem 4.4 provides a solution $(\tilde{X}_s) \in \bigcap_{p>1} L^p([0, t], \mathbb{D}^{1,p})$ of the equation

$$\tilde{X}_s = \tilde{G} + \int_0^s a(\tilde{X}_r)\,\mathrm{d}W_r + \int_0^s b(\tilde{X}_r)\,\mathrm{d}r \quad \text{a.e., } s \in [0, t]\ ,$$

which is unique in the class of all processes whose restriction to any bounded ball $A \subset \Omega$ belongs to $\mathscr{D}_t(A)$.

Since $\tilde{G} = G$ on $B_{r_1'}(0)(\subset \{|\xi| \leqq r_1\})$, for $r_1' = (1/C)\ln r_1 - 1$, $(\tilde{X}_s) \in \mathscr{D}_t(B_{r_1'}(0))$ is a solution of the Skorohod SDE (4.1) on $B_{r_1'}(0)$ with initial value G. Hence, for a given bounded ball $A = B_r(h) \subset \Omega$ choose $r_1' > r + |h|_{L^2([0,1])}$: Then $A \subset B_{r_1'}(0)$, and $(\tilde{X}_s)$ is a solution of (4.1) on A and belongs to $\mathscr{D}_t(A)$. In order to complete the proof note that the uniqueness of the solution $(\tilde{X}_s) \in \mathscr{D}_t(A)$ on $B_{r-3t\sup_x|a'(x)|}(h)$ follows from Theorem 3.4.

6 Existence of a density of X_s

Finally, at the end of this section we assume again (G) in order to deduce a criterion for the absolute continuity of the solution $X_s \in \mathbb{D}^{1,*}$ of the Skorohod SDE (4.1) with respect to the Lebesgue measure for any $s \in [0,1]$. For this, we will use Theorem 7 of [3]:

Proposition 4.14 *If* $F \in \mathbb{D}^{1,2}$ *and* $\int_0^1 |D_r F|^2 \mathrm{d}r > 0$ a.e., *then* F *has a density with respect to the Lebesgue measure.*

Now we can state:

Proposition 4.15 *Let* a *be a* $C^3(R^1)$*-function with bounded derivatives of the first three orders,* b *be* $C^2(R^1)$*-functions with bounded derivatives of first and second order, assume* (G) *and let* (X_s) *denote the solution of* (4.1) *presented in Theorem* 4.2. *Then, for any* $s \in (0,1)$, X_s *has a density if*

$$(4.23)\qquad P\left(\left\{\int_s^1 |D_r G|^2 \mathrm{d}r = 0\right\} \cap \left\{(D_r G) \in C^1([0,s]),\quad D_0 G = -a(G),\right.\right.$$
$$\left.\left.\frac{\mathrm{d}}{\mathrm{d}s} D_s G|_{s=0} = -[b + \tfrac{1}{2}aa', a](G)\right\}\right) = 0 .$$

Here, $[.,.]$ *denotes the Lie brackets.*

Proof. Since X_s has the form

$$X_s = Y_s(G) \circ A_s,\quad s \in [0,1],$$

and $P \circ [A_s]^{-1}$ is equivalent to the Wiener measure P, the random variable X_s has a density if and only if $Y_s(G)$ has a density. So we apply Proposition 4.14 to $Y_s(G)$. For this, note that $Y_s(G) \in \mathbb{D}^{1,*}$, and

$$D_r[Y_s(G)] = (\partial_x Y_s)(G) D_r G + a(Y_r(G)) \frac{(\partial_x Y_s)(G)}{(\partial_x Y_r)(G)} I_{\{r \leqq s\}},\quad r \in [0,1] .$$

On the other hand, by the non-anticipative Itô formula we see that, for any $x \in R^1$,

$$(4.24)\qquad a(Y_r(x))\partial_x Y_r(x)^{-1} = a(x) + \int_0^1 \left[b + \frac{1}{2}aa', a\right](Y_v(x))\partial_x Y_v(x)\,\mathrm{d}v ,$$

for all $r \in [0,1]$, a.e.

Since both sides are continuous relative to x, we can substitute G, and we obtain

$$D_r[Y_s(G)]$$
$$= (\partial_x Y_s)(G)\left\{D_r G + \left(a(G) + \int_0^r \left[b + \frac{1}{2}aa', a\right](Y_v(G))(\partial_x Y_v)(G)\mathrm{d}v\right) I_{\{r \leqq s\}}\right\} .$$

Hence,

$$\left\{\int_0^1 |D_r[Y_s(G)]|^2 \mathrm{d}r = 0\right\} = \left\{\int_s^1 |D_r G|^2 \mathrm{d}r = 0\right\}$$
$$\cap \left\{\int_0^s \left|D_r G + a(G) + \int_0^r \left[b + \frac{1}{2}aa', a\right](Y_v(G))(\partial_x Y_v)(G)\mathrm{d}v\right|^2 \mathrm{d}r = 0\right\} .$$

On this set the mapping $r \mapsto D_r G$ has a modification which belongs to $C^1([0, s])$, takes the value $-a(G)$ for $r = 0$, and its derivative $(\mathrm{d}/\mathrm{d}r) D_r G$ has the value $-[b + \frac{1}{2} aa', a](G)$ in $r = 0$. This provides the desired result.

Of course, by the Taylor expansion of $a(Y_s(x))\partial_x Y_s(x)^{-1}$ computed by Kusuoka and Stroock in [4] one can improve condition (4.23) by the following requirement: With probability zero, $(D_s G)$ has the same Taylor expansion as $-(a(Y_s(G))\partial_x Y_s(G)^{-1})$. However, the price for such a statement consists in the more restrictive assumption that $a, b \in C^\infty(R^1)$ have bounded derivatives of all orders and the initial value G is a smooth random variable.

Appendix

We present the proof of Proposition 3.3.

Proof. Existence of a solution. Due to Proposition 3.2 there is a process $(X_s) \in L^1([0, t] \times \Omega)$ *with* $X_s \in L^1(\Omega)$, $(\sigma_r X_r I_{[0,s]}(r)) \in \mathrm{Dom}\,\delta$ and

$$\delta(\sigma X I_{[0,s]}) = X_s - X_0 - \int_0^s b_r X_r \mathrm{d}r \quad \text{a.e.,} \quad s \in [0, t] \,.$$

Then it follows from Lemma 2.4 that the restriction $(\tilde{X}_s)$ of (X_s) to A is in $L^1([0, t] \times A)$, $\tilde{X}_s \in L^1(A)$ and

$$\delta_A(\tilde{\sigma} \tilde{X} I_{[0,s]}) = \tilde{X}_s - \tilde{X}_0 - \int_0^s \tilde{b}_r \tilde{X}_r \mathrm{d}r \quad \text{a.e. on } A, \quad s \in [0, t] \,,$$

i.e., $(\tilde{X}_s)$ is a solution of (3.4).

The proof of the uniqueness requires the following auxiliary statements (cf. Proposition 2.1 and Lemma 3.1 in [1]):

Lemma A.1 *Let* $(\sigma_s) \in L^\infty([0, t], \mathbb{D}^{1,\infty})$. *Then there exists a sequence of smooth step processes*

$$\left(\sigma_s^n = \sum_{k=1}^{2^n} F_{n,k} I_{[(k-1)2^{-n}, k2^{-n})}(s) \right), \quad F_{n,k} \in \mathcal{S} \,, \quad k = 1, 2, \ldots, 2^n, \; n = 1, 2, 3, \ldots$$

with the following properties:

(i) (σ_s^n) *converges to* (σ_s) *in* $L^2([0, t], \mathbb{D}^{1,2})$, *and*

(ii) $\|\sigma^n\|_{L^\infty([0,t] \times \Omega)} \leqq \|\sigma\|_{L^\infty([0,t] \times \Omega)}$,
$\|D\sigma^n\|_{L^\infty([0,t]^2 \times \Omega)} \leqq 1 + \|D\sigma\|_{L^\infty([0,t]^2 \times \Omega)}$, $\quad n = 1, 2, 3 \ldots$

For a given process $(\sigma_s) \in L^\infty([0, t], \mathbb{D}^{1,\infty})$ fix such a sequence $((\sigma_s^n))$, and denote by $A_s^n : \Omega \to \Omega$ the transformation associated to (σ_s^n) by (3.2). Then we can state:

Lemma A.2 *For any* $F \in \mathcal{S}$ *we have, with the above notations*:

(i) $(F(A_s^n)) \in L^\infty([0, t], \mathbb{D}^{1,\infty})$ *is such that* $F(A_s^n) \in \mathcal{S}$, *for all* $s \in [0, t]$.

(ii) *The mapping* $r \mapsto D_s[F(A_r^n)]$ *is continuous for* a.e. $(s, \omega) \in [0, t] \times \Omega$, *and we can set*

$$D_s[F(A_s^n)] = \lim_{r \to s} D_s[F(A_r^n)] \,, \quad s \in [0, t] \,,$$

The process $(D_s[F(A_s^n)])$ *belongs to* $L^\infty([0,t],\mathbb{D}^{1,\infty})$, *and*

$$\text{(A.1)}\qquad \|D_s[F(A_s^n)]\|_{L^\infty(\Omega)} \leqq \|DF\|_{L^\infty([0,t]\times\Omega)} \exp\{2(1+\|D\sigma\|_{L^\infty([0,t]^2\times\Omega)})\}, \quad s\in[0,t].$$

(iii) *The mapping* $r \mapsto F(A_r^n)$ *is* a.e. *absolutely continuous with respect to the Lebesgue measure*

$$\text{(A.2)}\qquad \frac{\mathrm{d}}{\mathrm{d}s} F(A_s^n) = -\sigma_s^n \cdot D_s[F(A_s^n)], \quad \text{a.e.}$$

Finally, under the above assumptions we recall the following fact of the proof of Theorem 3.1 [1]:

Lemma A.3. *For any* $F \in \mathscr{S}$, *it holds*

$$F(A_s) = L^2(\Omega) - \lim_{n\to\infty} F(A_s^n), \quad s \in [0,t].$$

Now we are able to show the uniqueness of the solution of (3.4). For this we use the notations introduced above:

Proof of Proposition 3.3. Uniqueness. Fix an arbitrary $\varepsilon > 0$, and any $C_0^\infty(R^1)$-function ψ which has all its values between 0 and 1, takes the value 1 in the interval $[-(r-\varphi(t))+\varepsilon, (r-\varphi(t))-\varepsilon]$ and has its support inside $[-(r-\varphi(t))+(\varepsilon/2), (r-\varphi(t))-(\varepsilon/2)]$. Set $H = \psi(\|W(\theta_h)\|)$ and let $F \in \mathscr{S}$. Then, for any $s \in [0,t]$, and any natural n the random variable $HF(A_s^n)$ belongs to $\mathbb{D}^{1,\infty}(A)$, and (3.4) provides

$$\text{(A.3)}\qquad \int_A \tilde{X}_s HF(A_s^n)\mathrm{d}P = \int_A \tilde{G}HF(A_s^n)\mathrm{d}P + \int_A \left(\int_0^s \tilde{\sigma}_r \tilde{X}_r D_r[HF(A_s^n)]\mathrm{d}r\right)\mathrm{d}P + \int_A \left(\int_0^s \tilde{b}_r \tilde{X}_r HF(A_s^n)\mathrm{d}r\right)\mathrm{d}P.$$

By virtue of Lemma A.2 we have

$$F(A_s^n) = F - \int_0^s \sigma_v^n D_v[F(A_v^n)]\,\mathrm{d}v,$$

and, for $r \leqq s$,

$$D_r[HF(A_s^n)] = D_r[HF(A_r^n)] - \int_r^s D_r[\sigma_v^n H D_v[F(A_v^n)]]\,\mathrm{d}v.$$

Thus, the right-hand side of (A.3) takes the form

$$\text{(A.4)}\qquad \int_A \tilde{G}HF\mathrm{d}P + \int_A \left(\int_0^s \tilde{\sigma}_r \tilde{X}_r D_r[HF(A_r^n)]\mathrm{d}r\right)\mathrm{d}P + \int_A \left(\int_0^s \tilde{b}_r \tilde{X}_r HF(A_r^n)\mathrm{d}r\right)\mathrm{d}P$$
$$- \int_0^s \left\{ \int_A \tilde{G}H\sigma_v^n D_v[F(A_v^n)]\mathrm{d}P + \int_A \int_0^v \tilde{\sigma}_r \tilde{X}_r D_r[H\sigma_v^n D_v[F(A_v^n)]]\,\mathrm{d}r\,\mathrm{d}P + \int_A \int_0^v \tilde{b}_r \tilde{X}_r H\sigma_v^n D_v[F(A_v^n)]\mathrm{d}r\,\mathrm{d}P \right\} \mathrm{d}v.$$

Applying (3.4) to the random variable $H\sigma_v^n D_v[F(A_v^n)] \in \mathbb{D}^{1,\infty}(A)$, we see that $\int_A(\int_0^s \tilde{X}_r \sigma_r^n H D_r[F(A_r^n)]\mathrm{d}r)\,\mathrm{d}P$ is equal to the integral in the last two lines of (A.4), i.e.,

$$\int_A \tilde{X}_s HF(A_s^n)\,\mathrm{d}P - \int_A \tilde{G}HF\,\mathrm{d}P - \int_A\left(\int_0^s \tilde{b}_r\tilde{X}_r HF(A_r^n)\mathrm{d}r\right)\mathrm{d}P$$
$$= \int_A\left(\int_0^s (\tilde{\sigma}_r - \sigma_r^n)\tilde{X}_r D_r[HF(A_r^n)]\,\mathrm{d}r\right)\mathrm{d}P$$
$$+ \int_A\left(\int_0^s \sigma_r^n \tilde{X}_r F(A_r^n) D_r H\,\mathrm{d}r\right)\mathrm{d}P\,.$$

Since $F \in \mathscr{S}$ is bounded, Lemma A.3 allows to apply the dominating convergence theorem to the left-hand side of this equation. So we see that the limit of the left-hand side equals

$$\int_A \tilde{X}_s HF(A_s)\mathrm{d}P - \int_A \tilde{G}HF\,\mathrm{d}P - \int_A\left(\int_0^s \tilde{b}_r\tilde{X}_r HF(A_r)\mathrm{d}r\right)\mathrm{d}P\,.$$

Taking into account that $(\tilde{\sigma}_r)$ coincides on A with (σ_r) we derive from Lemmata A.1(ii) and A.2(ii) that the assumptions required for the dominating convergence theorem are satisfied for the first integral on the right-hand side, too. Hence Lemma A.1(i) implies that

$$\lim_{n\to\infty} \int_A\left(\int_0^s (\tilde{\sigma}_r - \sigma_r^n)\tilde{X}_r D_r[HF(A_r^n)]\,\mathrm{d}r\right)\mathrm{d}P = 0\,.$$

The same arguments provide

$$\lim_{n\to\infty} \int_A\left(\int_0^s \sigma_r^n\tilde{X}_r F(A_r^n) D_r H\,\mathrm{d}r\right)\mathrm{d}P = \int_A\left(\int_0^s \tilde{\sigma}_r\tilde{X}_r F(A_r) D_r H\,\mathrm{d}r\right)\mathrm{d}P\,.$$

Consequently,

$$\text{(A.5)}\qquad \int_A \tilde{X}_s HF(A_s)\mathrm{d}P = \int_A \tilde{G}HF\,\mathrm{d}P + \int_A\left(\int_0^s \tilde{b}_r\tilde{X}_r HF(A_r)\mathrm{d}r\right)\mathrm{d}P$$
$$+ \int_A\left(\int_0^s \tilde{\sigma}_r\tilde{X}_r F(A_r) D_r H\,\mathrm{d}r\right)\mathrm{d}P\,.$$

Since supp $H \subset B_{r-\varphi(t)}(h)$ and the transformation $A_s\omega = \omega - \int_0^{s\wedge\cdot} \sigma_r(A_{r,s}\omega)\mathrm{d}r$ maps $B_{r-\varphi(t)}(h)$ into A, a Girsanov transformation in (A.5) yields:

$$\text{(A.6)}\qquad \int_A \tilde{X}_s(T_s)H(T_s)F\mathscr{L}_s\mathrm{d}P = \int_A \tilde{G}HF\,\mathrm{d}P$$
$$+ \int_A\left(\int_0^s \tilde{b}_r(T_r)\tilde{X}_r(T_r)H(T_r)F\mathscr{L}_r\mathrm{d}r\right)\mathrm{d}P$$
$$+ \int_A\left(\int_0^s \tilde{\sigma}_r(T_r)\tilde{X}_r(T_r)(D_r H)(T_r)\mathscr{L}_r F\,\mathrm{d}r\right)\mathrm{d}P\,.$$

Here T_s denotes the inverse transformation to A_s, and $\mathscr{L}_s$ is the density of A_s. All the integrals

$$\int_A |\tilde{X}_s(T_s)\mathscr{L}_s| \mathrm{d}P = \int_A |\tilde{X}_s| \mathrm{d}P, \tag{A.7}$$

$$\int_A \left(\int_0^s |\tilde{b}_r(T_r)\tilde{X}_r(T_r)\mathscr{L}_r| \mathrm{d}r \right) \mathrm{d}P = \int_A \left(\int_A^s |\tilde{b}_r \tilde{X}_r| \mathrm{d}r \right) \mathrm{d}P$$

and

$$\int_A \left(\int_0^s |\tilde{\sigma}_r(T_r)\tilde{X}_r(T_r)\mathscr{L}_r| \mathrm{d}r \right) \mathrm{d}P = \int_A \left(\int_0^s |\tilde{\sigma}_r \tilde{X}_r| \mathrm{d}r \right) \mathrm{d}P$$

are finite, and $H \in \mathbb{D}^{1,\infty}(A)$. Hence, (A.6) does not only hold for all $F \in \mathscr{S}$, but also for all $F \in L^\infty(\Omega)$. Let now F be any element of $L^\infty(\Omega)$ with support in $B_{r-2\varphi(t)-\varepsilon}(h)$. In this ball $H(T_s)$ takes only the value 1, and the local property of D,

$$D_r H = 0 \quad \text{a.e. on } \{H = 1\} (\supset B_{r-\varphi(t)-\varepsilon}(h))$$

implies

$$(D_r H)(T_r) = 0 \quad \text{a.e. on } B_{r-2\varphi(t)-\varepsilon}(h) \; .$$

Hence, we can conclude from (A.5) and (A.6) that, for any $s \in [0, t]$,

$$\tilde{X}_s(T_s)\mathscr{L}_s = \tilde{G} + \int_0^s \tilde{b}_r(T_r)\tilde{X}_r(T_r)\mathscr{L}_r \mathrm{d}r \quad \text{a.e. on } B_{r-2\varphi(t)-\varepsilon}(h) \; ,$$

i.e.,

$$\tilde{X}_s(T_s)\mathscr{L}_s = \tilde{G} \exp\left\{ \int_0^s \tilde{b}_r(T_r) \mathrm{d}r \right\} .$$

This shows that the process $(\tilde{X}_s(T_s))$ is unique in $B_{r-2\varphi(t)-\varepsilon}(h)$, and so is (X_s) in $B_{r-3\varphi(t)-\varepsilon}(h)$. Since $\varepsilon > 0$ is arbitrary, (X_s) must be unique in the ball $B_{r-3\varphi(t)}(h)$, too. This completes the proof.

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