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Random recursive construction of self-similar fractal measures. The noncompact case

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Summary. The self-similarity of sets (measures) is often defined in a constructive way. In the present paper it will be shown that the random recursive construction model of Falconer, Graf and Mauldin/Williams for (statistically) self-similar sets may be generalized to the noncompact case. We define a sequence of random finite measures, which converges almost surely to a self-similar random limit measure. Under certain conditions on the generating Lipschitz maps we determine the carrying dimension of the limit measure.

Preface

This paper was mainly inspired by the ideas of my friend and teacher Ulrich Zähle (14.1.1950–1.12.1989), who died too early. With him we lost a sincere man and excellent mathematician. In the last years he worked intensively on the field of fractals. He developed a general approach for describing self-affine random measures, and founded the bases for wide application of this nice theory. This paper would not be possible without the good cooperation with him.

0 Introduction

Fractals are sets with a highly irregular structure, for instance all sets of non-integer Hausdorff dimension. A general account of fractals was given by Mandelbrot [M]. In many cases the Hausdorff dimension is determined by the parameters of self-similarity. A theory of strictly self-similar compact sets has been developed by Hutchinson [H]. He showed that for every finite set of contractions $S_1, \ldots, S_N$ acting on $\mathbb{R}^d$ there is a unique invariant non-empty compact set K with $K = \bigcup_{i=1}^{N} S_i K$. Moreover, if the maps $S_1, \ldots, S_N$ are similarities he gave an open set condition under which the Hausdorff dimension of K is equal to α, where α is the unique number satisfying $\sum_{i=1}^{N} (\mathrm{Lip}\, S_i)^\alpha = 1$. Later on, Falconer [F2], Graf [G] and Mauldin and Williams [MW] independently investigated random compact fractal sets by randomizing each step in Hutchinson's construction. They showed that Hutchinson's result has a probabilistic counterpart.

The aim of the present paper is to generalize the random recursive construction model of Falconer, Graf and Mauldin and Williams (FGMW-model) to the non-compact case. For this it is useful to translate the FGMW-model in a measure-theoretical language. We can do this in the following way (cf. Sect. 1). Let μ be a probability distribution on the set of all N-tuples of contractions acting on $\mathbb{R}^d$. First, we choose an N-tuple $(S_1, \ldots, S_N)$ of random contractions according to μ, and arbitrary random finite initial measures $\psi_1, \ldots, \psi_N$, independent in $i \in \{i, \ldots, N\}$, and independent of $(S_1, \ldots, S_N)$, and set

$$\phi_1 = \sum_{i=1}^{N} (\mathrm{Lip}\, S_i)^\alpha \psi_i \circ S_i^{-1},$$

where $\mathrm{Lip}(S_i)$ are the Lipschitz constants of S_i and α is the unique number with $\mathbb{E} \sum_{i=1}^{N} (\mathrm{Lip}\, S_i)^\alpha = 1$. Now we choose, for every $i \in \{1, \ldots, N\}$, an N-tuple $(S_{i,1}, \ldots, S_{i,N})$ w.r.t. μ, and random finite measures $\psi_{i,1}, \ldots, \psi_{i,N}$, such that $\{(S_{i,1}, \ldots, S_{i,N}), (S_1, \ldots, S_N), \psi_i, \psi_{i,j} \colon i, j \in \{1, \ldots, N\}\}$ is a family of independent random elements, and set

$$\phi_2 = \sum_{i,j=1}^{N} (\mathrm{Lip}\, S_i)^\alpha (\mathrm{Lip}\, S_{i,j})^\alpha \psi_{i,j} \circ S_{i,j}^{-1} \circ S_i^{-1}.$$

We continue this process and obtain a sequence of random finite measures

$$\phi_n = \sum_{i_1, \ldots, i_n = 1}^{N} (\mathrm{Lip}\, S_{i_1})^\alpha \ldots (\mathrm{Lip}\, S_{i_1, \ldots, i_n})^\alpha \psi_{i_1, \ldots, i_n} \circ S_{i_1, \ldots, i_n}^{-1} \circ \ldots \circ S_{i_1}^{-1}, \qquad n = 1, 2, \ldots.$$

In Sect. 2 we investigate under which conditions the sequence (ϕ_n) converges almost surely to a random finite measure ϕ. Moreover, we show that the limit measure is independent of the choice of the startmeasures.

In Sect. 3 we show tha the limit measure ϕ has an analogous self-similarity property like the limit set in the FGMW-model, that means, if ϕ^i are copies of ϕ, independent of ϕ and $(S_1, \ldots, S_N)$, then the measure $\sum_{i=1}^{N} (\mathrm{Lip}\, S_i)^\alpha \phi^i \circ S_i^{-1}$ has the same distribution as ϕ.

In Sect. 4 the first two moment measures of ϕ are investigated.

The carrying dimension of the limit measure ϕ will be computed in Sect. 5. Thorem 5.8 gives conditions under which the carrying dimension of ϕ equals $d^* = \min\{d, \alpha\}$ a.s.

Finally, in Sect. 6 we give some examples satisfying the conditions of Theorem 5.8. In particular, we show that in the deterministic case these conditions are implied by Hutchinson's open set condition. Moreover, we get in the compact case a class of examples where the open set condition in the FGMW-model is not satisfied, but the dimensions do not change.

Throuhout the paper we use the following notations

$\mathbb{N}\{1, 2, \ldots\}$

$[\mathbb{R}^d, \mathscr{R}^d]$	– d-dimensional euclidean space with Borel σ-algebra $\mathscr{R}^d$ and euclidean norm $|\cdot|$.
$\mathscr{L}^d$	– Lebesgue measure on $\mathbb{R}^d$
$B(x, r)$	– the closed ball in $\mathbb{R}^d$ with centre x and radius r
$M=M(\mathbb{R}^d)$	– the family of all Radon measures on $[\mathbb{R}^d, \mathscr{R}^d]$
$m=m(\mathbb{R}^d)$	– the usual σ-algebra on M (see [K])
$C_b=C_b(\mathbb{R}^d)$	– the set of all continuous bounded functions f: $\mathbb{R}^d \to [0, \infty)$
$C_c=C_c(\mathbb{R}^d)$	– the set of all continuous functions f: $\mathbb{R}^d \to [0, \infty)$ with compact support
$\mathscr{L}_c=\mathscr{L}_c(\mathbb{R}^d)$	– the set of all Lipschitz functions f: $\mathbb{R}^d \to [0, \infty)$ with compact support
$\|f\|$	– the supremum norm of the function f

$\varphi(f)=\int f(x)\,\varphi(dx)$, $\varphi \in M$ f: $\mathbb{R}^d \to [0, \infty)$ measurable, in particular $\varphi(|\cdot|)=\int |x|\,\varphi(dx)$

$(\Omega, \mathscr{F}, \mathbb{P})$	– a complete probability space, over which all random variables will be defined
$\mathbb{E}X$	– the expectation of the random variable X.

1 Model

In this section we describe the underlying construction model of certain sequences of random finite measures which should converge to a self-similar measure with fractal carrying dimension. This construction is an analogue to the FGMW-model (cf. [PZ]) for the noncompact case, that means, in our case the measures generally do not have compact support. We first introduce the following notations. Let N be a positive integer and $D_n=\{1, \ldots, N\}^n$.

Furthermore denote by $D=\bigcup_{n=0}^{\infty} D_n$ the family of all finite sequences $\sigma=(\sigma_1, \ldots, \sigma_n)$ (including $\emptyset$) in $\{1, \ldots, N\}$, by $|\sigma|=n$ the length of $\sigma \in D_n$, by $\tau|n=(\tau_1, \ldots, \tau_{n \wedge |\tau|})$ the curtailment for $\tau \in D$ and by $\sigma * \tau=(\sigma_1, \ldots, \sigma_{|\sigma|}, \tau_1, \ldots, \tau_{|\tau|})$ the juxtaposition of σ and τ.

For a map S: $\mathbb{R}^d \to \mathbb{R}^d$ let $\operatorname{Lip}(S)=\sup\left\{\frac{|Sx-Sy|}{|x-y|}: x, y \in \mathbb{R}^d x \neq y\right\}$ be the Lipschitz constant for S which may be infinit.

S is called a contraction if $\operatorname{Lip}(S)<1$. By $\operatorname{Con}(\mathbb{R}^d)$ we denote the set of all contractions of $\mathbb{R}^d$. $\operatorname{Con}(\mathbb{R}^d)$ is a topological space; it is equipped with the topology of pointwise convergence. A map S: $\mathbb{R}^d \to \mathbb{R}^d$ is a similarity if $|Sx-Sy|=\operatorname{Lip}(S)|x-y|$ for all $x, y \in \mathbb{R}^d$.

Let μ be the distribution of a random vector $(S_1, \ldots, S_N)$ of contracting maps, i.e., a probability measure on the Borel σ-algebra of $\operatorname{Con}(\mathbb{R}^d)^N$.

1.1. Lemma ([F2], [G] 6.1.) (i) $\int \sum_{i=1}^{N} \operatorname{Lip}(S_i)^\beta \mu(\mathrm{d}(S_1, \ldots, S_N))$ *is a strongly monotone decreasing function in* $\beta \geqq 0$.

(ii) *There exists a unique* $\alpha \geqq 0$ *with*

$$\int \sum_{i=1}^{N} \operatorname{Lip}(S_i)^\alpha \mu(d(S_1, \ldots, S_N)) = 1. \quad \square$$

1.2. We now define a sequence of random measures as follows. For every $\sigma \in D$ we choose an N-tuple $(S_{\sigma * 1}, \ldots, S_{\sigma * N})$ of random contractions distributed according to μ and a random finite measure ψ_σ. We suppose that $\{(S_{\sigma * 1}, \ldots, S_{\sigma * N}), \psi_\sigma : \sigma \in D\}$ is a family of independent random elements. For $\sigma \in D$ denote

$$\rho_\sigma = \operatorname{Lip}(S_\sigma),$$
$$T_\sigma = S_{\sigma|1} \circ S_{\sigma|2} \circ \ldots \circ S_\sigma \quad \text{and}$$
$$1_\sigma = \prod_{i=1}^{|\sigma|} \rho_{\sigma|i}.$$

The random measures ϕ_n $n = 1, 2, \ldots$, which form our model, are defined through

$$\phi_n = \sum_{\sigma \in D_n} 1_\sigma^\alpha \psi_\sigma \circ T_\sigma^{-1} \qquad n = 1, 2, \ldots,$$

where α is the unique number from 1.1. (ii).

In the next section we investigate under which (sufficient) conditions a random finite limit measure for the sequence (ϕ_n) exists.

2 Convergence

For random variables $\xi_1, \ldots, \xi_n$ denote by $\sigma(\xi_1, \ldots, \xi_n)$ the generated σ-algebra. Let $\mathscr{F}_n = \sigma\left(S_\tau : \tau \in \bigcup_{i=1}^{n} D_i\right)$ and $\tilde{\mathscr{F}}_n = \sigma\left(\psi_\tau, S_\tau : \tau \in \bigcup_{i=1}^{n} D_i\right)$ $n = 1, 2, \ldots$. For shortness we write $S_{\alpha,n}$ for $\sum_{\sigma \in D_n} 1_\sigma^\alpha$ $n = 1, 2, \ldots$.

Throughout the paper we suppsoe the following.

2.1. Conditions. (i) $\alpha > 0$

(ii) $\sup_n \mathbb{E} S_{\alpha,n}^2 < \infty$

(iii) $\mathbb{E}|S_i(0)| < \infty$ for $i = 1, \ldots, N$ (This implies $\mathbb{E}|S_i(x)| < \infty$.)

(iv) There exists a constant $K \geqq 0$ such that $\mathbb{E}\psi_\sigma(|\cdot|) \leqq K$ for all $\sigma \in D$, and for any $\sigma \in D$, $\psi_\sigma(\mathbb{R}^d) = 1$ a.s.

2.2. Lemma ([PZ] 3.1.) $S_{\alpha,n}$ *forms a martingale w.r.t. the filtration* $\mathbb{F} = (\mathscr{F}_n)_{n=1,2,\ldots}$ *and therefore there exists a random variable Z such that*

(i) $S_{\alpha,n} \to Z$ a.s. *for* $n \to \infty$

(ii) $\mathbb{E}Z = 1$

(iii) $\mathbb{E}Z^2 < \infty$. $\square$

2.3. Corollary *From* 1.2. *and* 2.2. *it follows immediately that* $\phi_n(\mathbb{R}^d)$ *converges* a.s. *to* Z. $\square$

The next lemma is a reformulation of the L^2-convergence theorem for L^2-bounded matingales.

2.4. Lemma ([R] 8.12) *Let* $\xi_1, \xi_2, \ldots$ *be a sequence of quadratic integrable random elements in* $[\mathbb{R}, \mathscr{R}]$ *and* $\xi_1^* = \xi_1 - \mathbb{E}\xi_1$, $\xi_n^* = \xi_n - \mathbb{E}[\xi_n | \sigma(\xi_1, \ldots, \xi_{n-1})]$ $n = 2, 3, \ldots$.

Then the sum $\sum_{k=1}^{\infty} \xi_k^*$ *converges* a.s. *if* $\sum_{k=1}^{\infty} \operatorname{Var} \xi_k^*$ *is finite.* □

The aim of this section is to prove the almost sure convergence of the sequence (ϕ_n) (see 1.2.) w.r.t. the weak topology on M. A sequence (φ_n) of finite measures converges *weakly* to a measure $\varphi \in M(\varphi_n \xrightarrow{w} \varphi)$ iff $\varphi_n(f) \xrightarrow{n\to\infty} \varphi(f)$ for all $f \in C_b(\mathbb{R}^d)$. Analogously, a sequence (φ_n) of local finite measures converges *vaguely* to a measure $\varphi \in M(\varphi_n \xrightarrow{v} \varphi)$ iff $\varphi_n(f) \xrightarrow{n\to\infty} \varphi(f)$ for all $f \in C_c(\mathbb{R}^d)$. For details see [K]. There the following is proved.

2.5. Lemma ([K], 15.7.6.) *Let* $\varphi, \varphi_1, \varphi_2, \ldots \in M$ *and denote by* $\mathscr{B}^d$ *the ring of all bounded Borel sets of* $\mathscr{R}^d$. *Then the following statements are aquivalent.*

(i) $\varphi_n \xrightarrow{w} \varphi$

(ii) $\varphi_n \xrightarrow{v} \varphi$ *and* $\inf_{B \in \mathscr{B}^d} \limsup_n \varphi_n(B^c) = 0$

(iii) $\varphi_n \xrightarrow{v} \varphi$ *and* $\varphi_n(\mathbb{R}^d) \to \varphi(\mathbb{R}^d)$. □

Using the tightness of $\mathscr{L}(\mathbb{R}^d)$ in $C(\mathbb{R}^d)$, the triangle inequality and 2.5 we obtain a criterion for weak convergence.

2.6. Lemma *A sequence* (φ_n) *from* M *converges weakly to a measure* $\varphi \in M$ *iff* $\varphi_n(f) \to \varphi(f)$ *for all* $f \in \mathscr{L}_c$ *and* $\inf_{B \in \mathscr{B}^d} \limsup_n \varphi_n(B^c) = 0$. □

The next lemma is the key for proving the convergence.

2.7. Lemma *For any function* $f \in \mathscr{L}_c$ *the sequence* $(\phi_n(f))_{n=1,2,\ldots}$ *forms almost surely a Cauchy sequence.*

Proof. Let $f \in \mathscr{L}_c$ with $\|f\| \leqq C$. By 1.2. we have a.s. for $n > m$:

$$\begin{aligned}
&|\phi_n(f) - \phi_m(f)| \\
&= \Big| \sum_{\sigma \in D_n} 1_\sigma^\alpha \psi_\sigma(f \circ T_\sigma) - \sum_{\tau \in D_m} 1_\tau^\alpha \psi_\tau(f \circ T_\tau) \Big| \\
&= \left| \sum_{j=m}^{n-1} \sum_{\sigma \in D_j} 1_\sigma^\alpha \left(\sum_{i=1}^{N} \rho_{\sigma * i}^\alpha \psi_{\sigma * i}(f \circ T_{\sigma * i}) - \psi_\sigma(f \circ T_\sigma) \right) \right| \\
&= \left| \sum_{j=m}^{n-1} \sum_{\sigma \in D_j} 1_\sigma^\alpha \left(\sum_{i=1}^{N} \rho_{\sigma * i}^\alpha \psi_{\sigma * i}(f \circ T_{\sigma * i}) - \sum_{i=1}^{N} \rho_{\sigma * i} \psi_\sigma(f \circ T_\sigma) \right.\right. \\
&\quad \left.\left. + \sum_{i=1}^{N} \rho_{\sigma * i}^\alpha \psi_\sigma(f \circ T_\sigma) - \psi_\sigma(f \circ T_\sigma) \right) \right|.
\end{aligned}$$

Defining

$$\xi_j = \sum_{\sigma \in D_j} 1_\sigma^\alpha \psi_\sigma(f \circ T_\sigma) \cdot \left(\sum_{i=1}^{N} \rho_{\sigma * i}^\alpha - 1 \right)$$

and

$$\eta_j = \sum_{\sigma \in D_{j+1}} 1_\sigma^\alpha [\psi_\sigma(f \circ T_\sigma) - \psi_{\sigma|j}(f \circ T_{\sigma|j})]$$

we obtain

$$|\phi_n(f) - \phi_m(f)| = \left| \sum_{j=m}^{n-1} (\eta_j + \xi_j) \right|.$$

Thus we see that $(\phi_n(f))$ forms a.s. a Cauchy sequence if $\left|\sum_{j=1}^{\infty}(\eta_j+\xi_j)\right|$ is a.s. finite. Using the triangle inequality and the symmetry of ξ_j it is enough to show that

(a) $$\sum_{j=1}^{\infty}|\eta_j|<\infty \quad \text{a.s. and}$$

(b) $$\sum_{j=1}^{\infty}\xi_j<\infty \quad \text{a.s.}$$

For (a) it suffices to show that $\mathbb{E}\sum_{j=1}^{\infty}|\eta_j|<\infty$. Indeed, using the monotone convergence theorem we get

$$\begin{aligned}
\mathbb{E}\sum_{j=1}^{\infty}|\eta_j| &= \sum_{j=1}^{\infty}\mathbb{E}|\eta_j| \\
&\leqq \sum_{j=1}^{\infty}\mathbb{E}\sum_{\sigma\in D_j}1_\sigma^\alpha\sum_{i=1}^{N}\rho_{\sigma*i}|\psi_{\sigma*i}(f\circ T_{\sigma*i})-f(T_\sigma 0)+f(T_\sigma 0)-\psi_\sigma(f\circ T_\sigma)| \\
&\leqq \sum_{j=1}^{\infty}\mathbb{E}\sum_{\sigma\in D_j}1_\sigma^\alpha\sum_{i=1}^{N}\rho_{\sigma*i}^\alpha\left(\int|f(T_{\sigma*i}x)-f(T_\sigma 0)|\,\psi_{\sigma*i}(dx)+\int|f(T_\sigma 0)|\,\psi_\sigma(dx)\right) \\
&\leqq \operatorname{Lip}(f)\sum_{j=1}^{\infty}\mathbb{E}\sum_{\sigma\in D_j}1_\sigma^{\alpha+1}\sum_{i=1}^{N}\rho_{\sigma*i}^\alpha(\psi_{\sigma*i}(|\cdot|\circ S_{\sigma*i})+\psi_\sigma(|\cdot|)) \\
&\leqq \operatorname{Lip}(f)\sum_{j=1}^{\infty}\sum_{\sigma\in D_j}\mathbb{E}1_\sigma^{\alpha+1}\sum_{i=1}^{N}(\mathbb{E}\psi_{\sigma*i}(|\cdot|)+\mathbb{E}\psi_\sigma(|\cdot|)+\mathbb{E}|S_{\sigma*i}(0)|).
\end{aligned}$$

From 2.1. (iii) and (iv) it follows that there exists a constant $\tilde{K}$ such that

$$\begin{aligned}
\mathbb{E}\sum_{j=1}^{\infty}|\eta_j| &\leqq \tilde{K}\sum_{j=1}^{\infty}\sum_{\sigma\in D_j}\mathbb{E}1_\sigma^{\alpha+1} \\
&= \tilde{K}\sum_{j=1}^{\infty}\left(\sum_{i=1}^{N}\mathbb{E}\rho_i^{\alpha+1}\right)^j<\infty \qquad \text{(by 1.1. (i)).}
\end{aligned}$$

For (b) we will use 2.4. By definition of $(\xi_i)_{i=1,2,\dots}$ it is clear that $\sigma(\xi_1,\dots,\xi_{n-1})\subset\tilde{\mathscr{F}}_n$. Moreover, we get as a corollary of the required independence

$$\mathbb{E}\xi_j=\sum_{\sigma\in D_j}\mathbb{E}1_\sigma^\alpha\psi_\sigma(f\circ T_\sigma)\cdot\mathbb{E}\left(\sum_{i=1}^{N}\rho_{\sigma*i}^\alpha-1\right)=0$$

and

$$\begin{aligned}
&\mathbb{E}[\xi_n|\sigma(\xi_1,\dots,\xi_{n-1})] \\
&\quad=\mathbb{E}[\mathbb{E}(\xi_n|\tilde{\mathscr{F}}_n|\,\sigma(\xi_1,\dots,\xi_{n-1})] \\
&\quad=\mathbb{E}\left[\sum_{\sigma\in D_n}1_\sigma^\alpha\psi_\sigma(f\circ T_\sigma)\,\mathbb{E}\left(\sum_{i=1}^{N}\rho_{\sigma*i}^\alpha-1\right)|\sigma(\xi_1,\dots,\xi_{n-1})\right] \\
&\quad=0.
\end{aligned}$$

According to 2.4. it suffices to show that $\sum_{j=1}^{\infty} \operatorname{Var} \xi_j < \infty$. Indeed, using again the independence and 2.1. (ii) and (iii) we infer

$$\begin{aligned}
\sum_{j=1}^{\infty} \operatorname{Var} \xi_j &= \sum_{j=1}^{\infty} \mathbb{E}\xi_j^2 \\
&= \sum_{j=1}^{\infty} \mathbb{E}\Big(\sum_{\sigma\in D_j} 1_\sigma^\alpha \psi_\sigma(f\circ T_\sigma)\Big(\sum_{i=1}^{N} \rho_{\sigma * i}^\alpha - 1\Big)\Big)^2 \\
&= \sum_{j=1}^{\infty} \sum_{\sigma\in D_j} \mathbb{E} 1_\sigma^{2\alpha} (\psi_\sigma(f\circ T_\sigma))^2 \mathbb{E}\Big(\sum_{i=1}^{N} \rho_{\sigma * i}^\alpha - 1\Big)^2 \\
&\quad + \sum_{j=1}^{\infty} \sum_{\substack{\sigma,\tau\in D_j\\ \sigma\neq\tau}} \mathbb{E} 1_\sigma^\alpha 1_\tau^\alpha \psi_\sigma(f\circ T_\sigma)\psi_\tau(f\circ T_\tau) \mathbb{E}\Big(\sum_{i=1}^{N} \rho_{\sigma * i}^\alpha - 1\Big) \mathbb{E}\Big(\sum_{k=1}^{N} \rho_{\tau * k}^\alpha - 1\Big) \\
&\leqq N^2 \cdot C \sum_{j=1}^{\infty} \Big(\mathbb{E} \sum_{i=1}^{N} \rho_i^{2\alpha}\Big)^j + 0 \\
&< \infty. \quad \square
\end{aligned}$$

We now turn to the main result of this section.

2.8. Theorem (i) *The sequence* $(\phi_n)_{n=1,2,\ldots}$ *converges almost surely to a random finite measure* ϕ *with respect to the weak topology on* M.
(ii) *If there are two families* $\{\psi_\sigma\colon \sigma\in D\}$ *and* $\{\tilde{\psi}_\sigma\colon \sigma\in D\}$ *which satisfy* 2.1. *(iv), then the corresponding limit measures* ϕ *and* $\tilde{\phi}$ *are a.s. equal.*

Proof. (i) First we will show that (ϕ_n) converges a.s. to a random finite measure ϕ w.r.t. the vague topology.

Let $E_{\mathscr{L}_c}$ be a countable and dense subset of $\mathscr{L}_c$. We define $\Omega_0 = \{\omega\in\Omega\colon (\phi_n(\omega)(f))_{n=1,2,\ldots}$ is a Cauchy sequence for all $f\in E_{\mathscr{L}_c}\}$. In view of 2.7. we have $\mathbb{P}(\Omega_0)=1$. Using the completeness of the real numbers, we define for any $\omega\in\Omega_0$ a linear functional $1(\omega)\colon E_{\mathscr{L}_c} \to [0,\infty)$ by $1(\omega)(f) := \lim_{n\to\infty} \phi_n(\omega)(f), f\in E_{\mathscr{L}_c}$.

These functionals are continuous as a consequence of 2.1. (iii) and 2.2. because

$$\begin{aligned}
\|1(\omega)\| &= \sup_{f\in E_{\mathscr{L}_c}} \frac{|1(\omega)(f)|}{\|f\|} \\
&= \sup_{f\in E_{\mathscr{L}_c}} \lim_{n\to\infty} \frac{\phi_n(\omega)(f)}{\|f\|} \\
&\leqq \lim_{n\to\infty} \phi_n(\omega)(\mathbb{R}^d) \\
&= Z(\omega) < \infty \qquad \text{for } \mathbb{P}\text{-a.e. } \omega.
\end{aligned}$$

Thus we can extend these functionals to linear and continuous functionals $1(\omega)$: $C_c \to [0,\infty)$ and we obtain $\mathbb{P}(\phi_n(f) \xrightarrow{n\to\infty} 1(f)$ for all $f\in C_c) = 1$. Since the map $\omega \mapsto 1(\omega)$ is measurable and following [Fe] 2.5.2. we get a random measure

ϕ, putting $\phi(f)=1(f)$, such that $\mathbb{P}(\phi_n \xrightarrow{v} \phi)=1$. Using 2.5. it remains to show that $\inf_{B\in\mathscr{B}^d} \limsup_n \phi_n(B^c)=0$ a.s. This condition is easy to verify, since

$$\begin{aligned}
\phi_n(B(0,m)^c) &\leqq \frac{1}{m}\int |x|\,\phi_n(\mathrm{d}x)\\
&=\frac{1}{m}\sum_{\sigma\in D_n} 1_\sigma^\alpha \int |T_\sigma x|\,\psi_0(\mathrm{d}x)\\
&\leqq \frac{1}{m}\Big[\sum_{\sigma\in D_n} 1_\sigma^{\alpha+1}\,\psi_\sigma(|\cdot|)+\sum_{\sigma\in D_n} 1_\sigma^\alpha |T_\sigma 0|\Big]\\
&\leqq \frac{1}{m}\sum_{j=1}^{\infty}\Big(\sum_{\sigma\in D_j} 1_\sigma^{\alpha+1}\,\psi_0(|\cdot|)+\sum_{\sigma\in D_j} 1_{\sigma|j-1}^{\alpha+1}|S_\sigma 0|\Big).
\end{aligned}$$

The right hand side does not depend on n and tends a.s. to zero for $m\to\infty$, because $\sum_{j=1}^{\infty}\Big(\sum_{\sigma\in D_j} 1_\sigma^{\alpha+1}\,\psi_\sigma(|\cdot|)+\sum_{\sigma\in D_j} 1_{\sigma|j-1}^{\alpha+1}|S_\sigma 0|\Big)$ is a.s. finite, which follows as above from 1.1. and 2.1. (iii) and (iv). Thus we obtain $\lim_{m\to\infty}\limsup_n \phi_n(B(0,m)^c)=0$ a.s. and (i) is proved.

(ii) Let $\{\psi_\sigma\colon \sigma\in D\}$ and $\{\tilde\psi_\sigma\colon \sigma\in D\}$ be two families of independent random finite measures which satisfy 2.1. (iv). Denote $\phi_n=\sum_{\sigma\in D_n} 1_\sigma^\alpha\,\psi_\sigma\circ T_\sigma^{-1}$ and $\tilde\phi_n=\sum_{\sigma\in D_n} 1_\sigma^\alpha\,\tilde\psi_\sigma\circ T_\sigma^{-1}$. From (i) we obtain two random finite measures ϕ and $\tilde\phi$ (with mass Z) such that $\phi_n \xrightarrow{w} \phi$ a.s. and $\tilde\phi_n \xrightarrow{w} \tilde\phi$ a.s. Now we want to show that ϕ and $\tilde\phi$ are almost surely equal. For this, by 2.5. and 2.6. it is sufficient to show that $|\phi_n(f)-\tilde\phi_n(f)|$ tends to zero a.s. for all $f\in\mathscr{L}_c$ as $n\to\infty$. Indeed, using the triangle inequality and 2.1. (iv) we have a.s. for $f\in\mathscr{L}_c$

$$\begin{aligned}
&|\phi_n(f)-\tilde\phi_n(f)|\\
&\quad\leqq \sum_{\sigma\in D_n} 1_\sigma^\alpha |\psi_\sigma(f\circ T_\sigma)-\tilde\psi_\sigma(f\circ T_\sigma)|\\
&\quad\leqq \sum_{\sigma\in D_n} 1_\sigma^\alpha\Big(\Big|\int (f(T_\sigma x)-f(T_\sigma 0))\,\psi_0(\mathrm{d}x)\Big|+\Big|\int (f(T_\sigma 0)-f(T_\sigma x))\,\tilde\psi_\sigma(\mathrm{d}x)\Big|\Big)\\
&\quad\leqq \mathrm{Lip}(f)\sum_{\sigma\in D_n} 1_\sigma^{\alpha+1}(\psi_0(|\cdot|)+\tilde\psi_\sigma(|\cdot|))\\
&\quad\leqq \mathrm{Lip}(f)\,2KS_{\alpha+1,n}.
\end{aligned}$$

The right hand side tends to zero if $\sum_{n=1}^{\infty} S_{\alpha+1,n}$ is a.s. finite. The finiteness follows from 1.1. and the monotone convergence theorem, because $\mathbb{E}\sum_{n=1}^{\infty} S_{\alpha+1,n} = \sum_{n=1}^{\infty}\left(\mathbb{E}\sum_{i=1}^{N}\rho_i^{\alpha+1}\right)^n<\infty$. This completes the proof. □

A property which carries over from the start measures to the limit measure is the following.

2.9. Lemma *The limit measure ϕ has the property*

$$\mathbb{E}\phi(|\cdot|)<\infty.$$

Proof. Using 2.8. and 2.1. (iii) and (iv) we get

$$\begin{aligned}\mathbb{E}\phi(|\cdot|) &= \lim_{c\to\infty} \mathbb{E}(\phi(1_{\{y:|y|\leq c\}}(\cdot)|\cdot|))\\ &= \lim_{c\to\infty}\lim_{n\to\infty} \mathbb{E}(\phi_n(1_{\{y:|y|\leq c\}}(\cdot)|\cdot|))\\ &\leqq \lim_{n\to\infty} \mathbb{E}\phi_n(|\cdot|)\\ &= \lim_{n\to\infty} \mathbb{E}\sum_{\sigma\in D_n} 1_\sigma^\alpha \int |T_\sigma x - T_\sigma 0 + T_\sigma 0|\, \psi_\sigma(\mathrm{d}x)\\ &\leqq 0+\lim_{n\to\infty} \mathbb{E}\sum_{\sigma\in D_n} 1_\sigma^\alpha |T_\sigma 0|\\ &\leqq \lim_{n\to\infty} \sum_{j=1}^{n}\sum_{\sigma\in D_j} \mathbb{E}1_{\sigma|j-1}^{\alpha+1}\, \mathbb{E}|S_\sigma 0|\\ &\leqq \text{const} \sum_{j=1}^{\infty}\left(\mathbb{E}\sum_{i=1}^{N}\rho_i^{\alpha+1}\right)^{j-1}\\ &<\infty. \quad \square\end{aligned}$$

2.10. Remarks (i) All results from this section carry over without difficulties from $\mathbb{R}^d$ to any complete separable metric space.

It is also possible to take more general masses. That means, if we replace the masses 1_σ^α in 1.2. by $\prod_{i=1}^{\sigma} p_{\sigma|i}$, where the p_σ, $\sigma\in D$, are [0,1)-valued random variables with the same independence conditions as in 1.2. and $\mathbb{E}\sum_{i=1}^{N} p_{\sigma * i}=1$, then Theorem 2.8 remains valid if $\sup_n |\mathbb{E}\left(\sum_{\sigma\in D_n}\prod_{i=1}^{|\sigma|} p_{\sigma|i}\right)^2<\infty$. But in view of the next sections we restrict to the canonical masses.

(ii) The condition $\psi_\sigma(\mathbb{R}^d)=1$ a.s. for $\sigma\in D$ is not necessary. It may be replaced by the following conditions. For any $\sigma\in D_n$ and any $i=1\ldots N$, $\mathbb{E}[\psi_{\sigma * i}(\mathbb{R}^d)|\mathscr{F}_n]=\psi_\sigma(\mathbb{R}^d)$ holds and there exists a constant K such that $\mathbb{E}\psi_\sigma^2(\mathbb{R}^d)<K$ for all $\sigma\in D$.

(iii) If we suppose that the family $\{\psi_\sigma\colon \sigma\in D\}$ consist, of identically distributed random finite meaures then 2.1. (iv) can be replaced by the conditions $\mathbb{E}\phi_0(\mathbb{R}^d)^2<\infty$ and $\mathbb{E}\phi_0(|\cdot|)<\infty$.

3 Self similarity

In this section we will show that the limit measure ϕ (more exactly, its distribution) has an analogous self-similarity property as the limit set in the FGMW model (cf. [G] 4). We denote by $\mathscr{P}(M)$ the set of all Borel probability measures on $[M, \mathfrak{M}]$ and by P_μ the distribution law of the limit measure ϕ.

3.1. Definition Let μ be a Borel probability measure on $\mathrm{Con}(\mathbb{R}^d)^N$. A probability measure $P \in \mathscr{P}(M)$ is called *μ-self-similar* (μ-s.s.) if for any measurable function $f: M \to [0, \infty)$

$$\int f(\varphi)\, P(\mathrm{d}\varphi) = \iint f\left(\sum_{i=1}^{N} \rho_i^\alpha\, \varphi^i \circ S_i^{-1}\right) P^N(\mathrm{d}(\varphi^1, \ldots, \varphi^N))\, \mu(\mathrm{d}(S_1, \ldots, S_N)),$$

where P^N denotes the product measure of P on $[M, \mathfrak{M}]^N$.

3.2. We define the map $T_\mu: \mathscr{P}(M) \to \mathscr{P}(M)$ by

$$\int f(\varphi)(T_\mu Q)(\mathrm{d}\varphi) = \iint f\left(\sum_{i=1}^{N} \rho_i^\alpha\, \varphi^i \circ S_i^{-1}\right) Q^N(\mathrm{d}(\varphi^1, \ldots, \varphi^N))\, \mu(\mathrm{d}(S_1, \ldots, S_N))$$

for $f: M \to [0, \infty)$ measurable and $Q \in \mathscr{P}(M)$.

3.3. In this notation $P \in \mathscr{P}(M)$ is μ-s.s. if and only if $T_\mu P = P$.

3.4. Theorem *Let μ be a Borel probability measure on* $\mathrm{Con}(\mathbb{R}^d)^N$. *Then P_μ is μ-self-similar. Moreover, for any $Q \in \mathscr{P}(M)$ with $\int \varphi(|\cdot|)\, Q(\mathrm{d}\varphi) < \infty$ the sequence $T_\mu^n Q$ converges to P_μ w.r.t. the weak topology.*

With the help of 2.9. we infer the following uniqueness.

3.5. Corollary *P_μ is the unique μ-s.s. probability measure in the class of measures $Q \in \mathscr{P}(M)$ satisfying $\int \varphi(|\cdot|)\, Q(\mathrm{d}\varphi) < \infty$.* □

Proof of 3.4. As a corollary of 2.8 (ii) we may assume that the start measures $\{\phi_\sigma: \sigma \in D\}$ in the construction of P_μ are independent and identically disributed according to a probability measure $P_0 \in \mathscr{P}(M)$ with $\int \varphi(|\cdot|)\, P_0(\mathrm{d}\varphi) < \infty$. We denote by $P_n = P_n(P_0)$ the distribution of ϕ_n (cf. 1.2.) and by μ^D the product measure of μ. First we will show that P_μ is μ-s.s., i.e., that $T_\mu P_\mu = P_\mu$. Using 1.2. and 2.8. we obtain for any function $f \in C_c(\mathbb{R}^d)$

$$\begin{aligned}
\int f(\varphi)\, P_\mu(\mathrm{d}\varphi) &= \lim_{n \to \infty} \int f(\varphi)\, P_{n+1}(\mathrm{d}\varphi) \\
&= \lim_{n \to \infty} \iint f\Big(\sum_{\sigma \in D_{n+1}} 1_\sigma^\alpha\, \varphi_\sigma \circ T_\sigma^{-1}\Big)\, P_0^D(\mathrm{d}(\varphi_\sigma)_{\sigma \in D})\, \mu^D(\mathrm{d}(S_{\sigma * 1}, \ldots, S_{\sigma * N})_{\sigma \in D}) \\
&= \lim_{n \to \infty} \iiint f\left(\sum_{i=1}^{N} \rho_i^\alpha \sum_{\sigma \in D_n} {}_\rho\rho_{i*\sigma|1}^\alpha \cdot \ldots \cdot \rho_{i*\sigma}^\alpha (\varphi_{i*\sigma} \circ S_{i*\sigma}^{-1} \circ \ldots \circ S_{i*\sigma|1}^{-1})\right) P_0^D(\mathrm{d}(\varphi_\sigma)_{\sigma \in D}) \\
&\quad \cdot \mu^D(\mathrm{d}(S_{i*\sigma*1}, \ldots, S_{i*\sigma*N})_{\sigma \in D})\, \mu(\mathrm{d}(S_1, \ldots, S_N)).
\end{aligned}$$

Because of the boundedness of the integrand and the almost sure convergence we obtain.

$$\begin{aligned}
&\int f(\varphi)\, P_\mu(\mathrm{d}\varphi) \\
&= \int \lim_{n \to \infty} \int f\left(\sum_{i=1}^{N} \rho_i^\alpha\, \varphi^i \circ S_i^{-1}\right) P_n^N(\mathrm{d}(\varphi^1, \ldots, \varphi^N))\, \mu(\mathrm{d}(S_1, \ldots, S_N)) \\
&= \iint f\left(\sum_{i=1}^{N} \rho_i^\alpha\, \varphi^i \circ S_i^{-1}\right) P_\mu^N(\mathrm{d}(\varphi^1, \ldots, \varphi^N))\, \mu(\mathrm{d}(S_1, \ldots, S_N)) \\
&= \int f(\varphi)(T_\mu P_\mu)(\mathrm{d}\varphi).
\end{aligned}$$

Thus we have $T_\mu P_\mu = P_\mu$.

It remains to show that $T_\mu^n Q \xrightarrow{w} P_\mu$ for any $Q \in \mathscr{P}(M)$ with $\int \varphi(\|\cdot\|)\, Q(\mathrm{d}\varphi) < \infty$. Using induction and 2.8. it is easy to see that for any continuous bounded function $f\colon M \to [0, \infty)$

$$\begin{aligned}
&\lim_{n\to\infty} \int f(\varphi)(T_\mu^n Q)(\mathrm{d}\varphi)\\
&\quad = \lim_{n\to\infty} \iint f\Big(\sum_{\sigma\in D_n} 1_\sigma^\alpha\, \varphi^\sigma \circ T_\sigma^{-1}\Big)\, Q^D(\mathrm{d}(\varphi^\sigma)_{\sigma\in D})\, \mu^D(\mathrm{d}(S_{\sigma * 1}, \ldots, S_{\sigma * N})_{\sigma\in D})\\
&\quad = \lim_{n\to\infty} \int f(\varphi)\, P_n(Q)\, (\mathrm{d}\varphi)\\
&\quad = \int f(\varphi)\, P_\mu(\mathrm{d}\varphi), \qquad \text{and hence the theorem is proved. } \square
\end{aligned}$$

3.6. Remarks (i) In the proof of the preceding theorem we used the same techniques as in [G]4.
(ii) P_μ is μ-s.s., that means, if $(\phi^\sigma)_{\sigma\in D_n}$ are copies of ϕ, independent of ϕ and $(T_\sigma)_{\sigma\in D_n}$, then the random measures $\sum_{\sigma\in D_n} 1_\sigma^\alpha\, \phi^\sigma \circ T_\sigma^{-1}$, $n = 1, 2, \ldots$, have the same distribution as ϕ.
(iii) Our μ-self-similarity is a stochastic version of Hutchinsons self-similarity (cf. 6.1.) and generalizes the stochastic version of Graf (cf. 6.2.) to the noncompact case. If $\mathbb{R}^d$ is replaced by a compact subset then P_μ is the unique μ-self-similar distribution in $\mathscr{P}(M)$ without further conditions.
(iv) The self-similarity property also carries over to the case of general masses (cf. 2.10. (i)), replacing ρ_i^α by p_i. In particular, the hyperbolic iterated function system on a compact subset $K \subset \mathbb{R}^d$, which generated a unique invariant measure μ (see [G|H]) is contained in our model. (This is easy to see if we take $\mu = \delta_{(S_1, \ldots, S_N)}$ and $p_i = \delta_{m_i}$ with $m_i > 0$, $i = 1, \ldots, N$, and $\sum_{i=1}^N m_i = 1$.)

4 Moment measures

For further purposes it is necessary to know something about the structure of the first two moment measures of the limit measure ϕ.

The *intensity measure* Λ_ψ of a random measure ψ on $\mathbb{R}^d$ is defined on $\mathscr{R}^d$ by $\Lambda_\psi(B) = \mathbb{E}\psi(B)$, $B \in \mathscr{R}^d$. The *second order moment measure* of ψ is the measure $\Lambda_\psi^{(2)}$ on $\mathbb{R}^d \otimes \mathbb{R}^d$ defined by $\Lambda_\psi^{(2)}(\cdot) = \mathbb{E}(\psi \times \psi)(\cdot)$. For details see [K].

Intensity measures

For the intensity measures $\Lambda_n = \Lambda_{\phi_n}$ we can prove the following.

4.1. Theorem *If the conditions* 2.1.(ii)–(iv) *are satisfied, there exists a finite measure Λ_μ on $[\mathbb{R}^d, \mathscr{R}^d]$ with*

(i) $\Lambda_n \xrightarrow{w} \Lambda_\mu$
(ii) $\Lambda_\mu(\cdot) = \mathbb{E}\phi(\cdot)$, $\Lambda_\mu(\mathbb{R}^d) = 1$
(iii) Λ_μ *does not depend on the choice of the startmeasures.*

Proof. (iii) follows immediately from 2.8.(ii). (i) will be proved in an analogous way as in 2.8.(i).

We first show that $(\Lambda_n(f))_{n=1,2,\dots}$ forms a Cauchy sequence for $f\in\mathcal{L}_c$. Similarly as in 2.8.(i) we have for $n>m$

$$\begin{aligned}
&|\Lambda_n(f)-\Lambda_m(f)|\\
&=\left|\sum_{j=m}^{n-1}\mathbb{E}\sum_{\sigma\in D_j}1_\sigma^\alpha\left(\sum_{i=1}^{N}\rho_{\sigma*i}^\alpha[\psi_{\sigma*i}(f\circ T_{\sigma*i})-\psi_\sigma(f\circ T_\sigma)]\right.\right.\\
&\quad\left.\left.+\sum_{i=1}^{N}\rho_{\sigma*i}^\alpha\psi_\sigma(f\circ T_\sigma)-\psi_\sigma(f\circ T_\sigma)\right)\right|\\
&=\sum_{j=m}^{n-1}\sum_{\sigma\in D_{j+1}}\mathbb{E}1_\sigma^\alpha|\psi_\sigma(f\circ T_\sigma)-\psi_{\sigma|j}(f\circ T_{\sigma|j})|\\
&\quad+\sum_{j=m}^{n-1}\sum_{\sigma\in D_j}\left|\mathbb{E}1_\sigma^\alpha\psi_\sigma(f\circ T_\sigma)\,\mathbb{E}\left(\sum_{i=1}^{N}\rho_{\sigma*i}^\alpha-1\right)\right|\\
&\leqq\sum_{j=m}^{n-1}\sum_{\sigma\in D_{j+1}}\mathbb{E}1_\sigma^\alpha|\psi_\sigma(f\circ T_\sigma)-f(T_\sigma 0)+f(T_\sigma 0)-\psi_{\sigma|j}(f\circ T_{\sigma|j})|+0\\
&\leqq\mathrm{Lip}(f)\sum_{j=m}^{n-1}\sum_{\sigma\in D_j}\mathbb{E}1_\sigma^{\alpha+1}\sum_{i=1}^{N}(\mathbb{E}\psi_{\sigma*i}(|\cdot|)+\mathbb{E}\psi_\sigma(|\cdot|)+\mathbb{E}|S_{\sigma*i}0|)\\
&\leqq\text{const.}\sum_{j=m}^{n-1}\left(\mathbb{E}\sum_{i=1}^{N}\rho_i^{\alpha+1}\right)^j.
\end{aligned}$$

Using 1.1. we get that $|\Lambda_n(f)-\Lambda_m(f)|$ tends to zero for $n, m\to\infty$ and therefore $(\Lambda_n(f))$ forms a Cauchy sequence for $f\in\mathcal{L}_c$. We now define a linear functional $1: \mathcal{L}_c\to[0,\infty)$ by $1(f)=\lim_{n\to\infty}\Lambda_n(f), f\in\mathcal{L}_c$. This functional is continuous because $\|1\|=\sup_{f\in\mathcal{L}_c}\frac{|1(f)|}{\|f\|}\leqq 1$. Thus we can extend 1 to a linear continuous functional 1 on C_c and get a finite measure Λ_μ on $[\mathbb{R}^d, \mathcal{R}^d]$ with $\Lambda_\mu(f)=1(f), f\in C_c$ (cf. e.g. [Fe] 2.5.2). By construction we have $\Lambda_n\xrightarrow{v}\Lambda_\mu$.

For the weak convergence by 2.5. it is sufficient to show that $\lim_{n\to\infty}\limsup_{n\to\infty}\Lambda_n(B(0,m)^c)=0$. This condition is satisfied because

$$\begin{aligned}
\Lambda_n(B(0,m)^c)&\leqq\frac{1}{m}\Lambda_n(|\cdot|)\\
&=\frac{1}{m}\mathbb{E}\int|x|\,\phi_n(\mathrm{d}x)\\
&\leqq\frac{1}{m}\text{const.}\qquad\text{(cf. 2.8.i).}
\end{aligned}$$

Again by 2.5. we obtain $\Lambda_\mu(\mathbb{R}^d)=\lim_{n\to\infty}\Lambda_n(\mathbb{R}^d)=1$. For (ii) it remains to show that $\Lambda_\mu(\cdot)=\mathbb{E}\phi(\cdot)$.

This is an immediate consequence from the uniform boundedness of the second order moment measures. We have $\sup_n \Lambda_{\phi_n}^{(2)}(\mathbb{R}^d \times \mathbb{R}^d) = \sup_n \mathbb{E}\phi_n(\mathbb{R}^d)^2 = \sup_n \mathbb{E}S_{\alpha,n}^2 < \infty$. □

Like for the limit measure we have a self-similarity property for its intensity measure Λ_μ.

4.2. Theorem *Let μ be a Borel probability measure on* $\mathrm{Con}(\mathbb{R}^d)^N$. *Then Λ_μ is the unique measure φ in M with*

(i) $\varphi(\mathbb{R}^d) = 1$,
(ii) $\varphi(|\cdot|) < \infty$, *and*
(iii) $\varphi = \sum_{i=1}^{N} \mathbb{E}\rho_i^\alpha(\varphi \circ S_i^{-1})$.

Proof. Obviously, Λ_μ has the properties (i)–(iii), since (i) follows from 4.1.(ii), (ii) from 2.9., and (iii) from 3.4.(i). If we take a measure $\varphi \in M$ which satisfies (i)–(iii) and choose $\psi_\sigma = \varphi$, $\sigma \in D$, then using (iii) we have $\Lambda_n = \varphi$ for $n = 1, 2, \ldots$. From 4.1. we know that $\Lambda_n \xrightarrow{w} \Lambda_\mu$ and thus we obtain $\varphi = \Lambda_\mu$. □

Second order moment measures

The second order moment measure of the limit measure ϕ plays an important role for dimension estimations (see 5.6.).

For shortness we denote $\Lambda_{\phi_n}^{(2)}$, $n = 1, 2, \ldots$, and $\Lambda_{\psi_\sigma}^{(2)}$, $\sigma \in D$, by $\Lambda_n^{(2)}$ and $\Lambda_\sigma^{(2)}$, respectively. Recall that $\Lambda_n^{(2)}(B_1 \times B_2) = \mathbb{E}\phi_n(B_1)\,\phi_n(B_2)$ for B_1, $B_2 \in \mathscr{R}^d$. We define measures Λ_n^σ, $n = 1, 2, \ldots$, $\sigma \in D$, on $[\mathbb{R}^d, \mathscr{R}^d]$ by $\Lambda_n^\sigma(B) = \int \Lambda_n^x(B)\,\Lambda_\sigma(\mathrm{d}x)$, $B \in \mathscr{R}^d$, where Λ_n^x is the intensity measure of ϕ_n starting with $\psi_\sigma = \delta_x$, $x \in \mathbb{R}^d$. That means Λ_n^σ is the intensity measure of $\sum_{\tau \in D_n} 1_\tau^\alpha 1\,\psi_{\sigma * \tau} \circ T_\tau^{-1}$.

4.3. Lemma *For any measurable bounded function f from $\mathbb{R}^d \times \mathbb{R}^d$ into $[0, \infty)$ we get*

$$\begin{aligned}
&\int f(x, y)\,\Lambda_n^{(2)}(\mathrm{d}(x, y)) \\
&\quad = \sum_{k=1}^{n} \mathbb{E}\Bigg(\sum_{\sigma \in D_{k-1}} 1_\sigma^{2\alpha} \sum_{\substack{i,j=1 \\ i \neq j}} \rho_{\sigma * i}^\alpha \rho_{\sigma * j}^\alpha \iint f(T_{\sigma * i}x, T_{\sigma * j}y)\,\Lambda_{n-k}^{\sigma * i}(\mathrm{d}x)\,\Lambda_{n-k}^{\sigma * j}(\mathrm{d}y)\Bigg) \\
&\qquad + \mathbb{E} \sum_{\sigma \in D_n} 1_\sigma^{2\alpha} \int f(T_\sigma x, T_\sigma y)\,\Lambda_\sigma^{(2)}(\mathrm{d}(x, y)).
\end{aligned}$$

Proof. We first assume that $f = 1_{B_1 \times B_2}$ with B_1, $B_2 \in \mathscr{R}^d$. In this case we have

$$\begin{aligned}
\Lambda_n^{(2)}(B_1 \times B_2) = {} & \mathbb{E} \sum_{\sigma, \tau \in D_n} 1_\sigma^\alpha(\psi_\sigma \circ T_\sigma^{-1})(B_1)\, 1_\tau^\alpha(\psi_\tau \circ T_\tau^{-1})(B_2) \\
& + \mathbb{E} \sum_{\sigma \in D_n} 1_\sigma^{2\alpha}(\psi_\sigma \circ T_\sigma^{-1})(B_1)(\psi_\sigma \circ T_\sigma^{-1})(B_2).
\end{aligned}$$

We denote the two parts of the sum by S_1 and S_2. Using properties of the conditional expectation and the independence in the model we obtain

$$S_2 = \mathbb{E}\mathbb{E}\Big[\sum_{\sigma\in D_n} 1_\sigma^{2\alpha}(\psi_\sigma\circ T_\sigma^{-1})(B_1)(\psi_\sigma\circ T_\sigma^{-1})(B_2)|\mathscr{F}_n\Big]$$
$$= \mathbb{E}\sum_{\sigma\in D_n} 1_\sigma^{2\alpha} \Lambda_\sigma^{(2)}(T_\sigma^{-1}B_1\times T_\sigma^{-1}B_2).$$

$$S_1 = \sum_{k=1}^n \mathbb{E}\sum_{\sigma\in D_{k-1}} 1_\sigma^{2\alpha}\sum_{\substack{i,j=1\\ i\neq j}}^N \rho_{\sigma*i}^\alpha\rho_{\sigma*j}^\alpha \sum_{\tau\in D_{n-k}}\rho_{\sigma*i*\tau|1}^\alpha\cdots\rho_{\sigma*i*\tau}^\alpha$$
$$\cdot(\psi_{\sigma*i*\tau}\circ T_{\sigma*i*\tau})(B_1)\sum_{\nu\in D_{n-k}}\rho_{\sigma*j*\nu|1}^\alpha\cdots\rho_{\sigma*j*\nu}^\alpha(\psi_{\sigma*j*\nu}\circ T_{\sigma*j*\nu})(B_2)$$
$$= \sum_{k=1}^n \mathbb{E}\mathbb{E}\Big[\sum_{\sigma\in D_{k-1}} 1_\sigma^{2\alpha}\sum_{\substack{i,j=1\\ i\neq j}}^N \rho_{\sigma*i}^\alpha\rho_{\sigma*j}^\alpha \sum_{\tau\in D_{n-k}}\rho_{\sigma*i*\tau|1}^\alpha\cdots\rho_{\sigma*i*\tau}^\alpha$$
$$\cdot(\psi_{\sigma*i*\tau}\circ S_{\sigma*i*\tau}^{-1}\circ\ldots\circ S_{\sigma*i*\tau|1}^{-1})(T_{\sigma*i}^{-1}B_1)\sum_{\nu\in D_{n-k}}\rho_{\sigma*j*\nu|1}^\alpha\cdots\rho_{\sigma*j*\nu}^\alpha$$
$$\cdot(\psi_{\sigma*j*\nu}\circ S_{\sigma*j*\nu}^{-1}\circ\ldots\circ S_{\sigma*j*\nu|1}^{-1})(T_{\sigma*j}^{-1}B_2)|\mathscr{F}_k\Big]$$
$$= \sum_{k=1}^n \mathbb{E}\sum_{\sigma\in D_{k-1}} 1_\sigma^{2\alpha}\sum_{\substack{i,j=1\\ i\neq j}}^N \rho_{\sigma*i}^\alpha\rho_{\sigma*j}^\alpha \Lambda_{n-k}^{\sigma*i}(T_{\sigma*i}^{-1}B_1)\,\Lambda_{n-k}^{\sigma*j}(T_{\sigma*j}^{-1}B_2).$$

This leads to the assertion for our special f. The general case follows by standard measure theoretical arguments. □

4.4. Theorem *If* $\sup_n \mathbb{E}S_{\alpha,n}^3<\infty$ *and the conditions* 2.1. *are satisfied, then* $\Lambda_n^{(2)}$ *converges weakly to the second order moment measure* $\Lambda_\mu^{(2)}$ *of* ϕ *and*

$$\int f(x,y)\,\Lambda_\mu^{(2)}(\mathrm{d}(x,y))$$
$$= \sum_{k=1}^\infty \mathbb{E}\sum_{\sigma\in D_{k-1}} 1_\sigma^{2\alpha}\sum_{\substack{i,j=1\\ i\neq j}}^N \rho_{\sigma*i}^\alpha\rho_{\sigma*j}^\alpha \iint f(T_{\sigma*i}x, T_{\sigma*j}y)\,\Lambda_\mu(\mathrm{d}x)\,\Lambda_\mu(\mathrm{d}y)$$

for all measurable bounded functions $f\colon \mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$.

Proof. WE first show that $\Lambda_n^{(2)}\xrightarrow{v}\Lambda_\mu^{(2)}$. For this it is enough to show that

$$\lim_{n\to\infty}\int f(x)\,g(y)\,\Lambda_n^{(2)}(\mathrm{d}(x,y)) = \int f(x)\,g(y)\,\Lambda_\mu^{(2)}(\mathrm{d}(x,y))$$

for any functions $f, g\in C_c$. Let $\max\{\|f\|,\|g\|\}\leq C$. Using Lemma 4.3. we get

$$\lim_{n\to\infty}\int f(x)\,g(y)\,\Lambda_n^{(2)}(\mathrm{d}(x,y))$$
$$= \lim_{n\to\infty}\sum_{k=1}^n \mathbb{E}\sum_{\sigma\in D_{k-1}} 1_\sigma^{2\alpha}\sum_{\substack{i,j=1\\ i\neq j}}^N \int f(T_{\sigma*i}x)\,\Lambda_{n-k}^{\sigma*i}(\mathrm{d}x)\int g(T_{\sigma*j}y)\,\Lambda_{n-k}^{\sigma*j}(\mathrm{d}y)$$
$$+ \lim_{n\to\infty}\mathbb{E}\sum_{\sigma\in D_n} 1_\sigma^{2\alpha}\int f(T_\sigma x)\,g(T_\sigma y)\,\Lambda_\sigma^{(2)}(\mathrm{d}(x,y)).$$

We denote the summands by S_1 and S_2, respectively, and examine them separately. Together with 2.1.(iv) and 1.1. we have

$$\begin{aligned} S_2 &\leqq \lim_{n\to\infty} C\mathbb{E} \sum_{\sigma\in D_n} 1_\sigma^{2\alpha} \\ &= C \lim_{n\to\infty} \left(\mathbb{E} \sum_{i=1}^{N} \rho_i^{2\alpha}\right)^n \\ &= 0. \end{aligned}$$

Since $S_2 \geqq 0$ it follows $S_2 = 0$.

In order to compute S_1 we use 4.1., Lebesgue's dominated convergence theorem, and the independence required in the model. Thus we obtain

$$\begin{aligned} S_1 = \sum_{n=1}^{\infty} \lim_{n\to\infty} \mathbb{E}\Bigg(\sum_{\sigma\in D_{k-1}} 1_\sigma^{2\alpha} \sum_{\substack{i,j=1\\ i\neq j}}^{N} \rho_{\sigma*i}^{\alpha} \rho_{\sigma*j}^{\alpha} \int f(T_{\sigma*i}x)\, \Lambda_{n-k}^{\sigma*i}(\mathrm{d}x) \\ \cdot \int g(T_{\sigma*j}y)\, \Lambda_{n-k}^{\sigma*j}(\mathrm{d}y)\Bigg) \\ = \sum_{k=1}^{\infty} \mathbb{E} \sum_{\sigma\in D_{k-1}} 1_\sigma^{2\alpha} \sum_{\substack{i,j=1\\ i\neq j}}^{N} \rho_{\sigma*i}^{\alpha} \rho_{\sigma*j}^{\sigma} \int f(T_{\sigma*i}x)\, \Lambda_\mu(\mathrm{d}x) \int g(T_{\sigma*j}y)\, \Lambda_\mu(\mathrm{d}y). \end{aligned}$$

Hence, the vague convergence of $\Lambda_n^{(2)}$ to $\Lambda_\mu^{(2)}$ is proved. $\Lambda_\mu^{(2)}(\cdot) = \mathbb{E}(\phi\times\phi)(\cdot)$ follows from $\sup_n \mathbb{E}(\phi_\nu(\mathbb{R}^d)^3) = \sup_n S_{\alpha,n}^3 < \infty$. With the help of 2.5. we get $\Lambda_n^{(2)} \xrightarrow{w} \Lambda_\mu^{(2)}$, since

$$\Lambda_\mu^{(2)}(\mathbb{R}^d \times \mathbb{R}^d) = \mathbb{E}(\phi(\mathbb{R}^d)^2) = \mathbb{E}Z^2 \quad \text{and} \quad \lim_{n\to\infty} \Lambda_n^{(2)}(\mathbb{R}^d\times\mathbb{R}^d) = \mathbb{E}Z^2.$$

$\mathbb{E}Z^2 < \infty$ implies that $\Lambda_\mu^{(2)}$ is a finite measure on $\mathscr{R}^d \otimes \mathscr{R}^d$. □

In the next section we will see how to use the structure of the second order moment measure $\Lambda_\mu^{(2)}$ for determining the carrying dimension of the limit measure ϕ.

5 The carrying dimension

We denote by $\mathscr{H}^\beta$ the β-dimensional Hausdorff measure on $\mathbb{R}^d$ and by $\dim B$ the Hausdorff-Besicovitch dimension for $B \in \mathscr{R}^d$. (For details see [F1].) Further denote by diam B the diameter of B.

5.1. Definition (cf. [Z2]) A measure $\varphi \in M$ has the *carrying dimension* β (cardim $\varphi = \beta$) if the following conditions are satisfied.

(1) there exists a Borel set B with $\varphi(B^c) = 0$ and $\dim B \leqq \beta$ and
(2) $\varphi(B) > 0$ implies $\dim B \geqq \beta$.

5.2. *Remarks* (i) There exist measures having no carrying dimension. For example, measures which are supported by disjoint sets of different dimensions have no carrying dimension.
(ii) In general the carrying dimension of a measure φ is not greater than the dimension of the support. For example, let φ be a measure on $([0, 1], \mathscr{R}^1 \cap [0, 1])$ concentrated on the rational numbers. Then $\operatorname{cardim} \varphi = 0$ and $\dim \operatorname{spt} \varphi = 0$.

Upper bounds

The next lemma gives conditions under which one can get an upper bound for the Hausdorff dimension of a random set carrying a random measure ψ on $\mathbb{R}^d$. The proof of this lemma without the last step is completely the same as in [Z 1] 6.1.

5.3. Lemma. *Let ψ be a random measure on $\mathbb{R}^d$ and $\beta \geqq 0$. Suppose that for any bounded set $B \subseteqq \mathbb{R}^d$ and any $\varepsilon > 0$ there exists a sequence $(\Gamma_n) = (\Gamma_n(B, \varepsilon))$ of countable families of random closed subsets of $\mathbb{R}^d$, having diameter at most 1, such that*

$$\lim_{n \to \infty} \mathbb{P}(\psi(B \setminus \cup \Gamma_n) < \varepsilon) = 1 \quad \textit{and} \quad \mathbb{P}\left(\liminf_{n \to \infty} \sum_{\substack{K \in \Gamma_n \\ K \cap B \neq \emptyset}} (\operatorname{diam} K)^{\beta + \varepsilon} < \infty\right) = 1.$$

Then there exists an increasing sequence Ξ_n of random closed sets on $\mathbb{R}^d$ such that $\dim \Xi_n \leqq \beta$ a.s. *and* $\psi(\mathbb{R}^d \setminus \bigcup_{n \in \mathbb{N}} \Xi_n) = 0$ a.s.

Proof. Let $\varepsilon > 0$ and a bounded set $B \in \mathscr{R}^d$ be given. Then we can choose a subsequence $\mathbb{N}'$ of $\mathbb{N}$ such that $\limsup \psi(B \setminus \cup \Gamma_n(B, \varepsilon)) < \varepsilon$ a.s. and

$$\liminf \sum_{\substack{K \in \Gamma_n(B, \varepsilon) \\ K \cap B \neq \emptyset}} (\operatorname{diam} K)^{\beta + \varepsilon} < \infty \text{ a.s.}, n \in \mathbb{N}'.$$

Let $m \in \mathbb{N}$. Putting $B = B(0, m)$ and $\varepsilon = 2^{-m}$ we can choose for any $\delta > 0$ increasing subsequence (n_m) of $\mathbb{N}'$ such that

(1) $\psi(B \setminus \cup \Gamma_{n_m}(B(0, m), 2^{-m}) < 2^{-m+1}$ a.s. and

(2) $\liminf\limits_{m} \sum\limits_{\substack{K \in \Gamma_{n_m}(B(0,m), 2^{-m}) \\ K \cap B(0,m) \neq \emptyset}} (\operatorname{diam} K)^{\beta + \delta} < \infty$ a.s.

Defining $\Gamma_{\delta, m} = \cup \Gamma_{n_m}(B(0, m), 2^{-m})$ and

$$\Xi'_k = \bigcap_{1 \in \mathbb{N}} (\Gamma_{1/1, k+1} \cap B(0, k)), \qquad k \in \mathbb{N},$$

we obtain for any bounded $B \in \mathscr{R}^d$

$$\begin{aligned}\psi\left(B \setminus \bigcup_{k \in \mathbb{N}} \Xi'_k\right) &= \psi\left(\bigcap_k (B \setminus \Xi'_k)\right)\\ &\leqq \lim_{k \to \infty} \psi(B \setminus \Xi'_k)\\ &= \lim_{k \to \infty} \psi\left(B \setminus \bigcap_{1 \in \mathbb{N}} (\Gamma_{1/1, k+1} \cap B(0, k))\right)\\ &= \lim_{k \to \infty} \psi\left(B \bigcup_1 (B \setminus \Gamma_{1/1, k+1})\right)\\ &\leqq \lim_{k \to \infty} \sum_1 \psi(B \setminus \Gamma_{1/1, k+1})\\ &\leqq \lim_{k \to \infty} \sum_1 2^{-k-1+1} = 0 \qquad \text{by (1).}\end{aligned}$$

It remains to prove $\dim \Xi'_k \leqq \beta$. By the definition of the Hausdorff measure and (2) we have

$$\mathscr{H}^{\beta+\delta}(\Xi'_k \cap B) \leqq \liminf_{1 \to \infty} \sum_{\substack{K \in \Gamma_{1/1, k+1} \\ K \cap B \neq \emptyset}} (\operatorname{diam} K)^{\beta+\delta} < \infty$$

for all bounded $B \in \mathscr{R}^d$ and $\delta > 0$ and therefore $\dim \Xi'_k \leqq \beta$. Finally, we choose $\Xi_n = \bigcup_{k=1}^{n} \Xi'_k$ to obtain an increasing sequence. □

Lemma 5.3. gives us a very helpful condition for finding an upper bound for the carrying dimension of the limit measure ϕ.

5.4. Theorem *If the assumptions* 2.1.(i)–(iv) *are satisfied, then there exists an increasing sequence* Ξ_n *of random closed sets on* $\mathbb{R}^d$ *such that* $\dim \Xi_n \leqq \alpha$ a.s. *and* $\phi(\mathbb{R}^d \setminus \bigcup_{n \in \mathbb{N}} \Xi_n) = 0$ a.s. (*This implies* cardim $\phi \leqq \alpha$ a.s. *if the carrying dimension of* ϕ a.s. *exists.*)

Proof. For given bounded $B \in \mathscr{R}^d$ and $\varepsilon > 0$ we can choose a sequence (r_n) of real numbers with

(1) $\lim_{n \to \infty} r_n = \infty$ and

(2) there exists a constant $c < 1$ such that $r_n^{\alpha+\varepsilon}\left(\mathbb{E} \sum_{i=1}^{N} \rho_i^{\alpha+\varepsilon}\right)^n \leqq c^n$.

Now we choose an arbitrary point $x_0 \in \mathbb{R}^d$ and define families Γ_n of random closed sets Γ_n by $\Gamma_0 = B(x_0, 1/2)$ and

$$\Gamma_n = \{\Gamma_\sigma = B(T_\sigma x_0, 1_\sigma r_n/2) : \sigma \in D_n\}, \qquad n = 1, 2, \ldots.$$

Then it suffices to show that both conditions from 5.2. are satisfied for $\beta=\alpha$, and that the diameters of the element of Γ_n are at most 1 for $n \geqq n_0$. Using the Markov inequality we see that for the first condition it is sufficient to show that $\mathbb{E}\phi((\bigcup\Gamma_n)^c)$ tends to zero, because

$$\begin{aligned}\mathbb{P}(\phi(B\setminus\bigcup\Gamma_n)<\varepsilon) &\geqq 1-\mathbb{P}(\phi((\bigcup\Gamma_n)^c)>\varepsilon)\\ &\geqq 1-\mathbb{E}\phi((\bigcup\Gamma_n)^c)/\varepsilon \qquad \text{for any } \varepsilon \text{ and } B.\end{aligned}$$

With the help of the self-similarity for ϕ (cf. 4) and properties of the conditional expectation we obtain

$$\begin{aligned}\mathbb{E}\phi((\bigcup\Gamma_n)^c) &= \mathbb{E}\mathbb{E}[\phi((\bigcup\Gamma_n)^c)|\mathscr{F}_n]\\ &= \mathbb{E}\mathbb{E}[\sum_{\sigma\in D_n} 1_\sigma^\alpha(\phi^\sigma\circ T_\sigma^{-1})((\bigcup\Gamma_n)^c)|\mathscr{F}_n],\end{aligned}$$

where the ϕ^σ, $\sigma\in D_n$, have the same distribution as ϕ and are independent of $\mathscr{F}_n$. Thus we have

$$\mathbb{E}\phi((\bigcup\Gamma_n)^c)=\mathbb{E}\sum_{\sigma\in D_n} 1_\sigma^\alpha \Lambda_\mu(T_\sigma^{-1}((\bigcup\Gamma_n)^c)).$$

By the definition of (Γ_n) we may continue

$$\begin{aligned}\mathbb{E}\phi((\bigcup\Gamma_n)^c) &= \mathbb{E}\sum_{\sigma\in D_n} 1_\sigma^\alpha \Lambda_\mu(T_\sigma^{-1}(\bigcup_{\tau\in D_n}\Gamma_\tau)^c)\\ &\leqq \mathbb{E}\sum_{\sigma\in D_n} 1_\sigma^\alpha \Lambda_\mu(T_\sigma^{-1}(\Gamma_\sigma^c))\\ &= \mathbb{E}\sum_{\sigma\in D_n} 1_\sigma^\alpha \Lambda_\mu(T_\sigma^{-1}B(T_\sigma x_0, 1_\sigma r_{n/2})^c)\\ &\leqq \mathbb{E}\sum_{\sigma\in D_n} 1_\sigma^\alpha \Lambda_\mu(B(x_0, r_{n/2})^c)\\ &= \Lambda_\mu(B(x_0, r_{n/2})^c),\end{aligned}$$

which tends to zero for $n\to\infty$ (use (1)).

For the second condition in 5.2. it suffices to show that $\mathbb{E}\liminf_n \sum_{K\in\Gamma_n} (\operatorname{diam} K)^{\alpha+\varepsilon}<\infty$. For, by Fatou's Lemma and (2) we have

$$\begin{aligned}\mathbb{E}\liminf_n \sum_{K\in\Gamma_n} (\operatorname{diam} K)^{\alpha+\varepsilon} &= \mathbb{E}\liminf_n \sum_{\sigma\in D_n} (\operatorname{diam}\Gamma_\sigma)^{\alpha+\varepsilon}\\ &= \mathbb{E}\liminf_n \sum_{\sigma\in D_n} r_n^{\alpha+\varepsilon} 1_\sigma^{\alpha+\varepsilon}\\ &\leqq \liminf_n r_n^{\alpha+\varepsilon}\left(\mathbb{E}\sum_{i=1}^N \rho_i^{\alpha+\varepsilon}\right)^n\\ &<\infty.\end{aligned}$$

Finally, from (2) it follows that there exists an $n_0 \in \mathbb{N}$ such that for any $\sigma \in \{1, \ldots, N\}^{\mathbb{N}}$ the diameters $1_{\sigma|n} \cdot r_n$ do not exeed 1 for $n \geqq n_0$, since

$$\sum_{n=1}^{\infty} \mathbb{P}\left(1_{\sigma|n} \geqq \frac{1}{r_n}\right) < \infty \quad \text{implies } \mathbb{P}\left(\bigcap_{n_0=1}^{\infty} \bigcup_{n \geqq n_0} \left\{1_{\sigma|n} \geqq \frac{1}{r_n}\right\}\right) = 0$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{P}\left(1_{\sigma|n} \geqq \frac{1}{r_n}\right) &\leqq \sum_{n=1}^{\infty} \mathbb{P}\Big(\sum_{\sigma \in D_n} 1_\sigma^{\alpha+\varepsilon} r_n^{\alpha+\varepsilon} \geqq 1\Big) \\ &\leqq \sum_{n=1}^{\infty} r_n^{\alpha+\varepsilon} \left(\mathbb{E} \sum_{i=1}^{N} \rho_i^{\alpha+\varepsilon}\right)^n \\ &\leqq \sum_{n=1}^{\infty} c^n \\ &< \infty \quad \text{holds.}\end{aligned}$$

Thus the theorem is proved. □

5.5. *Remark* Up to this point all results remain valid if we replace $\mathbb{R}^d$ by any complete separable metric space.

Lower bounds

The proof of cardim $\phi \geqq \alpha$ is more delicate and makes use of the structure of the second order moment measure $\Lambda_\mu^{(2)}$. The underlying theory was developed by O. Frostmann and U. Zähle.

5.6. Lemma (cf. [Z1] 6.3.) *Let* ψ *be a random measure on* $\mathbb{R}^d$, $\beta > 0$, $r > 0$, *and* $B_n \uparrow \mathbb{R}^d$. *Suppose*

$$\int |x-y|^{-\beta} 1_{B_n}(x)\, 1\{|x-y| < r\}\, \Lambda_\psi^{(2)}(\mathrm{d}(x, y)) < \infty \quad \textit{for any } n \in \mathbb{N}.$$

Then the implication

$$\dim B \geqq \beta \quad \textit{whenever } \psi(B) > 0\, B \in \mathscr{R}^d,$$

almost surely holds. □

5.7. Corollary *If the condition in* 5.6. *is satisfied we obtain* cardim $\psi \geqq \beta$ a.s. *if* cardim ϕ a.s. *exists.* □

5.8. Theorem *Suppose that in addition to* 2.1. *the following conditions are satisfied*

(i) $S_1, \ldots, S_N$ *are similarities* μ-a.s.,

(ii) $\sup_n \mathbb{E} S_{\alpha,n}^3 < \infty$ *and*

(iii) $\mathbb{E} \iint |S_i x - S_j y|^{-\beta} \Lambda_\mu(\mathrm{d}x)\, \Lambda_\mu(\mathrm{d}y) < \infty$ *for* $i, j \in \{1 \ldots N\}$ $i \neq j$ *and* $\beta \leqq \min\{\mathrm{d}, \alpha\}$.

Then we have cardim $\phi \geqq \min\{\mathrm{d}, \alpha\}$ a.s.

Proof. Using the structure of the second moment measure $\Lambda_\mu^{(2)}$ (cf. 4.4.) and the independence from 1.2. we get

$$\int |x-y|^{-\beta} 1_{B_n}(x)\, 1\{|x-y|<r\}\, \Lambda_\mu^{(2)}(\mathrm{d}(x,y))$$
$$\leqq \int |x-y|^{-\beta} \Lambda_\mu^{(2)}(\mathrm{d}(x,y))$$
$$= \sum_{k=1}^{\infty} \mathbb{E} \sum_{\sigma \in D_{k-1}} 1_\sigma^{2\alpha} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \rho_{\sigma * i}^{\alpha} \rho_{\sigma * j}^{\alpha} \iint |T_{\sigma * i} x - T_{\sigma * j} y|^{-\beta} \Lambda_\mu(\mathrm{d}x)\, \Lambda_\mu(\mathrm{d}y)$$
$$= \sum_{k=1}^{\infty} \mathbb{E} \sum_{\sigma \in D_{k-1}} 1_\sigma^{2\alpha-\beta} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \rho_{\sigma * i}^{\alpha} \rho_{\sigma * j}^{\alpha} \iint |S_{\sigma * i} x - S_{\sigma * j} y|^{-\beta} \Lambda_\mu(\mathrm{d}x)\, \Lambda_\mu(\mathrm{d}y)$$
$$\leqq \text{const.} \sum_{k=1}^{\infty} \mathbb{E} \sum_{\sigma \in D_{k-1}} 1_\sigma^{2\alpha-\beta}$$
$$= \text{const.} \sum_{k=1}^{\infty} \left(\mathbb{E} \sum_{i=1}^{N} \rho_i^{2\alpha-\beta} \right)^{k-1}$$
$$< \infty \quad \text{for } \beta < \alpha.$$

Now the theorem follows immediately from 5.6. and 5.7. □

Together with Theorem 5.3. we get the following main result of this paper.

5.9. Theorem *If the conditions* 2.1.(i)–(iv) *and* 5.8.(i)–(iii) *are satisfied, then the carrying dimension of the limit measure* ϕ *is equal to* $\mathrm{d}^* = \min\{\mathrm{d}, \alpha\}$. □

5.10. Remark The condition 5.8.(iii) is on the first view complicated and unhandly, but we learned that it is sufficient to know the structure of the intensity measure of the random limit measure ϕ. The condition is not to hard for the deterministic case. We will show in the next section, that Hutchinsons open set condition implies 5.8.(iii).

6 Examples

6.1 Hutchinsons self-similarity sets

A theory of strictly self-similar compact sets has been developed by Hutchinson (cf. [H]). He proved the following.

Theorem (Hutchinson) (1) *For every finite set of contractions* $S_1, \ldots, S_N$ *of a complete metric space* (X, ρ) *there exists a unique self-similar measure* φ *with compact support* K, i.e.

$$\varphi = \sum_{i=1}^{N} (\mathrm{Lip}\, S_i)^\alpha \varphi \circ S_i^{-1}$$

where α *is the unique number for which* $\sum_{i=1}^{N} (\mathrm{Lip}\, S_i)^\alpha = 1$.

(2) *Let X be a compact subset of $\mathbb{R}^d$ and suppose that for $S_1, \ldots, S_N$ the following conditions are satisfied*
(a) *The map S_i is a similarity, $i=1, \ldots, N$*
(b) *Open set condition:*
There exists an open set $0 \subset X$, such that

$$\bigcup_{i=1}^{N} S_i 0 \subset 0 \quad \text{and} \quad S_i 0 \cap S_j 0 = \emptyset \quad \text{for } i \neq j.$$

Then the Hausdorff dimension of the invariant set K equals α. □

We see that (1) is a special case of Theorems 2.8. and 3.4. taking $\mu = \delta_{(S_1, \ldots, S_N)}$ (point measure). The additional conditions are only restrictions for the noncompact case. Now we show that (2) is a special case of Theorem 5.9. To do this we have to check that Hutchinsons open set condition implies 5.8.(iii).

In the deterministic case 5.8.(iii) is equivalent to the existence of a constant c, such that $\int |x-y|^{-\beta} \varphi(\mathrm{d}x) < c$ for all $y \in \mathbb{R}^d$ and all $\beta < \alpha$. By choosing a sequence (r_k) of positive real numbers with $r_1 = 1$ and $r_n \xrightarrow{n \to \infty} 0$ we obtain

$$\begin{aligned} \int |x-y|^{-\beta} \varphi(\mathrm{d}x) &\leq \varphi(\mathbb{R}^d) + \int 1\{|x-y|<1\} |x-y|^{-\beta} \varphi(\mathrm{d}x) \\ &= \sum_{k=1}^{\infty} \int 1_{B(y,r_k) \setminus B(y,r_{k+1})}(x) |x-y|^{-\beta} \varphi(\mathrm{d}x) + \varphi(\mathbb{R}^d) \\ &\leq \sum_{k=1}^{\infty} r_{k+1}^{-\beta} [\varphi(B(y,r_k)) - \varphi(B(y,r_{k+1}))] + \varphi(\mathbb{R}^d) \\ &= \frac{\varphi(B(y,1))}{r_2^{\beta}} + \sum_{k=2}^{\infty} \left(\frac{r_k^{\beta}}{r_{k+1}^{\beta}} - 1 \right) \frac{\varphi(B(y,r_k))}{r_k^{\alpha}} r^{\alpha-\beta} + \varphi(\mathbb{R}^d). \end{aligned}$$

Following Hutchinson ([H] S. 738) we see that the open set condition implies the existence of a constant $\tilde{c}$, such that $\frac{\varphi(B(y,r_k))}{r_k^{\alpha}} \leq \tilde{c}$ for all y and k and α. This constant is independent of (r_k) and therefore we can choose (r_k), such that $\sum_{k=2}^{\infty} \left(\frac{r_k^{\beta}}{r_{k+1}^{\beta}} - 1 \right) r_k^{\alpha-\beta}$ is finite. Hence, we get a constant c such that $\int |x-y| \varphi(\mathrm{d}x) < c$ for all y and all $\beta < \alpha$.

6.2 *The recursive construction model of Falconer, Graf and Mauldin-Williams*

The idea of this model is to form a probabilistic counterpart to Hutchinsons self-similar sets. The main results are summarized in the following.

Theorem (cf. [G]) *Let K be a compact subset of $\mathbb{R}^d$ with $K = \overline{\operatorname{int} K}$ and μ be a Borel probability measure on* $\operatorname{Con}(K)^N$. *For every $\sigma \in D$ we choose an N-tuple $(S_{\sigma * 1}, \ldots, S_{\sigma * N})$ of contractions* w.r.t. *μ and a random set K_σ. Suppose that $\{(S_{\sigma * 1}, \ldots, S_{\sigma * N}), K_\sigma \colon \sigma \in D\}$ is a family of independent random elements. Then we have:*

(1) *There exists* a.s. *the limit set* (w.r.t. *Hausdorff metric*)

$$\Xi = \lim_{n\to\infty} \bigcup_{\sigma\in D_n} T_\sigma(K_\sigma).$$

(2) *Ξ is statistically self-similar, that means, if $(\Xi^\sigma)_{\sigma\in D}$ are copies of Ξ, independent of Ξ and $(T_\sigma)_{\sigma\in D_n}$, then the random sets $\bigcup_{\sigma\in D_n} T_\sigma(\Xi^\sigma)$ have the same distribution as Ξ, $n=1, 2, \ldots$.*

(3) *If* μ-a.e. $(S_1, \ldots, S_N)$ *are similarities and* $S_i(\text{int}\, K)\cap S_j(\text{int}\, K)=\emptyset$ μ-a.s. *for* $i\neq j$, *then the Hausdorff dimension of the limit set Ξ equals almost surely α, where α is the unique number with* $\mathbb{E}\sum_{i=1}^N (\text{Lip}\, S_i)^\alpha = 1$. □

If we translate this model in the measure-theoretical language we see that our model contains the FGHW-model. (1) and (2) are special cases of Theorem 2.8. and 3.4. if we restrict to the compact case.

(3) is a special result of Theorem 5.9. under some stronger assumptions. Suppose that μ-a.a. $(S_1, \ldots, S_N)$ are similarities and there exists a constant $c>0$ such that $\text{dist}(S_iK, S_jK)\geqq c$ μ-a.s. for $i, j=\{1, \ldots, N\}$, $i\neq j$. Then the condition 5.8.(iii) is satisfied and therefore the carrying dimension of the random limit measure almost surely equals α. This is easy to see because of

$$\begin{aligned}
&\mathbb{E}\iint |S_ix-S_jx|^{-\beta} 1\{|S_ix-S_jy|<1\}\, \Lambda_\mu(\mathrm{d}x)\, \Lambda_\mu(\mathrm{d}y)\\
&\leqq \mathbb{E}\iint 1_K(x)\, 1_K(y) |S_ix-S_jy|^{-\beta} \Lambda_\mu(\mathrm{d}x)\, \Lambda_\mu(\mathrm{d}y)\\
&\leqq c^{-\beta}\Lambda_\mu(K)^2\\
&<\infty \qquad \text{for } \beta>0.
\end{aligned}$$

However, these conditions are only sufficient. Section 6.4. gives examples, also for the compact case, for self-similar measures with carrying dimension α where $S_i(\text{int}\, K)\cap S_j(\text{int}\, K)=\emptyset$ for $i\neq j$ is not satisfied.

6.3 Self-similar measures with absolutely continuous intensity measures

6.3.1. Proposition *Suppose that the conditions in* 5.8. *are satisfied and replace* 5.8.*(iii) by*

(i) *Λ_μ has a bounded density f_μ* w.r.t. $\mathscr{L}^d$

(ii) $\mathbb{E}\rho_i^{-1}<\infty$ *for* $i=1, \ldots, N$

Then cardim $\phi=\min\{\mathrm{d}, \alpha\}$ a.s.

Proof. Let $f_\mu\leqq c$. Now the proposition is an obvious consequence of Theorem 5.8. since

$$\begin{aligned}
&\mathbb{E}\iint 1\{|S_ix-S_jy|<1\}\, |S_ix-S_jy|^{-\beta}\Lambda_\mu(\mathrm{d}x)\, \Lambda_\mu(\mathrm{d}y)\\
&\quad= \mathbb{E}\rho_i^{-1}\iint 1\{|w|<1\}\, |w|^{-\beta} f_\mu(S_i^{-1}(w+S_jy)) f_\mu(y)\, \mathscr{L}^d(\mathrm{d}y)\, \mathscr{L}^d(\mathrm{d}w)\\
&\quad\leqq c\cdot\mathbb{E}\rho_i^{-1}\int 1\{|w|<1\}\, |w|^{-\beta}\mathscr{L}^d(\mathrm{d}w)\\
&\quad<\infty \qquad \text{for } \beta<d. \quad \square
\end{aligned}$$

6.3.2. Proposition *Suppose that the conditions in Theorem* 5.8. *are satisfied and replace* 5.8.*(iii) by*

(i) *Λ_μ has a bounded density f_μ* w.r.t. $\mathscr{L}^d$

(ii) *S_i and S_j are independent for $i, j \in \{1, \dots, N\}$ $i \neq j$.*
Then cardim $\phi = \min\{d, \alpha\}$ a.s.
Proof. Using the self-similarity of Λ_μ we get

$$\mathbb{E} \sum_{\substack{i,j=1 \\ i \neq j}} \rho_i^\alpha \rho_j^\alpha \iint 1\{|S_i x - S_j y| < 1\} |S_i x - S_j y|^{-\beta} \Lambda_\mu(\mathrm{d}x) \Lambda_\mu(\mathrm{d}y)$$

$$\leq \iint 1\{|x-y|<1\} |x-y|^{-\beta} \left(\mathbb{E} \sum_{i=1}^{N} \rho_i^\alpha \Lambda_\mu \circ S_i^{-1}\right)(\mathrm{d}x) \left(\mathbb{E} \sum_{j=1}^{N} \rho_j^\alpha \Lambda_\mu \circ S_j^{-1}\right)(\mathrm{d}y)$$

$$= \iint 1\{|x-y|<1\} |x-y|^{-\beta} \Lambda_\mu(\mathrm{d}x) \Lambda_\mu(\mathrm{d}y).$$

By the same arguments as in 6.3.1. the right hand side is finite. □
Remarks (1) The boundedness of the density f_μ can be replaced by restrictions on the increase of f_μ in a neighbourhood of the origin.
(2) Generally, it is very difficult to check whether the limit measure Λ_μ is absolutely continuous w.r.t. $\mathscr{L}^d$. But in the special cases it is possible to show this, for example if the translations of the similarities are distributed w.r.t. a stable distribution ($d=1$). However, it sufficies to require that the translations of the similarities have a bounded density w.r.t. $\mathscr{L}^d$. This case will be investigated in the next subsection.

6.4 Similarities with absolutely continuous translation distributions

It is well known that any similarity S acting on $\mathbb{R}^d$ can be written as follows: $Sx = \rho_S 0_S x + \eta_S$, where $\eta_S \in \mathbb{R}^d$, $\rho_S = \mathrm{Lip}(S)$ and 0_S is an orthogonal $d \times d$ matrix. So we can prove the following.

Proposition *Suppose that the conditions in Theorem* 5.8. *are satisfied and replace* 5.8.*(iii) by*
(i) *S_i and S_j are independent for $i \neq j$ $i, j \in \{1, \dots, N\}$*
(ii) *$(\rho_i 0_i)$ and η_i are independent for each $i \in \{1, \dots, N\}$*
(iii) *η_i has a bounded density g_i* w.r.t. $\mathscr{L}^d$.
Then cardim $\phi = \min\{d, \alpha\}$ a.s.

Proof. Denote by $\mathscr{H}_i$ the distribution of η_i, let $g_i \leq c$ and use the required independence in order to get

$$\mathbb{E} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \rho_i^\alpha \rho_j^\alpha \iint 1\{|S_i x - S_j y| < 1\} |S_i x - S_j y|^{-\beta} \Lambda_\mu(\mathrm{d}x) \Lambda_\mu(\mathrm{d}y)$$

$$= \mathbb{E} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \rho_i^\alpha \rho_j^\alpha \iint 1\{|\rho_i 0_i x - \rho_j 0_j y + u - v| < 1\} |\rho_i 0_i x - \rho_j 0_j y + u - v|^{-\beta}$$

$$\cdot \mathscr{H}_i(\mathrm{d}u) \mathscr{H}_j(\mathrm{d}v) \Lambda_\mu(\mathrm{d}x) \Lambda_\mu(\mathrm{d}y)$$

$$\leq \mathbb{E} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \rho_i^\alpha \rho_j^\alpha c \iiiint 1\{|w|<1\} |w|^{-\beta} \mathscr{L}^d(\mathrm{d}w)\, g_j(v)\, \mathscr{L}^d(\mathrm{d}v) \Lambda_\mu(\mathrm{d}x) \Lambda_\mu(\mathrm{d}y)$$

$$\leq c \int |w|^{-\beta} 1\{|w|<1\} \mathscr{L}^d(\mathrm{d}w)$$

$$< \infty \quad \text{for } \beta < d.$$

The proposition follows from Theorem 5.8. □

Remarks (1) (i)–(iii) can be weakend by analogous assumptions on the conditional distributions.
(2) 6.4. provides a class of examples where the "open set condition" in the FGMW-model is not satisfied but the carrying dimension of the self-similar measure ϕ is almost surely equal to α.
(3) An open problem is to find further and more general explicit conditions on $S_1, \ldots, S_N$ under which 5.8.(iii) is satisfied.

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