Probab. Theory Relat. Fields 102, 289–311 (1995)

Probability Theory and Related Fields

Large time behavior of interface solutions to the heat equation with Fisher-Wright white noise

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Received: 8 November 1993 / In revised form: 16 August 1994

Summary. The one-dimensional heat equation driven by Fisher–Wright white noise is studied. From initial conditions with compact support, solutions retain this compact support and die out in finite time. There exist interface solutions which change from the value 1 to the value 0 in a finite region. The motion of the interface location is shown to approach that of a Brownian motion under rescaling. Solutions with a finite number of interfaces are approximated by a system of annihilating Brownian motions.

Mathematics Subject Classification: 60H15

1 Description of results

We consider in this paper the equation

$$\dot{u} = \tfrac{1}{2}\Delta u + |u(1-u)|^{1/2}\dot{W}, \tag{1}$$

where $\dot{W}$ is a space–time white noise on $[0, \infty) \times \mathbb{R}$. The equation arises from a model of population genetics (see [9]). It also arises as a hydrodynamic limit from the long range voter process (see [5]). We consider only solutions for which $u_t(x) \in [0,1]$ for all t, x. Let $\mathscr{C} = (f\colon \mathbb{R} \to [0,1]$ continuous) with the topology of uniform convergence on compacts. The existence of continuous $\mathscr{C}$ valued solutions (with possibly random initial conditions) can be established using the methods of Shiga [8] or Reimers [7]. Uniqueness in law (and the strong Markov property) follows from a duality relation where the dual process is a system of coalescing Brownian motions (see [9]).
In Sect. 2 we use the duality relation to calculate certain moments needed in later sections. In Sect. 3 we prove a compact support property. Define for $f \in \mathscr{C}$,

$$R(f) = \sup(x\colon f(x) > 0), \qquad L(f) = \inf(x\colon f(x) < 1)$$

and let $\mathscr{C}_I = (f \in \mathscr{C}\colon\ -\infty < L(f) < R(f) < \infty)$. We show that if $u_0 \in \mathscr{C}_I$ then $u_t \in \mathscr{C}_I$ for all t. We call the region lying between $L(u_t)$ and $R(u_t)$ an interface. In [6] solutions with values in $\mathscr{C}_I$ are studied and it is shown that the shape of the interface converges to a unique stationary law concentrated on $\mathscr{C}_I$. The aim of this paper is to describe the evolution of the position of this interface.

In Sect. 4 we consider initial conditions in $\mathscr{C}_I$ and show that, under Brownian rescaling, the motion of the position of the wavefront $(t \to R(u_{n^2t})/n)$ converges to that of a Brownian motion as $n \to \infty$. In Sect. 5 we show that solutions that initially have compact support die out in finite time. We then consider solutions started from initial conditions with a finite number of interfaces. The solution for all time consists of intervals where it equals 0 or 1 separated by interfaces. When two interfaces collide they annihilate each other precisely because solutions with compact support die out. Using this we show that (again under Brownian rescaling) the motion of the interface locations approaches the motion of a system of annihilating Brownian motions.

We remark that the proofs require the exact moment formulae given by duality and do not immediately generalise to similar equations for which in general no such formulae exist.

Notation. For $f, g\colon \mathbb{R} \to \mathbb{R}$ we write (f, g) for the integral $\int f(x) g(x)\, dx$ whenever this is defined. We use the same notation (μ, f) when μ is a measure on $\mathbb{R}$. M will be the space of Radon measures on $\mathbb{R}$ with the vague topology: $\mu_n \to \mu$ if and only if $(\mu_n, \phi) \to (\mu, \phi)$ for all $\phi \in C_c$ the space of continuous functions on $\mathbb{R}$ with compact support. For a function $f\colon [0, \infty) \times \mathbb{R} \to \mathbb{R}$ we write $f', \Delta f$ for spatial derivatives and $\dot{f}$ for the partial derivative in time. $\|f\|_p$ denotes the L^p norm for $p \in [1, \infty]$. We write $p_t(x)$ for the Brownian transition density and P_t for the Brownian semigroup. We write $L_t^0(Y)$ for the local time at 0 of a semimartingale Y. We have the convention that $\inf(\emptyset) = +\infty$. Finally C will denote a quantity whose dependence will be indicated but whose exact value is unimportant and may vary from line to line.

2 Duality and moments

The dual process is a system of Brownian particles where each pair coallesces at rate $\frac{1}{2}$ measured by their intersection local time. Here is a construction. Fix $\underline{z} = (z_1, \dots, z_m) \in \mathbb{R}^m$. Let $(B^i)_{i=i, \dots, m}$ be independent Brownian motions with $B_0^i = z_i$ under $E_{\underline{z}}$. Let $(e_{i,j})_{i,j=1, \dots, m}$ be an independent collection of exponential mean 2 variables. Write $l_{i,j}$ for the local time of $B^i - B^j$ at zero. We shall define for each particle $i \geqslant 2$ a coallescence time T_i and a coallescence partner α_i. The definition is inductive on i. Set $T_1 = \infty$. If $T_1, \dots, T_k$ are already defined then let

$$T_{k+1,j} = \inf(t\colon l_{k+1,j}(t \wedge T_j) \geqq e_{k+1,j}),$$

$$T_{k+1} = \min(T_{k+1,j}\colon j = 1, \dots, k),$$

$$\alpha_k = \min(j \in \{1, \dots, k\}\colon T_{k+1} = T_{k+1,j}).$$

The duality relation becomes: for any solution u to (1) started at $f \in \mathscr{C}$,

$$E\left(\prod_{i=1}^{m} u_t(z_i)\right) = E_z\left(\prod_{i\in(1,\dots,m):T_i>t} f(B_t^i)\right).$$

We now use this to establish two moment bounds that are needed during the remaining sections.

Lemma 2.1 *Let u be a solution to* (1) *started at $f \in \mathscr{C}_I$.*
a)

$$E(\int u_t(z)(1-u_t(z+x))\,dz) \to 1 + (x \vee 0) \quad \text{as } t \to \infty$$

and the expectation is bounded by $1 + (x \vee 0) + (R(f) - L(f))$ *for all* t.
b) If $f \leqq \boldsymbol{I}(0, M)$ then for $|x| \leqq 1$

$$E(\int u_t(z)(1-u_t(z+x))\,dz) \leqq CMt^{-1/2}.$$

c) For $\varepsilon > 0$ there exists $C(\varepsilon) < \infty$ such that whenever $|z_1 - z_4| \vee |z_2 - z_3| \leqq 1$ and $d := \min(z_1, z_4) - \max(z_2, z_3) \geqq 0$ then

$$E(\int u_t(z_1+x)u_t(z_2+x)(1-u_t(z_3+x))(1-u_t(z_4+x))\,dx)$$
$$\leqq C(\varepsilon)d^{-2(1-\varepsilon)}(R(f) - L(f) + 1) \quad \text{for all } t.$$

d)

$$E(\int\int u_t(x)(1-u_t(x))u_t(y)(1-u_t(y))\,dx\,dy) \leqq C(f) \quad \text{for all } t.$$

Proof. a) From the duality relation we have

$$E(\int u_t(z)(1-u_t(z+x))\,dz)$$
$$= \int E_{(z,z+x)}(f(B_t^1) - \prod_{i\in(1,2):\,T_i>t} f(B_t^i))\,dz$$
$$= \int E_{(z,z+x)}(f(B_t^1)(1-f(B_t^2))\boldsymbol{I}(t<T_2))\,dz$$
$$= E_{(0,x)}(\int f(z+B_t^1)(1-f(z+B_t^2))\,dz\,\boldsymbol{I}(t<T_2))$$
$$= E_{(0,x)}(\int f(z+B_t^1)(1-f(z+B_t^2))\,dz\,e^{-l_{1,2}(t)/2})$$
$$\leqq E_{(0,x)}(\int \boldsymbol{I}(z+B_t^1 \leqq R(f), z+B_t^2 \geqq L(f))\,dz\,e^{-l_{1,2}(t)/2})$$
$$\leqq E_{(0,x)}((R(f) - L(f) + (B_t^2 - B_t^1)_+)e^{-l_{1,2}(t)/2})$$
$$= E_{(0,x)}((R(f) - L(f))e^{-l_{1,2}(t)/2}) + (x \vee 0)$$
$$+ E_{(0,x)}\left(\int_0^t e^{-l_{1,2}(s)/2}((\tfrac{1}{2}) - (\tfrac{1}{2})(B_s^2 - B_s^1)_+)l_{1,2}(ds)\right)$$
$$+ E_{(0,x)}\left(\int_0^t e^{-l_{1,2}(s)/2}\operatorname{sgn}(B_s^2 - B_s^1)\,d(B_s^2 - B_s^1)\right)$$
$$= E_{(0,x)}((R(f) - L(f))e^{-l_{1,2}(t)/2}) + (x \vee 0)$$
$$+ E_{(0,x)}(1 - e^{-l_{1,2}(t)/2}) \to (x \vee 0) + 1 \quad \text{as } t \to \infty,$$

where we have used integration by parts and Tanaka's formula in the fifth equality. The upper bound follows from the penultimate line. A lower bound is obtained in exactly the same way and together these prove convergence.
b) As in part a) we have

$$E(\int u_t(z)(1-u_t(z+x))\,dz) = E_{(0,x)}(\int f(z+B_t^1)(1-f(z+B_t^2))\,dz\,e^{-l_{1,2}(t)/2})$$
$$\leqq M E_{(0,x)}(e^{-l_{1,2}(t)/2})$$
$$= M E_{(x)}(e^{-L^0_{2t}(B^1)/2}).$$

Using the Levy equivalence, $L_t^0(B^1) \overset{\mathcal{D}}{=} \sup_{s \leqq t} B_s^1$ under $P_{(0)}$, an easy calculation bounds the exponential moment

$$E_{(x)}(e^{-L_t^0(B^1)/2}) \leqq C t^{-1/2} \quad \text{for all } |x| \leqq 1. \tag{2}$$

c) We expand the product $u_t(z_1)u_t(z_2)(1-u_t(z_3))(1-u_t(z_4)))$ and use the duality relation on each of the four resulting products. To recombine them on one probability space we rewrite

$$E(u_t(z_1)u_t(z_2)u_t(z_4)) = E_{(z_1,z_2,z_4)}\left(\prod_{i\in(1,2,3):T_i>t} f(B_t^i)\right)$$
$$= E_{\underline{z}}(f(B_t^1)\tilde{f}(t,B_t^2)\hat{f}(t,B_t^3,B_t^4)),$$

where

$$\tilde{f}(t,B_t^i) = \begin{cases} f(B_t^i) & \text{if } t < T_i, \\ 1 & \text{if } t \geqq T_i, \end{cases}$$

$$\hat{f}(t,B_t^3,B_t^4) = \begin{cases} f(B_t^4) & \text{if } t < T_4, \\ f(B_t^3) & \text{if } T_4 \leqq t < T_3, \alpha(4)=3, \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$E(u_t(z_1)u_t(z_2)(1-u_t(z_3))(1-u_t(z_4)))$$
$$= E_{\underline{z}}(f(B_t^1)\tilde{f}(t,B_t^2)(1-\tilde{f}(B_t^3)-\hat{f}(t,B_t^3,B_t^4)+\tilde{f}(B_t^3)\tilde{f}(B_t^4)))$$
$$= E_{\underline{z}}(f(B_t^1)f(B_t^2)(1-f(B_t^3))(1-f(B_t^4))\boldsymbol{I}(t<T_2\wedge T_3\wedge T_4)$$
$$+ f(B_t^1)(1-f(B_t^3))(1-f(B_t^4))\boldsymbol{I}(T_2\leqq t<T_3\wedge T_4)$$
$$+ f(B_t^1)f(B_t^2)(1-f(B_t^3))\boldsymbol{I}(T_4\leqq t<T_2\wedge T_3,\alpha(4)=3)$$
$$+ f(B_t^1)(1-f(B_t^3))\boldsymbol{I}(T_2\vee T_4\leqq t<T_3,\alpha(4)=3)),$$

where the last equality comes by exhaustively considering each case. Let $f_a(x)=\boldsymbol{I}(x\leqq a)$. Then $f_L\leqq f\leqq f_R$ and we may then bound this last expression by

$$E_{\underline{z}}(f_R(B_t^1)(1-f_L(B_t^3))\boldsymbol{I}(t<T_3)\boldsymbol{I}((t<T_4)\cup(t\geqq T_4,\alpha(4)=3))).$$

Then

$$\begin{aligned}
&E(\int u_t(z_1+x)u_t(z_2+x)(1-u_t(z_3+x))(1-u_t(z_4+x))\,dx)\\
&\quad\leqq E_z(\int f_R(x+B_t^1)(1-f_L(x+B_t^3))\,dx\,I(t<T_3)\\
&\qquad\qquad I((t<T_4)\cup(t\geqq T_4,\alpha(4)=3)))\\
&\quad= E_z((R(f)-L(f)+B_t^3-B_t^1)_+ I(t<T_3)\\
&\qquad\qquad I((t<T_4)\cup(t\geqq T_4,\alpha(4)=3))).
\end{aligned} \tag{3}$$

Recall now the hypotheses on the initial positions $z_1, \dots, z_4$. Let $\tau_{i,j} = \inf(t\colon B_t^i = B_t^j)$ and define $\tau = \tau_{1,2} \wedge \tau_{1,3} \wedge \tau_{2,4} \wedge \tau_{3,4}$. The pair of particles (B^1, B^4) remains to the right of the pair (B^2, B^3) upto the time τ. So

$$\begin{aligned}
&I(t<T_3)I((t<T_4)\cup(t\geqq T_4,\alpha(4)=3))\\
&\leqq I(l_{1,4}(t\wedge\tau)<e_{1,4}, l_{2,3}(t\wedge\tau)<e_{2,3})\\
&\leqq I(t\wedge\tau\leqq d^{2(1-\varepsilon)}) + I(l_{1,4}(d^{2(1-\varepsilon)})<e_{1,4}, l_{2,3}(d^{2(1-\varepsilon)})<e_{2,3}\tau>d^{2(1-\varepsilon)}).
\end{aligned}$$

Letting $\mathscr{G}_\tau = \sigma(B_s^i\colon s \leqq \tau, i = 1, \dots, 4)$ we have by arguing as in part a) that $E((B_t^3 - B_t^1)_+ I(t < T_3)|\mathscr{G}_\tau) \leqq 1$. So, continuing from (3),

$$\begin{aligned}
&E(\int u_t(z_1+x)u_t(z_2+x)(1-u_t(z_3+x))(1-u_t(z_4+x))\,dx)\\
&\leqq E_z((R(f)-L(f)+B_t^3-B_t^1)_+ I(t\leqq d^{2(1-\varepsilon)}))\\
&\quad + E_z(I(\tau\leqq d^{2(1-\varepsilon)})E_z(R(f)-L(f)+(B_t^3-B_t^1)_+ I(t<T_3)|\mathscr{G}_\tau))\\
&\quad + E_z(I(l_{1,4}(d^{2(1-\varepsilon)})<e_{1,4}, l_{2,3}(d^{2(1-\varepsilon)})<e_{2,3}\tau>d^{2(1-\varepsilon)})E_z(R(f)-L(f)\\
&\quad + (B_t^3-B_t^1)_+ I(t<T_3)|\mathscr{G}_\tau))\\
&\leqq E_z((R(f)-L(f)+B^3_{d^{2(1-\varepsilon)}}-B^1_{d^{2(1-\varepsilon)}})_+)\\
&\quad + (R(f)-L(f)+1)P_z(\tau\leqq d^{2(1-\varepsilon)}) + (R(f)-L(f)+1)\\
&\quad \times E_z(\exp(-l_{1,4}(d^{2(1-\varepsilon)})/2))E_z(\exp(-l_{2,3}(d^{2(1-\varepsilon)})/2)).
\end{aligned} \tag{4}$$

The last two expectations in (4) are bounded using (2) by $Cd^{-(1-\varepsilon)}$. The first two terms are easily bounded using Brownian crossing probabilities by $C(R(f) - L(f) + 1)d^{-2}$ completing part c). Part d) follows from parts a) and c). □

3 Compact support property

We shall need control of $\sup_x u_t(x)$. Given the Green's function representation

$$u_t(x) = P_t f(x) + \int_0^t \int p_{t-s}(x-y)\,|u_s(y)(1-u_s(y))|^{1/2}\,dW_{y,s}$$

this can be done by controlling $\sup_x N_t(x)$ where

$$N_t(x) = \int_0^t \int p_{t-s}(x-y)\,|u_s(y)(1-u_s(y))|^{1/2}\,dW_{y,s}.$$

This in turn is done by controlling all increments $|N_t(x) - N_t(y)|$ for x, y in dyadic grids (as in say the proof of the modulus of continuity of Brownian motion). Estimates of the sort in the following lemma occur in several papers ([4, 10]) but since none are quite suited to our needs we prove another.

Lemma 3.1 *For* $\varepsilon, t > 0, A \in \mathbb{R}$

$$P(|N_s(x)| \geqq \varepsilon, \exists s \leqq t, x \geqq A) \leqq C\varepsilon^{-20}(t \vee t^{22})(f, P_t I(A, \infty)).$$

Proof. We use the estimates, for $0 < s < t$, $x, y \in \mathbb{R}$,

$$\int_0^t \int (p_{t-s}(x-z) - p_{t-s}(y-z))^2 \, dz \, ds \leqq C|x-y| \wedge t^{1/2},$$

$$\int_0^s \int (p_{t-r}(x-z) - p_{s-r}(x-z))^2 \, dz \, dr \leqq C|t-s|^{1/2} \wedge s^{1/2}.$$

Applying Burkholder's and then Holder's inequalities we have, taking $p \geqq 2$,

$$\begin{aligned}
&E(|N_t(x) - N_t(y)|^{2p}) \\
&\leqq C(p) E\left(\left(\int_0^t \int (p_{t-s}(x-z) - p_{t-s}(y-z))^2 u_s(z) \, dz \, ds\right)^p\right) \\
&\leqq C(p)(|x-y| \wedge t^{1/2})^{p-1} \\
&\quad \times E\left(\int_0^t \int (p_{t-s}(x-z) - p_{t-s}(y-z))^2 u_s^p(z) \, dz \, ds\right) \\
&\leqq C(p)(|x-y| \wedge t^{1/2})^{p-1} \\
&\quad \times E\left(\int_0^t (t-s)^{-1/2} \int (p_{t-s}(x-z) + p_{t-s}(y-z)) u_s(z) \, dz \, ds\right) \\
&\leqq C(p)(|x-y| \wedge t^{1/2})^{p-1} t^{1/2} (f, p_t(x - \cdot) + p_t(y - \cdot)).
\end{aligned}$$

Similarly, for $0 \leqq s < t$,

$$\begin{aligned}
&E(|N_t(x) - N_s(x)|^{2p}) \\
&\leqq C(p) E\left(\left(\int_s^t \int p_{t-r}^2(x-z) u_r(z) \, dz \, dr\right)^p\right) \\
&\quad + C(p) E\left(\left(\int_0^s \int (p_{t-r}(x-z) - p_{s-r}(x-z))^2 u_r(z) \, dz \, dr\right)^p\right) \\
&\leqq C(p)\left(\int_s^t \int p_{t-r}^2(x-z) \, dz \, dr\right)^{p-1} E\left(\int_s^t \int p_{t-r}^2(x-z) u_r(z) \, dz \, dr\right) \\
&\quad + C(p)(|t-s| \wedge s)^{(p-1)/2} \\
&\quad \times E\left(\int_0^s \int (p_{t-r}(x-z) - p_{s-r}(x-z))^2 u_r(z) \, dz \, dr\right) \\
&\leqq C(p)|t-s|^{(p-1)/2} t^{1/2} (f, p_t(x - \cdot) + p_s(x - \cdot)).
\end{aligned}$$

Define $x_n^j = t_n^j = j2^{-n}$ for $j \in Z$, $n \in \mathbb{N}$. Define the events

$$A^1_{j,k,n}(\varepsilon) = \{|N_{t_n^j}(x_n^{k+1}) - N_{t_n^j}(x_n^k)| \geqq \varepsilon 2^{-n/10}\}$$

$$A^2_{j,k,n}(\varepsilon) = \{|N_{t_n^j}(x_n^k) - N_{t_n^{j-1}}(x_n^k)| \geqq \varepsilon 2^{-n/10}\}.$$

Set $n_0 = \inf(n \geqq 1\colon 2^{-n} \leqq t^{1/2})$. Then

$$\begin{aligned}
&\sum_{n \geqq n_0} \sum_{1 \leqq j \leqq 2^n t} \sum_{k \geqq 2^n A} P(A^1_{j,k,n}(\varepsilon)) \\
&\leqq \sum_{n \geqq n_0} \sum_{1 \leqq j \leqq 2^n t} \sum_{k \geqq 2^n A} \varepsilon^{-2p} 2^{np/5} E(|N_{t_n^j}(x_n^{k+1}) - N_{t_n^j}(x_n^k)|^{2p}) \\
&\leqq C(p)\varepsilon^{-2p} t^{1/2} \sum_{n \geqq n_0} 2^{n(2-(4p/5))} \sum_{1 \leqq j \leqq 2^n t} \\
&\quad \times \sum_{k \geqq 2^n A} 2^{-n}(f, P_{t_n^j}(x_n^k - \cdot) + p_{t_n^j}(x_n^{k+1} - \cdot)) \\
&\leqq C(p)\varepsilon^{-2p} t^{1/2} \sum_{n \geqq n_0} 2^{n(2-(4p/5))} \sum_{1 \leqq j \leqq 2^n t} \\
&\quad \times \left(\int f(x) \left(\int_A^\infty p_{t_n^j}(x-y)\,dy + 2^{-n}(2\pi t_n^j)^{-1/2} \boldsymbol{I}(x \geqq A - 2^{-n}) \right) dx \right) \\
&\leqq C(p)\varepsilon^{-2p} t^{1/2} \sum_{n \geqq n_0} 2^{n(2-(4p/5))} \\
&\quad \times \sum_{1 \leqq j \leqq 2^n t} ((f, P_t \boldsymbol{I}(A, \infty))(t_n^j/t)^{-1/2} + (t_n^j)^{-1/2}(f, (A - 2^{-n_0}, \infty))) \\
&\leqq C(p)\varepsilon^{-2p}(t \vee t^{1/2}) \sum_{n \geqq n_0} 2^{n(3-(4p/5))}(f, P_t \boldsymbol{I}(A, \infty)) \sum_{1 \leqq j \leqq 2^n t} 2^{-n}(t_n^j)^{-1/2} \\
&\leqq C(p)\varepsilon^{-2p}(t^{3/2} \vee t)(f, P_t \boldsymbol{I}(A, \infty)) \quad \text{if } p > \tfrac{15}{4}.
\end{aligned}$$

The same bound (with a different constant) holds when A^1 is replaced by A^2 provided that we take $p > \frac{25}{3}$. Define

$$A(\varepsilon) = \bigcup_{n \geqq n_0} \bigcup_{1 \leqq j \leqq 2^n A} \bigcup_{k \geqq 2^n A} A^1_{j,k,n}(\varepsilon) \cup A^2_{j,k,n}(\varepsilon).$$

Then taking $p = 10$ we have $P(A(\varepsilon)) \leqq C(t \vee t^{3/2})\varepsilon^{-20}(f, P_t \boldsymbol{I}(A, \infty))$. On the set $A^c(\varepsilon)$ we may estimate $|N_s(x)|$ for $s \leqq t$, $x \geqq A$ by an infinite sum of increments over neighbouring dyadics in the usual manner. Moreover we need at most $2t$ increments over step length 2^{-n_0} and two steps (one in space and one in time) of length 2^{-n} for $n \geqq 2$. So

$$|N_s(x)| \leqq 2t\varepsilon 2^{-1/10} + 2 \sum_{n \geqq 2} \varepsilon 2^{-n/10} \leqq c_0 \varepsilon (1 + t).$$

The set $A(c_0^{-1}\varepsilon(1+t)^{-1})$ then leads to the desired result. □

We now establish a compact support property by considering the Laplace functional of a solution, adapting the method used for super-Brownian motion in Dawson et al. [2].

Proposition 3.2 *Let u be a solution to (1) such that $R(u_0) \leqq 0$. Then for all $t \geqq 0$, $b \geqq 4t^{1/2}$*

$$P\left(\sup_{s \leqq t} R(u_s) \geqq b\right) \leqq C(t^{-1/2} \vee t^{23}) e^{-b^2/16t}.$$

Remark. By considering $1-u$ we obtain a corresponding result about L_t.

Proof. Fix $\psi: \mathbb{R} \to [0,1]$ continuous, integrable and with $(x\colon \psi(x) > 0) = (0, \infty)$. Let $\psi_b(x) = \psi(x-b)$. For $0 < a < b$ define stopping times

$$\tau_a = \inf(t \geqq 0\colon u_t(x) \geqq \tfrac{1}{2}, \exists x \geqq a), \qquad \rho_b = \inf(t \geqq 0\colon (u_t, \psi_b) > 0).$$

Fix t and let $(\phi_s^\lambda(x)\colon s \in [0,t],\, x \in \mathbb{R})$ be the unique non-negative bounded solution to

$$\begin{cases} -\dot{\phi}^\lambda = \frac{1}{2}\Delta\phi^\lambda - \frac{1}{4}(\phi^\lambda)^2 + \lambda\psi_b, \\ \phi_t^\lambda \equiv 0 \end{cases}$$

The existence and uniqueness for this equation is discussed in [3]. Comparing with the solution to the same equation without the $-\frac{1}{4}(\phi^\lambda)^2$ term shows that $\phi_s^\lambda(x) \leqq \lambda \int_0^{t-s} P_{t-s-r}\psi_b(x)\,dr$ and is thus integrable. The function $h(x) = 3(b-x)^{-2}$ solves $\frac{1}{2}\Delta h = \frac{1}{4}h^2$ on $(-\infty, b)$. Arguing as in the proof of the maximum principle shows that $\phi_s^\lambda(x) \leqq 3(b-x)^{-2}$ for all $x < b$, $s \leqq t$, $\lambda > 0$. Using the Feynman-Kac representation for ϕ^λ as in [2] Lemma 3.5 we have for any $r \in (x, b)$

$$\begin{aligned} \phi_s^\lambda(x) &\leqq 3(b-r)^{-2} P_x(\inf(q\colon B_1(q) = r) \leqq t-s) \\ &\leqq 6(b-r)^{-2} P_0(B_1(t-s) \geqq r-x) \end{aligned}$$

by the reflection principle. Supposing that $b \geqq 4t^{1/2}$, $x \leqq b - 2t^{1/2}$ we choose $r = b - t^{1/2}$ to find

$$\phi_s^\lambda(x) \leqq 6t^{-1} e^{-(b-x)^2/8t} \quad \forall s \leqq t. \tag{5}$$

Ito's formula gives

$$\begin{aligned} &d\left(\exp\left(-(u_s, \phi_s^\lambda) - \lambda \int_0^s (u_r, \psi_b)\,dr\right)\right) \\ &\quad = \exp\left(-(u_s, \phi_s^\lambda) - \lambda \int_0^s (u_r, \psi_b)\,dr\right)(|u_s(x)(1-u_s(x))|^{1/2} \phi_s^\lambda(x)\,dW_{x,s} \\ &\qquad + (u_s, -\dot{\phi}_s^\lambda - \tfrac{1}{2}\Delta\phi_s^\lambda - \lambda\psi_b) + \tfrac{1}{2}(u_s(1-u_s), (\phi_s^\lambda)^2)\,ds). \end{aligned}$$

So, using the integrability of ϕ_s^λ to show the stochastic integral is a martingale,

$$\begin{aligned} &E\left(1 - \exp\left(-(u_{t\wedge\tau_a}, \phi_{t\wedge\tau_a}^\lambda) - \lambda \int_0^{t\wedge\tau_a} (u_r, \psi_b)\,dr\right)\right) \\ &\quad = E(1 - \exp(-(u_0, \phi_0^\lambda))) \\ &\qquad + E\left(\int_0^{t\wedge\tau_a} \exp\left(-(u_s, \phi_s^\lambda) - \lambda \int_0^s (u_r, \psi_b)\,dr\right)\left(\tfrac{1}{4}u_s - \tfrac{1}{2}u_s(1-u_s), (\phi_s^\lambda)^2\right) ds\right) \\ &\quad \leqq E(1 - e^{-(u_0, \phi_0^\lambda)}) + E\left(\int_0^t (\tfrac{1}{4}u_s \boldsymbol{I}(-\infty, a), (\phi_s^\lambda)^2)\,ds\right). \end{aligned} \tag{6}$$

As $\lambda \to \infty$ so $\phi^\lambda \uparrow \phi^\infty$ with $\phi_s^\infty(x) = \infty$ for $x \geqq b, s < t$. Letting $\lambda \to \infty$ in (6) gives

$$P(\rho_b < \tau_a \wedge t) \leqq \lim_{\lambda \to \infty} E\left(1 - \exp\left(-(u_{t \wedge \tau_a}, \phi^\lambda_{t \wedge \tau_a}) - \lambda \int_0^{t \wedge \tau_a} (u_r, \psi_b)\, dr\right)\right)$$
$$\leqq E(1 - e^{-(u_0, \phi_0^\infty)}) + E\left(\int_0^t (\tfrac{1}{4} u_s \boldsymbol{I}(-\infty, a), (\phi_s^\infty)^2)\, ds\right)$$
$$\leqq \int_{-\infty}^0 \phi_0^\infty(x)\, dx + \int_0^t \int_{-\infty}^a \int_{-\infty}^0 \tfrac{1}{4} p_s(x-y)(\phi_s^\infty(x))^2\, dy\, dx\, ds.$$

Choosing $a = b/2$ and using the bounds in (5) we have for $b \geqq 4t^{1/2}$

$$P(\rho_b < \tau_{b/2} \wedge t)$$
$$\leqq \int_{-\infty}^0 6t^{-1} e^{-(b-x)^2/8t}\, dx + \int_0^t \int_{-\infty}^{b/2} \int_{-\infty}^0 p_s(x-y) 9t^{-2} e^{-(b-x)^2/4t}\, dy\, dx\, ds.$$
$$\leqq 24 b^{-1} e^{-b^2/8t} + 9t^{-1} \int_{-\infty}^{b/2} e^{-(b-x)^2/4t}\, dx$$
$$\leqq C b^{-1} e^{-b^2/16t}$$
$$\leqq C t^{-1/2} e^{-b^2/16t}. \tag{7}$$

But from Lemma 3.1a) we have for $b \geqq 4t^{1/2}$

$$P(\tau_{b/2} \leqq t) = P(P_s f(x) + N_s(x) \geqq \tfrac{1}{2}, \exists x \geqq b/2, s \leqq t)$$
$$\leqq P(P_t \boldsymbol{I}(-\infty, 0)(b/2) + N_s(x) \geqq \tfrac{1}{2}, \exists x \geqq b/2, s \leqq t)$$
$$\leqq P(N_s(x) \geqq \tfrac{1}{2} - P_0(B_1(1) \geqq 2), \exists x \geqq b/2, s \leqq t)$$
$$\leqq C(t \vee t^{22})(\boldsymbol{I}(-\infty, 0), P_t \boldsymbol{I}(b/2, \infty))$$
$$\leqq C(t \vee t^{22}) t^{1/2} e^{-b^2/8t},$$

which combined with (7) completes the proof. □

Corollary 3.3 *Let u be a solution to (1) with $u_0 \in \mathscr{C}_I$. Then the path $t \to R(u_t)$ is, almost surely, right continuous with left limits and at jump times $R(u_t) \leqq \lim_{s \uparrow t} R(u_s)$.*

Proof. We first use Lemma 3.2 to obtain a one-sided modulus of continuity for $t \to R(u_t)$. Define $t_j^n = j 2^{-n}$ and

$$\tau(j, k, n) = \inf(t \geqq t_j^n \colon R(u_t) \leqq R(u_{t_j^n}) - k 2^{-n/4}) \wedge t_{j+1}^n$$
$$A(j, k, n) = \left(\sup_{t \in [\tau(j,k,n), t_{j+1}^n]} R(u_s) \geqq R(u_{\tau(j,k,n)}) + 2^{-(n/4)+3} \right).$$

Applying Lemma 3.2 at the stopping time $\tau(j, k, n)$ shows that $P(A(j, k, n)) \leqq C 2^{n/2} \exp(-2^{n/2})$. For any K, L Borel Cantelli provides a $n_0 = n_0(K, L, \omega) < \infty$ almost surely so that $\omega \in A^c(j,k,n)$ for all $n \geqq n_0$, $0 \leqq j \leqq K2^n$, $0 \leqq k \leqq L2^{n/4}$. Also by lemma 3.2 we have

$$P\left(\bigcap_K \bigcup_L (-L/2 \leqq L(u_s) \leqq R(u_s) \leqq L/2 \text{ for all } s \leqq K)\right) = 1 .$$

Fix ω with $R(u_t) \in [-L/2, L/2]$ for all $t \leqq K$ and $n_0(K, L, \omega) < \infty$. Fix also $n \geqq n_0, j \leqq K2^n$. Choose $k \leqq L2^{n/4}$ such that

$$\inf_{t \in [t_j^n, t_{j+1}^n]} R(u_t) \in [R(u_{t_j^n}) - k2^{n/4}, R(u_{t_j^n}) - (k+1)2^{n/4}) .$$

From the definition of $A(j,k,n)$ and $\tau(j,k,n)$ we have

$$R(u_{t_{j+1}^n}) - R(u_s) \leqq 2^{-(n/4)+4} \quad \text{for all } s \in [t_j^n, t_{j+1}^n] .$$

From the definition of $A(j,0,n)$ we have

$$R(u_s) - R(u_{t_j^n}) \leqq 2^{-(n/4)+3} \quad \text{for all } s \in [t_j^n, t_{j+1}^n] .$$

These combine to show give the modulus $R(u_t) - R(u_s) \leqq C(t-s)^{1/4}$ for all $0 \leqq s \leqq t \leqq K$ with $t - s \leqq 2^{-n_0}$.

The modulus immediately implies that $\limsup_{s \downarrow t} R(u_s) \leqq R(u_t)$. The continuity of $(t,x) \to u_t(x)$ implies

$$\liminf_{s \to t} R(u_s) \geqq R(u_t) . \tag{8}$$

So $t \to R(u_t)$ is right continuous. Also if $\limsup_{s \uparrow t} R(u_s) > \liminf_{s \uparrow t} R(u_s)$ we may obtain a contradiction to the modulus of continuity. So left limit exists and the fact that jumps are backwards follows from (8). □

The last lemma in this section gives control on the width of the interface. In Mueller and Tribe [6] the moments in Lemma 2.1 are used to show that for a solution u started at $f \in \mathscr{C}_I$ the width satisfies for $\alpha \in [0,1)$

$$E(|R(u_t) - L(u_t)|^\alpha) \leqq C(f, \alpha) < \infty \quad \text{for all } t \geqq 0 . \tag{9}$$

We now combine this with Lemma 3.2 to control the width over finite time intervals.

Lemma 3.4 *For u a solution to (1) started at $f \in \mathscr{C}_I$ we have for $p \in (\frac{1}{3}, \frac{1}{2}]$, $q \in (0, 3p-1)$, $t > 0$*

$$P\left(\sup_{s \leqq t} R(u_s) - L(u_s) \geqq t^p\right) \leqq C(f, p, q) t^{-q} .$$

Proof. We may take $t \geqq 1$. Choose $\alpha \in [0,1)$ such that $1 + q < (2+\alpha)p$ and set $\beta = 1 + q - \alpha p$. Let $s_j = jt^\beta/144$. By Chebychev and the moments (9) we have $P(R(u_{s_j}) - L(u_{s_j}) \geqq t^p/3) \leqq C(f) t^{-\alpha p}$. From Lemma 3.2 we have $P(\sup_{t \in [s_j, s_{j+1}]} R(u_t) - R(u_{s_j}) \geqq t^p/3) \leqq C(p,q) t^{-1}$. A similar estimate holds for the left hand edge. Combining these gives

$$P\left(\sup_{t \in [s_j, s_{j+1}]} R(u_t) - L(u_t) \geqq t^p\right) \leqq C(f, p, q) t^{-\alpha p} .$$

The interval $[0,t]$ is contained inside the union of $Ct^{1-\beta}$ of the intervals $[s_j, s_{j+1}]$ and Boole's inequality gives the result. □

4 A single wavefront

Throughout this section u is a solution to (1) started at $f \in \mathscr{C}_I$. Define

$$v_t^{(n)}(x) = u_{n^2 t}(nx). \tag{10}$$

Lemma 4.1 *For $\phi \in C_c$ the processes $((v_t^{(n)}, \phi)\colon t \geqq 0)_{n=1,2,\dots}$ are tight.*

Proof. Rescaling the Green's function representation gives, for integrable ϕ,

$$(v_t^{(n)}, \phi) = (v_0^{(n)}, P_t\phi) + n^{1/2} \int_0^t \int |v_s^{(n)}(x)(1 - v_s^{(n)}(x))|^{1/2} P_{t-s}\phi(x)\, d\bar{W}_{x,s} \tag{11}$$

for some new white noise $\bar{W}$. Note that all terms in (11) are continuous in t. The first term on the right hand side of (11) converges to $(\boldsymbol{I}(-\infty, 0], P_t\phi)$. We shall check the Kolmogorov tightness criterion for the stochastic integral in (11). For $0 < s < t$

$$n^{1/2} \int_0^t \int |v_r^{(n)}(x)(1 - v_r^{(n)}(x))|^{1/2} P_{t-r}\phi(x)\, d\bar{W}_{x,r}$$

$$- n^{1/2} \int_0^s \int |v_r^{(n)}(x)(1 - v_r^{(n)}(x))|^{1/2} P_{s-r}\phi(x)\, d\bar{W}_{x,r}$$

$$= n^{1/2} \int_s^t \int |v_r^{(n)}(x)(1 - v_r^{(n)}(x))|^{1/2} P_{t-r}\phi(x)\, d\bar{W}_{x,r}$$

$$+ n^{1/2} \int_0^s \int |v_r^{(n)}(x)(1 - v_r^{(n)}(x))|^{1/2} (P_{t-r}\phi(x) - P_{s-r}\phi)\, d\bar{W}_{x,r}.$$

Using the moment bounds from Lemma 2.1d)

$$E\left(\left(n^{1/2} \int_s^t \int |v_r^{(n)}(x)(1 - v_r^{(n)}(x))|^{1/2} P_{t-r}\phi(x)\, d\bar{W}_{x,r}\right)^4\right)$$

$$\leqq C \|\phi\|_\infty^4 E\left(\left(\int_s^t \int n v_r^{(n)}(x)(1 - v_r^{(n)}(x))\, dr\, dx\right)^2\right)$$

$$\leqq C(\phi)(t-s) E \int_s^t \left(\int u_{n^2 r}(x)(1 - u_{n^2 r}(x))\, dx\right)^2 dr$$

$$\leqq C(\phi, f)(t-s)^2.$$

Using the bound $\|P_t\phi - P_s\phi\|_\infty \leqq \|\phi\|_\infty(|t-s|s^{-1} \wedge 1)$ we also have

$$E\left(\left(n^{1/2}\int_0^s\int |v_r^{(n)}(x)(1-v_r^{(n)}(x))|^{1/2}(P_{t-r}\phi(x) - P_{s-r}\phi)\,d\bar{W}_{x,r}\right)^4\right)$$
$$\leqq C(\phi)E\left(\left(\int_0^s\int u_{n^2r}(nx)(1-u_{n^2r}(x))\,dx(|t-s|^2(s-r)^{-2}\wedge 1)\,dr\right)^2\right)$$
$$\leqq C(\phi)(t-s)\int_0^s E\left(\left(\int u_{n^2r}(nx)(1-u_{n^2r}(x))\,dx\right)^2\right)(|t-s|^2(s-r)^{-2}\wedge 1)\,dr$$
$$\leqq C(\phi,f)(t-s)^2.$$

This checks Kolmogorov's criterion and completes the proof. □

Theorem 4.2 *a) Given $\varepsilon > 0$, $T < \infty$ then for all sufficiently large n there is a coupling of processes $(\bar{u}_t, B_t\colon t \geqq 0)$ with B a Brownian motion started at 0, $\bar{u}$ a solution to (1) started at f and*

$$P\left(\sup_{t\leqq T}|(R(u_{n^2t})/n) - B_t| \vee |(L(u_{n^2t})/n) - B_t| \geqq \varepsilon\right) \leqq \varepsilon.$$

b) If u is a solution to (1) *started at f then the processes $(R(u_{n^2t})/n\colon t \geqq 0)_{n=1,2,\ldots}$ converge in distribution to a Brownian motion started at* 0.

Proof. Part b) follows directly from part a). The key step in proving part a) is to show that the measure valued processes $(v_t^{(n)}(x)\,dx\colon t \geqq 0)_{n=1,2,\ldots}$, as defined in (10), converge in distribution as continuous M valued processes and that the limit has the law of the process

$$\mu_t(dx) = \boldsymbol{I}(x \leqq B_t)\,dx \tag{12}$$

where B_t is a standard Brownian motion started at 0.

Lemma 4.1 and the fact that $|v_t^{(n)}(x)| \leqq 1$ imply that the M valued processes $(v_t^{(n)}(x)\,dx\colon t \geqq 0)_{n=1,2,\ldots}$ are tight (see [1], 3.6.4). We extract a convergent subsequence, which we continue to label $v^{(n)}$. By changing probability space we may take measure valued processes $\mu^{(n)}$, μ with $\mu_\cdot^{(n)} \stackrel{\mathcal{D}}{=} v_\cdot^{(n)}(x)\,dx$ and $\mu^{(n)} \xrightarrow{\text{a.s.}} \mu$. Thus for any $t \geqq 0$ and ϕ continuous with compact support we have

$$\sup_{s\leqq t}|(\mu_s^{(n)},\phi) - (\mu_s,\phi)| \to 0. \tag{13}$$

Note that $\sup_{t\geqq 0}\mu_t^{(n)}(\phi) \leqq \|\phi\|_1$ a.s. Using this we may extend the convergence in (13) to continuous integrable ϕ. Note also that $\mu_0 = \boldsymbol{I}(x \leqq 0)\,dx$ a.s. Suppose $\phi_s(x)$ is smooth, bounded and $\sup_{s\in[0,t]}|\phi_s| \vee |\Delta\phi_s| \vee |\dot{\phi}_s|$ is integrable. Then

$$(\mu_s^{(n)},\phi_s) = (\mu_0^{(n)},\phi_0) + \int_0^s (\mu_r^{(n)}, \tfrac{1}{2}\Delta\phi_r + \dot{\phi}_r)\,dr + m_s^{(n)}(\phi) \quad \text{for } s \leqq t,$$

$m_s^{(n)}(\phi)$ is a continuous martingale with

$$[m^{(n)}(\phi)]_s \stackrel{\mathcal{D}}{=} \int_0^s\int n\phi_r^2(x)u_{n^2r}(nx)(1-u_{n^2r}(nx))\,dx\,dr.$$

Note that $|m_s^{(n)}(\phi)| \leqq (1+s)\|\sup_{s\in[0,t]}|\phi_s| \vee |\Delta\phi_s| \vee |\dot{\phi}_s|\|_1$. We may pass to the limit $n \to \infty$ to see that

$$(\mu_s, \phi_s) = (\mu_0, \phi_0) + \int_0^s (\mu_r, \tfrac{1}{2}\Delta\phi_r + \dot{\phi}_r)\,dr + m_s(\phi) \quad \text{for } s \leqq t$$

$$m_s(\phi) \text{ is a continuous martingale.}$$

We shall now identify the brackets process of $m_t(\phi)$ for certain ϕ. The crucial lemma is as follows.

Lemma 4.3 *Fix $\psi \geqq 0$, smooth, of compact support and with $\int \psi(x)\,dx = 1$. Let $\phi_s = P_{t-s}\psi$. Then as $n \to \infty$*

$$\int_0^t \int n\phi_s^2(x) u_{n^2s}(nx)(1-u_{n^2s}(nx))\,dx\,ds - \int_0^t \int 2\phi_s(x)\phi_s'(x) u_{n^2s}(nx)\,dx\,ds \overset{L^2}{\to} 0.$$

We delay the proof of this lemma to the end of this section.

Choose $\psi_1, \ldots, \psi_n \in C_c$ and $0 \leqq r_1 < \cdots < r_n \leqq r \leqq s$. For $\phi_s(x)$ as in Lemma 4.3

$$\begin{aligned}
&E((m_s(\phi) - m_r(\phi))^2 (\mu_{r_1}, \psi_1) \ldots (\mu_{r_n}, \psi_n)) \\
&= \lim_{n\to\infty} E((m_s^{(n)}(\phi) - m_r^{(n)}(\phi))^2 (\mu_{r_1}^{(n)}, \psi_1) \ldots (\mu_{r_n}^{(n)}, \psi_n)) \\
&= \lim_{n\to\infty} E(([m^{(n)}(\phi)]_s - [m^{(n)}(\phi)]_r)(\mu_{r_1}^{(n)}, \psi_1) \ldots (\mu_{r_n}^{(n)}, \psi_n)) \\
&= \lim_{n\to\infty} E\left(\int_r^s \int n\, u_{n^2q}(nx)(1-u_{n^2q}(nx))\, \phi_q^2(x)\,dx\,dq\,(v_{r_1}^{(n)}, \psi_1) \ldots (v_{r_n}^{(n)}, \psi_n) \right) \\
&= \lim_{n\to\infty} E\left(\int_r^s \int 2\phi_q(x)\phi_q'(x) u_{n^2q}(nx)\,dx\,dq\,(v_{r_1}^{(n)}, \psi_1) \ldots (v_{r_n}^{(n)}, \psi_n) \right) \\
&= E\left(\int_r^s (\mu_s, 2\phi_q\phi_q')\,dq\,(\mu_{r_1}, \psi_1) \ldots (\mu_{r_n}, \psi_n) \right)
\end{aligned}$$

using Lemma 4.3 in the penultimate step. This calculates the brackets process and shows that the limit point μ satisfies the following martingale problem: For all $\psi \geqq 0$, smooth, of compact support and with $\int \psi(x)\,dx = 1$

$$(\mu_s, P_{t-s}\psi) = (\boldsymbol{I}(-\infty, 0), P_t\psi) + m_s(\psi) \quad \text{for } s \leqq t,$$

$$m_s(\psi) \text{ is a continuous martingale with}$$

$$[m_s(\psi)] = \int_0^s (\mu_r, 2P_{t-r}\psi P_{t-r}\psi')\,dr \quad \text{for } s \leqq t.$$

Applying Ito's formula shows that μ_t given by (12) also satisfies this martingale problem. It remains to show that uniqueness of solutions holds. By polarisation show that

$$[m_s(\psi), m_s(\tilde{\psi})] = \int_0^s (\mu_r, P_{t-r}\psi P_{t-r}\tilde{\psi}' + P_{t-r}\tilde{\psi} P_{t-r}\psi')\,dr.$$

Applying Ito's formula gives

$$E((\mu_t,\psi_1)\dots(\mu_t,\psi_k)) = \prod_{i=1}^{k}(I(-\infty,0),P_t\psi_i)$$

$$+ \sum_{i,j=1;i\neq j}^{k} E\left(\int_0^t (\mu_s, P_{t-s}\psi_i P_{t-r}\psi_j' + P_{t-r}\psi_j P_{t-r}\psi_i') \prod_{k\neq i,j}(\mu_s,P_{t-s}\psi_k)\,ds\right). \tag{14}$$

The identity $E((\mu_t,\psi)) = (I(-\infty,0),P_t\psi)$ can be extended by a monotone class argument to hold for all non-negative ψ. Similarly using induction and (14) the moments $E((\mu_t,\psi_1)\dots(\mu_t,\psi_k))$ are determined. Since $\mu_t(\psi) \leqq \|\psi\|_1$ these moments determine the distribution of μ_t and, as usual, uniqueness of the one dimensional distributions implies uniqueness for the martingale problem. This completes the proof of convergence as measure valued processes.

To obtain the coupling stated in part a) we fix $\varepsilon \in (0,1)$, $T<\infty$ and choose $k \geqq 1$ such that $P_{(0)}(\sup_{t\leqq T} B_t^1 > k) \leqq \varepsilon$. We also choose $0 \leqq \phi \in C_c$ with $(\phi>0)=(0,1)$ and define $\psi_k(x) = ((x+k+1)\wedge(k+1-x)\wedge 1)_+$, an approximation to $I(-k,k)$. Since $\mu^{(n)} \overset{\mathscr{D}}{=} v^{(n)}$ it has a jointly continuous density $\bar{v}_t^{(n)}(x)$ and defining $\bar{u}_t(x) = \bar{v}_{t/n^2}^{(n)}(x/n)$ then, by extending the probability space to define a suitable white noise, $\bar{u}$ is a solution to (1) started at f. The process $\bar{B}_t = \sup(x\colon \mu_t((x,\infty))>0)$ has the finite dimensional distributions of a Brownian motion. Letting $B_t = \limsup(\bar{B}_s\colon s\leqq t, s\in Q, s\to t)$ produces a Brownian motion started at 0 such that $\mu_t(x)\,dx = I(x\leqq B_t)\,dx$, $\forall t\geqq 0$. We consider sufficiently large n so that $R(\bar{v}_0^{(n)}), L(\bar{v}_0^{(n)}) \in [-1,1]$. Then

$$P\left(\sup_{t\leqq T} R(\bar{v}_t^{(n)}) > k\right) = P\left(\sup_{t\leqq T}(\mu^{(n)},\phi(\cdot - k)) > 0\right)$$

$$\to P\left(\sup_{t\leqq T}(\mu,\phi(\cdot - k))>0\right)$$

$$= P_0\left(\sup_{t\leqq T} B_t > k\right) \leqq \varepsilon .$$

So for sufficiently large n, using Lemma 3.4 and (13), we have

$$P\left(\sup_{t\leqq T}|R(\bar{u}_{n^2t})/n| \vee |L(\bar{u}_{n^2t})/n| \vee |B_t| \geqq k\right) \leqq 5\varepsilon,$$

$$P\left(\sup_{t\leqq T}|R(\bar{u}_{n^2t}) - L(\bar{u}_{n^2t})|/n \geqq \varepsilon\right) \leqq \varepsilon,$$

$$P\left(\sup_{t\leqq T}|(\bar{v}_t^{(n)},\psi_k) - (I(x\leqq B_t),\psi_k)| \geqq \varepsilon\right) \leqq \varepsilon .$$

On the complement of the union of these three sets we have

$$\begin{aligned}R(\bar{u}_{n^2t})/n &= R(\bar{v}_t^{(n)})\\ &\leqq L(\bar{v}_t^{(n)}) + \varepsilon\\ &= (I(x \leqq L(\bar{v}_t^{(n)})), \psi_k) - (k + \tfrac{1}{2}) + \varepsilon\\ &\leqq (\bar{v}_t^{(n)}, \psi_k) - (k + \tfrac{1}{2}) + \varepsilon\\ &\leqq (\boldsymbol{I}(x \leqq B_t), \psi_k) - (k + \tfrac{1}{2}) + 2\varepsilon\\ &= B_t + 2\varepsilon .\end{aligned}$$

Similarly $L(\bar{u}_{n^2t})/n \geqq B_t - 2\varepsilon$ which gives the desired coupling. □

Proof of Lemma 4.3 Fix ψ, ϕ_s as in the statement of the lemma. Let X^1, X^2 be independent variables with density $\psi(x)$ and independent of the Brownian motion $\underline{B}$. Define $f^{(n)}(x) = f(nx)$.

Lemma 4.4

$$\left| E\left(\int_0^t \int n\phi_s^2(x) u_{n^2s}(nx)(1 - u_{n^2s}(nx))\, dx\, ds \right) - 2\int_0^t \int P_s f(x) \phi_s(x) \phi_s'(x)\, dx\, ds \right|$$
$$\leqq 2\iint P_t\psi(x) P_t\psi(y) f^{(n)}(x)(1 - f^{(n)}(y)) \boldsymbol{I}(y \leqq x)\, dy\, dx + |e(n,t)|$$

where $e(n,t)$ is independent of f and $1 \geqq |e(n,t)| \to 0$ as $n \to \infty$.

Proof of Lemma 4.4 Using duality as in Lemma 2.1a) we have

$$\begin{aligned}&E\left(\int_0^t \int n u_{n^2s}(nx)(1 - u_{n^2s}(nx)) \phi_s^2(x)\, dx\, ds \right)\\ &= E\left(\int_0^t \int n u_{n^2(t-s)}(nx)(1 - u_{n^2(t-s)}(nx)) \phi_{(t-s)}^2(x)\, dx\, ds \right)\\ &= \int_0^t \int \phi_{t-s}^2(x) E_{\underline{0}}(f(nx + B^1_{n^2(t-s)})(1 - f(nx + B^2_{n^2(t-s)}))ne^{-l_{1,2}(n^2(t-s))/2})\, dx\, ds\\ &= \int_0^t \int (P_s\psi(x))^2 E_{\underline{0}}(f^{(n)}(x + B^1_{(t-s)})(1 - f^{(n)}(x + B^2_{(t-s)}))ne^{-nl_{1,2}(t-s)/2})\, dx\, ds\\ &= E_{\underline{0}}\left(\int_0^t f^{(n)}(X^1 + B^1_t)(1 - f^{(n)}(X^2 + B^2_t)) \right.\\ &\qquad \left. \times ne^{-n(L_t^0 - L_s^0)(X^1 + B^1 - X^2 - B^2)/2} \tfrac{1}{2} dL_s^0(X^1 + B^1 - X^2 - B^2) \right)\\ &= E_{\underline{0}}(f^{(n)}(X^1 + B^1_t)(1 - f^{(n)}(X^2 + B^2_t))(1 - e^{-nL_t^0(X^1 + B^1 - X^2 - B^2)/2}))\\ &= E_{\underline{0}}(f^{(n)}(X^1 + B^1_t)(1 - f^{(n)}(X^2 + B^2_t))\boldsymbol{I}(\tau \leqq t)) + e(n,t), \qquad (15)\end{aligned}$$

where $\tau = \inf(t\colon X^1 + B^1_t = X^2 + B^2_t)$ and

$$|e(n,t)| \leqq E(|1 - e^{-nL_t^0(X^1 + B^1 - X^2 - B^2)/2} - \boldsymbol{I}(\tau \leqq t)|) \to 0 \quad \text{as } n \to \infty .$$

Let $f_0(x) = \boldsymbol{I}(x \leqq 0)$. Note that $f^{(n)} \to f_0$ as $n \to \infty$.

$$\begin{aligned}
&E_{\underline{0}}(f_0(X^1 + B_t^1)(1 - f_0(X^2 + B_t^2))\boldsymbol{I}(\tau \leqq t)) \\
&\quad = P_{\underline{0}}(X^1 + B_t^1 \leqq 0, X^2 + B_t^2 \geqq 0, \tau \leqq t)) \\
&\quad = 2P_{\underline{0}}(X^1 + B_t^1 \leqq 0, X^2 + B_t^2 \geqq 0, X^2 \leqq X^1) \quad \text{(reflection principle)} \\
&\quad = 2P_{\underline{0}}(B_t^2 \leqq X^2 \leqq X^1 \leqq B_t^1) \\
&\quad = E_{\underline{0}}\left(\left(\int_{B_t^2}^{B_t^1} \psi(z)\,dz\right)^2 \boldsymbol{I}(B_t^1 \geqq B_t^2)\right) \\
&\quad = \tfrac{1}{2}E_{\underline{0}}\left(\left(\int_{B_t^2}^{B_t^1} \phi_t(z)\,dz\right)^2\right).
\end{aligned}$$

This is a smooth function of (t, B_t^1, B_t^2) and may be expanded by Ito's formula. This leads, after some calculation, to

$$\begin{aligned}
&= E_{\underline{0}}\left(\int_0^t \tfrac{1}{2}\phi_s^2(B_s^1) + \tfrac{1}{2}\phi_s^2(B_s^2)\,ds\right) \\
&= \int_0^t \int p_s(x)\phi_s^2(x)\,dx\,ds \\
&= 2\int_0^t \int P_s f_0(x)\phi_s(x)\phi_s'(x)\,dx\,ds.
\end{aligned}$$

By replacing $\psi(y)$ by $\psi(y + a)$ we see that the same equality still holds if f_0 is replaced by $f_a(x) = \boldsymbol{I}(x \leqq a)$. For $f \in \mathscr{C}_I$ smooth we have

$$\begin{aligned}
&2\int_0^t \int P_s f(x)\phi_s(x)\phi_s'(x)\,dx\,ds \\
&\quad = -2\int_{-\infty}^{\infty} f'(a)\int_0^t \int P_s f_a(x)\phi_s(x)\phi_s'(x)\,dx\,ds\,da. \\
&\quad = -\int_{-\infty}^{\infty} f'(a)E_{\underline{0}}(f_a(X^1 + B_t^1)(1 - f_a(X^2 + B_t^2))\boldsymbol{I}(\tau \leqq t))\,da \\
&\quad = -E_{\underline{0}}\left(\int_{-\infty}^{\infty} f'(a)\boldsymbol{I}(X^1 + B_t^1 \leqq a \leqq X^2 + B_t^2, \tau \leqq t)\,da\right) \\
&\quad = E_{\underline{0}}((f(X^1 + B_t^1) - f(X^2 + B_t^2))\boldsymbol{I}(\tau \leqq t, X^2 + B_t^2 \geqq X^1 + B_t^1)).
\end{aligned}$$

The same equality must then also hold without the smoothness assumption on f. Then

$$\begin{aligned}
&\left|E_{\underline{0}}(f(X^1 + B_t^1)(1 - f(X^2 + B_t^2))\boldsymbol{I}(\tau \leqq t)) - 2\int_0^t \int P_s f(x)\phi_s(x)\phi_s'(x)\,dx\,ds\right| \\
&\quad = E_{\underline{0}}(f(X^2 + B_t^2)(1 - f(X^1 + B_t^1))\boldsymbol{I}(\tau \leqq t, X^2 + B_t^2 \geqq X^1 + B_t^1)) \\
&\qquad + E_{\underline{0}}(f(X^1 + B_t^1)(1 - f(X^2 + B_t^2)\boldsymbol{I}(\tau \leqq t, X^2 + B_t^2 < X^1 + B_t^1))
\end{aligned}$$

$$\leqq 2\mathrm{E}_0(f(X^1+B_t^1)(1-f(X^2+B_t^2))I(X^2+B_t^2 \leqq X^1+B_t^1))$$
$$= 2\iint P_t\psi(x)P_t\psi(y)f(x)(1-f(y))I(y \leqq x)\,dy\,dx\,. \tag{16}$$

Combining (15) and (16) completes the proof of Lemma 4.4. □

We can now finish the proof of Lemma 4.3. Let $\mathscr{F}_t = \sigma(u_s\colon s \leqq t)$.

$$\left\| \int_0^t \int \phi_s^2(x) n u_{n^2 s}(nx)(1-u_{n^2 s}(nx))\,dx\,ds - 2\phi_s(x)\phi_s'(x)u_{n^2 s}(nx)\,dx\,ds \right\|_2$$
$$= 2E\int_0^t \left(\int n\phi_s^2(x)u_{n^2 s}(nx)(1-u_{n^2 s}(nx)) - 2\phi_s(x)\phi_s'(x)u_{n^2 s}(nx)\,dx \right)$$
$$\times \left(\int_s^t \int n\phi_r^2(y)u_{n^2 r}(ny)(1-u_{n^2 r}(ny)) - 2\phi_r(y)\phi_r'(y)u_{n^2 r}(ny)\,dy\,dr \right) ds$$
$$\leqq 2E\int_0^t Z_s E\left(\int_s^t \int n\phi_r^2(y)u_{n^2 r}(ny)(1-u_{n^2 r}(ny)) \right.$$
$$\left. - 2\phi_r(y)\phi_r'(y)u_{n^2 r}(ny)\,dy\,dr \,\middle|\, \mathscr{F}_{n^2 s} \right) ds\,, \tag{17}$$

where $0 \leqq Z_s \leqq C(\phi)(1+\int u_{n^2 s}(x)(1-u_{n^2 s}(x))\,dx)$ is square integrable by Lemma 2.1d). We use the Markov property and Lemma 4.4 to bound the conditional expectation in (17) by

$$2\iint P_{t-s}\psi(x)P_{t-s}\psi(y)u_{n^2 s}(nx)(1-u_{n^2 s}(ny))I(y \leqq x)\,dx\,dy + e(n,t-s)\,. \tag{18}$$

Note that both terms in (18) are bounded. Substituting the bound (18) into (17) will produce a term that vanishes as $n \to \infty$ provided that the terms in (18) converge to zero in probability as $n \to \infty$. This is immediate for $e(n,t-s)$ by Lemma 4.4. For the first term in (18) we have, using the duality again,

$$E\left(\iint P_{t-s}\psi(x)P_{t-s}\psi(y)u_{n^2 s}(nx)(1-u_{n^2 s}(ny))I(y \leqq x)\,dx\,dy \right)$$
$$= \iint P_{t-s}\psi(x)P_{t-s}\psi(y)I(y \leqq x)E_{(nx,ny)}(f(B_{n^2 s}^1)(1-f(B_{n^2 s}^2))e^{-l_{1,2}(n^2 s)/2})\,dx\,dy$$
$$= \iint P_{t-s}\psi(x)P_{t-s}\psi(y)I(y \leqq x)E_{(x,y)}(f^{(n)}(B_s^1)(1-f^{(n)}(B_s^2))e^{-nl_{1,2}(s)/2})\,dx\,dy$$
$$\leqq \iint P_{t-s}\psi(x)P_{t-s}\psi(y)I(y \leqq x)$$
$$\times E_{(x,y)}(I(B_s^1 \leqq R(f)/n, B_s^2 \geqq L(f)/n)e^{-nl_{1,2}(s)/2})\,dx\,dy$$
$$\leqq \iint P_{t-s}\psi(x)P_{t-s}\psi(y)I(y \leqq x)(E_{(x,y)}(I(\tau_{1,2} < s)e^{-nl_{1,2}(s)/2})$$
$$+ P_{(x)}(B_s^1 \in [L(f)/n, R(f)/n]))\,dx\,dy\,.$$

where $\tau_{1,2} = \inf(t\colon B_t^1 = B_t^2)$. Now the integral converges to zero by the dominated convergence theorem which finishes the proof of Lemma 4.3. □

5 Multiple wavefronts

In this section we fix $f_i \in \mathscr{C}_I$, $i = 1, \ldots, 2l$ and $a_1 < a_2 < \cdots < a_{2l} \in R$. We consider the initial conditions

$$f^n(x) = \sum_{i=1}^{2l} (-1)^i f_i(x - na_i).$$

We consider only n large enough that $na_k + R(f_k) < na_{k+1} + L(f_{k+1})$ for $k = 1, \ldots, 2l - 1$ and so in particular $f^n \in \mathscr{C}$. This section contains the proof of the following result.

Theorem 5.1 *Let u^n be a solution to (1) started at f^n. Set $v_t^n(x) = u^n_{n^2 t}(nx)$. Then $v^n_\cdot(x)\,dx$ converge in distribution as $n \to \infty$ as continuous M valued processes. The limit has the law of*

$$v_t^\infty(x)\,dx = \sum_{i=1}^{2l} (-1)^i \boldsymbol{I}(x \leqq X_i(t))\,dx, \tag{19}$$

where $(X_1, \ldots, X_{2l})$ is a system of annihilating Brownian motions started at $(a_1, \ldots, a_{2l})$. (Here we have the convention that $X_i(t) = -\infty$ for values of t after the annihilation of particle i).

We have been unable to follow the method of proof used in Theorem 4.2 since we cannot find a suitably simple martingale problem for the system of annihilating Brownian motions. Instead we use the following argument: the $2l$ wavefronts move independently until they begin to overlap. For large n the positions of the wavefronts move approximately as Brownian motions so with large probability the first collision will be between exactly two wavefronts and the others will still be separated by a distance $O(n)$. By Lemma 3.4 the colliding wavefronts will have total width $O(n^{1-\eta})$ for some $\eta > 0$. By Lemma 5.2 below these two colliding wavefronts will die in time $O(n^{2-2\eta})$ which is too quick for the other wavefronts to have interfered or for another collision to have occurred. After the first annihilation we are left with $2l - 2$ wavefronts and the argument can be repeated.

Lemma 5.2 *For all solutions to (1) started at $f \leqq \boldsymbol{I}(0, M)$ and all $t \geqq 3$*

$$P((u_t, 1) > 0) \leqq CMt^{-1/2}.$$

Proof. The proof is in two steps. First we use moment estimates to show that the total mass $(u_s, 1)$ is likely to be small at time $s = t - 2$. Then we estimate the Laplace transform $E(\exp(-\lambda(u_t, 1)))$ and letting $\lambda \to \infty$ bound the probability of complete extinction.

Define a smoothed density $\bar{u}_t(x) = \int_{-1/2}^{1/2} u_t(x + z)\,dz$ and set $I_t = (\bar{u}_t(1 - \bar{u}_t), 1)$. Note that $\bar{u}_t(x)$ is zero for all large $|x|$ by the compact support property. Also, $|\bar{u}_t'(x)| \leqq 1$ for all x and if there exists x with $\bar{u}_t(x) \geqq \frac{1}{2}$ this implies that $I_t \geqq \frac{1}{8}$. Conversely if $\bar{u}_t(x) \leqq \frac{1}{2}$ for all x then

$$(u_t, 1) = (\bar{u}_t, 1) \leqq 2(\bar{u}_t(1 - \bar{u}_t), 1) = 2I_t.$$

Hence $(u_t, 1)I(I_t \leqq \frac{1}{8}) \leqq 2I_t$. The moments in Lemma 2.1 b). imply that $E(I_t) \leqq CMt^{-1/2}$ which gives control on the total mass at time t and completes the first step.

Let $\tau = \inf(s \geqq t-1 \colon \sup_x u_s(x) \geqq \frac{1}{2})$. The process $\exp(-4(t-s)^{-1}(u_s, 1))$ for $s \in [0, t)$ has drift

$$\exp(-4(t-s)^{-1}(u_s,1))(-4(t-s)^{-2}(u_s,1) + 8(t-s)^{-2}(u_s(1-u_s),1))$$

which is nonnegative for $s \in [t-1, \tau \wedge t)$. So

$$\begin{aligned}
&P((u_t,1)=0) + P(\tau < t, I_{t-2} \leqq \tfrac{1}{8})\\
&\quad \geqq P((\bar{u}_t,1)=0, \tau \geqq t, I_{t-2} \leqq \tfrac{1}{8})\\
&\qquad + E(\exp(-4(t-\tau)^{-1}(u_\tau,1))I(\tau < t, I_{t-2} \leqq \tfrac{1}{8}))\\
&\quad \geqq \lim_{s \to t} E(\exp(-4(t-(\tau \wedge s))^{-1}(u_{\tau \wedge s},1))I(I_{t-2} \leqq \tfrac{1}{8}))\\
&\quad \geqq E(\exp(-4(u_{t-1},1))I(I_{t-2} \leqq \tfrac{1}{8})).
\end{aligned}$$

Rearranging gives

$$\begin{aligned}
&P((u_t,1) > 0)\\
&\quad \leqq P(I_{t-2} \geqq \tfrac{1}{8}) + E((1-\exp(-4(u_{t-1},1)))I(I_{t-2} \leqq \tfrac{1}{8})) + P(\tau < t, I_{t-2} \leqq \tfrac{1}{8})\\
&\quad \leqq P(I_{t-2} \geqq \tfrac{1}{8}) + E(4(u_{t-2},1)I(I_{t-2} \leqq \tfrac{1}{8})) + P(\tau < t, I_{t-2} \leqq \tfrac{1}{8}). \qquad (20)
\end{aligned}$$

Note that on $(I_{t-2} \leqq \frac{1}{8}) \subseteq ((u_{t-2}, 1) \leqq \frac{1}{4})$ we have $P_s u_{t-2}(x) \leqq \frac{1}{4}$ for all $s \geqq 1$. So

$$\begin{aligned}
P(\tau < t, I_{t-2} \leqq \tfrac{1}{8}) &\leqq P(|u_{t-2+s}(x) - P_s u_{t-2}(x)| \geqq \tfrac{1}{4}, \exists s \leqq 2, x \in \mathbb{R}, I_{t-2} \leqq \tfrac{1}{8})\\
&\leqq CE((u_{t-2},1)I(I_{t-2} \leqq \tfrac{1}{8}))
\end{aligned}$$

by Lemma 3.1. Combining this with (20) and step one gives

$$\begin{aligned}
P((u_t,1) > 0) &\leqq P(I_{t-2} \geqq \tfrac{1}{8}) + CE((u_{t-2},1)I(I_{t-2} \leqq \tfrac{1}{8}))\\
&\leqq CE(I_{t-2})\\
&\leqq CM(t-2)^{-1/2}. \qquad \square
\end{aligned}$$

Proof of Theorem 5.1. Given $\varepsilon_0 > 0, T < \infty$, $m \geqq 1, \phi_1, \ldots \phi_m \in C_c$ with $\|\phi_i\|_\infty \leqq 1$ we shall show, for sufficiently large n, there exists a solution u to (1) started at f^n and a process v^∞ arising from annihilating Brownian motions $(X_1, \ldots, X_{2l})$ as in (19) such that

$$P\left(\sup_{t \leqq T} |(v_t^n, \phi_j) - (v_t^\infty, \phi_j)| \geqq \varepsilon_0, \exists j = 1, \ldots, m\right) \leqq \varepsilon_0. \qquad (21)$$

This implies the desired convergence in distribution.

Given $\varepsilon > 0$, for sufficiently large n we may, by Theorem 4.2, find a probability space with the following variables: independent solutions $(\bar{u}^k\colon k = 1, \ldots, 2l)$ to

(1), started at f_k, driven by independent white noises W^k and Brownian motions $\bar{B}^k$ such that $P(A^c) \leqq \varepsilon$ where

$$A = \{ |n^{-1}R(\bar{u}^k_{n^2t}) - \bar{B}^k_t| \vee |n^{-1}L(\bar{u}^k_{n^2t}) - \bar{B}^k_t| \leqq \varepsilon \\ \text{for } k = 1, \ldots, 2k, t \leqq T \}. \tag{22}$$

We fix such an n and solutions ($\bar{u}^k$: $k = 1, \ldots, 2l$) (suppressing the dependence on n). Define for $k = 1, \ldots, 2l$

$$u^k_t(x) = \bar{u}^k_t(x - na_k), \qquad B^k_t = \bar{B}^k_t + a_i .$$

We now construct our solution u to (1) started at f^n by using the processes (u^k: $k = 1, \ldots, 2l$) as the basic ingredients. We seem to need a fair amount of notation alas. We shall define stopping times $T_1, \ldots, T_l$ that mark the times of successive collisions and subsets $(1, \ldots, 2l) = S_0 \supset S_1 \supset \cdots \supset S_l = \emptyset$ that list the labels of the remaining wavefronts after each collision. S_k will have $2l - 2k$ elements that we list in increasing order as $s(k,1) < \cdots < s(k, 2l - 2k)$. Define $T_0 = 0$ and for $k = 1, \ldots, l$

$$\begin{aligned}
T^{k,i} &= \inf(t \geqq T^{k-1}: R_t(u^{s(k-1,i)}) = L_t(u^{s(k-1,i+1)})) \quad \text{for } i \leqq |S_{k-1}| - 1 \\
T^k &= \inf(T^{k,i}: i = 1, \ldots, 2l - 2k + 1) \\
J^k &= \inf(i: T^k = T^{k,i}) \\
a(k) &= s(k-1, J^k) \\
b(k) &= s(k-1, J^k + 1) \\
S_k &= S_{k-1} - \{a(k), b(k)\} .
\end{aligned}$$

When a collision occurs we shall use a new independent solution of (1) to follow the annihilation of the two colliding wavefronts. For $k = 1, \ldots, l$ let $\bar{w}^k$ be solutions to (1) driven by an independent white noises $\bar{W}^k$ and with initial conditions

$$\bar{w}^k_0 = u^{b(k)}_{T^k} - u^{a(k)}_{T^k} .$$

Let $w^k_{t-T^k} = \bar{w}^k_t$ for $t \geqq T^k$, $k = 1, \ldots, l$ and let $w^0 \equiv 0$. Fixing $\varepsilon \in (0,1]$ we plan that the colliding pairs of wave fronts die out in time $n^2\varepsilon$. So we define a process $\bar{u}_t$ taking the values

$$\begin{cases}
\sum_{j=1}^{2l-2k}(-1)^{s(k,j)} u^{s(k,j)}_t + (-1)^{b(k)} w^k_t & \text{on } [T^k, (T^k + n^2\varepsilon) \wedge T^{k+1}) \text{ for } 0 \leqq k \leqq l, \\
\sum_{j=1}^{2l-2k}(-1)^{s(k,j)} u^{s(k,j)}_t & \text{on } [(T^k + n^2\varepsilon) \wedge T^{k+1}, T^{k+1}),\ 0 \leqq k \leqq l-1, \\
0 & \text{on } [T^l + n^2\varepsilon, \infty).
\end{cases}$$

The process $\bar{u}$ will be the desired solution on the set where nothing goes wrong. We shall modify the definition whenever the annihilating pairs of wavefronts live too long ($S^k_2 > T^k + n^2\varepsilon$ below), collide with another wavefront ($S^k_1 < T^k + n^2\varepsilon$) or when another collision occurs during their annihilation ($T^{k+1} < T^k + n^2\varepsilon$). Define

$$S_1^k = \inf(t \geqq T^k\colon R(w_t^k) \geqq L(u_t^{s(k-1,J^k+2)}) \text{ or } L(1-w_t^k) \leqq R(u_t^{s(k-1,J^k-1)})),$$

$$S_2^k = \inf(t \geqq T^k\colon (w_t^k, 1) = 0),$$

$$S = \inf(S_1^k \wedge T^{k+1}\colon S_1^k \wedge T^{k+1} < T^k + n^2\varepsilon) \wedge \inf(T^k + n^2\varepsilon\colon T^k + n^2\varepsilon < S_2^k).$$

(We need to define $R(u_t^{s(k-1,-1)}) = -\infty$, $L_t(u_t^{s(k-1,2l-2k+3)}) = \infty$ to ensure S_1^k is well defined).

The stopping time S is the first time something goes wrong. Let $\tilde{u}$ be a solution to (1) driven by yet another independent white noise $\tilde{W}$ and with starting condition $\tilde{u}_0 = \bar{u}_S$. Then we define

$$u_t = \bar{u}_t I(t < S) + \tilde{u}_{t-S} I(t \geqq S).$$

Then it is possible to check that u is a solution to (1) started at f^n.

Let $(X_t^k\colon k = 1, \ldots, 2l)$ be the system of annihilating Brownian motions induced by $(B^k\colon k = 1, \ldots, 2l)$ and let v^∞ be the induced measure valued process by the recipe (19). Also let $v_t^n(x) = u_{n^2t}(nx)$. We shall now check that (21) is satisfied which will finish the proof.

We define various good sets:

$$B_1 = \{(w^k_{T^k+n^2\varepsilon}, 1) = 0, \quad \forall k = 1, \ldots, l\},$$

$$B_2 = \{(R(w_t^k) - R(w_{T^k}^k)) \vee (L(1 - w_{T^k}^k) - L(1 - w_t^k)) \leqq n\varepsilon$$
$$\forall k = 1, \ldots, l, \quad \forall t \in [T^k, T^k + n^2\varepsilon]\},$$

$$B = B_1 \cap B_2.$$

Define $S^{i,j} = \inf(t \geqq 0\colon B_t^j - B_t^i \leqq 2\varepsilon)$ and $\bar{S}^{i,j} = \inf(t \geqq 0\colon B_t^j - B_t^i \leqq -2\varepsilon)$ and

$$C_1 = \{[S^{i,j}, \bar{S}^{i,j} + \varepsilon] \text{ are disjoint intervals for all } 1 \leqq i < j \leqq 2l\},$$

$$C_2 = \{|B_t^k - B_s^i| \wedge |B_t^k - B_s^j| > 3\varepsilon, \quad \forall k \neq i,j, \forall s,t \in [S^{i,j}, \bar{S}^{i,j} + \varepsilon],$$
$$\forall i,j\colon 1 \leqq i < j \leqq 2l\},$$

$$C_3 = \{|B_t^i - B_t^j| \leqq \varepsilon^{1/3}, \quad \forall t \in [S^{i,j}, \bar{S}^{i,j} + \varepsilon], \forall i,j\colon 1 \leqq i < j \leqq 2l\},$$

$$C = C_1 \cap C_2 \cap C_3.$$

To prove (21) it is enough to prove the following two claims.

Claim I. On the set $A \cap B \cap C$ we have $|(v_t^n, \phi) - (v_t^\infty, \phi)| \leqq C\varepsilon^{1/3}$ for $t \leqq T$ and $\phi \in (\phi_1, \ldots, \phi_m)$.

Claim II. $\lim_{\varepsilon \to 0} \limsup_{n \to \infty} P(A^c \cup B^c \cup C^c) = 0$.

We choose ε small so that $|a_i - a_{i-i}| > 2\varepsilon$. Thus at time zero, on the set A, the wavefronts are separated and in the same order as the Brownian motions. On the set A we have $n^{-2}T^{i,j} \in [S^{i,j}, \bar{S}^{i,j}]$. So on $A \cap C_1$ we may match the sequence of collisions $T^1, \ldots, T^l$ exactly with the successive collisions in the annihilating Brownian motions. In particular the kth annihilation is between the Brownian particles $B^{a(k)}, B^{b(k)}$ and occurs during the interval $[S^{a(k),b(k)}, \bar{S}^{a(k),b(k)}]$.

We now check Claim I. On $A \cap C_1$ we have $T^{k+1} \geqq T^k + n^2\varepsilon$. On B_1 we have $\inf(T^k + n^2\varepsilon\colon T^k + n^2\varepsilon < S_2^k) = \infty$. On $A \cap C_2 \cap B_2$ we have for $t \in [T^k, T^k + n^2\varepsilon]$, $t \leqq T$

$$\begin{aligned} n^{-1}R(w_t^k) &\leqq n^{-1}R(w_{T^k}^k) + \varepsilon \\ &= n^{-1}R(u_{T^k}^{b(k)}) + \varepsilon \\ &\leqq B_{n^{-2}T^k}^{b(k)} + 2\varepsilon \\ &< B_{n^{-2}t}^{s(k-1, J^k+2)} - \varepsilon \\ &\leqq n^{-1}L(u_t^{s(k-1, J^k+2)}), \end{aligned}$$

and similarly $L(1 - w_t^k) > R(u_t^{s(k-1, J^k-1)})$. Thus $S_1^k > T^k + n^2\varepsilon$ and this paragraph has then checked that on $A \cap B \cap C$ we have $S = \infty$ and the solution u agrees with the process $\bar{u}$ constructed from the independent parts $(u^j\colon j = 1, \ldots, 2l)$.

We work now on the set $A \cap B \cap C$. Fix ϕ with $\|\phi\|_\infty \leqq 1$. For $t \in [\bar{S}^{a(k),b(k)} + \varepsilon, S^{a(k+1),b(k+1)}]$, $k = 0, \ldots, l-1$ we have

$$\begin{aligned} (v_t^n, \phi) &= (u_{n^2t}(n\,\cdot), \phi) \\ &= (\bar{u}_{n^2t}(n\,\cdot), \phi) \\ &= \sum_{j=1}^{2l-2k} (-1)^{s(k,j)} (u_{n^2t}^{s(k,j)}(n\,\cdot), \phi) \\ &= \sum_{j=1}^{2l-2k} (-1)^{s(k,j)} (\bar{u}_{n^2t}^{s(k,j)}(n\,\cdot), \phi(\cdot + na_{s(k,j)})). \end{aligned}$$

Also for such t

$$(v_t^\infty, \phi) = \sum_{j=1}^{2l-2k} (-1)^{s(k,j)} (I(x \leqq \bar{B}_t^{s(k,j)}), \phi(\cdot + na_{s(k,j)})).$$

So $|(v_t^\infty, \phi) - (v_t^n, \phi)| \leqq 2l\varepsilon$ by the definition of A.

For $t \in [S^{a(k),b(k)}, \bar{S}^{a(k),b(k)} + \varepsilon]$, $k = 0, \ldots, l$ there are two possible extra errors:

$$\begin{aligned} |(w_{n^2t}^k(n\,\cdot), \phi)| &\leqq n^{-1}(R(w_{n^2t}^k) - L(1 - w_{n^2t}^k)) \\ &\leqq 2\varepsilon + n^{-1}(R(w_{T^k}^k) - L(1 - w_{T^k}^k)) \\ &= 2\varepsilon + n^{-1}(R(u_{T^k}^{b(k)}) - L(u_{T^k}^{a(k)})) \\ &\leqq 4\varepsilon + (B_{n^{-2}T^k}^{b(k)} - B_{n^{-2}T^k}^{a(k)}) \\ &\leqq \varepsilon^{1/3} + 4\varepsilon \end{aligned}$$

and

$$\left| \int_{X_t^{a(k)}}^{X_t^{b(k)}} \phi(x)\,dx \right| \leqq |X_t^{b(k)} - X_t^{a(k)}| \leqq \varepsilon^{1/3},$$

where in both bounds we use the definition of C_3. This proves Claim I. To prove Claim II we fix $\frac{1}{2} > \bar{p} > p > \frac{1}{3}$ and $0 < q < 3p - 1$. Define

$$\tilde{B}_1 = \{(w^k_{T^k + n^{4p}}, 1) = 0, \forall k = 1, \dots, l\}$$

$$B_3 = \{R(\bar{u}^k_t) - L(\bar{u}^k_t) \leqq n^{2p} T^p, \forall k = 1, \dots, 2l, t \leqq n^2 T\}.$$

From Lemma 3.4 we have $P(B_3) \leqq C((f_i)_i, p, q, T) n^{-2q}$. On the set B_3 we have $R(w^k_{T^k}) - L(w^k_{T^k}) \leqq 2n^{2p} T^p$ so from lemma 5.2 we find $P(\tilde{B}^c_1 \cap B_3) \leqq C(T) n^{2(p-\bar{p})}$. From Lemma 3.2 we have $P(B^c_2 \cap \tilde{B}_1) \leqq C(l) n^{92\bar{p}}$ $\exp(-\varepsilon^2 n^{2(1-2\bar{p})}/16)$. Then since $\tilde{B}_1 \subseteq B_1$ for large n we have

$$\limsup_{n\to\infty} P(B^c) \leqq \limsup_{n\to\infty} P(B^c_3) + P(\tilde{B}^c_1 \cap B_3) + P(B^c_2 \cap \tilde{B}_1) = 0.$$

$P(C^c)$ is independent of n and it follows from elementary properties of Brownian motions that $P(C^c) \to 0$ as $\varepsilon \to 0$. Together with (22) this completes Claim II and the proof. □

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