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Probability Theory and Related Fields

A critical case for Brownian slow points*

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Summary. Let X_t be a Brownian motion and let $S(c)$ be the set of reals $r \geqq 0$ such that $|X_{r+t} - X_r| \leqq c\sqrt{t}$, $0 \leqq t \leqq h$, for some $h = h(r) > 0$. It is known that $S(c)$ is empty if $c < 1$ and nonempty if $c > 1$, a.s. In this paper we prove that $S(1)$ is empty a.s.

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1. Introduction

Let X_t be a Brownian motion and let

$$S(c) = \{r \geqq 0\colon \text{ there exists } h > 0 \text{ such that } |X_{r+t} - X_r| \leqq c\sqrt{t},\ 0 \leqq t \leqq h\}.$$

$S(c)$ is the set of "slow points" with parameter c. For every $r \in S(c)$, a piece of the path of Brownian motion lies within c times a square root boundary just after r. As is well known, the law of the iterated logarithm implies that after any fixed time r the next piece of the Brownian motion path does not lie in any multiple of a square root boundary, almost surely. Nevertheless, slow points exist for some values of c. Kahane [K1, K2] showed that $S(c) \neq \emptyset$, a.s. provided c is sufficiently large. Dvoretzky [D] showed that $S(1/4)$ is empty. Independently, Davis [Da] and Greenwood and Perkins [GP] showed that $S(c)$ was empty if $c < 1$ and nonempty if $c > 1$. Davis and Perkins [DP] examined a number of critical cases for Brownian slow points (e.g., asymmetric square root boundaries, two-sided (in time) boundaries), but left unresolved the question of whether $S(1)$ is empty or not. They did show that if $S(1)$ is nonempty, it must be at most countable. For additional information on slow points, see [BP, P].

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Our main result is the following theorem.

Theorem 1.1. *With probability one* $S(1) = \emptyset$.

The present article is motivated not only by the desire to record the solution to an open problem about slow points but to present a new argument which seems to be applicable to other "critical" case questions as well.

In Sect. 2 we derive a number of estimates on the densities of Ornstein–Uhlenbeck processes and on the exit probabilities from an interval. These are all either well known or extensions of known results using standard methods. Rather than working with square root boundaries, it is necessary for us to work with boundaries of the form $t \mapsto 2 + \sqrt{t}$, and Sect. 3 is devoted to developing the appropriate estimates. The method we use is an adaptation of one of Novikov [N]. Novikov's paper deals with moving boundaries up to but not including the critical case $t^{1/2}$, and our results in Sect. 3 may be of independent interest. The main work is done in Sect. 4. We define approximate slow points. If A_j represents the event that there is an approximate slow point in the interval $[j, j+1)$, then we estimate $\mathbb{P}(A_k | A_j)$ and $\mathbb{P}(A_k \cap A_p | A_j)$. A standard second moment argument then tells us that $\mathbb{P}(\bigcup_{k=j+1}^{n} A_k | A_j)$ is bounded below by a constant independent of n. Unfortunately, we need that constant to be close to 1; it is necessary to iterate the estimates, which makes the proof considerably more complicated. Finally in Sect. 5 we show that our estimates on approximate slow points imply that $S(1)$ is empty.

The letter c with subscripts will denote constants whose exact values are unimportant. We begin numbering anew at each new proposition. The distribution of Brownian motion starting from x will be denoted $\mathbb{P}^x$. We will often write $\mathbb{P}$ for $\mathbb{P}^0$.

2. Ornstein–Uhlenbeck processes

We begin by recording some known facts about Ornstein–Uhlenbeck processes and their connection with Brownian motions. Let X_t be one-dimensional Brownian motion. Let

$$Z_t = e^{-t/2} X(e^t) . \tag{2.1}$$

Starting the Ornstein–Uhlenbeck process Z_t at $Z_0 = z$ is then the same thing as starting the Brownian motion at $X_1 = z$. The probability that the reflected Brownian motion $|X_t|$ starting from z at time 1 lies under the curve $t \mapsto \sqrt{t}$ on the interval $[1, s]$ is the same as the probability that the reflected Brownian motion $|X_t|$ starting from z at time 0 lies under the curve $t \mapsto \sqrt{1+t}$ on the interval $[0, s-1]$. With these facts in mind, we see that

$$\mathbb{P}^z(|Z_u| \leqq 1, 0 \leqq u \leqq T) = \mathbb{P}^z(|X_t| \leqq \sqrt{1+t}, 0 \leqq t \leqq e^T - 1) . \tag{2.2}$$

Integration by parts in (2.1) shows that Z_t satisfies the stochastic differential equation

$$Z_t = Z_0 + W_t - \int_0^t \frac{Z_s}{2} \, ds , \tag{2.3}$$

where W_t is another one-dimensional Brownian motion. The solution to this SDE is unique. The law of Z_t is that of the diffusion on the line with infinitesimal generator $\mathscr{A}f(x) = (\frac{1}{2})(f''(x) - xf'(x))$ started at $Z_0 = X_1$.

$\mathscr{A}$ is a symmetric operator with respect to the measure $m(dx) = 2e^{-x^2/2}\,dx$. The transition densities with respect to m for Z_t killed on exiting $[-b, b]$ can be written

$$p(t,x,y) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\varphi_i(x)\varphi_i(y)\,, \tag{2.4}$$

where the series converges absolutely and uniformly, $0 < \lambda_1 < \lambda_2 < \ldots$, the φ_i are C^2 and vanish at $-b$ and b, $\varphi_1 > 0$ on $(-b, b)$, $\varphi_1'(-b) > 0$, $\varphi_1'(b) < 0$, $\int_{-b}^{b} \varphi_i^2(x)m(dx) = 1$, and $\mathscr{A}\varphi_i(x) = -\lambda_i\varphi_i(x)$. Moreover, $\lambda_1 = 1$ when $b = 1$. See Knight [Kn] and Perkins [P].

We will need the following estimate.

Proposition 2.1. *Let $\varepsilon > 0$. There exists t_0 such that if $b \in (\frac{1}{2}, 2)$ and $t > t_0$, then*

$$\left| \frac{p(t,x,y)}{e^{-\lambda_1 t}\varphi_1(x)\varphi_1(y)} - 1 \right| < \varepsilon, \quad |x|, |y| \leqq b\,.$$

Proof. First, we get a lower bound on $\varphi_1'(-b)$ that is valid for all $b \in (\frac{1}{2}, 2)$. Note φ_1' can equal 0 in $(-b, b)$ only at local maxima; for if $\varphi_1' = 0$ at x_0, then the equation

$$\varphi_1''(x) = x\varphi_1'(x) - 2\lambda_1\varphi_1(x) \tag{2.5}$$

evaluated at x_0 shows that $\varphi_1''(x_0)$ is strictly negative since $\lambda_1, \varphi_1 > 0$. By the symmetry of $\mathscr{A}$ about 0, φ_1 is symmetric. So $\varphi_1'(0) = 0$ and 0 is a local maximum. Therefore $\varphi_1' \geqq 0$ in $(-b, 0)$, hence φ_1 is nondecreasing on $(-b, 0)$. Eq. (2.5) shows that φ_1'' is negative on $(-b, 0)$ and so φ_1' decreases on this interval. Using the symmetry of φ_1 we have

$$1 = \int_{-b}^{b} \varphi_1^2(x)m(dx) = 2\int_{-b}^{0} \varphi_1^2(x)m(dx) \leqq 4b\|\varphi_1\|_\infty^2\,.$$

Since

$$|\varphi_1(x)| = |\varphi_1(x) - \varphi_1(-b)| = \left|\int_{-b}^{x} \varphi_1'(y)\,dy\right|$$

$$\leqq 2b\|\varphi_1'\|_\infty \leqq 2b\varphi_1'(-b)\,,$$

we obtain $\varphi_1'(-b) \geqq (16b^3)^{-1/2} \geqq 1/12$.

Second, we get upper bounds on φ_i and φ_i'. As a function of b, λ_1 is smallest when b is largest ([CH]). So there exists $c_1 > 0$ independent of $b \in (\frac{1}{2}, 2)$ such that $\lambda_i \geqq \lambda_1 \geqq c_1$. From $\mathscr{A}\varphi_i = -\lambda_i\varphi_i$, we see that

$$|\varphi_i''(x)| \leqq 2|\varphi_i'(x)| + 2\lambda_i|\varphi_i(x)|\,. \tag{2.6}$$

Integration by parts shows that if $f \in C^2[-b, b]$ and $f(-b) = f(b) = 0$, then

$$\int_{-b}^{b} (f')^2\,dx = -\int_{-b}^{b} f''f\,dx\,,$$

so by the Cauchy–Schwarz inequality,

$$\|f'\|_2^2 \leqq \|f''\|_2\|f\|_2 ,$$

where $\|f\|_2$ denotes $(\int_{-b}^{b} |f|^2\, dx)^{1/2}$. From this and (2.6) we obtain

$$\|\varphi_i'\|_2^2 \leqq (2\|\varphi_i'\|_2 + 2\lambda_i\|\varphi_i\|_2)\|\varphi_i\|_2 .$$

Since

$$\|\varphi_i\|_2^2 = \int_{-b}^{b} \varphi_i^2\, dx \leqq c_2 \int_{-b}^{b} \varphi_i^2 m(dx) = c_2 ,$$

we conclude

$$\|\varphi_i'\|_2^2 \leqq 2c_2^{1/2}\|\varphi_i'\|_2 + 2c_2\lambda_i ,$$

which implies $\|\varphi_i'\|_2 \leqq c_3\lambda_i^{1/2}$. Now by the Cauchy–Schwarz inequality

$$|\varphi_i(x)| = \left|\int_{-b}^{x} \varphi_i'(y)\, dy\right| \leqq c_4 \int_{-b}^{b} |\varphi_i'(y)|^2\, dy \leqq c_5\lambda_i ,$$

or

$$\|\varphi_i\|_\infty \leqq c_5\lambda_i . \tag{2.7}$$

Set $r = (2\|f\|_\infty/\|f''\|_\infty)^{1/2} \wedge 1$ and let $x \in [-b, b]$. By the mean value theorem on the interval $[-b \vee (x-r), b \wedge (x+r)]$, there exists a point x^* in the interval such that $|f'(x^*)| \leqq 2\|f\|_\infty/r$, while we also have $|f'(x) - f'(x^*)| \leqq r\|f''\|_\infty$. Hence $\|f'\|_\infty \leqq 2\|f\|_\infty/r + r\|f''\|_\infty$. With our choice of r and the fact that $(u+v)^2 \leqq 2u^2 + 2v^2$, we get the inequality

$$\|f'\|_\infty^2 \leqq 8\|f\|_\infty^2/r^2 + 2r^2\|f''\|_\infty^2 \leqq (8\|f''\|_\infty + 16\|f\|_\infty)(\|f\|_\infty) .$$

From this, (2.6) and (2.7) we have

$$\|\varphi_i'\|_\infty^2 \leqq (16\|\varphi_i'\|_\infty + (16\lambda_i + 16)\|\varphi_i\|_\infty)\|\varphi_i\|_\infty \leqq c_6(\|\varphi_i'\|_\infty + \lambda_i^2)\lambda_i .$$

Therefore

$$\|\varphi_i'\|_\infty \leqq c_7\lambda_i^{3/2} \leqq c_8\lambda_i^2 . \tag{2.8}$$

Third, we get an upper bound on $|\varphi_i(x)|/\varphi_1(x)$. From (2.6)–(2.8), $\|\varphi_1''\|_\infty \leqq c_9$. Since $\varphi_1'(-b) \geqq \frac{1}{12}$, then $\varphi_1'(x) \geqq \frac{1}{24}$ if $x + b \leqq \frac{1}{24}c_9$. Using the fact that φ_1 is nondecreasing on $(-b, 0)$ and using symmetry to deal with positive x, we see then that

$$\varphi_1(x) \geqq c_{10}(b - |x|) , \tag{2.9}$$

where c_{10} does not depend on b. From (2.8), we obtain

$$|\varphi_i(x)| \leqq c_{11}\lambda_i^2(b - |x|) , \tag{2.10}$$

and we therefore have

$$|\varphi_i(x)|/\varphi_1(x) \leqq c_{12}\lambda_i^2 .$$

Finally, to conclude the proof, note that as a function of b, each λ_i is continuous and decreases as b increases [CH]. So there exists i_0 independent of b such that if $i \geqq i_0$, then $\lambda_i > 2\lambda_1$. We also deduce that there exists $\delta > 0$ independent of b such that $\lambda_i - \lambda_1 > \delta$ for all i. We have

$$p(t,x,y) = e^{-\lambda_1 t}\varphi_1(x)\varphi_1(y)\left[1 + \sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)t}\frac{\varphi_i(x)}{\varphi_1(x)}\frac{\varphi_i(y)}{\varphi_1(y)}\right] .$$

If $i \leqq i_0$, then

$$e^{-(\lambda_i-\lambda_1)t}\left|\frac{\varphi_i(x)}{\varphi_1(x)}\frac{\varphi_i(y)}{\varphi_1(y)}\right| \leqq c_{12}^2\lambda_i^4 e^{-(\lambda_i-\lambda_1)t} \leqq c_{12}^2\lambda_{i_0}^4 e^{-\delta t} .$$

This goes to 0 as $t \to \infty$. On the other hand, note from (2.4) that $p(s,x,x)$ is decreasing in s. So

$$\begin{aligned}
&\left|\sum_{i=i_0+1}^{\infty} e^{-(\lambda_i-\lambda_1)t}\frac{\varphi_i(x)}{\varphi_1(x)}\frac{\varphi_i(y)}{\varphi_1(y)}\right| \\
&\quad \leqq c_{12}^2 \sum e^{-\lambda_i t/2}\lambda_i^4 \leqq c_{12}^2\left(\sup_{\lambda \geqq 0} \lambda^4 e^{-\lambda t/4}\right)\sum e^{-\lambda_i t/4} \\
&\quad \leqq c_{12}^2 (16/t)^4 e^{-4} \int_{-b}^{b} p(t/4,x,x)m(dx) ,
\end{aligned}$$

which also tends to 0 as $t \to \infty$. □

There are a number of consequences of this proposition. For $b > 0$, let

$$\tau_b = \inf\{t \geqq 0 \colon \ |Z_t| \geqq b\} .$$

Proposition 2.2. *Let $b \in (\frac{1}{2}, 2)$. There exist c_1, c_2, and $v_1 > 1$ such that if $t \geqq v_1$, then*

$$c_1\varphi_1(x)e^{-\lambda_1 t} \leqq \mathbb{P}^x(\tau_b \in dt)/dt \leqq c_2\varphi_1(x)e^{-\lambda_1 t} .$$

Proof. By the proof of Proposition 2.1, we have

$$\begin{aligned}
\mathbb{P}^x(\tau_b > t) &= \mathbb{P}^x(|Z_s| < b, 0 \leqq s \leqq t) = \int_{-b}^{b} p(t,x,y)m(dy) \\
&= \sum_{i=1}^{\infty} e^{-\lambda_i t}\varphi_i(x)\int_{-b}^{b} \varphi_i(y)m(dy) .
\end{aligned}$$

Differentiating with respect to t,

$$\mathbb{P}^x(\tau_b \in dt)/dt = \sum \lambda_i e^{-\lambda_i t}\varphi_i(x)\int_{-b}^{b} \varphi_i(y)m(dy) .$$

Very similarly to the last part of the proof of Proposition 2.1, we see that the first term, $\lambda_1 e^{-\lambda_1 t}\varphi_1(x)\int \varphi_1(y)m(dy)$, is the dominant term when t is large. □

We fix for the rest of the paper a number $v_1 > 1$ which satisfies Proposition 2.2.

Proposition 2.3. *Let $b \in (\frac{1}{2}, 2)$. There exist c_1 and c_2 such that if $t \geqq v_1$, then*

$$c_1(b - |x|)e^{-\lambda_1 t} \leqq \mathbb{P}^x(\tau_b > t) \leqq c_2(b - |x|)e^{-\lambda_1 t}.$$

Proof. This follows from integrating the result of Proposition 2.2 and using (2.9) and (2.10) with $i = 1$. □

Proposition 2.4. *There exists c_1 such that if $u \geqq t \geqq v_1$ and $x \in (-1, 1)$, then*

$$\mathbb{P}^x(|Z_t| \leqq 1/2 | \tau_1 > u) \geqq c_1.$$

Proof. By the Markov property at time t,

$$\begin{aligned}\mathbb{P}^x(|Z_t| \leqq 1/2, \tau_1 > u) &= \mathbb{E}^x[\mathbb{P}^{Z_t}(\tau_1 > u - t); \tau_1 > t, |Z_t| \leqq 1/2] \\ &= \int_{-1}^{1} \int_{-1/2}^{1/2} p(t, x, y) p(u - t, y, z) m(dy) m(dz).\end{aligned}$$

By Proposition 2.2, this is greater than

$$c_2 \varphi_1(x) e^{-t} \int_{-1}^{1} \int_{-1/2}^{1/2} \varphi_1(y) p(u - t, y, z) m(dy) m(dz). \tag{2.11}$$

If $u - t < v_1$ and $y \in [-\frac{1}{2}, \frac{1}{2}]$, then

$$\int_{-1}^{1} p(u - t, y, z) m(dz) \geqq \mathbb{P}^y(\tau_1 > v_1) \geqq c_3 \geqq c_3 e^{-(u-t)}.$$

If $u - t \geqq v_1$ and $y \in [-\frac{1}{2}, \frac{1}{2}]$, then by Proposition 2.2

$$\int_{-1}^{1} p(u - t, y, z) m(dz) \geqq c_4 e^{-(u-t)} \varphi_1(y) \int_{-1}^{1} \varphi_1(z) m(dz) \geqq c_5 e^{-(u-t)}.$$

So in either case, (2.11) is greater than

$$c_6 \varphi_1(x) e^{-t} \int_{-1/2}^{1/2} \varphi_1(y) e^{-(u-t)} m(dy) \geqq c_7 \varphi_1(x) e^{-u}. \tag{2.12}$$

On the other hand,

$$\begin{aligned}\mathbb{P}^x(\tau_1 > u) &= \int_{-1}^{1} p(u, x, y) m(dy) \\ &\leqq c_8 e^{-u} \varphi_1(x) \int_{-1}^{1} \varphi_1(y) m(dy) \leqq c_9 e^{-u} \varphi_1(x).\end{aligned} \tag{2.13}$$

Taking the ratio of (2.12) and (2.13) proves the proposition. □

Proposition 2.5. *Let* $b \in (\frac{1}{2}, 2)$ *and* $t \leqq 20v_1$. *Then there exists* c_1 *such that*

$$\mathbb{P}^x(\tau_b > t) \leqq c_1 \frac{b - |x|}{\sqrt{t}}, \quad x \in (-b, b).$$

Proof. Define a probability measure $\mathbb{Q}$ on $\mathscr{F}_t$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}^x} = \exp\left(\frac{1}{2}\int_0^t Z_s\, dW_s - \frac{1}{8}\int_0^t Z_s^2\, ds\right), \tag{2.14}$$

where W_t is defined by (2.3) and is a Brownian motion under $\mathbb{P}^x$. By the Girsanov theorem, $Z_t = W_t - \int_0^t Z_s/2\, ds$ is a martingale under $\mathbb{Q}$ with the same quadratic variation as that of W under $\mathbb{P}^x$, namely t. So by Lévy's theorem, Z_t is a Brownian motion under $\mathbb{Q}$.

On the set $\{\tau_b > t\}$, we have $\int_0^t Z_s^2\, ds \leqq tb^2 \leqq 20v_1 b^2 \leqq 80v_1$. Also, using (2.3) and Itô's lemma,

$$\int_0^t Z_s\, dW_s = \int_0^t Z_s\, dZ_s + \frac{1}{2}\int_0^t Z_s^2\, ds = \frac{Z_t^2 - Z_0^2 - t}{2} + \frac{1}{2}\int_0^t Z_s^2\, ds. \tag{2.15}$$

On the set $\{\tau_b > t\}$, the right-hand side of (2.15) is bounded by $(2b^2 + t)/2 + 10b^2 v_1 \leqq 4 + 50v_1$. Therefore the exponent in (2.14) is bounded in absolute value by $K = 2 + 35v_1$.

We then have

$$\mathbb{P}^x(\tau_b > t) = \int_{\{\tau_b > t\}} \frac{d\mathbb{P}^x}{d\mathbb{Q}} d\mathbb{Q} \leqq e^K \mathbb{Q}(\tau_b > t).$$

Since Z_t is a Brownian motion under $\mathbb{Q}$, a well-known estimate says that $\mathbb{Q}(\tau_b > t) \leqq c_2(b - |x|)/\sqrt{t}$, which completes the proof. □

Proposition 2.6. *Let* $b \in (\frac{1}{2}, 2)$ *and* $1 \leqq t \leqq 20v_1$. *There exists* c_1 *such that if* $x \in (-b, b)$,

$$\mathbb{P}^x\left(\sup_{s \leqq t} Z_s < b\right) \leqq c_1(b - x).$$

Proof. Let $B = \{Z(\tau_b) = -b, \tau_b \leqq t\}$. On the set $B, |Z_s| \leqq b$ if $s \leqq \tau_b$. So as in Proposition 2.5,

$$M_t = \frac{1}{2}\int_0^t Z_s\, dW_s - \frac{1}{8}\int_0^t Z_s^2\, ds$$

is bounded in absolute value by a constant K depending only on v_1 when $t \leqq \tau_b$. B is in the σ-field $\mathscr{F}_{\tau_b}$, hence

$$\mathbb{P}^x(B) = \int_B \frac{d\mathbb{P}^x}{d\mathbb{Q}} d\mathbb{Q} = \int_B e^{-M_{\tau_b}}\, d\mathbb{Q} \leqq e^K \mathbb{Q}(B).$$

Under $\mathbb{Q}$ the process Z_t is a Brownian motion, thus $\mathbb{Q}(B)$ is less than the probability that a Brownian motion started at x hits $-b$ before b, which is

$(b-x)/2b$. Since $\{\sup_{s\leqq t} Z_s < b\} \subseteq \{\tau_b > t\} \cup B$, our result follows from this estimate together with the result of Proposition 2.5. □

Proposition 2.7. *Let $b \in (\frac{1}{2}, 2)$ and $a \in (\frac{1}{2}, b)$. There exists c_1 such that if $x \in [-b, a]$ and $v_1 \leqq t_1 \leqq 20v_1$, then*

$$\mathbb{P}^x(\tau_a \leqq 2t_1, \tau_b > 2t_1) \leqq c_1(b-a)\,.$$

Proof. By Proposition 2.3 and the strong Markov property at time τ_a,

$$\mathbb{P}^x(\tau_a \leqq t_1, \tau_b > 2t_1) \leqq \mathbb{P}^a(\tau_b > t_1) \leqq c_2(b-a)\,. \tag{2.16}$$

On the other hand, by the strong Markov property at τ_a and Proposition 2.5,

$$\begin{aligned}\mathbb{P}^x(2t_1 \geqq \tau_a > t_1, \tau_b > 2t_1) &= \int_{t_1}^{2t_1} \mathbb{P}^a(\tau_b > 2t_1 - s)\mathbb{P}^x(\tau_a \in ds) \\ &\leqq \int_{t_1}^{2t_1} \frac{c_3(b-a)}{\sqrt{2t_1 - s}} \mathbb{P}^x(\tau_a \in ds)\,. \end{aligned} \tag{2.17}$$

By Proposition 2.2, $\mathbb{P}^x(\tau_a \in ds) \leqq c_4\, ds$ for $s \geqq t_1$. With (2.17) this shows that

$$\mathbb{P}^x(2t_1 \geqq \tau_a > t_1, \tau_b > 2t_1) \leqq c_5(b-a)\,.$$

Adding to (2.16) proves our result. □

Remark. 2.8. For any r, s, and $T, \mathbb{P}^x(|X_t| \leqq \sqrt{r+t}, s \leqq t \leqq T)$ is largest when $x = 0$. To see this, convert this to an equivalent statement about the Ornstein–Uhlenbeck process Z_t. Since Z_t is symmetric about 0, this expression is easily seen to be largest for $x = 0$.

Proposition 2.9. *There exist c_1 and c_2 such that if $T \geqq v_1$ and $x \in (-\frac{1}{2}, \frac{1}{2})$, then*

$$c_1/T \leqq \mathbb{P}^x(|X_t| \leqq \sqrt{t}, 1 \leqq t \leqq T) \leqq c_2/T\,.$$

Proof. Let Z_t be defined by (2.1). For the upper bound, by the Markov property and Remark 2.8,

$$\begin{aligned}\mathbb{P}^z(|X_t| \leqq \sqrt{t}, 1 \leqq t \leqq T) &= \mathbb{E}^z\mathbb{P}^{X_1}(|X_t| \leqq \sqrt{1+t}, 0 \leqq t \leqq T-1) \\ &\leqq \mathbb{E}^z\mathbb{P}^0(|X_t| \leqq \sqrt{1+t}, 0 \leqq t \leqq T-1)\,.\end{aligned}$$

By (2.2) and Proposition 2.3, this is equal to

$$\mathbb{P}^0(|Z_t| \leqq 1, 0 \leqq t \leqq \log T) \leqq c_3 e^{-\lambda_1 \log T} = c_3/T\,,$$

recalling that $\lambda_1 = 1$ when $b = 1$.

For the lower bound, by the Markov property,

$$\begin{aligned}\mathbb{P}^z(|X_t| \leqq \sqrt{t}, 1 \leqq t \leqq T) &\geqq \mathbb{P}^z(|X_t| \leqq \sqrt{t}, 1 \leqq t \leqq T, |X_1 - X_0| \leqq 1/4) \\ &= \mathbb{E}^z[\mathbb{P}^{X_1}(|X_t| \leqq \sqrt{1+t}, 0 \leqq t \leqq T-1); \\ &\qquad |X_1 - X_0| \leqq 1/4]\,. \end{aligned} \tag{2.18}$$

If $|y| \leqq 3/4$, then

$$\mathbb{P}^y(|X_t| \leqq \sqrt{1+t}, 0 \leqq t \leqq T-1) = \mathbb{P}^y(|Z_t| \leqq 1, 0 \leqq t \leqq \log T)$$
$$\geqq c_4 \varphi_1(y) e^{-\lambda_1 \log T} \geqq c_5/T$$

by (2.2) and Proposition 2.3. If $X_0 = z$, $|z| \leqq 1/2$ and $|X_1 - X_0| \leqq 1/4$, then we have $|X_1| \leqq 3/4$. Therefore the right hand side of (2.18) is bigger than

$$(c_5/T)\mathbb{P}^z(|X_1 - X_0| \leqq 1/4) \geqq c_6/T . \quad \square$$

Proposition 2.10. *There exist c_1 and c_2 such that if $|z| \leqq \sqrt{s}/2$, then*

$$c_1 s/T \leqq \mathbb{P}^z(|X_t| \leqq \sqrt{t}, s \leqq t \leqq T) \leqq c_2 s/T .$$

Proof. This follows from Proposition 2.9 by scaling. Note

$$\mathbb{P}^z(|X_t| \leqq \sqrt{t}, s \leqq t \leqq T) = \mathbb{P}^z(|X_{us}| \leqq \sqrt{us}, s \leqq us \leqq T)$$
$$= \mathbb{P}^z(|X_{us}/\sqrt{s}| \leqq \sqrt{u}, 1 \leqq u \leqq T/s) . \tag{2.19}$$

If $Y_u = X_{us}/\sqrt{s}$, then Y_u is another Brownian motion and the right hand side of (2.19) equals

$$\mathbb{P}^{z/\sqrt{s}}(|Y_u| \leqq \sqrt{u}, 1 \leqq u \leqq T/s) .$$

We now apply Proposition 2.9. $\square$

Remark. 2.11. From Proposition 2.3 we derive

$$\mathbb{P}^x(|X_u| \leqq \sqrt{1+u}/2; |X_s| \leqq \sqrt{1+s}, 0 < s < u)$$
$$\geqq c_1 \mathbb{P}^x(|X_s| \leqq \sqrt{1+s}, 0 < s < u)$$

if $|x| < 1$ by arguments similar to those of Proposition 2.9.

3. Moving boundaries

We need some estimates on moving boundaries. We adapt a method of Novikov [N].

Suppose $f \in C^2[0, \infty)$ and there exists $\kappa_1 > 1$ such that

(a) $\kappa_1^{-1} \leqq f(t) \leqq \kappa_1, \quad t \in [0, \infty)$;

(b) $|f(t) - 1|\sqrt{t} \leqq \kappa_1, \quad t \in [1, \infty)$;

(c) $|f'(t) t^{3/2}| \leqq \kappa_1, \quad t \in [1, \infty)$;

(d) $|f''(t) t^{5/2}| \leqq \kappa_1, \quad t \in [1, \infty)$.

(e) $f = 1$ *in a neighborbood of* 0 . (3.1)

The assumptions (3.1)(b)–(d) could be weakened, but they are good enough for our purposes. In our applications, the value of $f(t)$ for $t \in [0, 1)$

will usually be immaterial, and we can change f to be smooth there and identically 1 for $t \leqq \frac{1}{2}$ without any loss of generality.

Proposition 3.1. (a) *Suppose f satisfies* (3.1). *If $f(t) \leqq 1$ for all t or $f(t) \geqq 1$ for all t, there exist c_1 and c_2 such that for $T \geqq v_1$*

$$c_1/T \leqq \mathbb{P}(|X_t| \leqq f(t)\sqrt{t}, 1 \leqq t \leqq T) \leqq c_2/T .$$

The constants c_1 and c_2 depend on f only through κ_1.
(b) *Suppose $r \in [1, \infty)$, $b \in [0, 2]$, and $a \in [0, \frac{1}{2}]$. Let*

$$f_{a,b,r}(t) = \min(1, b/\sqrt{t} + \sqrt{1 + r/t} - a\sqrt{r/t}) .$$

There exists c_3, not depending on a, b, or r, such that

$$\mathbb{P}(|X_t| \leqq f_{a,b,r}(t)\sqrt{t}, 1 \leqq t \leqq T) \geqq c_3/T .$$

Proof. Let

$$F(t) = f(t) \exp\left(\frac{1}{2}\int_0^t \frac{1}{u}\left[1 - \frac{1}{(f(u))^2}\right] du\right) , \tag{3.2}$$

$$h(t) = \int_0^t \frac{1}{(F(s))^2}\, ds , \tag{3.3}$$

and

$$Y_t = F(t)\int_0^t [F(s)]^{-1}\, dX_s . \tag{3.4}$$

By the Itô product formula,

$$dY_t = dX_t + \frac{Y_t}{F(t)} F'(t)\, dt . \tag{3.5}$$

Define a new probability measure $\mathbb{Q}$ by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathscr{F}_T} = \exp\left(-\int_0^T \frac{F'(s)}{F(s)} Y_s\, dX_s - \frac{1}{2}\int_0^T \left(\frac{F'(s)}{F(s)}\right)^2 Y_s^2\, ds\right) . \tag{3.6}$$

Under $\mathbb{P}$, $Y_t - \int_0^t Y_s F'(s)/F(s)\, ds$ is a martingale, so by Girsanov's theorem, Y_t is a martingale under $\mathbb{Q}$. The quadratic variation of Y_t is the same under both measures, namely $\langle Y\rangle_T = \langle X\rangle_T = t$, and Y_t is continuous. By Lévy's theorem, Y_t is a Brownian motion under $\mathbb{Q}$.

Let A be the event $\{|Y_t| \leqq f(t)\sqrt{t}, 1 \leqq t \leqq T\}$. Note

$$\mathbb{P}(A) = \mathbb{E}_{\mathbb{Q}}\left[1_A \frac{d\mathbb{P}}{d\mathbb{Q}}\right] .$$

Later on in the proof we will bound the exponent in $d\mathbb{Q}/d\mathbb{P}$ in absolute value by K. So then

$$e^{-K}\mathbb{Q}(A) \leqq \mathbb{P}(A) \leqq e^{K}\mathbb{Q}(A) . \tag{3.7}$$

The law of Y_t under $\mathbb{Q}$ is the same as the law of X_t under $\mathbb{P}$, hence

$$\mathbb{Q}(A) = \mathbb{P}(|X_t| \leqq f(t)\sqrt{t}, 1 \leqq t \leqq T),$$

the quantity we are attempting to estimate.

From (3.4) we have

$$\begin{aligned} &\mathbb{P}(|Y_t| \leqq f(t)\sqrt{t}, 1 \leqq t \leqq T) \\ &\quad = \mathbb{P}\left(\left|\int_0^t [F(s)]^{-1} dX_s\right| \leqq f(t)\sqrt{t}/F(t), 1 \leqq t \leqq T\right). \end{aligned} \tag{3.8}$$

Let $W_t = \int_0^{h^{-1}(t)} [F(s)]^{-1}\, dX_s$. W_t is a continuous martingale that is also a Gaussian process. The variance of $W_u - W_t$ is $\int_{h^{-1}(t)}^{h^{-1}(u)} [F(s)]^{-2}\, ds = u - t$, so W_t is a Brownian motion. Let H be the inverse of h. Then the right-hand side of (3.8) is

$$\mathbb{P}\left(|W_t| \leqq \frac{f(H(t))\sqrt{H(t)}}{F(H(t))}, h(1) \leqq t \leqq h(T)\right). \tag{3.9}$$

From the definition of F we have

$$\frac{F'}{F} = \frac{f'}{f} + \frac{1}{2u} - \frac{1}{2uf^2},$$

which leads to

$$\left[u\left(\frac{f(u)}{F(u)}\right)^2\right]' = \frac{1}{F(u)^2},$$

or after integrating,

$$u\left(\frac{f(u)}{F(u)}\right)^2 = h(u) + c_4.$$

Since both sides are 0 when $u = 0$, then $c_4 = 0$. Taking square roots of both sides and setting $u = H(t)$, we have

$$\frac{f(H(t))}{F(H(t))}\sqrt{H(t)} = \sqrt{t}.$$

By the definition of F and (3.1)(a), (b), (e), there exist constants c_5 and c_6 such that

$$c_5/\kappa_1 \leqq c_5 f(t) \leqq F(t) \leqq c_6 f(t) \leqq \kappa_1 c_6,$$

hence

$$t/\kappa_1^2 c_6^2 \leqq h(t) \leqq \kappa_1^2 t/c_5^2.$$

Moreover, if $f(t) \geqq 1$ for all t, then $F(t) \geqq f(t) \geqq 1$, so $h(t) \leqq t$ for all t. This and Proposition 2.9 implies that the right hand side of (3.9) is bounded above by

$$\mathbb{P}(|W_t| \leqq \sqrt{t}, 1 \leqq t \leqq T/\kappa_1^2 c_6^2) \leqq c_7/T.$$

Also, if $f(t) \geqq 1$ for all t, then

$$\mathbb{P}(|X_t| \leqq f(t)\sqrt{t}, 1 \leqq t \leqq T) \geqq \mathbb{P}(|X_t| \leqq \sqrt{t}, 1 \leqq t \leqq T) \geqq c_8/T$$

by Proposition 2.9. Similarly, if $f(t) \leqq 1$ for all t, then $h(t) \geqq t$ for all t and the right hand side of (3.9) is bounded below by

$$\mathbb{P}(|W_t| \leqq \sqrt{t}, 1 \leqq t \leqq T\kappa_1^2/c_5^2) \geqq c_9/T .$$

Also, if $f(t) \leqq 1$ for all t, then

$$\mathbb{P}(|X_t| \leqq f(t)\sqrt{t}, 1 \leqq t \leqq T) \leqq \mathbb{P}(|X_t| \leqq \sqrt{t}, 1 \leqq t \leqq T) \leqq c_{10}/T$$

by Proposition 2.9.

To finish the proof of (a), it remains to bound the exponent of (3.6) on the set A. Using (2.3) and Itô's lemma,

$$\begin{aligned}
&-\int_0^T \frac{F'(s)}{F(s)} Y_s\, dX_s - \frac{1}{2}\int_0^T \left(\frac{F'}{F}\right)^2 Y_s^2\, ds \\
&= -\int_0^T \frac{F'}{F} Y_s\, dY_s + \frac{1}{2}\int_0^T \left(\frac{F'}{F}\right)^2 Y_s^2\, ds \\
&= -\frac{1}{2}\int_0^T \frac{F'}{F}\, d(Y_s^2 - s) + \frac{1}{2}\int_0^T \left(\frac{F'}{F}\right)^2 Y_s^2\, ds \\
&= \frac{1}{2}\left(-Y_T^2 \frac{F'(T)}{F(T)} + Y_0^2 \frac{F'(0)}{F(0)} + \int_0^T Y_s^2 \left(\frac{F'}{F}\right)' ds \right. \\
&\qquad \left. + \int_0^T \frac{F'}{F}\, ds + \int_0^T \left(\frac{F'}{F}\right)^2 Y_s^2\, ds\right) \\
&= \frac{1}{2}\left(-Y_T^2 \frac{F'(T)}{F(T)} + Y_0^2 \frac{F'(0)}{F(0)} + \int_0^T Y_s^2 \frac{F''(s)}{F(s)}\, ds + \log(F(T)/F(0))\right) .
\end{aligned}$$

We will show that the last expression is bounded by a constant independent of T. The expression is continuous and equal to 0 for small T since $F(t) = f(t) = 1$ and $F'(t) = F''(t) = 0$ if t is sufficiently small (see (3.1)(e)).

Let $\psi(t)$ denote the exponent in (3.2). By (3.1)(a)–(b),

$$|\psi'(t)| = \frac{|f(t) - 1|(f(t) + 1)}{2t(f(t))^2} \leqq c_{11} t^{-3/2} .$$

Since $f(t) = 1$ for t small by (3.1)(e), it follows that $\sup_t |\psi(t)| < \infty$, and hence that F is bounded above and below by positive constants. Because

$$\psi'' = -\frac{(f-1)(f+1)}{2t^2 f^2} + \frac{f'}{t f^3} ,$$

(3.1)(a)–(c) show that $|\psi''(t)| \leqq c_{12} t^{-5/2}$. Our estimates have to hold only on the set A so we may assume that $|Y_s| \leqq f(s)\sqrt{s}$ for $0 \leqq s \leqq T$. We have $F' = f' e^{\psi} + f\psi' e^{\psi}$, so

$$\left| Y_T^2 \frac{F'(T)}{F(T)} \right| \leqq c_{13}(T)(T^{-3/2}) \leqq c_{13} .$$

The second term $Y_0^2 F'(0)/F(0)$ is equal to 0 because $Y_0 = 0$. Using (3.1)(d) and the formula

$$F'' = f''e^{\psi} + 2f'\psi'e^{\psi} + f(\psi')^2 e^{\psi} + f\psi''e^{\psi},$$

we obtain $|F''(t)| \leqq c_{14} t^{-5/2}$. Hence

$$\left| \int_1^T Y_s^2 \frac{F''(s)}{F(s)} \, ds \right| \leqq c_{15} \int_1^{\infty} s \cdot s^{-5/2} \, ds < \infty,$$

where the bound is independent of T. Finally, F has been shown to be bounded above and below and therefore $\log(F(T)/F(0))$ is bounded as well.

To prove (b) we proceed as in the proof of (a) above. We will only outline the new elements of the proof. If t_0 is the point where $b + \sqrt{r+t} - a\sqrt{r} = \sqrt{t}$, a calculation shows that $t_0 \geqq (9/16)r$. Note $f_{a,b,r}(t) = 1$ for $t \leqq t_0$. Then

$$\begin{aligned} \int_0^t \frac{1}{u} \left| 1 - \frac{1}{(f_{a,b,r}(u))^2} \right| du &\leqq \int_{t_0}^{\infty} \frac{1}{u} \frac{|f_{a,b,r}(u) - 1|(f_{a,b,r}(u) + 1)}{(f_{a,b,r}(u))^2} \, du \\ &\leqq c_{16}\sqrt{r} \int_{r/2}^{\infty} \frac{1}{u} \frac{1}{\sqrt{u}} \, du, \end{aligned} \tag{3.10}$$

which is bounded independently of t, r, b, and a. It follows that there exists c_{17} such that $h(t) \leqq c_{17} t$ for all t. As we saw above, $h(t) \geqq t$. So

$$\mathbb{P}(|W_t| \leqq \sqrt{t}, h(1) \leqq t \leqq h(T)) \geqq \mathbb{P}(|W_t| \leqq \sqrt{t}, 1 \leqq t \leqq c_{17}T) \geqq c_{18}/T.$$

F is not in C^1, but if we approximate F in a suitable way and take a limit, we see that $\mathbb{Q}(A) \geqq e^{-K}\mathbb{P}(A)$, where K is a bound for

$$\begin{aligned} \frac{1}{2} \Bigg(-Y_T^2 \frac{F'(T)}{F(T)} + Y_0^2 \frac{F'(0)}{F(0)} + \int_{t_0}^{T} Y_s^2 \frac{F''(s)}{F(s)} \, ds + [F'(t_0+) - F'(t_0-)] \\ \times Y_{t_0}^2 + \log(F(T)/F(0)) \Bigg). \end{aligned}$$

Using the fact that $F(t) = f(t) = 1$ and $F'(t) = F''(t) = 0$ for $t < t_0$, we bound this by a quantity independent of a, b, and r in a manner similar to that used in (3.10). □

Remark. 3.2. The same proof shows that if (3.1) holds, there exists c_1 such that

$$\mathbb{P}^0(|X_t| \leqq f(t)\sqrt{t}, s \leqq t \leqq T) \leqq c_1 s/T. \tag{3.11}$$

One can similarly generalize Proposition 3.1(b).

Proposition 3.3. *There exists c_1 such that if $1 \leqq s \leqq 5s + 2v_1 \leqq T$, then*

$$\mathbb{P}(|X_t| \leqq 2 + \sqrt{s+t}, 0 \leqq t \leqq T) \leqq c_1 s/T.$$

Proof. If $s \leqq v_1$, then

$$\mathbb{P}(|X_t| \leqq 2+\sqrt{s+t}, 0 \leqq t \leqq T) \leqq \mathbb{P}(|X_t| \leqq 2+\sqrt{s}+\sqrt{t}, 0 \leqq t \leqq T) \\ \leqq \mathbb{P}(|X_t| \leqq 2+v_1+\sqrt{t}, 1 \leqq t \leqq T),$$

and the last probability can be estimated by Proposition 3.1(a) with $f(t) = 1 + (2+v_1)/\sqrt{t}$ for $t \geqq 1$.

Suppose now that $s \geqq v_1$. We write

$$\begin{aligned}\mathbb{P}(|X_t| &\leqq 2+\sqrt{s+t}, 0 \leqq t \leqq T) \\ &\leqq \mathbb{P}(|X_s| \leqq 2+\sqrt{2s}; |X_t| \leqq 2+\sqrt{s+t}, s \leqq t \leqq T) \\ &\leqq \int_{-2-\sqrt{3s}}^{2+\sqrt{3s}} \mathbb{P}^0(X_s \in du)\mathbb{P}^u(|X_t| \leqq 2+\sqrt{2s+t}, 0 \leqq t \leqq T-s).\end{aligned} \tag{3.12}$$

Next we have for $|y| \leqq 3\sqrt{s}$,

$$\begin{aligned}\mathbb{P}^y(|X_t| &\leqq 2+\sqrt{s+t}, s \leqq t \leqq T) \\ &\geqq \int_{-2-\sqrt{3s}}^{2+\sqrt{3s}} \mathbb{P}^y(X_s \in du)\mathbb{P}^u(|X_t| \leqq 2+\sqrt{2s+t}, 0 \leqq t \leqq T-s).\end{aligned} \tag{3.13}$$

Since $\mathbb{P}^y(X_s \in du) = (2\pi s)^{-1/2} e^{-(u-y)^2/2s}\, du$, we see there exists c_2 such that

$$\mathbb{P}^0(X_s \in du) \leqq c_2 \mathbb{P}^y(X_s \in du), \quad |y| \leqq 3\sqrt{s}, |u| \leqq 2+\sqrt{3s}.$$

So combining (3.12) and (3.13), if $|y| \leqq 3\sqrt{s}$,

$$\mathbb{P}(|X_t| \leqq 2+\sqrt{s+t}, 0 \leqq t \leqq T) \leqq c_2 \mathbb{P}^y(|X_t| \leqq 2+\sqrt{s+t}, s \leqq t \leqq T). \tag{3.14}$$

Then we have

$$\begin{aligned}\mathbb{P}^0(|X_t| &\leqq 2+\sqrt{t}, 2s \leqq t \leqq T+s) \\ &\geqq \int_{-3\sqrt{s}}^{3\sqrt{s}} \mathbb{P}^0(X_s \in dy)\mathbb{P}^y(|X_t| \leqq 2+\sqrt{s+t}, s \leqq t \leqq T) \\ &\geqq c_2^{-1} \int_{-3\sqrt{s}}^{3\sqrt{s}} \mathbb{P}^0(X_s \in dy)\mathbb{P}^0(|X_t| \leqq 2+\sqrt{s+t}, 0 \leqq t \leqq T) \\ &= c_3 \mathbb{P}^0(|X_t| \leqq 2+\sqrt{s+t}, 0 \leqq t \leqq T).\end{aligned} \tag{3.15}$$

By Remark 3.2, the left hand side of (3.15) is bounded by $c_4 s/(T+s) \leqq c_4 s/T$. □

Proposition 3.4. *Suppose $s \geqq 1$ and $T \geqq 4s + 10v_1$. There exist c_1 and c_2 such that if $0 \leqq y \leqq 2+\sqrt{s}$, then*

$$c_1\left(1 - \frac{y-2}{\sqrt{s}}\right)\frac{s}{T} \leqq \mathbb{P}^y(|X_t| \leqq 2+\sqrt{s+t}, 0 \leqq t \leqq T, |X_T| \leqq \sqrt{s+T}/2)$$

and

$$\mathbb{P}^y(|X_t| \leqq 2 + \sqrt{s+t}, 0 \leqq t \leqq T) \leqq c_2 \left(1 - \frac{y-2}{\sqrt{s}}\right) \frac{s}{T}.$$

Proof. For the lower bound,

$$\begin{aligned}
&\mathbb{P}^y(|X_t| \leqq 2 + \sqrt{s+t}, 0 \leqq t \leqq T, |X_T| \leqq \sqrt{s+T}/2) \\
&\quad \geqq \mathbb{P}^y(|X_t| \leqq 2 + \sqrt{s}, 0 \leqq t \leqq s, |X_s| \leqq \sqrt{s}/4 ; \\
&\qquad |X_t| \leqq 2 + \sqrt{s+t}, s \leqq t \leqq T, |X_T| \leqq \sqrt{s+T}/2) \\
&\quad \geqq \mathbb{E}^y[\mathbb{P}^{X_s}(|X_t| \leqq 2 + \sqrt{2s+t}, 0 \leqq t \leqq T-s, |X_{T-s}| \leqq \sqrt{T}/2) ; \\
&\qquad |X_s| \leqq \sqrt{s}/4, |X_t| \leqq 2 + \sqrt{s}, 0 \leqq t \leqq s] .
\end{aligned} \tag{3.16}$$

If $|z| \leqq \sqrt{s}/4$,

$$\begin{aligned}
&\mathbb{P}^z(|X_t| \leqq 2 + \sqrt{2s+t}, 0 \leqq t \leqq T-s; |X_{T-s}| \leqq \sqrt{T}/2) \\
&\quad \geqq \mathbb{P}^z(|X_t| \leqq \sqrt{2s+t}, 0 \leqq t \leqq T-s; |X_{T-s}| \leqq \sqrt{T}/2) \\
&\quad \geqq c_3 \mathbb{P}^z(|X_t| \leqq \sqrt{2s+t}, 0 \leqq t \leqq T-s) \\
&\quad \geqq c_4 s/T
\end{aligned} \tag{3.17}$$

by scaling, Remark 2.11 and Proposition 2.3. Therefore the right hand side of (3.16) is greater than

$$c_4 \frac{s}{T} \mathbb{P}^y(|X_s| \leqq \sqrt{s}/4, |X_t| \leqq 2 + \sqrt{s}, 0 \leqq t \leqq s).$$

But

$$\mathbb{P}^y(|X_t| \leqq 2 + \sqrt{s}, 0 \leqq t \leqq s) \geqq c_5 \frac{2 + \sqrt{s} - y}{\sqrt{s}} = c_5 \left(1 - \frac{y-2}{\sqrt{s}}\right),$$

and given that $|X_t|$ remains less than $2 + \sqrt{s}$ until time s, there is positive probability that $|X_s| \leqq \sqrt{s}/4$.

For the other inequality we have

$$\begin{aligned}
&\mathbb{P}^y(|X_t| \leqq 2 + \sqrt{s+t}, 0 \leqq t \leqq T) \\
&\quad \leqq \mathbb{P}^y(X_t \leqq 2 + \sqrt{s+t}, 0 \leqq t \leqq 3s; \\
&\qquad |X_t| \leqq 2 + \sqrt{s+t}, 3s \leqq t \leqq T) .
\end{aligned} \tag{3.18}$$

Using the Markov property at time $3s$ and Remark 2.8, the right hand side of (3.18) is bounded by

$$\mathbb{P}^{y-2}(X_t \leqq \sqrt{s+t}, 0 \leqq t \leqq 3s) \mathbb{P}^0(|X_t| \leqq 2 + \sqrt{4s+t}, 0 \leqq t \leqq T - 3s) . \tag{3.19}$$

The probability that X_t started at $y-2$ at time 0 stays under the curve $\sqrt{s+t}$ for $t \in [0, 3s]$ is the same as the probability that X_t starting at $y-2$ at time s stays under the curve $\sqrt{t}$ for $t \in [s, 4s]$. Defining Z_t by (2.1), this is the same as the probability that Z_t starting at $(y-2)/\sqrt{s}$ at time $\log s$ stays below 1 for $t \in [\log s, \log 4s]$. Thus the first factor is equal to

$$\mathbb{P}^{(y-2)/\sqrt{s}}(Z_t \leqq 1, 0 \leqq t \leqq \log 4) \leqq c_6 \left(1 - \frac{y-2}{\sqrt{s}}\right) \tag{3.20}$$

by Proposition 2.6. The second factor in (3.19) is bounded by

$$\mathbb{P}^0(|X_t| \leqq 2 + \sqrt{4s+t}, 1 \leqq t \leqq T - 3s) \leqq c_7 \frac{4s}{T-3s} \leqq c_8 \frac{s}{T}$$

by Proposition 3.3. Combining this with (3.20) gives the upper bound. □

4. Approximate slow points

Let $U = e^{10v_1}$, where v_1 is defined following Proposition 2.2. For $0 \leqq j \leqq n$, define the event

$$A_j = \{|X_t - X_j| \leqq 2 + \sqrt{t-j}, j+1 \leqq t \leqq U_n\}. \tag{4.1}$$

When the event A_j occurs, we say X_t has an approximate slow point at time j. Let $\beta > 0$ be arbitrary. Our goal is to show $\mathbb{P}(\bigcup_{j=1}^n A_j) \leqq \beta$ when n is sufficiently large. We do that by getting a suitable estimate on $\mathbb{P}(A_j \cap A_{j+1}^c \cap \ldots \cap A_n^c)$. We start by using induction to construct a finite sequence of pairs $(j_1, k_1), \ldots, (j_I, k_I)$ which have some special properties and are such that $j < j_1 < k_1 < \ldots < j_I < k_I < n$. Let

$$B_i = A_{j_i}^c \cap \ldots \cap A_{k_i}^c. \tag{4.2}$$

We will show there exists a $c_1 > 0$ and $\rho \in (0,1)$ such that

$$\mathbb{P}(A_0 \cap B_1 \cap \ldots \cap B_i) \leqq c_1 \rho^i / n. \tag{4.3}$$

Let us proceed with the $i = 1$ case. We will also concentrate primarily on the case $j = 0$ and then point out how the case of general j follows from this special case. The right hand side of the following proposition has also been proved in Sect. 3 of [DP].

Proposition 4.1. *There exist κ_2 and κ_3 such that*

$$\kappa_2/n \leqq \mathbb{P}(A_j) \leqq \kappa_3/n.$$

Proof. We use the Markov property at time j and translation invariance to get

$$\begin{aligned}\mathbb{P}(A_j) &= \mathbb{E}[\mathbb{P}^{X_j}(|X_t - X_0| \leqq 2 + \sqrt{t}, 1 \leqq t \leqq Un - j)] \\ &= \mathbb{P}(|X_t| \leqq 2 + \sqrt{t}, 1 \leqq t \leqq Un - j).\end{aligned}$$

The upper bound and lower bound now follow by using Proposition 3.1(a) with $f(t) = 1 + 2/\sqrt{t}$ for $t \geqq 1$. □

Proposition 4.2. *There exists* c_1 *such that if* $k < n$, *then*

$$\mathbb{P}(A_0 \cap A_k) \geqq c_1/nk .$$

Proof. By Remark 2.11 and Proposition 2.9, if $k \geqq v_1$, then

$$\mathbb{P}(|X_t| \leqq \sqrt{t}, 1 \leqq t \leqq k, \text{ and } |X_k|/\sqrt{k} \leqq 1/2) \geqq c_2 \mathbb{P}(|X_t| \leqq \sqrt{t}, 1 \leqq t \leqq k) \geqq c_3/k .$$

If $k < v_1$,

$$\mathbb{P}(|X_t| \leqq \sqrt{t}, 1 \leqq t \leqq k, \text{ and } |X_k|/\sqrt{k} \leqq 1/2) \geqq \mathbb{P}(|X_t| \leqq 1, 0 \leqq t \leqq k) \geqq c_4 \geqq c_4/k .$$

So using the Markov property at time k, it suffices to show there exists c_5 such that if $|y| \leqq \sqrt{k}/2$, then

$$\mathbb{P}^0(|X_t| \leqq 2 + \sqrt{t+k} - |y| \text{ and } |X_t| \leqq \sqrt{t}, 1 \leqq t \leqq Un - k) \geqq c_5/n . \quad (4.4)$$

By symmetry we can assume without loss of generality that $y \geqq 0$. We will show (4.4) when y is largest, namely $\sqrt{k}/2$; the same proof works for every smaller y.

Suppose $k \geqq 4$. The curves $t \mapsto 2 + \sqrt{t+k} - \sqrt{k}/2$ and $t \mapsto \sqrt{t}$ intersect at a point $t_0 \geqq 9k/16$. Let $f(t)$ be equal to 1 for $0 \leqq t \leqq t_0$ and equal to $(2 + \sqrt{k+t} - \sqrt{k}/2)/\sqrt{t}$ for $t_0 \leqq t \leqq Un - k$. Our result follows by Proposition 3.1(b).

The case $k \leqq 3$ must be dealt with separately, but is quite easy and is left to the reader. □

Proposition 4.3. *There exists* c_1 *such that if* $k + p \leqq n$, *then*

$$\mathbb{P}(A_0 \cap A_k \cap A_{k+p}) \leqq c_1/nkp .$$

Proof. We have

$$\begin{aligned} \mathbb{P}(A_0 \cap A_k \cap A_{k+p}) \leqq \mathbb{P}(&|X_t| \leqq 2 + \sqrt{t}, 1 \leqq t \leqq k ; \\ &|X_t - X_k| \leqq 2 + \sqrt{t-k}, k+1 \leqq t \leqq k+p ; \\ &|X_t - X_{k+p}| \leqq 2 + \sqrt{t-(k+p)}, k+p+1 \\ &\quad \leqq t \leqq Un) . \end{aligned}$$

By the Markov property at times k and $k+p$ and Remark 2.8, we bound the above probability by

$$\mathbb{P}(|X_t| \leqq 2 + \sqrt{t}, 1 \leqq t \leqq k) \times \mathbb{P}(|X_t| \leqq 2 + \sqrt{t}, 1 \leqq t \leqq p) \times \mathbb{P}(|X_t| \leqq 2 + \sqrt{t}, 1 \leqq t \leqq Un - (k+p)) .$$

Using Proposition 3.1, this in turn is bounded above by

$$(c_2/k)(c_2/p)(c_2/(Un - (k+p)) \leqq c_3/nkp . \quad \square$$

Let $j_1 = 1$.

Proposition 4.4. *There exists $\rho' \in (0,1)$ and a positive integer k_1 such that*

$$\mathbb{P}(A_0 \cap B_1) < \kappa_3 \rho'/n\,,$$

where B_1 is defined in terms of k_1 by (4.2).

Proof. Define a new probability measure $\mathbb{Q}$ by

$$\mathbb{Q}(E) = \mathbb{P}(E|A_0) = \frac{\mathbb{P}(E \cap A_0)}{\mathbb{P}(A_0)}\,.$$

Since $\mathbb{P}(A_0) > 0$, this is a well defined measure. From Propositions 4.2 and 4.3 we have $\mathbb{Q}(A_k) \geqq c_1/k$ and $\mathbb{Q}(A_k \cap A_p) \leqq c_2/k(p-k)$. Without loss of generality we may assume $c_1 \leqq 1$ and $c_2 \geqq 1$.

Let $N_r = \sum_{m=1}^{r} 1_{A_m}$. Then

$$\mathbb{E}_{\mathbb{Q}} N_r = \sum_{m=1}^{r} \mathbb{Q}(A_m) \geqq c_1 \sum_{m=1}^{r} 1/m \geqq c_1 \log r$$

and

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}} N_r^2 &= \sum_{m=1}^{r} \mathbb{Q}(A_m) + 2 \sum_{\substack{m,\, p=1 \\ m<p}}^{r} \mathbb{Q}(A_m \cap A_p) \\
&\leqq \mathbb{E}_{\mathbb{Q}} N_r + 2 \sum_{m<p} \frac{c_2}{m(p-m)} \leqq \mathbb{E}_{\mathbb{Q}} N_r + 2c_2(1+\log r)^2 \\
&\leqq \mathbb{E}_{\mathbb{Q}} N_r + 8c_2 \log^2 r \leqq \mathbb{E}_{\mathbb{Q}} N_r + 8c_2 c_1^{-2} (\mathbb{E}_{\mathbb{Q}} N_r)^2 \\
&\leqq (1 + 8c_2 c_1^{-2})(\mathbb{E}_{\mathbb{Q}} N_r)^2 \leqq 9c_2 c_1^{-2} (\mathbb{E}_{\mathbb{Q}} N_r)^2
\end{aligned}$$

as long as $r \geqq 3$ is big enough so that $c_1 \log r > 2$. Let $k_1 = r$ and $c_3 = 9c_2c_1^{-2}$.

By the Cauchy–Schwarz inequality,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}} N_r &\leqq 1 + \mathbb{E}_{\mathbb{Q}}[N_r; N_r > 1] \\
&\leqq 1 + (\mathbb{E}_{\mathbb{Q}} N_r^2)^{1/2} \mathbb{Q}(N_r > 1)^{1/2} \\
&\leqq 1 + c_3^{1/2} (\mathbb{E}_{\mathbb{Q}} N_r) \mathbb{Q}(N_r > 1)^{1/2}\,,
\end{aligned}$$

so

$$\mathbb{E}_{\mathbb{Q}} N_r/2 \leqq c_3^{1/2} (\mathbb{E}_{\mathbb{Q}} N_r) \mathbb{Q}(N_r > 1)^{1/2}\,,$$

or

$$\mathbb{Q}(N_r > 1) \geqq 1/(4c_3)\,.$$

Let $\rho' = 1 - 1/(40c_3)$. If $N_r > 1$, then B_1^c occurs. Thus $\mathbb{Q}(B_1) < \rho'$ and using Proposition 4.1, we get our desired estimate. □

Remark. 4.5. Note for future reference that the proof of Proposition 4.4 shows that if

$$c_1 = \inf_{1 \leqq m \leqq n} [m\mathbb{P}(A_0 \cap A_m)/\mathbb{P}(A_0)] \wedge 1\,, \tag{4.5}$$

$$c_2 = \sup_{1 \leqq m \leqq p \leqq n} [m(p-m)\mathbb{P}(A_0 \cap A_m \cap A_p)/\mathbb{P}(A_0)] \vee 1\,, \tag{4.6}$$

then

$$\rho' = 1 - 1/[40(9c_2c_1^{-2})] = 1 - c_1^2/(360c_2).$$

Define

$$D_j(N) = \{|X_t - X_j| \leqq 2 + \sqrt{t-j}, j+1 \leqq t \leqq N\}.$$

Proposition 4.6. *Let k be given and let $\varepsilon > 0$. There exists $N_0 > k$ (not depending on n) such that if $j \leqq k$, $N \geqq N_0$, and n is sufficiently large, then*

$$\mathbb{P}(A_0 \cap D_j(N) \cap A_j^c) \leqq \varepsilon/n.$$

Proof. Let $T = \inf\{t > j+1 : |X_t - X_j| > 2 + \sqrt{t-j}\}$. On $D_j(N) \cap A_j^c, N \leqq T \leqq Un$. If events A_0 and $\{N \leqq T \leqq n\}$ hold, we have $|X_T - X_j| \geqq 2 + \sqrt{T-j}$ and $|X_j| \leqq 2 + \sqrt{j}$. Therefore

$$|X_T| \geqq \sqrt{T-j} - \sqrt{j}.$$

$\mathbb{P}^y(|X_t| \leqq 2 + \sqrt{s+t}, 0 \leqq t \leqq (U-1)n)$ will be largest when $|y|$ is smallest. If $T \in [2^m N \wedge n, 2^{m+1} N \wedge n]$, then

$$\begin{aligned}&\mathbb{P}^{X_T}(|X_t| \leqq 2 + \sqrt{T+t}, 0 \leqq t \leqq (U-1)n)\\ &\qquad \leqq \mathbb{P}^{\sqrt{T-j}-\sqrt{j}}(|X_t| \leqq 2 + \sqrt{T+t}, 0 \leqq t \leqq (U-1)n). \quad (4.7)\end{aligned}$$

By Proposition 3.4, the right hand side of (4.7) is bounded by

$$c_2\left(1 - \frac{\sqrt{T-j} - \sqrt{j} - 2}{\sqrt{T}}\right)\frac{T}{n} \leqq c_3(j(2^{m+1}N \wedge n))^{1/2}/n.$$

So using the strong Markov property at time T and Proposition 3.1,

$$\begin{aligned}&\mathbb{P}(A_0, T \in [2^m N, 2^{m+1}N])\\ &\quad \leqq \mathbb{P}(|X_t| \leqq 2 + \sqrt{t}, 1 \leqq t \leqq 2^m N \wedge n, T \in [2^m N \wedge n, 2^{m+1} N \wedge n],\\ &\qquad\quad |X_{T+t}| \leqq 2 + \sqrt{T+t}, 0 \leqq t \leqq (U-1)n)\\ &\quad \leqq \mathbb{E}[\mathbb{P}^{X_T}(|X_t| \leqq 2 + \sqrt{T+t}, 0 \leqq t \leqq (U-1)n);\\ &\qquad\quad T \in [2^m N \wedge n, 2^{m+1} N \wedge n],\\ &\qquad\quad |X_t| \leqq 2 + \sqrt{t}, 1 \leqq t \leqq 2^m N \wedge n]\\ &\quad \leqq \frac{c_3(j(2^{m+1}N \wedge n))^{1/2}}{n} \frac{c_4}{2^m N \wedge n}\\ &\quad = \frac{c_5\sqrt{j}}{n(2^m N \wedge n)^{1/2}}.\end{aligned}$$

If we now sum this over m from 0 to the first integer greater than $(\log n - \log N)/\log 2$,

$$\mathbb{P}(A_0, N \leqq T \leqq n) \leqq c_6\sqrt{k}/n\sqrt{N} \leqq c_6\sqrt{k}/n\sqrt{N_0}.$$

Next we look at the event $A_0 \cap \{T \in [n, Un]\}$. For this to hold, first, $|X_t|$ must lie under the curve $t \mapsto 2 + \sqrt{t}$ for $t \in [1, n]$; second, $|X_t|$ must lie under the curve $t \mapsto 2 + \sqrt{t}$ for $t \in [n, Un]$; and third, for some y with $|y| \leqq 2 + \sqrt{j}$ (namely, $y = X_j$), X_t hits at least one of the curves $t \mapsto y \pm (2 + \sqrt{t-j})$ for some $t \in [n, Un]$. We give the proof for y as small as possible, that is, $y = -2 - \sqrt{j}$; essentially the same proof works for every larger y. Using the Markov property at time n and Proposition 3.1 (a) with $f(t) = 1 + 2/\sqrt{t}$ for $t \geqq 1$,

$$\begin{aligned}
&\mathbb{P}(A_0, T \in [n, Un]) \\
&\leqq \left(\frac{c_7}{n}\right) \sup_{|z| \leqq 2+\sqrt{n}} \mathbb{P}^z(|X_t| \leqq 2 + \sqrt{n+t}, 0 \leqq t \leqq (U-1)n\,; \\
&\qquad X_t > \sqrt{n+t-j} - \sqrt{j} \text{ for some } t \in [0, (U-1)n]) \\
&\leqq \left(\frac{c_7}{n}\right) \sup_{|z| \leqq 2+\sqrt{n}} \mathbb{P}^z(X_t \text{ hits } a\sqrt{n+t} \text{ for some } t \in [0, (U-1)n]\,, \\
&\qquad \text{but does not hit } b\sqrt{n+t} \text{ for } t \in [0, (U-1)n])\,, \qquad (4.8)
\end{aligned}$$

where $a = \sqrt{1 - j/n} - \sqrt{j/n}$ and $b = 1 + 2/\sqrt{n}$. The probability on the right hand side of (4.8) is the probability that a Brownian motion started at z at time n hits the curve $a\sqrt{t}$ but not $b\sqrt{t}$ before time Un. Using (2.1), this is the same as the probability that Z_t started at $z/\sqrt{n}$ at time $\log n$ hits the level a but not b before time $\log(Un)$. So

$$\mathbb{P}(A_0, T \in [n, Un]) \leqq \left(\frac{c_7}{n}\right) \sup_{|z| \leqq b} \mathbb{P}^z(\tau_a \leqq \log U, \tau_b \geqq \log U)\,.$$

By Proposition 2.7 and the inequality $\sqrt{1 - j/2n} \geqq 1 - j/2n$,

$$\mathbb{P}(A_0, T \in [n, Un]) \leqq c_8 \sqrt{j}/n\sqrt{n}\,.$$

So if we take N_0 large enough, we get our result provided n is sufficiently large. □

We are now ready to complete the induction step. We suppose we have selected $j_1, k_1, \ldots, j_i, k_i$ and we are to construct j_{i+1}, k_{i+1}.

Let $C_i(N) = D_1(N) \cup D_2(N) \cup \ldots \cup D_{k_i}(N)$. We will write B for $B_1 \cap \ldots \cap B_i$.

Proposition 4.7. *Let $i \geqq 1$. There exists $\rho'' \in (0,1)$ independent of i such that if N is any integer larger than $2k_i$, then there exist integers j_{i+1} and k_{i+1} (not depending on n) satisfying $2k_i < N < j_{i+1} < k_{i+1}$ so that*

$$\mathbb{P}(A_0 \cap C_i^c(N) \cap B \cap B_{i+1}) < \rho'' \mathbb{P}(A_0 \cap C_i^c(N) \cap B)$$

for n sufficiently large.

Proof. Let

$$\begin{aligned}
R &= \{|X_t| \leqq 2 + \sqrt{t}, 1 \leqq t \leqq N\}\,, \\
S &= \{|X_t| \leqq 2 + \sqrt{t}, N \leqq t \leqq Un\}\,, \\
S' &= \{|X_t + X_0| \leqq 2 + \sqrt{t+N}, 0 \leqq t \leqq Un - N\}\,, \\
A'_j &= \{|X_t + X_{j-N}| \leqq 2 + \sqrt{t - (j-N)}, j - N + 1 \leqq t \leqq Un - N\}\,.
\end{aligned}$$

Let N be any integer larger than $2k_i$ and let $j_{i+1} = 8N$. Suppose $j \geqq j_{i+1}$. We assume n is large enough so that $Un > 16N$.

By Proposition 3.4 there exist c_1 and c_2 such that if $|y| \leqq 2 + \sqrt{N}$,

$$c_1 \left(1 - \frac{|y| - 2}{\sqrt{N}}\right) \frac{N}{Un - N} \leqq \mathbb{P}^y(S')$$

$$\leqq c_2 \left(1 - \frac{|y| - 2}{\sqrt{N}}\right) \frac{N}{Un - N}. \tag{4.9}$$

To estimate $\mathbb{P}^y(S' \cap A'_j)$ from below, notice that by translation invariance and the Markov property, this is greater than the product of the following two factors:

(i) the probability that a Brownian motion started at y at time N lies between the curves $\pm\sqrt{t}$ until time j with $|X_j| \leqq \sqrt{j}/2$, and

(ii) the probability that after time j, the Brownian motion lies between square root boundaries centered at X_j up until time Un while at the same time remaining between $\pm\sqrt{t}$.

The probability in (i) is the same as

(i′) the probability that a Brownian motion started at y at time 0 lies between the curves $\pm\sqrt{t+N}$ until time $j - N$ with $|X_{j-N}| \leqq \sqrt{j-N}/2$.

Using Proposition 3.4 this probability is bounded below by

$$c_3 \left(1 - \frac{|y| - 2}{\sqrt{N}}\right) \frac{N}{j - N}.$$

Factor (ii) may be estimated using Proposition 3.1(b) – the lower bound here is a constant multiple of $1/(Un - j)$. We see that

$$\mathbb{P}^y(S' \cap A'_j) \geqq c_4 \left(1 - \frac{|y| - 2}{\sqrt{N}}\right) \frac{N}{j - N} \times \frac{1}{Un - j} \geqq c_4 \left(1 - \frac{|y| - 2}{\sqrt{N}}\right) \frac{N}{jn}.$$

Comparing with (4.9), there exists c_5 such that

$$\mathbb{P}^y(S' \cap A'_j) \geqq c_5 \mathbb{P}^y(S')/j.$$

Note c_5 can be chosen to be independent of N. Similarly, there exists c_6 such that

$$\mathbb{P}^y(S' \cap A'_j \cap A'_k) \leqq c_6 \mathbb{P}^y(S')/j(k - j).$$

Observe that $C_i^c(N) \cap B = C_i^c(N) \in \mathscr{F}_N$, the σ-field up to time N. By the Markov property at time N,

$$\begin{aligned}
\mathbb{P}(A_0 \cap B \cap C_i^c(N) \cap A_j) &= \mathbb{P}(R \cap B \cap C_i^c(N) \cap S \cap A_j) \\
&= \mathbb{E}[\mathbb{P}^{X_N}(S' \cap A'_j); R \cap B \cap C_i^c(N)] \\
&\geqq c_5 \mathbb{E}[\mathbb{P}^{X_N}(S'); R \cap B \cap C_i^c(N)]/j \\
&= c_5 \mathbb{P}(R \cap B \cap C_i^c(N) \cap S)/j \\
&= c_5 \mathbb{P}(A_0 \cap B \cap C_i^c(N))/j.
\end{aligned}$$

Similarly,

$$\mathbb{P}(A_0 \cap B \cap C_i^c(N) \cap A_j \cap A_k) \leqq c_6 \mathbb{P}(A_0 \cap B \cap C_i^c(N))/j(k - j).$$

Choose r large enough so that $\log r \geqq 6$ and

$$\sum_{m=j_{i+1}}^{r} \frac{1}{m} \geqq \log r/2 .$$

Let $k_{i+1} = r$. Proceeding just as in Proposition 4.4, there exists ρ'' such that

$$\mathbb{P}(A_0 \cap C_i^c(N) \cap B \cap B_{i+1}) < \rho'' \mathbb{P}(A_0 \cap C_i^c(N) \cap B) .$$

Note that ρ'' is independent of N, j_{i+1} and k_{i+1} (cf. Remark 4.5). □

Let $\rho = \max(\rho', \rho'')$.

Proposition 4.8. *There exist* $j_1, k_1, \ldots, j_i, k_i$ *(not depending on* n*) satisfying* $1 = j_1 < k_1 < j_2 < k_2 < \ldots < j_i < k_i$ *such that for* n *sufficiently large,*

$$\mathbb{P}(A_0 \cap B_1 \cap \ldots \cap B_i) < \kappa_3 \rho^i / n . \tag{4.10}$$

Proof. We use induction. The case $i = 1$ is Proposition 4.4. Suppose we have (4.10) holding for i and we want to prove it holds for $i + 1$. Write B for $B_1 \cap \ldots \cap B_i$.

Take

$$\varepsilon = [\kappa_3 \rho^i / n - \mathbb{P}(A_0 \cap B)]/4k_i$$

and choose N_0 as in Proposition 4.6 with $k = k_i$. Then if $N \geqq N_0$,

$$\mathbb{P}(A_0 \cap B \cap C_i(N)) \leqq \mathbb{P}\left(\bigcup_{j=1}^{k_i} (A_0 \cap D_j(N) \cap A_j^c) \right)$$

$$\leqq k_i \sup_{j \leqq k_i} \mathbb{P}(A_0 \cap D_j(N) \cap A_j^c) \leqq \varepsilon k_i / n . \tag{4.11}$$

By Proposition 4.7, taking $N = \max(N_0, 2k_i + 1)$, we can find j_{i+1} and k_{i+1} such that

$$\begin{aligned} \mathbb{P}(A_0 \cap B \cap B_{i+1} \cap C_i^c(N)) &< \rho \mathbb{P}(A_0 \cap B \cap C_i^c(N)) \\ &\leqq \rho \mathbb{P}(A_0 \cap B) . \end{aligned} \tag{4.12}$$

By (4.11),

$$\mathbb{P}(A_0 \cap B \cap B_{i+1} \cap C_i(N)) \leqq \rho[\kappa_3 \rho^i / n - \mathbb{P}(A_0 \cap B)]/4 . \tag{4.13}$$

Adding (4.12) and (4.13) and using the induction hypothesis,

$$\mathbb{P}(A_0 \cap B \cap B_{i+1}) < \rho \left[\frac{\kappa_3 \rho^i / n}{4} + \frac{3\mathbb{P}(A_0 \cap B)}{4} \right] < \kappa_3 \rho^{i+1} / n ,$$

which is (4.10) for $i + 1$. □

Proposition 4.9. *Let* $\beta > 0$. *For* n *sufficiently large,*

$$\mathbb{P}\left(\bigcup_{j=1}^{n} A_j \right) \leqq \beta .$$

Proof. Take I so that

$$\mathbb{P}(A_0 \cap B_1 \cap \ldots \cap B_I) \leqq \beta/2n\,. \tag{4.14}$$

Let $\gamma = \beta/2\kappa_3$. Take n large enough so that $\gamma n \geqq 2k_I$. By applying translation invariance to the estimate (4.14), we see that the proofs of Propositions 4.2–4.8 are still valid as long as $j \leqq n(1-\gamma)$ and we obtain

$$\mathbb{P}(A_j \cap A_{j+1}^c \cap \ldots \cap A_n^c) \leqq \beta/2n\,.$$

On the other hand, if $j \geqq (1-\gamma)n$, then by Proposition 4.1 we have

$$\mathbb{P}(A_j \cap A_{j+1}^c \cap \ldots \cap A_n^c) \leqq \mathbb{P}(A_j) \leqq \kappa_3/n\,.$$

We then obtain

$$\begin{aligned}\mathbb{P}\left(\bigcup_{j=1}^{n} A_j\right) &= \mathbb{P}\left(\bigcup_{j=1}^{n}(A_j \cap A_{j+1}^c \cap \ldots \cap A_n^c)\right)\\ &\leqq \sum_{j=1}^{n(1-\gamma)} \beta/2n + \sum_{j=n(1-\gamma)+1}^{n} \kappa_3/n\\ &\leqq \beta/2 + \kappa_3\gamma n/n = \beta\,.\end{aligned}$$

The proposition is proved. □

5. Slow points

In this section we prove that "critical" slow points do not exist. Let

$$E(n) = \{\text{there exists } r \in [0,n) \text{ such that } |X_t - X_r| \leqq \sqrt{t-r}, r \leqq t \leqq 2Un\}\,.$$

Proposition 5.1. *If* $\mathbb{P}(S(1) \neq \phi) > 0$, *then there exists* c_1 *such that* $\mathbb{P}(E(n)) > c_1$ *for all* n.

Proof. If $\mathbb{P}(S(1) \neq \phi) > 0$, then there exists an integer m such that Brownian motion has a slow point in the interval $[m, m+1)$ with positive probability. By translation invariance, Brownian motion has a slow point in $[0,1)$ with positive probability. There must exist a rational h such that the event

$$\{\text{there exists } r \in [0,1) \text{ such that } |X_t - X_r| \leqq \sqrt{t-r}, r \leqq t \leqq r+h\}$$

has positive probability. As in the first two lines of this proof, there exists $c_1 > 0$ such that

$$\mathbb{P}(\text{there exists } r < h/2U \text{ such that } |X_t - X_r| \leqq \sqrt{t-r}, r \leqq t < r+h) \geqq c_1\,.$$

An easy scaling argument (cf. Proposition 2.10) shows that if X_t has a slow point at time r, then for all a, X_{at} has one at time ar. Applying this with $a = 2Un/h$, we conclude that with probability at least c_1 that there exists $r < n$ such that $X_t - X_r$ lies within square root boundaries for a time at least $2Un$. But that event is $E(n)$. □

Recall the statement of our main result.

Theorem 1.1. *With probability one* $S(1) = \emptyset$.

Proof. Suppose not. Then by Proposition 5.1 there exists $c_1 > 0$ such that $\mathbb{P}(E(n)) \geqq c_1$ for all n. If $\omega \in E(n)$, then there exists $r < n$ such that $|X_t - X_r| \leqq \sqrt{t-r}$ for $r \leqq t \leqq 2Un$. (r, of course, depends on ω.) Let j be the first integer greater than r. Then

$$|X_j - X_r| \leqq \sqrt{j-r} \leqq 1 .$$

If $Un \geqq t \geqq j+1$, then

$$\begin{aligned} |X_t - X_j| &\leqq |X_t - X_r| + |X_r - X_j| \\ &\leqq \sqrt{t-r} + 1 \leqq \sqrt{1+t-j} + 1 \leqq 2 + \sqrt{t-j} . \end{aligned}$$

This means that $\omega \in A_j$. So $E(n) \subseteq \bigcup_{j=1}^{n} A_j$. Let $\beta = c_1/2$. By Proposition 4.9, the probability of $E(n)$ must be less than $c_1/2$ if n is sufficiently large, a contradiction. Therefore we must have $\mathbb{P}(S(1) \neq \phi) = 0$. □

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