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Entropy of First Return Partitions of a Markov Chain

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Summary. We consider the first return time distributions for each state in a Markov chain and show that finiteness of entropy of these distributions is a class property for recurrent and transient classes.

1. Introduction

In this note, we answer affirmatively the question raised in [2] concerning the finiteness of entropy for the first return time distributions of Markov chains as a class property. The interest of our result lies in the null recurrent and transient classes since it is known that the finite mean return time of a positive recurrent state implies that the first return distribution has finite entropy. On the other hand, it is easy to construct Markov chains whose first return distribution to a given state has infinite entropy; indeed, it is possible to construct a chain with any given first return distribution to a fixed state, cf. [1] p. 64.

In Section 2, we derive some bounds on entropy which are applied in Section 3 to prove our probabilistic result.

2. Preliminaries

Let $(\Omega, \mathscr{B}, \mu)$ be a σ-finite measure space. To a partition $\mathscr{A}=\{A_i\}_{i=0}^{\infty}$ of a set A, $\mu(A)<\infty$, is associated a sequence $f=\{f_i\}_{i=0}^{\infty}$ with $f_i=\mu(A_i)$ $i=0, 1, \ldots$. The *entropy* of f is the *entropy* of $\mathscr{A}$

$$H(f)=H(\mathscr{A})=-\sum_{i=1}^{\infty} f_i \log f_i. \tag{1}$$

(The base of the logarithm is usually taken to be 2; $0 \log 0=0$; there are no difficulties in definition (1) since at most a finite number of terms can be negative.) The *norm* $|f|=\sum f_i$; the convolution of f and g is $f*g$, i.e., $(f*g)_n=\sum_{i=0}^{n} f_{n-i} g_i$ and f^{*k} is the k-convolution of f with itself.

Lemma 1. *Let f, g be sequences. Then there is a constant C, depending only on $|f|$, $|g|$ and the base of the logarithm such that*

$$\operatorname{Max}(H(f), H(g))-C \leqq H(f+g) \leqq H(f)+H(g), \tag{2}$$

in particular

$$H(f+g)<\infty \qquad \text{if and only if } H(f)<\infty \text{ and } H(g)<\infty.$$

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Proof. The function $-\log x$ is decreasing and $-x\log x$ is increasing for $x\in[0, 1/e]$. For each n,

$$-(f_n+g_n)\log(f_n+g_n)=-f_n\log(f_n+g_n)-g_n\log(f_n+g_n)$$
$$\leqq -f_n\log f_n - g_n\log g_n,$$

while for n sufficiently large, $f_n+g_n\in[0, 1/e]$ and

$$-\max(f_n, g_n)\log\max(f_n, g_n)\leqq -(f_n+g_n)\log(f_n+g_n).$$

Lemma 2. *If f and g are sequences, then*

$$\max(f_{i_0}H(g), g_{j_0}H(f))-C\leqq H(f*g)\leqq |f|\,H(g)+|g|\,H(f), \tag{3}$$

where f_{i_0}, g_{j_0} are arbitrary non zero elements of f, g. In particular, we conclude $H(f*g)<\infty$ if and only if $H(f)<\infty$ and $H(g)<\infty$.

Proof. By the monotonicity of the logarithm function,

$$H(f*g)=-\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}f_i g_{n-i}\right)\log\left(\sum_{i=0}^{n}f_i g_{n-i}\right)$$
$$\leqq \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}f_i g_{n-i}\log f_i g_{n-i}\right).$$

Now interchanging the order of summation yields

$$-\sum_{i=0}^{\infty}\sum_{n=i}^{\infty}f_i g_{n-i}(\log f_i+\log g_{n-i})=|g|\,H(f)+|f|\,H(g).$$

The lower bound is obtained by noting that for i_0, n sufficiently large

$$-\left(\sum_{i=0}^{n}f_i g_{n-i}\right)\log\left(\sum_{i=0}^{n}f_i g_{n-i}\right)\geqq -f_{i_0}g_{n-i_0}\log f_{i_0}g_{n-i_0}$$
$$\geqq -f_{i_0}g_{n-i_0}\log g_{n-i_0}$$

which summed on n gives us $f_{i_0}H(g)-C$. We similarly can obtain a bound involving $H(f)$.

Lemma 3. *If we consider the k-fold convolution of f, f^{*k}, then*

$$H(f^{*k})\leqq k\,|f|^{k-1}H(f). \tag{4}$$

Proof. We prove (4) by induction, noting that for $k=1$ we have equality. If (4) holds for some positive integer k, it follows from Lemma 2 that

$$H(f^{*(k+1)})=H(f*f^{*k})\leqq|f^{*k}|\,H(f)+|f|\,H(f^{*k})$$
$$=(k+1)\,|f|^k H(f).$$

Lemma 4. *If $|f|<1$, then $H(f)<\infty$ implies*

$$H\left(\sum_{k=0}^{\infty}f^{*k}\right)\leqq\frac{H(f)}{(1-|f|)^2}<\infty.$$

Proof. The result follows from (2), (4) and $\sum ka^{k-1}=(1-a)^{-2}$.

3. Main Result

We now apply the preceeding lemmas to obtain the following proposition. (We follow the standard notation and terminology of [1].)

Proposition. *The finiteness of the entropy of first return distributions* $f_{kk} = \{f_{kk}^n\}_{n=1}^{\infty}$ *is a class property for Markov chains.*

Proof. Let the states i and j communicate. It is easily verified probabilistically that for any two states h, k

$$f_{kk}^n = {}_h f_{kk}^n + ({}_k f_{kh} * f_{hk})(n) = {}_h f_{kk}^n + \left({}_k f_{kh}^* \left(\sum_{m=0}^{\infty} {}_k f_{hh}^{*m}\right) * {}_h f_{hk}\right)(n).$$

If $H(f_{ii}) < \infty$, our lemmas imply that (i) $H({}_j f_{ii}) < \infty$, (ii) $H({}_i f_{ij}) < \infty$, (iii) $H({}_i f_{jj}) < \infty$ and (iv) $H({}_j f_{ji}) < \infty$. Since i and j communicate, we assert that $|{}_j f_{ii}| < 1$. From Lemma 4 we conclude that

$$H\left(\sum_{m=0}^{\infty} {}_j F_{ii}^{*m}\right) < \infty.$$

This together with (ii) and (iv) implies

$$H\left({}_j f_{ji} * \left(\sum_{m=0}^{\infty} {}_j f_{ii}^{*m}\right) * {}_i f_{ij}\right) < \infty,$$

which together with (iii) completes the proof.

4. Remark

It is natural to ask whether our result can be extended to a more general transformation τ acting on a σ-finite measure space $(\Omega, \mathscr{B}, \mu)$. Using the "sky-scraper" construction of Kakutani [3], it is easy to construct a measure space $(\Omega, \mathscr{B}, \mu)$ and an automorphism τ on Ω such that there are two sets of finite measure with the first return partition of one having finite entropy while the first return partition of the other has infinite entropy.

References

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