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The Necessity that a Conditional Decision Procedure be almost everywhere Admissible*

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1. Introduction

Theorem 1, stated below, includes a number of statistical problems in which it may be desirable to consider conditional procedures. It is our aim to obtain a complete class theorem covering such conditional procedures.

We will first state the assumptions needed, then Theorem 1.1. We conclude this section with some discussion. Section 2 contains a proof of Theorem 1.1, and Section 3 contains applications to the study of invariant tests.

We assume X and Y are complete separable metric spaces with $\mathfrak{J}_X$ the σ-algebra of Borel subsets of X, and $\mathfrak{J}_Y$ the σ-algebra of Borel subsets of Y. We assume μ is a regular totally σ-finite measure on $\mathfrak{J}_X$, and that if $\omega\in\Omega$, λ_ω is a probability measure on $\mathfrak{J}_Y$ dominated by the probability measure λ. Given is a set $\{f_\omega, \omega\in\Omega\}$ of conditional density functions on $X\times Y$. It is assumed that if $\omega\in\Omega$ then f_ω is jointly measurable. It is assumed that if $y\in Y$ then $f_\omega(\cdot, y)\in L_1(X, \mathfrak{J}_X, \mu)$ and $\int f_\omega(x, y)\,\mu(dx)=1$.

Of the parameter and decision spaces it is assumed that Ω is a separable metric space and that the decision space $\mathfrak{D}$ is separable locally compact metric space. We require that both spaces be complete in their respective metrics. Loss will be measured by a continuous function $W: \mathfrak{D}\times\Omega\to[0,\infty)$. We assume that if $\omega\in\Omega$ then $\lim_{t\to\infty} W(t,\omega)=\infty$. We assume there exists a partition of Ω into subsets $\Omega_1, \ldots, \Omega_k$ such that on each Ω_i, the risk function

$$(1.1)\qquad r(\omega, \delta, y)=\iint W(t,\omega)\,\delta(dt, x, y)\,f_\omega(x, y)\,\mu(dx)$$

is a continuous function of ω, this being true for all $y\in Y$ and all randomized decision functions δ. Lastly, if $r(\omega,\delta)$ is defined by

$$(1.2)\qquad r(\omega,\delta)=\int r(\omega,\delta,y)\,\lambda_\omega(dy),$$

then we suppose $\sup_\omega |r(\omega,\delta_1)-r(\omega,\delta_2)|=0$ implies

$$(\mu\times\lambda)(\{(x,y)\,|\,\delta_1(\cdot,x,y)\neq\delta_2(\cdot,x,y)\})=0.$$

Theorem 1.1. *Let the above hypotheses hold. If the (randomized) decision procedure δ is admissible then for almost all $y[\lambda]$ the decision procedure $\delta(\cdot,\cdot,y)$ is admissible for the loss function W and family of density functions $\{f_\omega(\cdot, y), \omega\in\Omega\}$.*

In the case the measure λ_ω do not depend on ω, then introduction of the variable y is equivalent to introducing randomization before the experiment in the sense of Wald and Wolfowitz [10]. We have

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Corollary 1.2. *Given the hypotheses of Theorem* 1.1, *if the admissible decision procedure* $\delta(\cdot,\cdot,\cdot)$ *involves randomization before the experiment then for almost all* $y[\lambda]$ *the decision procedures* $\delta(\cdot,\cdot,y)$ *are admissible for the problem.*

In case the conditional densities $f_\omega(\cdot, y)$ are independent of the variable y, then introduction of the variable y is equivalent to taking another independently distributed observation. We obtain

Corollary 1.3. *Given the hypotheses of Theorem* 1.1, *if the decision procedure* $\delta(\cdot,\cdot,\cdot)$ *depends on independently distributed observations* x, y *then for almost all* $y[\lambda]$ $\delta(\cdot,\cdot,y)$ *is an admissible procedure depending on* x.

The conclusion of Corollary 1.3 is very similar to the idea of hyperadmissibility discussed by Hanuras [2].

If the family of density functions $f_\omega \dfrac{d\lambda_\omega}{d\lambda}$ is an exponential family then for a suitable choice of ω we obtain the complete class theorem of Matthes and Traux [5].

A large number of problems which arise in multivariate analysis are problems in which a locally compact transformation group $\mathfrak{I}$ acts on $\Omega \times X \times Y$. A combination of Theorem 1.1 with assumptions about $\mathfrak{I}$ leads to Theorem 3.1, a complete class theorem for nonrandomized invariant tests. We apply Theorem 3.1 to the multivariate T^2 test.

2. Proof of Theorem 1.1.

In this section we assume δ is a given statistical procedure such that $\lambda(\{y \mid \delta(\cdot,\cdot,y)$ is inadmissible$\})>0$. Measurability of the set in question is proven in Lemma 2.2.

We have assumed $\mathfrak{D}$ is a complete separable metric space. We let $\mathfrak{B}$ be the σ-algebra of Borel sets of $\mathfrak{D}$ and define $C_0(\mathfrak{D}, \mathbb{R})$ to be the Banach space of continuous functions on $\mathfrak{D}$ to $\mathbb{R}$ with limits at ∞, that is, if $f \in C_0(\mathfrak{D}, \mathbb{R})$ then there exists a number $f(\infty)$ such that if $\varepsilon>0$, for some compact subset $C \subset \mathfrak{D}$, $t \notin C$ implies $|f(\infty)-f(t)|<\varepsilon$. We write $f(\infty)=\lim_{t\to\infty} f(t)$, and in this notation our hypothesis about W states, if $\omega \in \Omega$, $W(\infty,\omega)=\lim_{t\to\infty} W(t,\omega)=\infty$.

Lemma 2.1. *Let* $\{\omega_n, n \geqq 1\}$ *be a countable dense subset of* Ω *such that if* $1 \leqq i \leqq k$ *then* $\{\omega_n, n\geqq 1\} \cap \Omega_i$ *is dense in* Ω_i. *Given* $\varepsilon>0$ *there exists a compact set* $C \subset Y$ *such that* $\varepsilon+\lambda(C)>\lambda(Y)$ *and such that the maps*

$$
\begin{aligned}
&y \to \int W(t,\omega_n)\,\delta(dt,x,y)\,f_{\omega_n}(x,y)\,\mu(dx),\\
&n \geqq 1,\\
&y \to \int g(t)\,\delta(dt,x,y)\,f_{\omega_n}(x,y)\,\mu(dx),\\
&n \geqq 1,\ g \in C_0(\mathfrak{D},\mathbb{R}),\\
&y \to \int g(t)\,\delta(dt,x,y)\,h(x)\,\mu(dx),\\
&g \in C_0(\mathfrak{D},\mathbb{R}),\ h \in L_1(X,\mathfrak{I}_X,\mu),\\
&y \to \int h(x)\,(f_{\omega_n}(x,y))^{\frac{1}{2}}\,\mu(dx),\\
&h \in L_2(X,\mathfrak{I}_X,\mu),
\end{aligned}
\tag{2.1}
$$

are continuous maps of C *to* $\mathbb{R}$.

Proof. Let $\{g_n, n \geqq 1\}$ be a countable dense subset of $C_0(\mathfrak{D}, \mathbb{R})$ and $\{h_n, n \geqq 1\}$ be a countable dense subset of $L_1(X, \mathfrak{F}_X, \mu) \cap L_2(X, \mathfrak{F}_X, \mu)$. Then the second, third and fourth functions in (2.1) are uniform limits of functions having the forms

$$(2.2) \qquad \begin{aligned} k_{1mn}(y) &= \int g_m(t)\, \delta(dt, x, y)\, f_{\omega_n}(x, y)\, \mu(dx), && m, n \geqq 1. \\ k_{2mn}(y) &= \int g_m(t)\, \delta(dt, x, y)\, h_n(x)\, \mu(dx), && m, n \geqq 1. \\ k_{3mn}(y) &= \int h_m(x) \big(f_{\omega_n}(x, y)\big)^{\frac{1}{2}}\, \mu(dx), && m, n \geqq 1. \end{aligned}$$

Suppose for each pair m, n the compact set C_{mn} satisfies

$$(2.3) \qquad \varepsilon (m+n+1)^{-1}\, 2^{-(m+n+1)} + \lambda(C_{mn}) > \lambda(Y) = 1,$$

and such that on C_{mn} the functions $k_{1mn}, k_{2mn}, k_{3mn}$ are continuous, together with the first functions of (2.1). Then

$$(2.4) \qquad \lambda\Big(\bigcap_{m,n} C_{mn}\Big) + \varepsilon \geqq \lambda(Y) = 1,$$

so that the set $\bigcap_{mn} C_{mn}$ satisfies the conclusion of the lemma for the functions (2.2), and by taking uniform limits, all functions (2.1).

By Lusin's theorem, cf. Munroe [6], p. 159, closed sets C_{mn} satisfying (2.3) exist. By Prohorov [7] compact sets $C'_{mn} \subset C_{mn}$ satisfying (2.3) exist. ||

Let $\{g_n, n \geqq 1\}$ and $\{h_n, n \geqq 1\}$ be the dense sequences described in the preceding proof. For any decision procedure $\varphi(\cdot, \cdot)$: $\mathfrak{B} \times X \to [0, 1]$, we define the metric

$$(2.5) \qquad \begin{aligned} d(\varphi_1, \varphi_2) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} &(\|g_i\|\, \|h_i\|)^{-1} 2^{-(i+j+1)} (i+j+1)^{-1} \\ &\cdot \Big| \iint g_i(t)\, h_j(x) \big(\varphi_1(dt, x) - \varphi_2(dt, x)\big)\, \mu(dx) \Big|. \end{aligned}$$

It is easy to show d is a metric and that on the set of decision procedures d metrizes the weak topology. In terms of the metric d we define

(2.6) $\mathfrak{R}_y$ = set of decision procedures φ defined on $\mathfrak{B} \times X$ such that φ is as good as $\delta(\cdot, \cdot, y)$.

$\mathfrak{R}_{y\varepsilon}$ = set of decision procedures φ defined on $\mathfrak{B} \times X$ such that $d(\varphi, \delta(\cdot, \cdot, y)) \geqq \varepsilon$ and such that $\varphi \in \mathfrak{R}_y$.

Lemma 2.2. *$\{y \mid \delta(\cdot, \cdot, y)$ is admissible$\}$ is measurable $\mathfrak{F}_Y$.*

Proof. We let $\{\omega_n, n \geqq 1\}$ be a parameter sequence such that $\{\omega_n, n \geqq 1\} \cap \Omega_i$ is dense in Ω_i, $i = 1, \ldots, k$. We let $V = \{v_m, m \geqq 1\}$ be an enumeration of all finite measures supported on finite subsets of $\{\omega_n, n \geqq 1\}$ such that v_m assigns only rational masses to points. If $n \geqq 1$ we let $V_n \subset V$ such that $v_m \in V_n$ if and only if $v_m(\{\omega_n\}) = 1$.

If $m \geqq 1$ we let $\psi^*_m(\cdot, \cdot, \cdot)$ be a Bayes decision procedure for v_m relative to the family of densities $\{f_\omega, \omega \in \Omega\}$ in $L_1(X \times Y, \mathfrak{F}_X \times \mathfrak{F}_Y, \mu \times \lambda)$. Our hypotheses are sufficient to guarantee ψ^*_m exists. By the theorem of Wald and Wolfowitz [10] there exists a family of nonrandomized procedures $\psi_m(\cdot, \cdot, \cdot)$: $X \times Y \times [0, 1] \to \mathfrak{D}$ such that for almost all $\alpha \in [0, 1]$ $\psi_m(\cdot, \cdot, \alpha)$ is a nonrandomized Bayes procedure. Thus for some $\alpha \in [0, 1]$, for almost all $x, y\, [\mu \times \lambda]$,

$$(2.7) \qquad \int W(\psi_m(x, y, \alpha), \omega)\, f_\omega(x, y)\, v_m(d\omega) = \inf_t \int W(t, \omega)\, f_\omega(x, y)\, v_m(d\omega).$$

By Fubini's theorem there exists a set $N_m \in \mathfrak{Z}_Y$ such that $\lambda(N_m)=0$ and if $y \notin N_m$ then $\psi_m(\cdot, y, \alpha)$ satisfies

$$\int W(\psi_m(x, y, \alpha), \omega) f_\omega(x, y)\, \mu(dx)\, v_m(d\omega) = \inf_\varphi \int W(t, \omega)\, \varphi(dt, x) f_\omega(x, y)\, \mu(dx)\, v_m(d\omega). \tag{2.8}$$

In the sequel we write $\psi_m(\cdot,\cdot): X \times Y \to \mathfrak{D}$ for a jointly measurable function satisfying (2.8), $y \notin \bigcup_{m=1}^{\infty} N_m$.

We apply the necessary and sufficient condition for admissibility of Stein-LeCam. See Farrell [1]. Because conditional risk functions are continuous, the given randomized procedure δ is admissible if and only if

(2.9) for all n, for all p, there exists m such that $v_m \in V_n$ and

$$\int \left[r(\omega, \delta(\cdot,\cdot, y), y) - r(\omega, \psi_m(\cdot, y), y)\right] v_m(d\omega) < 1/p.$$

The notation is that introduced in (1.1). The condition just written involves a countable number of conditions on measurable sets, so that the conclusion of the lemma follows. ||

Lemma 2.3. *Let* $Y_0 = \{y \mid \delta(\cdot,\cdot, y)$ *is inadmissible*$\}$. *Let* $Y_1 \subset Y_0$, $\lambda(Y_1) > 0$, *such that* Y_1 *is compact and on* Y_1 *the maps* (2.1) *are continuous. If* $y \in Y_1$ *then there exists* $\varepsilon(y) > 0$ *such that* $\mathfrak{R}_{y\varepsilon(y)}$ *is nonempty and compact in the weak topology on decision procedures. There exists a compact set* $Y_2 \subset Y_1$ *and* $\varepsilon > 0$ *such that if* $y \in Y_2$ *then* $\mathfrak{R}_{y\varepsilon}$ *is nonempty and compact.*

Proof. Since $\lim_{t\to\infty} W(t, \omega) = \infty$ for all $\omega \in \Omega$, it follows that $\mathfrak{R}_y$ is a weakly compact and a convex set. Further, if $y \in Y_0$ then $\mathfrak{R}_y$ contains some ψ essentially distinct from the given δ, so that for some $\varepsilon > 0$, $\mathfrak{R}_{y\varepsilon}$ is nonempty. It is clear from the metric d that if $\mathfrak{R}_{y\varepsilon} \neq \emptyset$ then $\mathfrak{R}_{y\varepsilon}$ is weakly compact.

Let $\varepsilon > 0$ be fixed. We let Y_1 be a compact subset of Y_0 such that $\lambda(Y_1) > 0$ and such that on Y_1 (2.1) holds. We prove $\{y \mid y \in Y_1$ and $\mathfrak{R}_{y\varepsilon} \neq \emptyset\}$ is a compact set by showing every sequence in this set has a convergent subsequence. Thus, let $\{y_n, n \geqq 1\} \subset \{y \mid y \in Y_1$ and $\mathfrak{R}_{y\varepsilon} \neq \emptyset\}$, let $\{y_{a_n}, n \geqq 1\}$ be a convergent subsequence, and let $y = \lim_{n\to\infty} y_{a_n}$. In $\mathfrak{R}_{y_{a_n}\varepsilon}$ we take ψ_{a_n}. Then

$$\int W(t, \omega_1)\, \psi_{a_n}(dt, x) f_{\omega_1}(x, y_{a_n})\, \mu(dx) \leqq \int W(t, \omega_1)\, \delta(dt, x, y) f_{\omega_1}(x, y_{a_n})\, \mu(dx), \tag{2.10}$$

$\omega_1 \in \{\omega_n, n \geqq 1\}$. Since by (2.1) the numbers on the right side of inequality (2.10) convergence, it follows that no mass escapes to ∞. Note that the last condition of (2.1) implies $f_{\omega_1}(\cdot, y_{a_n}) \to f_{\omega_1}(\cdot, y)$ in L_1. Thus, $\{\psi_{a_n}, n \geqq 1\}$ has a convergent subsequence $\{\psi_{b_n}, n \geqq 1\}$ converging weakly to ψ.

By the continuity assumption following (1.1), the continuity relations (2.1), and the definition of d given in (2.5), it follows at once that

$$d(\psi, \delta) \geqq \varepsilon \quad \text{and that if } \omega \in \Omega, \tag{2.11}$$

$$\int W(t, \omega)\, \psi(dt, x) f_\omega(x, y)\, \mu(d\omega) \leqq \int W(t, \omega)\, \delta(dt, x, y) f_\omega(x, y)\, \mu(d\omega).$$

Therefore $\psi \in \mathfrak{R}_{y\varepsilon}$ as was to be shown.

Since $Y_1 = \bigcup_{n=1}^{\infty} Y_1 \cap \{y \mid \mathfrak{R}_{y1/n} \neq \emptyset\}$, it follows there exists an integer n such that $\lambda(\{y \mid \mathfrak{R}_{y1/n} \neq \emptyset\}) > 0$. ||

Lemma 2.4. *There exists a compact subset $Y_3 \subset Y_2$, an $\varepsilon > 0$, and a function $\psi'(\cdot, \cdot, y) \in \mathfrak{R}_{y\varepsilon}$, $y \in Y_3$, such that $\lambda(Y_3) > 0$ and such that if $g \in C_0(\mathfrak{T}, R)$ and $h \in L_1(X, \mathfrak{J}_X, \mu)$ then*

(2.12) $$\int g(t)\, \psi'(dt, x, y)\, h(x)\, \mu(dx)$$

is a continuous function of y.

Proof. To simplify notation we let $\{k_n, n \geqq 1\}$ be an enumeration of the functions $\{g_i(\cdot)\, h_j(\cdot),\ i \geqq 1, j \geqq 1\}$. We make an induction on the subscript n. Choose $\varepsilon > 0$ as in Lemma 2.3 and let Y_2 be a compact set satisfying $\lambda(Y_2) > 0$, $Y_2 \subset \{y \mid \mathfrak{R}_{y\varepsilon} \neq \emptyset\}$, and on Y_2 the conditions (2.1) hold. Define

(2.13) $$s_1(y) = \inf\{\int k_1(t, x)\, \psi(dt, x)\, \mu(dx) \mid \psi \in \mathfrak{R}_{y\varepsilon}\}.$$

The infinum is attained. If $y_n \to y$, $y_n \in Y_2$, and $\psi_n \in \mathfrak{R}_{y_n\varepsilon}$, $\psi_n \to \psi$ weakly, then

(2.14) $$s_1(y) \leqq \liminf_{n \to \infty} s_1(y_n).$$

The function s_1 is a lower semicontinuous function, therefore s_1 is measurable. Since Y_2 is a compact set, by Lusin's theorem we may choose $Y_{31} \subset Y_2$ a compact set such that

(2.15) $\lambda(Y_2 - Y_{31}) < (\frac{1}{4})\, \lambda(Y_2)$ and on Y_{31}, s_1 is a continuous function.

Define

(2.16) $$\mathfrak{R}_{y\varepsilon 1} = \{\psi \mid \psi \in \mathfrak{R}_{y\varepsilon} \text{ and } s_1(y) = \int k_1(t, x)\, \psi(dt, x)\, \mu(dx)\}.$$

Proceed inductively. Suppose $\mathfrak{R}_{y\varepsilon 1}, \ldots, \mathfrak{R}_{y\varepsilon n}$, $Y_{31} \supset Y_{32} \supset \cdots \supset Y_{3n}$ have been defined. Let

(2.17) $$s_{n+1}(y) = \inf\{\int k_{n+1}(t, x)\, \psi(dt, x)\, \mu(dx) \mid \psi \in \mathfrak{R}_{y\varepsilon n}\}.$$

Then s_{n+1} is a lower semicontinuous function defined on the compact set Y_{3n} and we choose a compact subset $Y_{3(n+1)} \subset Y_{3n}$ such that $\lambda(Y_{3n} - Y_{3(n+1)}) < 2^{-(n+2)}\, \lambda(Y_2)$ and such that on $Y_{3(n+1)}$ the function s_{n+1} is continuous.

Define $Y_3 = \bigcap_{n=1}^{\infty} Y_{3n}$. Then $\lambda(Y_3) \geqq (\frac{1}{2})\, \lambda(Y_2)$. Let $\mathfrak{R}'_{y\varepsilon} = \bigcap_{n=1}^{\infty} \mathfrak{R}_{y\varepsilon n}$. Then $\mathfrak{R}'_{y\varepsilon}$ contains exactly one element ψ', $y \in Y_3$. For if ψ'_1 and $\psi'_2 \in \mathfrak{R}'_{y\varepsilon}$ then over a dense sequence of functions $\{g_i(\cdot)\, h_j(\cdot), i \geqq 1, j \geqq 1\}$,

(2.18) $$\int g_i(t)\, \psi'_1(dt, x)\, h_j(x)\, \mu(dx) = \int g_i(t)\, \psi'_2(dt, x)\, h_j(x)\, \mu(dx), \quad i \geqq 1,\ j \geqq 1.$$

Therefore ψ'_1 and ψ'_2 represent the same bilinear form. We let $\psi'(\cdot, \cdot, y)$ be the element of $\mathfrak{R}'_{y\varepsilon}$. If $n \geqq 1$ then $\psi'(\cdot, \cdot, y) \in \mathfrak{R}_{y\varepsilon n}$. Therefore $s_n(y) = \int k_n(t, x)\, \psi'(dt, x, y) \cdot \mu(dx)$ is continuous in y. Since this holds on a dense set of functions it follows that if $g \in C_0(\mathfrak{T}, \mathbb{R})$ and $h \in L_1(X, \mathfrak{J}_X, \mu)$ then $\int g(t)\, \psi'(dt, x, y)\, h(x)\, \mu(dx)$ is continuous. ||

Lemma 2.5. *Let Y_4 be a compact set and $\lambda(Y_4)>0$. Suppose ψ': $\mathfrak{B}\times X\times Y_4\to[0,1]$ satisfies*

(i) *if $y\in Y_4$ then $\psi'(\cdot,\cdot,y)$ is a conditional probability measure on $\mathfrak{B}\times X$;*

(ii) *if $g\in C_0(\mathfrak{T},\mathbb{R})$, $h\in L_1(X,\mathfrak{Z}_X,\mu)$ then $\int g(t)\,\psi'(dt,x,y)\,h(x)\,\mu(dx)$ is continuous in y.*

Then there exists a conditional probability measure π: $\mathfrak{T}\times X\times Y\to[0,1]$ such that if $g\in C_0(\mathfrak{T},\mathbb{R})$ and $h\in L_1(X\times Y,\mathfrak{Z}_X\times\mathfrak{Z}_Y,\mu\times\lambda)$ then

(2.19) $\int\lambda(dy)\int g(t)\,\psi'(dt,x,y)\,h(x,y)\,\mu(dx)=\iint g(t)\,\pi(dt,x,y)\,h(x,y)\,\mu(dx)\,\lambda(dy).$

Proof. The set of functions $h\in L_1(X\times Y,\mathfrak{Z}_X\times\mathfrak{Z}_Y,\mu\times\lambda)=L_1$ for which

$$\int g(t)\,\psi'(dt,x,y)\,h(x,y)\,\mu(dx)$$

is measurable in y is a monotone class of functions which is a vector space containing all functions $k_1(x)\,k_2(y)$, $k_1\in L_1(X,\mathfrak{Z}_X,\mu)$ and $k_2\in L_1(Y,\mathfrak{Z}_Y,\lambda)$. Therefore $\int\lambda(y)\int g(t)\,\psi'(dt,\lambda,y)\,h(x,y)\,\mu(dx)$ is well defined for all $g\in C_0(\mathfrak{T},\mathbb{R})$, $h\in L_1$ and clearly defines a continuous bilinear form b: $C_0(\mathfrak{T},\mathbb{R})\times L_1$. Thus b can be represented by a conditional probability measure π: $\mathfrak{B}\times X\times Y$ and π by definition satisfies (2.19). ||

Proof of Theorem 1.1. Suppose $Y_0=\{y\,|\,\varphi(\cdot,\cdot,y)$ is inadmissible$\}$.

By Lemma 2.2 the set Y_0 is measurable. We assume $\lambda(Y_0)>0$. By Lemmas 2.4 and 2.5 there exists an $\varepsilon>0$ and a compact set $Y_3\subset Y_0$ such that $\lambda(Y_3)>0$ and a conditional probability measure ψ: $\mathfrak{B}\times X\times Y\to[0,1]$ such that if $y\in Y_3$ then $\psi(\cdot,\cdot,y)\in\mathfrak{R}_{y\varepsilon}$. We define a decision procedure δ' as follows.

(2.20) If $y\in Y_3$ then $\delta'(\cdot,\cdot,y)=\psi(\cdot,\cdot,y)$;
if $y\notin Y_3$ then $\delta'(\cdot,\cdot,y)=\delta(\cdot,\cdot,y)$.

Then by construction, if $y\in Y$, and if $\omega\in\Omega$, then

(2.21) $\int W(t,\omega)\,\delta'(dt,x,y)\,f_\omega(x,y)\,\mu(dx)\leqq\int W(t,\omega)\,\delta(dt,x,y)\,f_\omega(x,y)\,\mu(dx).$

If both sides of (2.21) are integrated by the measure λ_ω we obtain, if $\omega\in\Omega$,

(2.22) $$r(\omega,\delta')\leqq r(\omega,\delta).$$

Since $\lambda(\{y\,|\,\delta(\cdot,\cdot,y)\neq\delta'(\cdot,\cdot,y)\})>0$, the uniqueness assumption following (1.2) implies there exists $\omega\in\Omega$ such that $r(\omega,\delta')<r(\omega,\delta)$.

3. Invariance

It is the purpose of this section to apply Theorem 1.1 to obtain a complete class theorem for invariant tests in the presence of nusiance parameters. A general result about conditional tests when $\left\{f_\omega\dfrac{d\lambda_\omega}{d\lambda},\ \omega\in\Omega\right\}$ is an exponential family has been given by Matthes and Traux [5] but these authors do not consider the invariance question. The question of characterizing a complete class in the presence of invariance has been considered by R. Schwartz, and our results have a relationship to the problems considered by Kiefer and Schwartz [3] and Schwartz [8, 9].

We take a problem somewhat more general than the exponential family problem. We let $\omega \in \Omega$ have the form $\omega = (\Theta, \tau)$. We suppose $\mathfrak{I}$ is a locally compact transformation group acting on $\Omega \times X \times Y$ by means of

$$t(\Theta, \tau, x, y) = (t_1 \Theta, t_2 \tau, t_3 x, t_4 y).$$

We suppose each function $(t, \Theta, \tau, x, y) \to t(\Theta, \tau, x, y)$ is jointly measurable in the five variables. We assume also

(3.1) if $M \in \mathfrak{J}_X$, if $t \in \mathfrak{I}$, then $\mu(t_3(M)) = 0$ if and only if $\mu(M) = 0$; if $M \in \mathfrak{J}_Y$, if $t \in \mathfrak{I}$, then $\lambda(t_4(M)) = 0$ if and only if $\lambda(M) = 0$.

(3.2) $(\Theta, t) \in \Omega$ and $t \in \mathfrak{I}$ then $(t_1 \Theta, t_2 \tau) \in \Omega$.

(3.3) The hypothesis set H_0 and alternative set H_1 are invariant under the group action.

(3.4) i) X is a finite dimensional Euclidean space and $\mathfrak{C}$ is the collection of closed convex subsets of X with interior together with the null set.

If C is a closed convex set we assume μ (boundary C) $= 0$.

(3.4) ii) If C_1 and C_2 are distinct closed convex sets in $\mathfrak{C}$ then $\mu(C_1 \oplus C_2) > 0$, where $\oplus$ stands for symmetric set difference.

(3.4) iii) To each prior probability measure on Ω there exists a Bayes test γ such that if $y \in Y$ then $\{x \mid \gamma(x, y) = 0\} \in \mathfrak{C}$.

(3.4) iv) If $\psi(\cdot, y)$ is a admissible test for H_0 *vs* H_1 relative to the family of density functions $\{f_\omega(\cdot, y), \omega \in \Omega\}$ then there exists $C \in \mathfrak{C}$ such that

$$\int |1 - \chi_C(x) - \psi(x, y)| \, \mu(dx) = 0.$$

(3.4) v) If $t \in \mathfrak{I}$ then $t_3 \mathfrak{C} = \mathfrak{C}$.

Theorem 3.1. *Suppose the hypotheses of Theorem* 1.1 *hold and that assumptions* (3.1)–(3.4) *hold. Let γ be an admissible invariant test. There exists a set value mapping $y \to C(y) \in \mathfrak{C}$ and an invariant test φ satisfying φ is nonrandomized and $C(y) = \{x \mid \varphi(x, y) = 0\}$ such that $\int |\gamma(x, y) - \varphi(x, y)| \, \mu(dx) \, \lambda(dy) = 0$. If $\mathfrak{I}$ acts singly transitively on Y then there exists an invariant function $t_{y3} x$ (defined below) and a set $C \in \mathfrak{C}$ such that if φ^* is defined by, $t_{y3} x \in C$ if and only if $\varphi^*(x, y) = 0$ and*

$$\varphi^{*2} = \varphi^*, \quad \textit{then} \quad \int |\gamma(x, y) - \varphi^*(x, y)| \, \mu(dx) = 0.$$

Proof. We begin by noting that an admissible test must be nonrandomized. For, by Theorem 1.1, if γ is admissible then for almost all $y [\lambda]$, the conditional tests $\gamma(\cdot, y)$ are admissible. By hypothesis (3.4) iv) $\gamma(\cdot, y)$ is nonrandomized. Thus, for almost all $y [\lambda]$, $\{x \mid \gamma(x, y) \neq \gamma^2(x, y)\}$ has μ measure zero. By Fubini's theorem, $\{(x, y) \mid \gamma(x, y) \neq \gamma^2(x, y)\}$ has $\mu \times \lambda$ measure zero.

Next, if γ is nonrandomized and $\gamma = \text{weak} \lim_{n \to \infty} \gamma_n$ then $\gamma = \lim_{n \to \infty}$ in measure γ_n. For let $h \in L_1$. We compute

$$\begin{aligned} &\lim_{n \to \infty} \int (\gamma(x, y) - \gamma_n(x, y))^2 \, h(x, y) \, \mu(dx) \, \lambda(dy) \\ &\quad \leqq 2 \int \gamma(x, y) \, h(x, y) \, \mu(dx) \, \lambda(dy) - 2 \int \gamma^2(x, y) \, h(x, y) \, \mu(dx) \, \lambda(dy) = 0. \end{aligned} \tag{3.5}$$

It is known that each admissible test γ is a weak limit of Bayes tests γ_n. See Farrell [1]. By the preceeding paragraph, and by taking a subsequence, we may suppose

(3.6) $$\gamma = \lim_{n\to\infty} \gamma_n \quad \text{with probability one}.$$

Then there exists a set $N \in \mathfrak{Z}_Y$, $\lambda(N)=0$, such that if $y \notin N$ then for almost all $x[\mu]$,

(3.7) $$\gamma(x, y) = \lim_{n\to\infty} \gamma_n(x, y).$$

We may choose the representative γ_n from the equivalence class of L_∞ functions to satisfy (3.4) iii). Thus let

(3.8) $$C_n(y) = \{x \mid \gamma_n(x, y) = 0\}.$$

We know from Farrell [1] that (3.7) and (3.8) require that if $y \notin N$ then there exists a set $C(y) \in \mathfrak{C}$ such that $C_n(y) \to C(y)$ at every interior and exterior point of $C(y)$. Therefore only the boundary of $C(y)$ is in question and

(3.9) $$\{x \mid \gamma(x, y) = 0\} \subset C(y), \qquad \{x \mid \gamma(x, y) = 1\} \supset X - C(y).$$

We redefine γ as follows for $y \notin N$.

(3.10) If $\mu(C(y)) = 0$ then define $\gamma^*(x, y) = 1$ for all x; if $\mu(C(y)) > 0$ then $\gamma^* = \gamma^{*2}$ and $C(y) = \{x \mid \gamma^*(x, y) = 0\}$.

We show that γ^* is a measurable function. First let $D(y) = \{x \mid \gamma(x, y) = 0\}$. Then $\mu(C(y)) = \mu(D(y))$ is $\mathfrak{Z}_Y$ measurable. Thus $N_1 = \{y \mid \mu(C(y)) = 0\}$ is measurable. We need not consider this case further. Suppose $y \notin N \cup N_1$. The problem is to take $\{(x, y) \mid \gamma(x, y) = 0, y \notin N \cup N_1\}$ and take the closure on each y-section of this set, showing that the resulting set is Borel. Since interior $C(y)$ = interior $D(y)$ and $C(y) \supset D(y)$, a simple set construction shows the desired result.

Therefore we may suppose every admissible test may be represented by a function γ satisfying $\gamma = \gamma^2$ and for almost all $y[\lambda]$, $\{x \mid \gamma(x, y) = 0\} \in \mathfrak{C}$.

We now adapt a proof of Lehmann [4], p. 225. We suppose ψ is an admissible invariant test, that γ satisfies the conditions of the preceeding paragraph, and that

(3.11) $$\int |\psi(x, y) - \gamma(x, y)| \mu(dx)\, \lambda(dy) = 0.$$

Thus we define $D(y) = \{x \mid \psi(x, y) = 0\}$ and $C(y) = \{x \mid \gamma(x, y) = 0\}$. We wish to replace γ by an invariant test à la Lehmann. First, note that for almost all $y[\lambda]$,

(3.12) $$\mu(D(y) \oplus C(y)) = 0.$$

We now show that for almost all $y[\lambda]$,

(3.13) $$t_3^{-1} C(t_4 y) = C(y).$$

Let $N_2 = \{y \mid \psi(\cdot, y)$ is inadmissible as a conditional test or (3.11) fails$\}$. Then if $t \in \mathfrak{Z}$, $\lambda(t_4^{-1} N_2 \cup N_2) = 0$, and if $y \notin t_4^{-1} N_2 \cup N_2$ then $\mu(D(t_4 y) \oplus C(t_4 y)) = 0$. Using (3.1), $\mu(t_3^{-1} D(t_4 y) \oplus t_3^{-1} C(t_4 y)) = \mu(D(y) \oplus t_3^{-1} C(t_4 y)) = 0$. Since $\mu(D(y) \oplus C(y))$

$=0$, it follows that $\mu(C(y)\oplus t_3^{-1}\,C(t_4\,y))=0$. Assumption (3.4) ii) then implies (3.13). And (3.13) implies that γ is almost invariant.

Let dt represent a probability measure on $\mathfrak{T}$ such that Haar measure and dt are mutually absolutely continuous. Following Lehmann let

$$X\times Y-N_3=\{(x,y)\,|\,\gamma(x,y)=\gamma(t_3\,x,t_4\,y)\text{ a.e. }[dt]\}. \tag{3.14}$$

We define a function h by

$$h(x,y,t)=\left|\int\gamma(t_3'\,x,t_4'\,y)\,dt'=\gamma(t_3\,x,t_4\,y)\right| \tag{3.15}$$

and define a set A by

$$A=\{(x,y)\,|\int h(x,y,t)\,dt=0\}. \tag{3.16}$$

Then A is an invariant set having measure 1. We define

$$\begin{aligned}\varphi(x,y)&=\int\gamma(t_3\,x,t_4\,y)\,dt, &&\text{if } (x,y)\in A,\\ &=0, &&\text{if } (x,y)\notin A.\end{aligned} \tag{3.17}$$

Then $\varphi(x,y)=\gamma(x,y)$ for almost all $(x,y)\,[\mu\times\lambda]$.

As previously shown, if $t\in\mathfrak{T}$ then for almost all $y\,[\lambda]$, $t_3^{-1}\,C(t_4\,y)=C(y)$. Thus we may choose $N_4\in\mathfrak{Y}_Y$, $\lambda(N_4)=0$, such that $y\notin N_4$ if and only if

$$\text{for almost all } s,t\in\mathfrak{T},\ \gamma(t_3\,x,t_4\,y)=\gamma(s_3\,x,s_4\,y). \tag{3.18}$$

As in Lehmann, op. cit., the set N_4 is invariant. Then,

$$\text{if } y\notin N_4,\ \varphi(x,y)=\int\gamma(t_3\,x,t_4\,y)\,dt=\chi_{C(s_4\,y)}(s_3\,x). \tag{3.19}$$

Therefore φ is an invariant test such that if $y\notin N_4$ then $\{x\,|\,\varphi(x,y)=0\}\in\mathfrak{C}$. That completes the proof of the first part.

Let then the group act in a singly transitive manner on Y. Take $y_0\in Y$. Let t_y be the element of $\mathfrak{T}$ such that $(t_y)_4(y)=y_0$. We define $f\colon X\times Y\to X$ by $f(x,y)=(t_y)_3(x)$. Then f is an invariant function and $f(x,y)\in C(y_0)$ if and only if $x\in C(y)$. ||

Example of the T^2-Problem. Let $X_1,\ldots,X_p$ be normal (Σ,Θ_1), $Y_1,\ldots,Y_q$ be normal (Σ,Θ_2) and $Z_1,\ldots,Z_r$ be normal $(\Sigma,0)$. We suppose these are $p+q+r$ independently distributed random vectors, each $1\times k$. The joint density is of exponential form. Let the transformations of the parameters be

$$(\Theta_1,\Theta_2,\Sigma)\to(A\,\Theta_1,A\,\Theta_2+a,A\,\Sigma\,A^T), \tag{3.20}$$

where A is a nonsingular lower triangular matrix with positive diagonal elements. If the testing problem is to test $\Theta_1=0$ against $\Theta_1\neq 0$ then the hypotheses of Theorem 3.1 are satisfied. Define

$$\begin{aligned}&y=q^{-1}\sum_{i=1}^{q}y_i,\qquad x=p^{-1}\sum_{i=1}^{p}x_i\quad\text{and}\\ &s=\sum_{i=1}^{p}x_i\,x_i^T+\sum_{i=1}^{q}y_i\,y_i^T+\sum_{i=1}^{r}z_i\,z_i^T-x\,x^T-y\,y^T.\end{aligned} \tag{3.21}$$

Let s_τ be the uniquely determined lower triangular matrix of positive diagonal elements such that $s_\tau s_\tau^T = s$. Then the function $t_{y3}x$ of the theorem may be taken to be the maximal invariant $(s_\tau)^{-1}x$, and every admissible test invariant under the triangular matrix group must be essentially of the form, accept H_0 if and only if $(s_\tau)^{-1}x \in C$, C a closed convex set, s_τ^T the transpose of s_τ.

We continue this example and consider tests invariant under the full linear group. Then in addition to the group of triangular matrices orthogonal transformations are allowed. With Θ_2 eliminated the group acts only on Σ and Θ_1 through matrix multiplication so $\Sigma \to A\Sigma A^T$ and $\Theta_1 \to A\Theta_1$. Let t_{y3} be as before and let t' correspond to transformation by A which is orthogonal. By Theorem 3.1 the acceptance region has the form, accept H_0 if and only if

$$x \in (t_3' t_{3s})^{-1} C(t_4' t_{4s} s) = t_{3s}^{-1} t_3'^{-1} C(t_4' s_0). \tag{3.22}$$

Let the choice of s_0 be the $k \times k$ identity matrix so that $t_4' s_0 = s_0$. Then the test satisfies $t_{3s} x \in UC$ for every $k \times k$ orthogonal matrix U. Because the family of distributions is complete, $UC = C$ follows (C is a closed convex set).

We summarize the discussion in a corollary.

Corollary 3.2. *In the T^2-problem, an admissible test invariant under translations on the nuisance parameters Θ_2 and under multiplication of lower triangular matrices must have essentially the form, accept H_0 if and only if $(s_\tau)^{-1} x \in C$, where C is a closed convex set. In addition, if the test is invariant under the full linear group then $UC = C$ for all $k \times k$ orthogonal matrices U.*

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