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Some Ratio Limit Theorems for a General State Space Markov Process*

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Summary. Consider a discrete time parameter Markov Process with stationary probability functions, a general state space X and the Harris recurrence condition. This then implies the existence and essential uniqueness of a sigma-finite stationary measure π. It is also assumed that the class of measurable sets $\mathscr{B}$ contains single point sets. Let $P^{(m)}(x, S)$ denote the m-step transition probability from x to $S \in \mathscr{B}$ and $p^{(m)}(x, \cdot)$, the component of $P^{(m)}(x, \cdot)$ which is absolutely continuous with respect to π. Let $\mathscr{S} = \{C\colon C \in \mathscr{B}$, for some $n \inf_{x, y \in C} p^{(n)}(x, y) > 0\}$ and $\mathscr{I} = \{n\colon \inf_{x, y \in C} p^{(n)}(x, y) > 0, C \in \mathscr{S}\}$. The paper here presented contains theorems of which the following is typical: *Theorem:* Let $S \in \mathscr{S}$ with $\pi(S) > 0$, measurable $B \subset S$, $\pi(B) > 0$ and $q \in B$ with $\lim_{m \to \infty} \int_B f(x) P^{(m+1)}(y, dx)/P^{(m)}(q, B) = \int_B f(x) \pi(dx)/\pi(B)$ uniformly in y, $y \in B$ for all non-negative measurable f. Then for all measurable $A \subset S$ with $\pi(A) > 0$, $k = 0, \pm 1, \pm 2, \ldots$ $\lim_{m \to \infty} P^{(m+k)}(x, A)/P^{(m)}(q, B) = \pi(A)/\pi(B)$ in measure π on S. If the g.c.d. $(\mathscr{I}) = 1$ and $\pi' \ll \pi$ with $\pi'(X) < \infty$ then the above limit holds in measure π' on X.

1. Introduction

The questions which we shall study are concerned with Markov Processes with a discrete time parameter, stationary transition probability functions and a general state space. In addition, we shall impose the recurrence condition assumed by Harris [6], referred to as condition C: There exists a sigma-finite measure μ defined on the space X and separable class of measurable sets $\mathscr{B}$ with $\mu(X) > 0$ for which $\mu(S) > 0$ implies that

$$P[\text{Visiting } S \text{ at some time} \mid \text{Starting from } x] = 1$$

for all $x \in X$, where P is the underlying probability measure. This then implies the existence of a unique (up to a constant factor) sigma-finite measure π on $(X, \mathscr{B})$ which will be discussed further in the next section.

In the ergodic theory of these processes, the convergence properties of $P^{(m)}(x, A)$, i.e., the m-step transition probability from x to A, have been investigated to quite an extent. The situation in which $\pi(X) < \infty$ is known through the works of Orey [14], and Jamison and Orey [12]. In these papers it has been shown that there exists a partition of X into measurable sets $C_0, C_1, \ldots, C_{d-1}, F$ where $d \geqq 1$ such that $\pi(F) = 0$ and for $x \in C_i$, $P(x, C_{i+1}) = 1$ where the subscripts are modulo d. Furthermore, if $\pi_k(A) = \pi(A \cap C_k)$ for $0 \leqq k \leqq d-1$ and if for any initial probability distribution φ we write

$$a_i(\varphi) = \lim_{m \to \infty} \int_X P^{(md)}(x, C_i)\, \varphi(dx), \qquad 0 \leqq i \leqq d-1$$

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then

$$\lim_{m\to\infty}\left[\int_X P^{(md+k)}(x,A)\,\varphi(dx)-d\sum_{i=0}^{d-1}a_i(\varphi)\,\pi_{k+1}(A)\right]=0$$

uniformly in sets A where the subscripts are integers modulo d, $0\leq k\leq d-1$.

The most recent results deal with the difficult case when $\pi(X)=\infty$. Under this assumption, Jain [10] has shown that $P^{(m)}(x,A)\to 0$ if $\pi(A)<\infty$. It is then only natural to consider the limit of the ratio of the probabilities.

In the context of a discrete space, Orey [15] considered the limit of

$$\frac{P^{(m+k)}(w,x)}{P^{(m)}(y,z)}$$

for all finite k and proved that it tended to $\pi(x)/\pi(z)$ under the assumption that

$$\frac{P^{(m+1)}(q,q)}{P^{(m)}(q,q)}\to 1.$$

This *strong ratio limit property* was discussed by Folkman and Port [3] who found equivalent conditions under which it held.

In Section 3 we prove a strong ratio limit theorem for a general state space, in particular, our results yield sufficient conditions for which

$$\lim_{m\to\infty}\frac{P^{(m+k)}(x,A)}{P^{(m)}(q,B)}=\frac{\pi(A)}{\pi(B)}\qquad\text{in measure }\pi'$$

where $\pi'\ll\pi$ and $\pi'(X)<\infty$. In the countable case, we find that our results strengthen those of Orey [15].

2. Definitions and Preliminary Results

X is taken to be a general state space and shall always be considered as such throughout the entire paper unless explicitly designated otherwise. Let $\mathscr{B}=\mathscr{B}(X)$ be a separable (countably generated) Borel field of subsets of X which contains single point sets. For $S\in\mathscr{B}$, let $\mathscr{B}(S)=\{T\colon T=S\cap B \text{ for some } B\in\mathscr{B}\}$.

Let the sequence $\{X_k, k\geq 0\}$ be a discrete parameter Markov Process with the stationary transition probability function $P(\cdot,\cdot)$ on the state space $(X,\mathscr{B})$.

For $m\geq 0$

$$P[\text{Going from } x \text{ to } S \text{ in } m \text{ steps without visiting } T]\equiv {}_TP^{(m)}(x,S).$$

Note that ${}_TP^{(k)}(x,S)=P^{(k)}(x,S)$ for $k=0, 1$ while for $m>1$

$${}_TP^{(m)}(x,S)=\int_{X-T}\overset{(m-1)}{\cdots}\int_{X-T}P(x,dx_1)\,P(x_1,dx_2)\dots P(x_{m-1},S)$$

$$P[\text{Ever reaching } S \text{ starting from } x]=\sum_{k=1}^{\infty}{}_SP^{(k)}(x,S).$$

In [7], Harris proved the following

Theorem. Let condition (C) hold. Then there exists a unique (up to a constant factor) sigma-finite measure π on $(X, \mathscr{B})$ such that

(1) $\mu \ll \pi$, i.e., $\pi(S)=0$ implies $\mu(S)=0$

(2) $\pi(S)=\int_X P^{(m)}(x, S)\,\pi(dx)$ for all $S\in\mathscr{B}$, $m\geqq 0$.

Furthermore

(3) $\pi(S)>0$ implies $\sum_{k=1}^{\infty} {}_S P^{(k)}(x, S)=1$ for all $x\in X$.

This theorem permits us to replace μ in condition (C) by π. π shall always refer to the "stationary" measure of this theorem whenever mentioned in this paper.

For $m\geqq 1$, let the absolutely continuous and singular component of $P^{(m)}(x,\cdot)$ with respect to π be denoted by $p^{(m)}(x,\cdot)$ and $P_0^{(m)}(x,\cdot)$ respectively. As is justified in Doob [2], we can assume that $P^{(m)}(\cdot,\cdot)$ is measurable in (x, y) on $(X\times X, \mathscr{B}\times\mathscr{B})$ and that for all integers $m\geqq 1$ and $k\geqq 1$

$$p^{(m+k)}(x, y)\geqq \int_X p^{(m)}(x, z)\, p^{(k)}(z, y)\,\pi(dz)$$

for all $x\in X$ and for all $y\in X$. Thus we may write for all $S\in\mathscr{B}$ and $m\geqq 1$

$$p^{(m)}(x, S)=\int_S p^{(m)}(x, y)\,\pi(dy)+P_0^{(m)}(x, S).$$

Let

$$\mathscr{S}=\{C\colon C\in\mathscr{B},\quad \inf_{x\in C,\, y\in C} p^{(n)}(x, y)>0 \text{ for some } n\}$$

and

$$\mathscr{I}(C)=\{n\colon \inf_{x\in C,\, y\in C} p^{(n)}(x, y)>0,\ C\in\mathscr{S}\}.$$

Whenever we refer to d in this paper, we shall mean the greatest common divisor of $\mathscr{I}(C)$. (Further on in this section, we show that d is independent of C.)

We use the notation

$$f_m(x)\xrightarrow[S]{\pi} f(x)$$

if $f_m(x)\to f(x)$ in measure π on S.

Proposition 2.1. *If $S\in\mathscr{B}$ with $\pi(S)>0$, then there exists $C\in\mathscr{S}$ such that $\pi(C)>0$ and $C\subset S$.*

Proof. See Theorem 2.1 of Orey [14].

Proposition 2.2. *There exists a unique integer d such that whenever $C\in\mathscr{S}$ with $\pi(C)>0$ then $\mathscr{I}(C)\neq\emptyset$, d is the greatest common divisor of $\mathscr{I}(C)$ and all sufficiently large multiples of d belong to $\mathscr{I}(C)$. Moreover, d is independent of C.*

Proof. This is done in Theorem 1 of Jain and Jamison [11] via Doob [2].

Theorem 2.1. *There exist sets $C_i \in \mathscr{S}$, $i=1,2,\ldots$ such that* $\pi\left(X-\bigcup_{i=1}^{\infty} C_i\right)=0$.

Proof. Assume $\pi(X)<\infty$. Let $\mathscr{S}_i=\{C\colon \pi(C\cap X_i)>0,\ C\in\mathscr{S}\}$ where $X_1=X$, $X_i=X-\bigcup_{k=1}^{i-1} C_k$ for $i=2,3,\ldots$ where $\{C_k\}$ will be determined later. Choose arbitrary $\varepsilon>0$. Let $\alpha_1=\sup\{\pi(C)\colon C\in\mathscr{S}_1\}$. Then there exists $C_1\in\mathscr{S}_1$ such that $\alpha_1\geqq\pi(C_1)>\alpha_1-\frac{\varepsilon}{2}$. For $k=2,3,\ldots$ use the following procedure: if $\pi\left(X-\bigcup_{i=1}^{k-1} C_i\right)=0$, let $a_j=0$ and $C_j=\emptyset$ for all $j\geqq k$; if $\pi\left(X-\bigcup_{i=1}^{k-1} C_i\right)>0$, then for $\alpha_k=\sup\{\pi(C\cap X_k)\colon C\in\mathscr{S}_k\}$ there exists $C_k\in\mathscr{S}_k$ such that $\alpha_k\geqq\pi(C_k)\geqq\alpha_k-\frac{\varepsilon}{2^k}$. Since $C_i\cap C_j=\emptyset$ for all $i\neq j$ by construction,

$$\pi\left(\bigcup_{i=1}^{k} C_i\right)=\sum_{i=1}^{k}\pi(C_i)>\sum_{i=1}^{k}\alpha_i-\varepsilon\sum_{i=1}^{k}2^{-i}.$$

Since $\pi\left(X-\bigcup_{i=1}^{k} C_i\right)\geqq 0$ for all k,

$$0\leqq\pi(X)-\pi\left(\bigcup_{i=1}^{\infty} C_i\right)<\pi(X)-\sum_{i=1}^{\infty}\alpha_i+\varepsilon.$$

Therefore

$$\sum_{i=1}^{\infty}\alpha_i<\pi(X)+\varepsilon<\infty.$$

Let $X_\infty=X-\bigcup_{i=1}^{\infty} C_i$ and $\mathscr{S}_\infty$ be defined accordingly. If $\pi(X_\infty)=\pi(X)-\pi\left(\bigcup_{i=1}^{\infty} C_i\right)>0$, then there exists a set $C'\in\mathscr{S}_\infty$, i.e., $\pi(C'\cap X_\infty)>0$. However, $\alpha_i\to 0$ implying that $\sup\{\pi(C\cap X_\infty)\colon C\in\mathscr{S}_\infty\}=0$, a contradiction. Therefore

$$\pi(X)=\pi\left(\bigcup_{i=1}^{\infty} C_i\right).$$

If $\pi(X)=\infty$, we may choose a measure π_0 such that $\pi_0(X)<\infty$ and for $S\in\mathscr{B}$, $\pi(S)=0$ if and only if $\pi_0(S)=0$. Using π_0 in the above argument, $\pi_0\left(X-\bigcup_{i=1}^{\infty} C_i\right)=0$ implying

$$\pi\left(X-\bigcup_{i=1}^{\infty} C_i\right)=0.$$

Theorem 2.2. *Let $C_i\in\mathscr{S}$ with $\pi(C_i)>0$, $i=1,2,\ldots$. Then if $d=1$, $\bigcup_{i=1}^{L} C_i\in\mathscr{S}$ for all finite $L>0$.*

Proof. Without loss of generality, we just have to show that $C_1\cup C_2\in\mathscr{S}$. $d=1$ implies that there exists k_i such that $n\in\mathscr{I}(C_i)$ for all $n\geqq k_i$, $i=1,2$. Let $k=\max(k_1,k_2)$. Let $\xi_i\in C_i$ and $\eta_i\in C_i$ for $i=1,2$. Then there exists $\varepsilon_i>0$ such that

$$\inf_{\xi_i\in C_i,\ \eta_i\in C_i} p^{(k)}(\xi_i,\eta_i)>\varepsilon_i.$$

We may choose integers M and N such that $P^{(M)}(x_1,C_2)>\delta_1$ for all $x_1\in A_1$ for some $A_1\in\mathscr{B}(C_1)$ with $\pi(A_1)>0$ and $P^{(N)}(x_2,C_1)>\delta_2$ for all $x_2\in A_2$ for some

$A_2 \in \mathscr{B}(C_2)$ with $\pi(A_2) > 0$, for some $\delta_i > 0$. Without loss of generality, let $M \geqq N$, hence, let $Q = 2k + M$. (If $N > M$, let $Q = 2k + N$.)

$$p^{(Q)}(\eta_1, \eta_2) \geqq \int_{x_2 \in C_2} \int_{x_1 \in A_1} p^{(k)}(\eta_1, x_1)\, P^{(M)}(x_1, dx_2)\, p^{(k)}(x_2, \eta_2)\, \pi(dx_1) > \varepsilon_1 \varepsilon_2 \delta_1 \pi(A_1),$$

$$p^{(Q)}(\eta_2, \eta_1) \geqq \int_{x_1 \in C_1} \int_{x_2 \in A_2} p^{(k)}(\eta_2, x_2)\, P^{(N)}(x_2, dx_1)\, p^{(k+M-N)}(x_1, \eta_1)\, \pi(dx_2)$$

$$> \varepsilon_1 \varepsilon_2 \delta_2 \pi(A_2),$$

$$p^{(Q)}(\xi_i, \eta_i) = p^{(2k+M)}(\xi_i, \eta_i) > \varepsilon_i, \quad i = 1, 2.$$

Therefore $\inf_{\xi \in C_1 \cup C_2,\, \eta \in C_1 \cup C_2} p^{(Q)}(\xi, \eta) > 0$ implying that

$$C_1 \cup C_2 \in \mathscr{S}.$$

3. An Individual Ratio Limit Theorem

The principal result of this section is Theorem 3.3 which represents the generalization of Orey's Theorem for discrete spaces ([15], Lemma 1). The techniques employed in order to prove Theorem 3.3 stem from the method used by Orey. The major part of this proof consisted of demonstrating the convergence of

$$A_m(x) = \frac{P^{(m)}(q, x)}{P^{(m)}(q, q)} = \frac{\sum_{k=1}^{m} P^{(m-k)}(q, q)\, {}_qP^{(k)}(q, x)}{P^{(m)}(q, q)} = \sum_{k=1}^{N} + \sum_{k=N+1}^{m}$$
$$= B_{N,m}(x) + C_{N,m}(x)$$

where $q \in X$, $x \in X$ and $m > N \geqq 1$. In order to do this, it followed that $\lim_{N \to \infty} \lim_{m \to \infty} C_{N,m}(x)$ had to be shown to exist. This situation was handled by finding $x \in X$ with $x \neq q$ and $j > 0$ such that ${}_qP^{(j)}(x, q) > 0$ implying that

$${}_qP^{(k)}(q, x) \leqq \frac{{}_qP^{(k+j)}(q, q)}{{}_qP^{(j)}(x, q)} \tag{$*$}$$

for all $k \geqq 1$. This then enabled him to show the required existence.

The main difficulty in generalizing this theorem was encountered with finding an appropriate analogy to $(*)$. This difficulty was overcome with the establishing of the propositions and lemmas preceding Theorem 3.1.

Proposition 3.1. *Let $S \in \mathscr{B}$ with $\pi(S) > 0$ and for some $\varepsilon > 0$ $\inf_{x \in S,\, y \in S} p^{(n)}(x, y) \geqq \varepsilon$. Then $\pi(S) < \infty$.*

Proof. For $x \in S$,

$$1 \geqq P^{(n)}(x, S) = \int_S P^{(n)}(x, dy) \geqq \int_S p^{(n)}(x, y)\, \pi(dy) \geqq \varepsilon \int_S \pi(dy) = \varepsilon\, \pi(S).$$

Therefore

$$\pi(S) \leqq \frac{1}{\varepsilon}.$$

Proposition 3.2. *Let $S \in \mathscr{B}$ with $\pi(S)>0$ and for some $\varepsilon>0$ $\inf_{x\in S,\, y\in S} p^{(n)}(x,y) \geqq \varepsilon$. Then for all $B \in \mathscr{B}(S)$ with $\pi(B)>0$ and for all $x \in S$ $\sum_{m=1}^{n} {}_B P^{(m)}(x,B) \geqq \varepsilon\,\pi(B)$.*

Proof.

$$P^{(n)}(x,B) = \sum_{m=1}^{n-1} \int_B {}_B P^{(m)}(x,dy)\, P^{(n-m)}(y,B) + {}_B P^{(n)}(x,B)$$

$$\leqq \sum_{m=1}^{n} {}_B P^{(m)}(x,B).$$

Hence

$$\sum_{m=1}^{n} {}_B P^{(m)}(x,B) \geqq P^{(n)}(x,B) \geqq \int_B p^{(n)}(x,y)\,\pi(dy) \geqq \varepsilon\,\pi(B).$$

Proposition 3.3. *Let $S \in \mathscr{B}$ with $\pi(S)>0$ and for some $\varepsilon>0$ $\inf_{x\in S,\, y\in S} p^{(n)}(x,y) \geqq \varepsilon$. For any $B \in \mathscr{B}(S)$ with $\pi(B)>0$, if*

$$S_1 = \left\{x\colon x\in S,\ {}_B P^{(1)}(x,B) \geqq \frac{\varepsilon\,\pi(B)}{n}\right\},$$

and

$$S_i = \left\{x\colon x \in S - \bigcup_{j=1}^{i-1} S_j,\ {}_B P^{(i)}(x,B) \geqq \frac{\varepsilon\,\pi(B)}{n}\right\} \qquad \text{for } i=2,3,\ldots,n$$

then it follows that:

1) $S_i \cap S_j = \emptyset$ *for all* $i \neq j$,

2) $\bigcup_{i=1}^{n} S_i = S$,

3) *for all $A \in \mathscr{B}(S)$ with $\pi(A)>0$, for all $x \in S$,*

$${}_B P^{(m)}(x,A) \leqq \frac{n}{\varepsilon\,\pi(B)} \sum_{i=0}^{n} {}_B P^{(m+i)}(x,B) \qquad \text{for all } m>0.$$

Proof. 1. Obvious.

2. $\bigcup_{i=1}^{n} S_i \subset S$. For $x \in S$, there exists a smallest k, $1 \leqq k \leqq n$ such that

$${}_B P^{(k)}(x,B) \geqq \frac{\varepsilon\,\pi(B)}{n}$$

by Proposition 3.2. Therefore, $x \in S_k$ implying that $S \subset \bigcup_{i=1}^{n} S_i$.

3. Let $A' = A - B$.

$$\sum_{i=1}^{n} {}_B P^{(m+i)}(x,B) \geqq \sum_{i=1}^{n} \int_{A'\cap S_i} {}_B P^{(m)}(x,dy)\, {}_B P^{(i)}(y,B)$$

$$\geqq \frac{\varepsilon\,\pi(B)}{n} \sum_{i=1}^{n} {}_B P^{(m)}(x, A'\cap S_i) = \frac{\varepsilon\,\pi(B)}{n}\, {}_B P^{(m)}(x,A').$$

${}_BP^{(m)}(x,B) \geqq {}_BP^{(m)}(x, A\cap B) \geqq \frac{\varepsilon\,\pi(B)}{n}\,{}_BP^{(m)}(x, A\cap B)$ since $1 \geqq \varepsilon\,\pi(B) \geqq \frac{\varepsilon\,\pi(B)}{n}$ by Proposition 3.2.

$$\begin{aligned}{}_BP^{(m)}(x,A) &= {}_BP^{(m)}(x,A') + {}_BP^{(m)}(x, A\cap B)\\ &\leqq \frac{n}{\varepsilon\,\pi(B)} \sum_{i=1}^{n} {}_BP^{(m+i)}(x,B) + \frac{n}{\varepsilon\,\pi(B)}\,{}_BP^{(m)}(x,B).\end{aligned}$$

Proposition 3.4. *Let $S\in\mathscr{B}$ with $\pi(S)>0$ and for some $\varepsilon>0$ $\inf_{x\in S,\, y\in S} p^{(n)}(x,y) \geqq \varepsilon$. For all $A\in\mathscr{B}(S)$ with $\pi(A)>0$, for all $B\in\mathscr{B}(S)$ with $\pi(B)>0$, for all $x\in S$*

$$\sum_{m=1}^{\infty} {}_BP^{(m)}(x,A) \leqq \frac{n(n+1)}{\varepsilon\,\pi(B)}.$$

Proof. ${}_BP^{(m)}(x,A) \leqq \frac{n}{\varepsilon\,\pi(B)} \sum_{i=0}^{n} {}_BP^{(m+i)}(x,B)$ for all $m>0$ by Proposition 3.3.

$$\begin{aligned}\sum_{m=1}^{\infty} {}_BP^{(m)}(x,A) &\leqq \frac{n}{\varepsilon\,\pi(B)} \sum_{m=1}^{\infty}\sum_{i=0}^{n} {}_BP^{(m+i)}(x,B)\\ &\leqq \frac{n}{\varepsilon\,\pi(B)} \sum_{i=0}^{n}\sum_{m=1}^{\infty} {}_BP^{(m)}(x,B) = \frac{n}{\varepsilon\,\pi(B)}\sum_{i=0}^{n} 1 = \frac{n(n+1)}{\varepsilon\,\pi(B)}.\end{aligned}$$

Definition. $W_{j,\varepsilon}^{B}(z) = \{x\colon {}_Bp^{(j)}(z,x) > \varepsilon,\ x\notin B\}$ where $B\in\mathscr{B}$.

Remark. By $\mathscr{L}_1^{+}(X,\mathscr{B},\pi)$, we shall mean the space of all non-negative, extended real-valued, π-integrable functions.

Lemma 3.1. *Let $B'\in\mathscr{B}$ and $\pi\big(W_{j,\varepsilon}^{B'}(z)\big)>0$ for some $j>0$, $\varepsilon>0$ and $z\in X$. Let $V\in\mathscr{B}\big(W_{j,\varepsilon}^{B'}(z)\big)$ with $\pi(V)>0$. If ρ is a probability measure on X such that $\rho\ll\pi$, and furthermore, if the Radon-Nikodym derivative of ρ with respect to π is bounded a.e. π on V, say by Q, then for all $m>0$ and for all $f\in\mathscr{L}_1^{+}(X,\mathscr{B},\pi)$*

$$\int_{x\in V}\int_{y\in X} f(y)\,{}_{B'}P^{(m)}(x,dy)\,\rho(dx) \leqq \frac{Q}{\varepsilon}\int_X f(y)\,{}_{B'}P^{(m+j)}(z,dy).$$

Proof. Let g be the Radon-Nikodym derivative of ρ with respect to π. Thus, $g\leqq Q$ a.e. π on V.

$$\begin{aligned}Q\int_X f(y)\,{}_{B'}P^{(m+j)}(z,dy) &\geqq Q\int_{y\in X} f(y)\int_{x\in V} {}_{B'}p^{(j)}(z,x)\,{}_{B'}P^{(m)}(x,dy)\,\pi(dx)\\ &\geqq \varepsilon Q\int_{y\in X} f(y)\int_{x\in V} {}_{B'}P^{(m)}(x,dy)\,\pi(dx)\\ &\geqq \varepsilon\int_{x\in V}\int_{y\in X} f(y)\,{}_{B'}P^{(m)}(x,dy)\,g(x)\,\pi(dx)\\ &\geqq \varepsilon\int_{x\in V}\int_{y\in X} f(y)\,{}_{B'}P^{(m)}(x,dy)\,\rho(dx).\end{aligned}$$

Lemma 3.2. *Let $A\in\mathscr{B}$ with $\pi(A)>0$. Then for all $z\in X$, there exists $m_z>0$ and $A_z^{*}\in\mathscr{B}(A)$ with $\pi(A_z^{*})>0$ such that*

$$P^{(m_z)}(z,A_z^{*}) = \int_{A_z^{*}} p^{(m_z)}(z,x)\,\pi(dx) > 0.$$

Proof. For any $z\in X$, suppose that for all m, $p^{(m)}(z,x)=0$ a.e. $x\in A$. Then

$$P^{(m)}(z,A)=P_0^{(m)}(z,A)+\int_A p^{(m)}(z,x)\,\pi(dx)=P_0^{(m)}(z,A).$$

Thus there exists $A_m(z)\in\mathscr{B}(A)$ with $\pi(A_m(z))=0$ such that $P_0^{(m)}(z,A)=P^{(m)}(z,A_m(z))$. This is true for all m, hence $P^{(m)}\left(z,A-\bigcup_{i=1}^{\infty}A_i(z)\right)=0$ for all m. Since we have a Harris Process and $\pi\left(A-\bigcup_{i=1}^{\infty}A_i(z)\right)=\pi(A)>0$, there exists some $m>0$ such that $P^{(m)}\left(z,A-\bigcup_{i=1}^{\infty}A_i(z)\right)>0$ which is a contradiction. Thus there exists $m_z>0$ and $A_z^*\in\mathscr{B}(A)$ with $\pi(A_z^*)>0$ such that

$$P^{(m_z)}(z,A_z^*)=\int_{A_z^*}p^{(m_z)}(z,x)\,\pi(dx)>0.$$

Lemma 3.3. *Let $A\in\mathscr{B}$ with $\pi(A)>0$ and $\pi(X-A)>0$. Then for all $B\in\mathscr{B}(X-A)$ with $\pi(B)>0$, there exists $z_0\in B$ and there exists $B'\in\mathscr{B}(B)$ with $\pi(B')>0$, $j>0$ and $\varepsilon>0$ such that*

$$\pi\left(A\cap W_{j,\varepsilon}^{B'}(z_0)\right)>0.$$

Proof. For any $z\in B$, Lemma 3.2 implies that there exists $A_z^*\in\mathscr{B}(A)$ with $\pi(A_z^*)>0$ and $m_z>0$ such that

$$P^{(m_z)}(z,A_z^*)=\int_{A_z^*}p^{(m_z)}(z,x)\,\pi(dx)>0.$$

Assume that for all $z\in B$, for all m, for all $B'\in\mathscr{B}(B)$ with $\pi(B')>0$ that ${}_{B'}p^{(m)}(z,x)=0$ a.e. $x\in A$.

For any $B'\in\mathscr{B}(B)$ with $\pi(B')>0$, $z\in B$, $k\geqq 1$ we have

$$P^{(k)}(z,B')=\int_{B'}p^{(k)}(z,x)\,\pi(dx)+P_0^{(k)}(z,B').$$

Therefore, there exists $B_k(z)\in\mathscr{B}(B')$ with $\pi(B_k(z))=0$ such that $P_0^{(k)}(z,B')=P^{(k)}(z,B_k(z))$. Hence

$$P^{(k)}(z,B'-B_k(z))=\int_{B'}p^{(k)}(z,x)\,\pi(dx).$$

We may thus choose $B_z^*\in\mathscr{B}\left(B'-\bigcup_{i=1}^{m_z-1}B_i(z)\right)$ with $\pi(B_z^*)>0$ and B_z^* as small as we like, the size to be determined later. We thus have

$$P^{(k)}(z,B_z^*)=\int_{B_z^*}p^{(k)}(z,x)\,\pi(dx)$$

for $k=1,2,\ldots,m_z-1$.

$$P^{(m_z)}(z,A_z^*)={}_{B_z^*}P^{(m_z)}(z,A_z^*)+\sum_{k=1}^{m_z-1}\int_{B_z^*}p^{(k)}(z,x)\,{}_{B_z^*}P^{(m_z-k)}(x,A_z^*)\,\pi(dx)$$

$${}_{B_z^*}P^{(m_z)}(z,A_z^*)=\int_{A_z^*}{}_{B_z^*}p^{(m_z)}(z,x)\,\pi(dx)=0\quad\text{by assumption}.$$

Therefore

$$P^{(m_z)}(z, A_z^*) = \sum_{k=1}^{m_z-1} \int_{B_z^*} p^{(k)}(z, x)\, {}_{B_z^*}P^{(m_z-k)}(x, A_z^*)\, \pi(dx).$$

If $B' - \bigcup_{i=1}^{m_z-1} B_i(z)$ has at least one atom, let us choose it for B_z^*, say b. (Note: D is an atom if $\pi(D)>0$ and $E\in\mathscr{B}(D)$ implies that $\pi(E)=0$ or $\pi(E-D)=0$. Since $\mathscr{B}$ is separable, it has a countable number of generators, say $\{B_i\}$. Consider the non-empty distinct sets of the form $\bigcap_{i=1}^{\infty} \bar{B}_i$ where $\bar{B}_i=B_i$ or B_i^c. Then, $D\supset \bigcap_{i=1}^{\infty} \bar{B}_i$ for some $\{\bar{B}_i\}$. Since point sets are measurable, $\bigcap_{i=1}^{\infty} \bar{B}_i=\{x\}$. For some $\alpha>0$, $\pi(D)=\alpha$. Thus for some $\bar{B}_i$, $i=1, 2, \ldots, n$ $\pi\left[D\cap\left(\bigcap_{i=1}^{n} \bar{B}_i\right)\right]=\alpha$ for all n. This implies that $\pi(\{x\})=\alpha$, i.e., the measure of an atom is concentrated at one point, the measure of the remaining points being zero. Hence we may think of an atom as essentially that one point.) Therefore

$$P^{(m_z)}(z, A_z^*) = \sum_{k=1}^{m_z-1} P^{(k)}(z, b)\, {}_bP^{(m_z-k)}(b, A_z^*)$$

$${}_bP^{(m_z-k)}(b, A_z^*) = {}_bP_0^{(m_z-k)}(b, A_z^*) + \int_{A_z^*} {}_bp^{(m_z-k)}(b, y)\, \pi(dy)$$

$$= {}_bP_0^{(m_z-k)}(b, A_z^*) \quad \text{by assumption.}$$

Thus there exists $A_k(b)\in\mathscr{B}(A_z^*)$ such that $\pi(A_k(b))=0$ and

$${}_bP_0^{(m_z-k)}(b, A_z^*) = {}_bP^{(m_z-k)}(b, A_k(b)).$$

Hence

$${}_bP^{(m_z-k)}(b, A_z^* - A_k(b))=0,$$

$$0<P^{(m_z)}(z, A_z^*) = P^{(m_z)}\left(z, A_z^* - \bigcup_{i=1}^{m_z-1} A_i(b)\right)$$

$$= \sum_{k=1}^{m_z-1} P^{(k)}(z, b)\cdot 0 = 0,$$

a contradiction.

If $B' - \bigcup_{i=1}^{m_z-1} B_i(z)$ has no atoms, then we may choose B_z^* sufficiently small with $\pi(B_z^*)>0$ such that

$$\sum_{k=1}^{m_z-1} \int_{B_z^*} p^{(k)}(z, x)\, \pi(dx) < P^{(m_z)}(z, A_z^*).$$

(Note: Our definition of separability and Theorem D, p. 56 of [5] imply the definition of separability of [16], p. 264 and hence the completeness of the subsequent metric space. We may choose a set $S\in\mathscr{B}\left(B' - \bigcup_{i=1}^{m_z-1} B_i(z)\right)$ with $0<\pi(S)<\infty$

since the measure is sigma-finite. Thus, the theorem on p. 264 of [16] applies implying the existence of a set $B_z^* \in \mathscr{B}(S)$ of arbitrarily small measure and $\pi(B_z^*)>0$.)

$$P^{(m_z)}(z, A_z^*) = \sum_{k=1}^{m_z-1} \int_{B_z^*} p^{(k)}(z,x)\, {}_{B_z^*}P^{(m_z-k)}(x, A_z^*)\, \pi(dx) < P^{(m_z)}(z, A_z^*),$$

a contradiction. Note that ${}_{B_z^*}P^{(m_z-k)}(x, A_z^*) \leqq 1$.

Hence, in either case, there exists some $j>0$, some $B' \in \mathscr{B}(B)$ with $\pi(B')>0$ and $z_0 \in B$ such that we have ${}_{B'}p^{(j)}(z_0, x)>0$ over a set $A' \in \mathscr{B}(A)$ with $0<\pi(A')<\infty$.

We may choose $\varepsilon>0$ sufficiently small such that

$$\int_{A'} {}_{B'}p^{(j)}(z_0, x)\, \pi(dx) > \varepsilon\, \pi(A'),$$

i.e.,

$$\int_{A'} [{}_{B'}p^{(j)}(z_0, x) - \varepsilon]\, \pi(dx) > 0.$$

Let $R = \{x\colon x \in A', {}_{B'}p^{(j)}(z_0, x) > \varepsilon\} = A' \cap W_{j,\varepsilon}^{B'}(z_0)$. If $\pi(R)=0$, then

$$\int_{A'} [{}_{B'}p^{(j)}(z_0, x) - \varepsilon]\, \pi(dx) = \int_R + \int_{A'-R} = \int_{A'-R} [{}_{B'}p^{(j)}(z_0, x) - \varepsilon]\, \pi(dx) \leqq 0$$

which is a contradiction.

Remark. If there are no atoms in B, then the result of Lemma 3.3 can be changed as follows: for all $z \in B$, there exists $B' \in \mathscr{B}(B)$ with $\pi(B')>0, j>0$ and $\varepsilon>0$ such that

$$\pi\big(A \cap W_{j,\varepsilon}^{B'}(z)\big) > 0.$$

Lemma 3.4. *Let $A \in \mathscr{B}$ with $\pi(A)>0$ and $\pi(X-A)>0$. Then for all $B \in \mathscr{B}(X-A)$ with $\pi(B)>0$, for $i=1,2,\ldots$ there exists $B_i \in \mathscr{B}(B)$ with $\pi(B_i)>0$ and there exists $z_i \in B$, $j_i>0$, $\varepsilon_i>0$ such that $\pi\left(A - \bigcup_{i=1}^{\infty} W_i\right) = 0$ where $W_i = W_{j_i,\varepsilon_i}^{B_i}(z_i) \cap A_i$, $A_1 = A$ and $A_j = A - \bigcup_{k=1}^{j-1} W_k$ for $j=2,3,\ldots$.*

Proof. Let $\mathscr{M}_i = \{W_{j,\varepsilon}^{B'}(z)\colon \pi\big(W_{j,\varepsilon}^{B'}(z) \cap A_i\big)>0$ where $j>0$, $\varepsilon>0$, $z \in B$, $B' \in \mathscr{B}(B)$, $\pi(B')>0\}$. $\mathscr{M}_i \neq \emptyset$ by Lemma 3.3 as long as $\pi(A_i)>0$. Choose arbitrary $\delta>0$. Let $\beta_k = \sup\{\pi\big(W_{j,\varepsilon}^{B'}(z) \cap A_k\big)\colon W_{j,\varepsilon}^{B'}(z) \in \mathscr{M}_k\}$. This proof now follows that of Theorem 2.1 letting δ, $\mathscr{M}_k$, W_k play an analogous role to ε, $\mathscr{S}_k$, C_k respectively.

Theorem 3.1. *Let $S \in \mathscr{B}$ with $\pi(S)>0$ and for some $\varepsilon>0$ $\inf_{x\in S, y\in S} p^{(n)}(x,y) \geqq \varepsilon$. Let there exist $B \in \mathscr{B}(S)$ with $\pi(B)>0$ and $q \in B$ for which*

$$\lim_{m\to\infty} \frac{P^{(m+1)}(y, B)}{P^{(m)}(q, B)} = 1$$

uniformly in y, $y \in B$. Let ρ be a probability measure on X such that $\rho \ll \pi$, and furthermore, let the Radon-Nikodym derivative of ρ with respect to π be bounded a.e. π on S. For $\pi(S-B)>0$, let $\mathscr{M} = \{W_{j,\varepsilon'}^{B'}(z)\colon \pi[W_{j,\varepsilon'}^{B'}(z) \cap (S-B)]>0$ where $j>0$, $\varepsilon'>0$, $z \in B$, $B' \in \mathscr{B}(B)$, $\pi(B')>0\}$. For $W_i \in \mathscr{M}$, $W_i' \in \mathscr{M}$ with $i=1,2,\ldots$ let $W \in \mathscr{B}\left(\bigcup_{i=1}^{L} W_i\right)$ with $\rho(W)>0$ and $W' \in \mathscr{B}\left(\bigcup_{i=1}^{M} W_i'\right)$ with $\rho(W')>0$ for any finite $L>0$, any finite

$M>0$. *Then for* $k=0, \pm 1, \pm 2, \ldots$

(a) $$\lim_{m\to\infty} \int_W \left| \frac{P^{(m+k)}(x,B)}{P^{(m)}(q,B)} - 1 \right| \rho(dx) = 0.$$

(b) $$\lim_{m\to\infty} \int_B \left| \frac{P^{(m+k)}(x,B)}{P^{(m)}(q,B)} - 1 \right| \rho(dx) = 0.$$

(c) $$\frac{P^{(m+k)}(x,B)}{P^{(m)}(q,B)} \xrightarrow[S]{\rho} 1.$$

(d) $$\lim_{m\to\infty} \frac{\int_W P^{(m+k)}(x,B)\,\rho(dx)}{\int_{W'} P^{(m)}(y,B)\,\rho(dy)} = \frac{\rho(W)}{\rho(W')}.$$

(e) *For* $B_i \in \mathscr{B}(B)$ *with* $\rho(B_i)>0$, $i=1,2$

$$\lim_{m\to\infty} \frac{\int_{B_1} P^{(m+k)}(x,B)\,\rho(dx)}{\int_{B_2} P^{(m)}(y,B)\,\rho(dy)} = \frac{\rho(B_1)}{\rho(B_2)}.$$

Proof. (a) Let $V_1 = W \cap W_1$ and $V_i = W \cap W_i - \bigcup_{k=1}^{i-1} W_k$ for $i=2,3,\ldots,L$. Thus

$$\lim_{m\to\infty} \int_W \left| \frac{P^{(m+k)}(x,B)}{P^{(m)}(q,B)} - 1 \right| \rho(dx) = \sum_{i=1}^{L} \lim_{m\to\infty} \int_{V_i} \left| \frac{P^{(m+k)}(x,B)}{P^{(m)}(q,B)} - 1 \right| \rho(dx).$$

Without loss of generality, we must show that

$$\lim_{m\to\infty} \int_{V_i} \left| \frac{P^{(m+k)}(x,B)}{P^{(m)}(q,B)} - 1 \right| \rho(dx) = 0.$$

$V_i \subset W_i = W_{j,\varepsilon'}^{B'}(z)$ for some $j>0$, $\varepsilon'>0$, $z\in B$, $B'\in\mathscr{B}(B)$, $\pi(B')>0$.

For $x\in S$, fixed k and $m>N\geqq 1$ we have

$$A_{m,k}(x) = \frac{P^{(m)}(x,B)}{P^{(m-k)}(q,B)} = \sum_{v=1}^{m-1} \int_{B'} \frac{{}_{B'}P^{(v)}(x,dy)\,P^{(m-v)}(y,B)}{P^{(m-k)}(q,B)} + \frac{{}_{B'}P^{(m)}(x,B)}{P^{(m-k)}(q,B)}$$

$$= \sum_{v=1}^{N} + \sum_{v=N+1}^{m} = B_{N,m,k}^{B'}(x) + C_{N,m,k}^{B'}(x),$$

$$\lim_{m\to\infty} \int_{V_i} |A_{m,k}(x)-1|\,\rho(dx) = \lim_{N\to\infty} \lim_{m\to\infty} \int_{V_i} |B_{N,m,k}^{B'}(x) + C_{N,m,k}^{B'}(x) - 1|\,\rho(dx)$$

$$\leqq \lim_{N\to\infty} \lim_{m\to\infty} \int_{V_i} |B_{N,m,k}^{B'}(x)-1|\,\rho(dx)$$

$$+ \lim_{N\to\infty} \lim_{m\to\infty} \int_{V_i} C_{N,m,k}^{B'}(x)\,\rho(dx).$$

Since $q \in B$, we may set $y=q$ in the hypothesis to get that

$$\lim_{m\to\infty} \frac{P^{(m+1)}(q,B)}{P^{(m)}(q,B)} = 1 \quad \text{implying} \quad \lim_{m\to\infty} \frac{P^{(m+k)}(q,B)}{P^{(m)}(q,B)} = 1 \qquad \text{for all } k.$$

$$\lim_{N\to\infty} \lim_{m\to\infty} B_{N,m,k}^{B'}(x) = \lim_{N\to\infty} \lim_{m\to\infty} \sum_{v=1}^{N} \int_{B'} \frac{{}_{B'}P^{(v)}(x,dy)\, P^{(m-v)}(y,B)}{P^{(m-k)}(q,B)}$$

$$= \sum_{v=1}^{\infty} \int_{B'} {}_{B'}P^{(v)}(x,dy) = \sum_{v=1}^{\infty} {}_{B'}P^{(v)}(x,B') = 1$$

for all $x \in S$. For fixed N, there exists $m_0(N)$ such that for all $m \geqq m_0$

$$\frac{P^{(m-i)}(y,B)}{P^{(m-k)}(q,B)} \leqq 2$$

for all $y \in B$, $i=1,2,\ldots,N$. For $m \geqq m_0$,

$$|B_{N,m,k}^{B'}(x) - 1| \leqq 2 \sum_{v=1}^{N} {}_{B'}P^{(v)}(x,B') + 1 \leqq 3.$$

Also

$$\lim_{m\to\infty} B_{N,m,k}^{B'}(x) = \sum_{v=1}^{N} {}_{B'}P^{(v)}(x,B').$$

Since $\rho(V_i) < \infty$, we may apply the Dominated Convergence Theorem twice to get that

$$\lim_{N\to\infty} \lim_{m\to\infty} \int_{V_i} |B_{N,m,k}^{B'}(x) - 1|\, \rho(dx) = \int_{V_i} \lim_{N\to\infty} \lim_{m\to\infty} |B_{N,m,k}^{B'}(x) - 1|\, \rho(dx) = 0.$$

$\lim_{m\to\infty} A_{m,k}(z) = 1$ for all $z \in B$ since

$$\lim_{m\to\infty} \frac{P^{(m)}(z,B)}{P^{(m-k)}(q,B)} = \lim_{m\to\infty} \frac{P^{(m)}(z,B)}{P^{(m-1)}(q,B)} \, \frac{P^{(m-1)}(q,B)}{P^{(m-k)}(q,B)} = 1.$$

Therefore $\lim_{N\to\infty} \lim_{m\to\infty} C_{N,m,k}^{B'}(x) = 0$ for all $z \in B$, for all $B' \in \mathscr{B}(B)$ with $\pi(B') > 0$.

$$\int_{V_i} C_{N,m,k}^{B'}(x)\, \rho(dx) = \int_{V_i} \left[\sum_{v=N+1}^{m-1} \int_{B'} \frac{{}_{B'}P^{(v)}(x,dy)\, P^{(m-v)}(y,B)}{P^{(m-k)}(q,B)} + \frac{{}_{B'}P^{(m)}(x,B)}{P^{(m-k)}(q,B)} \right] \rho(dx)$$

$$\leqq \frac{Q}{\varepsilon'} \left[\sum_{v=N+1}^{m-1} \int_{B'} \frac{{}_{B'}P^{(v+j)}(z,dy)\, P^{(m-v)}(y,B)}{P^{(m-k)}(q,B)} + \frac{{}_{B'}P^{(m+j)}(z,B)}{P^{(m-k)}(q,B)} \right]$$

for some $Q > 0$, applying Lemma 3.1.

For m sufficiently large,

$$\frac{P^{(m+j-k)}(q,B)}{P^{(m-k)}(q,B)} \leqq 1+\delta$$

given arbitrary $\delta > 0$ in which case

$$\begin{aligned}
&\int_{V_i} C^{B'}_{N,m,k}(x)\,\rho(dx) \\
&\leqq \frac{Q(1+\delta)}{\varepsilon'}\left[\sum_{v=N+j+1}^{m+j-1}\int_{B'} \frac{{}_{B'}P^{(v)}(z,dy)\,P^{(m+j-v)}(y,B)}{P^{(m+j-k)}(q,B)} + \frac{{}_{B'}P^{(m+j)}(z,B)}{P^{(m+j-k)}(q,B)}\right] \\
&\leqq \frac{Q(1+\delta)}{\varepsilon'}\, C^{B'}_{N+j,\,m+j,\,k}(z).
\end{aligned}$$

Hence $\lim_{N\to\infty}\lim_{m\to\infty}\int_{V_i} C^{B'}_{N,m,k}(x)\,\rho(dx)=0$ implying that

$$\lim_{m\to\infty}\int_{V_i} |A_{m,k}(x)-1|\,\rho(dx)=0.$$

(b) $$\lim_{m\to\infty}\int_B \left|\frac{P^{(m+k)}(x,B)}{P^{(m)}(q,B)}-1\right|\rho(dx)=\int_B \lim_{m\to\infty}\left|\frac{P^{(m+k)}(x,B)}{P^{(m)}(q,B)}-1\right|\rho(dx)=0.$$

(c) Choose the $W_i \in \mathscr{M}$ as defined in Lemma 3.4. Let $W=W'=\bigcup_{i=1}^{L} W_i$. (a) implies that $\frac{P^{(m+k)}(x,B)}{P^{(m)}(q,B)} \xrightarrow[W]{\rho} 1$. (b) implies $\frac{P^{(m+k)}(x,B)}{P^{(m)}(q,B)} \xrightarrow[B]{\rho} 1$. Lemma 3.4 implies that we may choose L sufficiently large such that for arbitrary $v>0$, $\pi(S-B-W)<\frac{v}{Q}$ implying that $\rho(S-B-W)\leqq Q\,\pi(S-B-W)<v$. Therefore

$$\frac{P^{(m+k)}(x,B)}{P^{(m)}(q,B)} \xrightarrow[S]{\rho} 1.$$

(d) (a) implies that $\lim_{m\to\infty}\int_W \frac{P^{(m+k)}(x,B)}{P^{(m)}(q,B)}\,\rho(dx)=\rho(W)$.

$$\begin{aligned}
&\lim_{m\to\infty}\frac{\int_W P^{(m+k)}(x,B)\,\rho(dx)}{\int_{W'} P^{(m)}(y,B)\,\rho(dy)} \\
&\quad=\lim_{m\to\infty}\frac{\int_W P^{(m+k)}(x,B)\,\rho(dx)}{P^{(m)}(q,B)}\;\frac{P^{(m)}(q,B)}{\int_{W'} P^{(m)}(y,B)\,\rho(dy)}=\frac{\rho(W)}{\rho(W')}.
\end{aligned}$$

(e) $$\lim_{m\to\infty}\frac{\int_{B_1} P^{(m+k)}(x,B)\,\rho(dx)}{\int_{B_2} P^{(m)}(y,B)\,\rho(dy)}=\lim_{m\to\infty}\frac{\int_{B_1}\frac{P^{(m+k)}(x,B)}{P^{(m-1)}(q,B)}\,\rho(dx)}{\int_{B_2}\frac{P^{(m)}(y,B)}{P^{(m-1)}(q,B)}\,\rho(dy)}=\frac{\rho(B_1)}{\rho(B_2)}.$$

Remark. The hypothesis of Theorem 3.1 may be further generalized by assuming that $q \in X$ and

$$\lim_{m\to\infty} \frac{P^{(m+1)}(q, B)}{P^{(m)}(q, B)} = 1$$

instead of $q \in B$.

Corollary 3.1. *Let there exist $B \in \mathscr{S}$ with $\pi(B) > 0$ and $q \in B$ for which*

$$\lim_{m\to\infty} \frac{P^{(m+1)}(y, B)}{P^{(m)}(q, B)} = 1$$

uniformly in y, $y \in B$. In addition, let $d = 1$. Let ρ be a probability measure on X such that $\rho \ll \pi$, and furthermore, let the Radon-Nikodym derivative of ρ with respect to π be bounded a.e. π on X. For $\pi(X - B) > 0$, let $\mathscr{M} = \{W_{j,\varepsilon'}^{B'}(z): \pi[W_{j,\varepsilon'}^{B'}(z) \cap (C - B)] > 0$ where $C \in \mathscr{S}$ with $\pi(C - B) > 0, j > 0, \varepsilon' > 0, z \in B, B' \in \mathscr{B}(B), \pi(B') > 0\}$. For $W_i \in \mathscr{M}$, $W_i' \in \mathscr{M}$ with $i = 1, 2, \ldots$ let $W \in \mathscr{B}\left(\bigcup_{i=1}^{L} W_i\right)$ with $\rho(W) > 0$ and $W' \in \mathscr{B}\left(\bigcup_{i=1}^{M} W_i'\right)$ with $\rho(W') > 0$ for any finite $L > 0$, any finite $M > 0$. Then for $k = 0, \pm 1, \pm 2, \ldots$

(a) $\displaystyle \lim_{m\to\infty} \int_W \left| \frac{P^{(m+k)}(x, B)}{P^{(m)}(q, B)} - 1 \right| \rho(dx) = 0.$

(b) $\displaystyle \frac{P^{(m+k)}(x, B)}{P^{(m)}(q, B)} \xrightarrow[X]{\pi'} 1$ *for any π' such that $\pi' \ll \pi$, $\pi'(X) < \infty$.*

(c) $\displaystyle \lim_{m\to\infty} \frac{\int_W P^{(m+k)}(x, B)\, \rho(dx)}{\int_{W'} P^{(m)}(y, B)\, \rho(dy)} = \frac{\rho(W)}{\rho(W')}.$

Proof. (a) and (c) Theorem 2.2 implies that $W \cup W' \cup B \in \mathscr{S}$ and hence Theorem 3.1 applies.

(b) Theorem 2.1 and (c) of Theorem 3.1 yield the desired result.

Theorem 3.2. *Let $S \in \mathscr{B}$ with $\pi(S) > 0$ and for some $\varepsilon > 0$ $\inf_{x\in S, y\in S} p^{(n)}(x, y) \geqq \varepsilon$. If there exists $B \in \mathscr{B}(S)$ with $\pi(B) > 0$ and $q \in S$ for which*

$$\lim_{m\to\infty} \int_B \frac{f(x)\, P^{(m+1)}(q, dx)}{P^{(m)}(q, B)} = \int_B \frac{f(x)\, \pi(dx)}{\pi(B)}$$

for all $f \in \mathscr{L}_1^+(B, \mathscr{B}(B), \pi)$ then for all $A \in \mathscr{B}(S)$ with $\pi(A) > 0$, for all $C \in \mathscr{B}(S)$ with $\pi(C) > 0, k = 0, \pm 1, \pm 2, \ldots$

$$\lim_{m\to\infty} \frac{P^{(m+k)}(q, A)}{P^{(m)}(q, C)} = \frac{\pi(A)}{\pi(C)}.$$

Proof. Without loss of generality (cf. Theorem 3.1), we just have to show that

$$\lim_{m\to\infty} \frac{P^{(m)}(q, A)}{P^{(m)}(q, B)} = \frac{\pi(A)}{\pi(B)}.$$

For $m > N \geqq 1$ we have

$$A_m(A) = \frac{P^{(m)}(q, A)}{P^{(m)}(q, B)} = \sum_{v=1}^{m-1} \int_B \frac{P^{(m-v)}(q, dx)\, {}_BP^{(v)}(x, A)}{P^{(m)}(q, B)} + \frac{{}_BP^{(m)}(q, A)}{P^{(m)}(q, B)}$$

$$= \sum_{v=1}^{N} + \sum_{v=N+1}^{m} = D_{N,m}(A) + E_{N,m}(A),$$

$$\lim_{N\to\infty} \lim_{m\to\infty} D_{N,m}(A) = \lim_{N\to\infty} \lim_{m\to\infty} \sum_{v=1}^{N} \int_B \frac{P^{(m-v)}(q, dx)\, {}_BP^{(v)}(x, A)}{P^{(m)}(q, B)},$$

$$\lim_{N\to\infty} \lim_{m\to\infty} D_{N,m}(A) = \sum_{v=1}^{\infty} \int_B \frac{{}_BP^{(v)}(x, A)\, \pi(dx)}{\pi(B)}$$

$$= \frac{\int_B \sum_{v=1}^{\infty} {}_BP^{(v)}(x, A)\, \pi(dx)}{\pi(B)} < \infty$$

by Proposition 3.4. Therefore $A_m(A)$ converges if and only if $\lim_{N\to\infty} \lim_{m\to\infty} E_{N,m}(A)$ exists. $A_m(B) = 1$ and $\lim_{N\to\infty} \lim_{m\to\infty} D_{N,m}(B) = 1$ implies that $\lim_{N\to\infty} \lim_{m\to\infty} E_{N,m}(B) = 0$.

$$E_{N,m}(A) = \sum_{v=N+1}^{m-1} \int_B \frac{P^{(m-v)}(q, dx)\, {}_BP^{(v)}(x, A)}{P^{(m)}(q, B)} + \frac{{}_BP^{(m)}(q, A)}{P^{(m)}(q, B)}$$

$$\leqq \sum_{v=N+1}^{m-1} \left[\frac{n}{\varepsilon\pi(B)} \sum_{i=0}^{n} \int_B \frac{P^{(m-v)}(q, dx)\, {}_BP^{(v+i)}(x, B)}{P^{(m)}(q, B)} \right]$$

$$+ \frac{n}{\varepsilon\pi(B)} \sum_{i=0}^{n} \frac{{}_BP^{(m+i)}(q, B)}{P^{(m)}(q, B)}$$

by Proposition 3.3. For m sufficiently large,

$$\frac{P^{(m+i)}(q, B)}{P^{(m)}(q, B)} \leqq (1+\delta)$$

given arbitrary $\delta > 0$, $i = 0, 1, \ldots, n$, in which case

$$E_{N,m}(A) \leqq \frac{n(1+\delta)}{\varepsilon\pi(B)} \sum_{i=0}^{n} \left[\sum_{v=N+i+1}^{m+i-1} \int_B \frac{P^{(m+i-v)}(q, dx)\, {}_BP^{(v)}(x, B)}{P^{(m+i)}(q, B)} + \frac{{}_BP^{(m+i)}(q, B)}{P^{(m+i)}(q, B)} \right]$$

i.e., $E_{N,m}(A) \leqq \dfrac{n(1+\delta)}{\varepsilon\pi(B)} \sum_{i=0}^{n} E_{N+i,m+i}(B)$.

Hence $\lim_{N\to\infty} \lim_{m\to\infty} E_{N,m}(A) = 0$. Therefore

$$\lim_{m\to\infty} A_m(A) = \frac{\int_B \sum_{v=1}^{\infty} {}_BP^{(v)}(x, A)\, \pi(dx)}{\pi(B)}.$$

It has been shown in Harris [7] that

$$\int_B \sum_{v=1}^{\infty} {}_B P^{(v)}(x, A)\, \pi(dx) = \pi(A)$$

hence,

$$\lim_{m\to\infty} \frac{P^{(m)}(q, A)}{P^{(m)}(q, B)} = \frac{\pi(A)}{\pi(B)}.$$

Corollary 3.2. *Let* $S \in \mathscr{B}$ *with* $\pi(S) > 0$ *and for some* $\varepsilon > 0$ $\inf_{x\in S,\, y\in S} p^{(n)}(x, y) \geqq \varepsilon$. *Also, let the probability measure be absolutely continuous. If there exists* $B \in \mathscr{B}(S)$ *with* $\pi(B) > 0$, $q \in S$ *and* $r \in B$ *for which*

$$\lim_{m\to\infty} \frac{p^{(m+1)}(q, x)}{p^{(m)}(q, r)} = 1$$

uniformly in x, $x \in B$, *then for all* $A \in \mathscr{B}(S)$ *with* $\pi(A) > 0$, *for all* $C \in \mathscr{B}(S)$ *with* $\pi(C) > 0$, $k = 0, \pm 1, \pm 2, \ldots$

$$\lim_{m\to\infty} \frac{P^{(m+k)}(q, A)}{P^{(m)}(q, C)} = \frac{\pi(A)}{\pi(C)}.$$

Proof. The results follow if

$$\lim_{m\to\infty} \int_B \frac{f(x)\, P^{(m+1)}(q, dx)}{P^{(m)}(q, B)} = \int_B \frac{f(x)\, \pi(dx)}{\pi(B)}$$

for all $f \in \mathscr{L}_1^+(B, \mathscr{B}(B), \pi)$.

$$\lim_{m\to\infty} \int_B \frac{f(x)\, P^{(m+1)}(q, dx)}{P^{(m)}(q, B)} = \lim_{m\to\infty} \frac{\int_B f(x) \dfrac{p^{(m+1)}(q, x)}{p^{(m)}(q, r)}\, \pi(dx)}{\int_B \dfrac{p^{(m)}(q, x)}{p^{(m-1)}(q, r)}\, \pi(dx)} \; \frac{p^{(m)}(q, r)}{p^{(m-1)}(q, r)}$$

$$= \frac{\int_B f(x)\, \pi(dx)}{\int_B \pi(dx)} \lim_{m\to\infty} \frac{p^{(m)}(q, r)}{p^{(m-1)}(q, r)} = \int_B \frac{f(x)\, \pi(dx)}{\pi(B)}.$$

Theorem 3.3. *Let* $S \in \mathscr{B}$ *with* $\pi(S) > 0$ *and for some* $\varepsilon > 0$ $\inf_{x\in S,\, y\in S} p^{(n)}(x, y) \geqq \varepsilon$. *Let there exist* $B \in \mathscr{B}(S)$ *with* $\pi(B) > 0$ *and* $q \in B$ *for which*

$$\lim_{m\to\infty} \int_B \frac{f(x)\, P^{(m+1)}(y, dx)}{P^{(m)}(q, B)} = \int_B \frac{f(x)\, \pi(dx)}{\pi(B)}$$

uniformly in y, $y \in B$, *for all* $f \in \mathscr{L}_1^+(B, \mathscr{B}(B), \pi)$. *Let* ρ *be a probability measure on* X *such that* $\rho \ll \pi$, *and furthermore, let the Radon-Nikodym derivative of* ρ *with respect to* π *be bounded a.e.* π *on* S. *For* $\pi(S - B) > 0$, *let* $\mathscr{M} = \{W_{j,\varepsilon'}^{B'}(z) \colon \pi[W_{j,\varepsilon'}^{B'}(z) \cap (S - B)] > 0$ *where* $j > 0$, $\varepsilon' > 0$, $z \in B$, $B' \in \mathscr{B}(B)$, $\pi(B') > 0\}$. *For* $W_i \in \mathscr{M}$, $W_i' \in \mathscr{M}$ *with* $i = 1, 2, \ldots$ *let* $W \in \mathscr{B}\left(\bigcup_{i=1}^{L} W_i\right)$ *with* $\rho(W) > 0$ *and* $W' \in \mathscr{B}\left(\bigcup_{i=1}^{M} W_i'\right)$ *with* $\rho(W') > 0$ *for* *any finite* $L>0$, *any finite* $M>0$. *Then for all* $A\in\mathscr{B}(S)$ *with* $\pi(A)>0$, *for all* $C\in\mathscr{B}(S)$ *with* $\pi(C)>0$, $k=0, \pm 1, \pm 2, \ldots$

(a) $$\lim_{m\to\infty}\int_W \left|\frac{P^{(m+k)}(x,A)}{P^{(m)}(q,B)}-\frac{\pi(A)}{\pi(B)}\right|\rho(dx)=0.$$

(b) $$\lim_{m\to\infty}\int_B \left|\frac{P^{(m+k)}(x,A)}{P^{(m)}(q,B)}-\frac{\pi(A)}{\pi(B)}\right|\rho(dx)=0.$$

(c) $$\frac{P^{(m+k)}(x,A)}{P^{(m)}(q,B)}\xrightarrow[S]{\rho}\frac{\pi(A)}{\pi(B)}.$$

(d) $$\lim_{m\to\infty}\frac{\int_W P^{(m+k)}(x,A)\,\rho(dx)}{\int_{W'} P^{(m)}(y,C)\,\rho(dy)}=\frac{\rho(W)\,\pi(A)}{\rho(W')\,\pi(C)}.$$

(e) *For* $B_i\in\mathscr{B}(B)$ *with* $\rho(B_i)>0$, $i=1,2$

$$\lim_{m\to\infty}\frac{\int_{B_1} P^{(m+k)}(x,A)\,\rho(dx)}{\int_{B_2} P^{(m)}(y,C)\,\rho(dy)}=\frac{\rho(B_1)\,\pi(A)}{\rho(B_2)\,\pi(C)}.$$

Proof. (a) Let $V_1=W\cap W_1$ and $V_i=W\cap W_i-\bigcup_{k=1}^{i-1} W_k$ for $i=2,3,\ldots,L$. Without loss of generality (cf. Theorem 3.1) we just have to show that

$$\lim_{m\to\infty}\int_{V_i}\left|\frac{P^{(m+k)}(x,A)}{P^{(m)}(q,B)}-\frac{\pi(A)}{\pi(B)}\right|\rho(dx)=0.$$

$V_i\subset W_i=W_{j,\varepsilon'}^{B'}(z)$ for some $j>0$, $\varepsilon'>0$, $z\in B$, $B'\in\mathscr{B}(B)$, $\pi(B')>0$.

For $y\in S$, fixed k and $m>N\geqq 1$ we have

$$\begin{aligned}A_{m,k}(y,A)&=\frac{P^{(m)}(y,A)}{P^{(m-k)}(q,B)}=\sum_{v=1}^{m-1}\int_B\frac{P^{(m-v)}(y,dx)\,{}_BP^{(v)}(x,A)}{P^{(m-k)}(q,B)}+\frac{{}_BP^{(m)}(y,A)}{P^{(m-k)}(q,B)}\\&=\sum_{v=1}^{N}+\sum_{v=N+1}^{m}=D_{N,m,k}(y,A)+E_{N,m,k}(y,A),\end{aligned}$$

$$\begin{aligned}\lim_{N\to\infty}\lim_{m\to\infty}D_{N,m,k}(y,A)&=\lim_{N\to\infty}\lim_{m\to\infty}\sum_{v=1}^{N}\int_B\frac{P^{(m-v)}(y,dx)\,{}_BP^{(v)}(x,A)}{P^{(m-k)}(q,B)}\\&=\sum_{v=1}^{\infty}\lim_{m\to\infty}\int_B {}_BP^{(v)}(x,A)\,\frac{P^{(m-v)}(y,dx)}{P^{(m-v-1)}(q,B)}\,\frac{P^{(m-v-1)}(q,B)}{P^{(m-k)}(q,B)}\\&=\sum_{v=1}^{\infty}\int_B\frac{{}_BP^{(v)}(x,A)\,\pi(dx)}{\pi(B)}=\frac{\pi(A)}{\pi(B)}\end{aligned}$$

uniformly in y, $y\in B$ (cf. Theorem 3.2).

$$\lim_{m\to\infty}A_{m,k}(y,B)=\lim_{m\to\infty}\int_B\frac{P^{(m)}(y,dx)}{P^{(m-1)}(q,B)}\,\frac{P^{(m-1)}(q,B)}{P^{(m-k)}(q,B)}=1$$

uniformly in y, $y\in B$. Therefore $\lim_{N\to\infty}\lim_{m\to\infty} E_{N,m,k}(y,B)=0$ uniformly in y, $y\in B$.

$$E_{N,m,k}(y,A)\leqq \frac{n(1+\delta)}{\varepsilon\,\pi(B)}\sum_{i=0}^{n} E_{N+i,m+i,k}(y,B)$$

(cf. Theorem 3.2) for m sufficiently large, given arbitrary $\delta>0$. Therefore

$$\lim_{N\to\infty}\lim_{m\to\infty} E_{N,m,k}(y,A)=0$$

uniformly in y, $y\in B$. We finally have that

$$\lim_{m\to\infty} A_{m,k}(y,A)=\frac{\pi(A)}{\pi(B)}$$

uniformly in y, $y\in B$.

For $y\in S$, fixed k and $m>N\geqq 1$ we have

$$\begin{aligned} A_{m,k}(y,A)&=\sum_{v=1}^{m-1}\int_{B'}\frac{{}_{B'}P^{(v)}(y,dx)\,P^{(m-v)}(x,A)}{P^{(m-k)}(q,B)}+\frac{{}_{B'}P^{(m)}(y,A)}{P^{(m-k)}(q,B)}\\ &=\sum_{v=1}^{N}+\sum_{v=N+1}^{m}=B^{B'}_{N,m,k}(y,A)+C^{B'}_{N,m,k}(y,A),\end{aligned}$$

$$\begin{aligned}&\lim_{m\to\infty}\int_{V_i}\left|A_{m,k}(y,A)-\frac{\pi(A)}{\pi(B)}\right|\rho(dy)\\ &\quad=\lim_{N\to\infty}\lim_{m\to\infty}\int_{V_i}\left|B^{B'}_{N,m,k}(y,A)+C^{B'}_{N,m,k}(y,A)-\frac{\pi(A)}{\pi(B)}\right|\rho(dy)\\ &\quad\leqq\lim_{N\to\infty}\lim_{m\to\infty}\int_{V_i}\left|B^{B'}_{N,m,k}(y,A)-\frac{\pi(A)}{\pi(B)}\right|\rho(dy)+\lim_{N\to\infty}\lim_{m\to\infty}\int_{V_i}C^{B'}_{N,m,k}(y,A)\,\rho(dy)\end{aligned}$$

$$\begin{aligned}\lim_{N\to\infty}\lim_{m\to\infty}B^{B'}_{N,m,k}(y,A)&=\lim_{N\to\infty}\lim_{m\to\infty}\sum_{v=1}^{N}\int_{B'}\frac{{}_{B'}P^{(v)}(y,dx)\,P^{(m-v)}(x,A)}{P^{(m-v)}(q,B)}\,\frac{P^{(m-v)}(q,B)}{P^{(m-k)}(q,B)}\\ &=\frac{\pi(A)}{\pi(B)}\sum_{v=1}^{\infty}\int_{B'}{}_{B'}P^{(v)}(y,dx)=\frac{\pi(A)}{\pi(B)}\end{aligned}$$

for all $y\in S$. For fixed N, there exists $m_0(N)$ such that for all $m\geqq m_0$

$$\frac{P^{(m-i)}(x,A)}{P^{(m-k)}(q,B)}\leqq\frac{\pi(A)}{\pi(B)}+1$$

for all $x\in B$, $i=1,2,\ldots,N$. For $m\geqq m_0$,

$$\left|B^{B'}_{N,m,k}(y,A)-\frac{\pi(A)}{\pi(B)}\right|\leqq\left(\frac{\pi(A)}{\pi(B)}+1\right)\sum_{v=1}^{N}{}_{B'}P^{(v)}(y,B')+\frac{\pi(A)}{\pi(B)}\leqq 2\,\frac{\pi(A)}{\pi(B)}+1.$$

Also

$$\lim_{m\to\infty}B^{B'}_{N,m,k}(y,A)=\frac{\pi(A)}{\pi(B)}\sum_{v=1}^{N}{}_{B'}P^{(v)}(y,B').$$

Since $\rho(V_i)<\infty$, we may apply the Dominated Convergence Theorem twice to get that

$$\lim_{N\to\infty}\lim_{m\to\infty}\int_{V_i}\left|B^{B'}_{N,m,k}(y,A)-\frac{\pi(A)}{\pi(B)}\right|\rho(dy)$$
$$=\int_{V_i}\lim_{N\to\infty}\lim_{m\to\infty}\left|B^{B'}_{N,m,k}(y,A)-\frac{\pi(A)}{\pi(B)}\right|\rho(dy)=0.$$

Since $\lim_{m\to\infty}A_{m,k}(z,A)=\frac{\pi(A)}{\pi(B)}$ for all $z\in B$, then $\lim_{N\to\infty}\lim_{m\to\infty}C^{B'}_{N,m,k}(z,A)=0$ for all $z\in B$, for all $B'\in\mathscr{B}(B)$ with $\pi(B')>0$.

$$\int_{V_i}C^{B'}_{N,m,k}(y,A)\,\rho(dy)\leqq\frac{Q(1+\delta)}{\varepsilon'}\,C^{B'}_{N+j,m+j,k}(z,A)$$

(cf. Theorem 1) for some $Q>0$, applying Lemma 3.1, for m sufficiently large, given arbitrary $\delta>0$.

Hence $\lim_{N\to\infty}\lim_{m\to\infty}\int_{V_i}C^{B'}_{N,m,k}(y,A)\,\rho(dy)=0$ implying that

$$\lim_{m\to\infty}\int_{V_i}\left|A_{m,k}(y,A)-\frac{\pi(A)}{\pi(B)}\right|\rho(dy)=0.$$

(b) $\lim_{m\to\infty}\int_B\left|\frac{P^{(m+k)}(x,A)}{P^{(m)}(q,B)}-\frac{\pi(A)}{\pi(B)}\right|\rho(dx)=\int_B\lim_{m\to\infty}\left|\frac{P^{(m+k)}(x,A)}{P^{(m)}(q,B)}-\frac{\pi(A)}{\pi(B)}\right|\rho(dx)=0.$

(c) Choose the $W_i\in\mathscr{M}$ as defined in Lemma 3.4. Let $W=W'=\bigcup_{i=1}^{L}W_i$. (a) implies that $\frac{P^{(m+k)}(x,A)}{P^{(m)}(q,B)}\xrightarrow[W]{\rho}\frac{\pi(A)}{\pi(B)}$. (b) implies that $\frac{P^{(m+k)}(x,A)}{P^{(m)}(q,B)}\xrightarrow[B]{\rho}\frac{\pi(A)}{\pi(B)}$. Lemma 3.4 implies that we may choose L sufficiently large such that for arbitrary $v>0$, $\pi(S-B-W)<\frac{v}{Q}$ implying that $\rho(S-B-W)\leqq Q\,\pi(S-B-W)<v$. Therefore

$$\frac{P^{(m+k)}(x,A)}{P^{(m)}(q,B)}\xrightarrow[S]{\rho}\frac{\pi(A)}{\pi(B)}.$$

(d) (a) implies that

$$\lim_{m\to\infty}\int_W\frac{P^{(m+k)}(x,A)}{P^{(m)}(q,B)}\,\rho(dx)=\rho(W)\frac{\pi(A)}{\pi(B)}.$$

$$\lim_{m\to\infty}\frac{\int_W P^{(m+k)}(x,A)\,\rho(dx)}{\int_{W'}P^{(m)}(y,C)\,\rho(dy)}$$
$$=\lim_{m\to\infty}\frac{\int_W P^{(m+k)}(x,A)\,\rho(dx)}{P^{(m)}(q,B)}\;\frac{P^{(m)}(q,B)}{\int_{W'}P^{(m)}(y,C)\,\rho(dy)}=\frac{\rho(W)\,\pi(A)}{\rho(W')\,\pi(C)}.$$

$$\text{(e)}\ \lim_{m\to\infty}\frac{\int\limits_{B_1} P^{(m+k)}(x,A)\,\rho(dx)}{\int\limits_{B_2} P^{(m)}(y,C)\,\rho(dy)}=\lim_{m\to\infty}\frac{\int\limits_{B_1}\frac{P^{(m+k)}(x,A)}{P^{(m-1)}(q,B)}\,\rho(dx)}{\int\limits_{B_2}\frac{P^{(m)}(y,C)}{P^{(m-1)}(q,B)}\,\rho(dy)}=\frac{\rho(B_1)\,\pi(A)}{\rho(B_2)\,\pi(C)}.$$

Remark. The hypothesis of Theorem 3.3 may be further generalized by assuming that $q\in X$ and

$$\lim_{m\to\infty}\frac{P^{(m+1)}(q,B)}{P^{(m)}(q,B)}=1$$

instead of $q\in B$.

Corollary 3.3. *Let $S\in\mathscr{B}$ with $\pi(S)>0$ and for some $\varepsilon>0$ $\inf\limits_{x\in S,\,y\in S}p^{(n)}(x,y)\geqq\varepsilon$. Also, let the probability measure be absolutely continuous. If there exists $B\in\mathscr{B}(S)$ with $\pi(B)>0$, $q\in B$ and $r\in B$ for which*

$$\lim_{m\to\infty}\frac{p^{(m+1)}(y,x)}{p^{(m)}(q,r)}=1$$

uniformly in x and y, $x\in B$, $y\in B$, then the results of Theorem 3.3 *follow.*

Proof. The results follow if

$$\lim_{m\to\infty}\int\limits_B\frac{f(x)\,P^{(m+1)}(y,dx)}{P^{(m)}(q,B)}=\int\limits_B\frac{f(x)\,\pi(dx)}{\pi(B)}$$

uniformly in y, $y\in B$, for all $f\in\mathscr{L}_1^+(B,\mathscr{B}(B),\pi)$. Let

$$I_m(y)=\left|\int\limits_B\frac{f(x)\,P^{(m+1)}(y,dx)}{P^{(m)}(q,B)}-\int\limits_B\frac{f(x)\,\pi(dx)}{\pi(B)}\right|\leqq\int\limits_B f(x)\left|\frac{p^{(m+1)}(y,x)}{P^{(m)}(q,B)}-\frac{1}{\pi(B)}\right|\pi(dx).$$

$$\lim_{m\to\infty}\frac{p^{(m+1)}(y,x)}{P^{(m)}(q,B)}=\lim_{m\to\infty}\frac{\frac{p^{(m+1)}(y,x)}{p^{(m-1)}(q,r)}}{\int\limits_B\frac{p^{(m)}(q,z)}{p^{(m-1)}(q,r)}\,\pi(dz)}=\frac{1}{\pi(B)}$$

uniformly in x and y, $x\in B$, $y\in B$.

Since $\int\limits_B\frac{f(x)\,\pi(dx)}{\pi(B)}<\infty$, we have that $\lim\limits_{m\to\infty}I_m(y)=0$ uniformly in y, $y\in B$, for all $f\in\mathscr{L}_1^+(B,\mathscr{B}(B),\pi)$.

Corollary 3.4. *Let there exist $B\in\mathscr{S}$ with $\pi(B)>0$ and $q\in B$ for which*

$$\lim_{m\to\infty}\int\limits_B\frac{f(x)\,P^{(m+1)}(y,dx)}{P^{(m)}(q,B)}=\int\limits_B\frac{f(x)\,\pi(dx)}{\pi(B)}$$

uniformly in y, $y\in B$, for all $f\in\mathscr{L}_1^+(B,\mathscr{B}(B),\pi)$. In addition, let $d=1$. Let ρ be a probability measure on X such that $\rho\ll\pi$, and furthermore, let the Radon-Nikodym *derivative of* ρ *with respect to* π *be bounded a.e.* π *on* X. *For* $\pi(X-B)>0$, *let* $\mathcal{M}=\{W_{j,\varepsilon'}^{B'}(z)\colon \pi[W_{j,\varepsilon'}^{B'}(z)\cap(C-B)]>0$ *where* $C\in\mathcal{S}$ *with* $\pi(C-B)>0, j>0, \varepsilon'>0, z\in B, B'\in\mathcal{B}(B), \pi(B')>0\}$. *For* $W_i\in\mathcal{M}$, $W_i'\in\mathcal{M}$ *with* $i=1,2,\ldots$ *let* $W\in\mathcal{B}\left(\bigcup_{i=1}^{L} W_i\right)$ *with* $\rho(W)>0$ *and* $W'\in\mathcal{B}\left(\bigcup_{i=1}^{M} W_i'\right)$ *with* $\rho(W')>0$ *for any finite* $L>0$, *any finite* $M>0$. *Then for all* $A\in\mathcal{S}$ *with* $\pi(A)>0$, *for all* $C\in\mathcal{S}$ *with* $\pi(C)>0$, $k=0, \pm 1, \pm 2, \ldots$

(a) $\displaystyle\lim_{m\to\infty}\int_W\left|\frac{P^{(m+k)}(x,A)}{P^{(m)}(q,B)}-\frac{\pi(A)}{\pi(B)}\right|\rho(dx)=0.$

(b) $\displaystyle\frac{P^{(m+k)}(x,A)}{P^{(m)}(q,B)}\xrightarrow[X]{\pi'}\frac{\pi(A)}{\pi(B)}$ *for any* π' *such that* $\pi'\ll\pi$, $\pi'(X)<\infty$.

(c) $\displaystyle\lim_{m\to\infty}\frac{\int_W P^{(m+k)}(x,A)\,\rho(dx)}{\int_{W'} P^{(m)}(y,C)\,\rho(dy)}=\frac{\rho(W)\,\pi(A)}{\rho(W')\,\pi(C)}.$

Proof. (a) and (c) Theorem 2.2 implies that $W\cup W'\cup B\in\mathcal{S}$ and hence Theorem 3.3 applies.

(b) Theorem 2.1 and (c) of Theorem 3.3 yield the desired result.

Corollary 3.5. *Let* X *be a discrete space and* $P^{(m)}(i,j)$ *be the m-step transition probabilities of a recurrent, irreducible, aperiodic* $(d=1)$ *Markov chain. If there exists* $q\in X$ *for which*

$$\lim_{m\to\infty}\frac{P^{(m+1)}(q,q)}{P^{(m)}(q,q)}=1$$

then for all $A\in\mathcal{S}$ *with* $\pi(A)>0$, *for all* $C\in\mathcal{S}$ *with* $\pi(C)>0$ *we have that for all* $x\in X$, *for all* $y\in X$, $k=0, \pm 1, \pm 2, \ldots$

$$\lim_{m\to\infty}\frac{P^{(m+k)}(x,A)}{P^{(m)}(y,C)}=\frac{\pi(A)}{\pi(C)}.$$

Proof. Theorem 2.1 states that $\pi\left(X-\bigcup_{i=1}^{\infty} C_i\right)=0$. Since $\pi(\{x\})>0$ for all $x\in X$, $X=\bigcup_{i=1}^{\infty} C_i$. (There are no non-empty null sets.) Therefore, for each $x\in X$, $x\in C_i$ for some i. Letting $\rho\equiv\pi$ and $B=\{q\}$ in Corollary 3.4, we get the desired results from (c). This follows since for any $x\in X$, ${}_qP^{(j)}(q,x)>\varepsilon$ for some $j>0$, $\varepsilon>0$ as we have a Harris Process and $\pi(\{x\})>0$, i.e., $x\in W_{j,\varepsilon}^{\{q\}}(q)$.

$$\lim_{m\to\infty}\frac{\int_{\{x\}} P^{(m+k)}(z,A)\,\pi(dz)}{\int_{\{y\}} P^{(m)}(z,C)\,\pi(dz)}=\lim_{m\to\infty}\frac{P^{(m+k)}(x,A)\,\pi(\{x\})}{P^{(m)}(y,C)\,\pi(\{y\})}=\frac{\pi(\{x\})\,\pi(A)}{\pi(\{y\})\,\pi(C)}.$$

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