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# Stationary Regenerative Phenomena

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*Summary.* Corresponding to any $p$-function $p$, a stationary version of the associated regenerative phenomenon is constructed for which the underlying "probability" measure may have infinite total mass (though it will always be $\sigma$-finite if $p$ is standard). As a trivial consequence, $p$ is a positive-definite function. The construction is generalised to quasi-Markov chains.

## 1. The Positive-Definiteness of $p$-Functions

According to Kendall [5] the transition probabilities $p_{ij}(\cdot)$ of a (standard) continuous-time Markov chain[1] admit a Fourier integral representation which in the diagonal case $i=j$ takes the form

$$p_{ii}(t)=p_{ii}(\infty)+\int_0^\infty \phi_i(\alpha)\cos\alpha t\,d\alpha, \tag{1}$$

for a non-negative function $\phi_i$ integrable on $(0,\infty)$. The proof makes use of rather deep Hilbert space methods, but a more elementary proof of a special case has been given by Feller and Orey [3].

The function $p_{ii}$ necessarily belongs to the class $\mathscr{P}$ of standard $p$-functions defined in [6], and Kendall's result is there extended to this class: every $p$ in $\mathscr{P}$ admits a representation

$$p(t)=p(\infty)+\int_0^\infty \phi(\alpha)\cos\alpha t\,d\alpha, \tag{2}$$

where $\phi$ is non-negative and integrable (Theorem 5).

It has been observed by Loynes [11] that a somewhat weaker form of Kendall's result could be obtained much more easily as follows. Suppose that the Markov chain admits an *invariant measure*, a collection of positive numbers $m_i$ (where $i$ runs over the countable state space $S$) such that

$$m_j=\sum_{i\in S} m_i\,p_{ij}(t) \tag{3}$$

for all $j\in S$ and all $t>0$. If also

$$\sum_{j\in S} m_j<\infty, \tag{4}$$

we can normalise the numbers $m_j$ to be a probability distribution over $S$, and then construct a stationary Markov chain $(X_t)$ with

$$\mathbf{P}(X_s=i)=m_i, \qquad \mathbf{P}(X_{s+t}=j\,|\,X_s=i)=p_{ij}(t) \tag{5}$$

[1] As usual, we follow the notation and terminology of Chung [1].

for $t>0$, $i, j \in S$. For any $i \in S$, define

$$\begin{aligned} Z_t &= 1 \quad \text{if } X_t = i, \\ &= 0 \quad \text{if } X_t \neq i; \end{aligned} \tag{6}$$

then $Z$ is a stationary stochastic process, and

$$\mathbf{E}(Z_s Z_t) = m_i\, p_{ii}(|t-s|). \tag{7}$$

It follows that, for any real numbers $t_\alpha$, $\xi_\alpha$ ($\alpha = 1, 2, \ldots, n$),

$$m_i \sum_{\alpha,\beta=1}^{n} p_{ii}(|t_\alpha - t_\beta|)\, \xi_\alpha \xi_\beta = \mathbf{E}\left\{\sum_{\alpha=1}^{n} Z_{t_\alpha} \xi_\alpha\right\}^2 \geqq 0. \tag{8}$$

Since $p_{ii}$ is continuous, Bochner's theorem implies the existence of a probability measure $\mu_i$ on $[0, \infty)$ such that

$$p_{ii}(t) = \int \cos \alpha t\, \mu_i(d\alpha). \tag{9}$$

This then establishes Kendall's result except for the absolute continuity of $\mu$ on $(0, \infty)$, which Loynes is able to deduce from the Wold decomposition.

Unfortunately this argument, so much simpler than those of [5] and [3], depends on the existence of a solution to (3) and (4), and it is known that this is possible only when

$$p_{ii}(\infty) = \lim_{t\to\infty} p_{ii}(t) > 0 \tag{10}$$

for all $i \in S$. Loynes pointed out however that the condition (4) is not really essential, and that the whole argument goes through without it, except that the underlying measure $\mathbf{P}$ will not then be totally finite. This does not however affect (7) ($\mathbf{E}$ being the operator of integration with respect to $\mathbf{P}$), and the rest of the argument goes through to establish (9).

This important remark extends the scope of Loynes's technique to all recurrent chains and even to some transient ones. However, not every Markov chain admits an invariant measure, a necessary and sufficient condition being given by the celebrated theorem of Harris and Veech [14]. In this paper we show how the difficulty may be avoided by concentrating on $Z$ rather than $X$. More precisely, it will be shown that a stationary version of $Z$ may always be constructed so long as the underlying measure is allowed to have infinite total mass.

It was argued in [6] and [8] that results about Markov chains which involve only one state[2] are most naturally expressed in terms of the theory of regenerative phenomena, and this principle will be respected. To do so has the incidental advantage of answering a non-trivial question in that theory (cf. [10]); are all $p$-functions, whether standard or not, necessarily positive-definite?

The arguments used in the next two sections are in fact of much wider applicability than will there appear, and may be used to construct stationary versions of many stochastic processes with a finite number of states.

[2] The corresponding arguments for a finite number of states will be found in § 6.

## 2. The Construction of Stationary Regenerative Phenomena

Let $\bar{\Omega}$ denote the compact Hausdorff product space $\{0, 1\}^R$, whose typical element is a function $\omega: R \to \{0, 1\}$ from the real line to the two-point set $\{0, 1\}$, let $o$ denote the zero function, and write $\Omega = \bar{\Omega} - \{o\}$. Then $\Omega$ is a locally compact Hausdorff space. We shall write

$$\Omega_t = \{\omega \in \Omega;\ \omega(t) = 1\} \tag{11}$$

for $t \in R$, and

$$\Omega_A = \bigcup_{t \in A} \Omega_t \tag{12}$$

for $A \subseteq R$. Then $\Omega_t$ is both open and compact, and $\Omega_R = \Omega$, so that the open sets $\Omega_t (t \in R)$ cover $\Omega$. It follows that every compact $K \subseteq \Omega$ is covered by a finite sub-covering, so that

$$K \subseteq \Omega_A \tag{13}$$

for some *finite* $A$.

Following Halmos [4], the Borel sets in $\Omega$ are the members of the $\sigma$-ring generated by the compact sets. Every Borel set is contained in a $\sigma$-compact set, so that every Borel set $B$ satisfies

$$B \subseteq \Omega_A \tag{14}$$

for some *countable* $A$. In particular, $\Omega$ is not itself a Borel set. We shall construct regular Borel measures on $\Omega$; that is, measures $\lambda$ on the $\sigma$-ring of Borel sets which are finite on compact sets and satisfy

$$\lambda(B) = \sup\{\lambda(K);\ K \subseteq B,\ K \text{ compact}\} \tag{15}$$

for all Borel sets $B$.

**Theorem 1.** *If $p$ is any $p$-function, there exists one and only one regular Borel measure $\lambda$ on $\Omega$ such that, for all $n \geqq 1$ and all $t_1 < t_2 < \cdots < t_n$,*

$$\lambda\{\omega;\ \omega(t_1) = \omega(t_2) = \cdots = \omega(t_n) = 1\} = p(t_2 - t_1)\, p(t_3 - t_2) \ldots p(t_n - t_{n-1}). \tag{16}$$

When $n = 1$, this equation is to be read as

$$\lambda\{\omega;\ \omega(t) = 1\} = 1. \tag{17}$$

*Proof.* By the definition of $p$-functions [6], there is a stochastic process $Z_t (t > 0)$ taking the values 0 and 1 such that, for $0 = t_0 < t_1 < \cdots < t_n$,

$$\mathbf{P}\{Z_{t_\alpha} = 1\ (\alpha = 1, 2, \ldots, n)\} = \prod_{\alpha=1}^{n} p(t_\alpha - t_{\alpha-1}).$$

Hence, by the Kakutani-Nelson theorem ([13] Theorem 1.1, "familière aux lecteurs de Bourbaki" [12]) there is a unique regular probability measure $\pi$ on the compact space $\Lambda = \{0, 1\}^{(0, \infty)}$ such that

$$\pi\{\omega;\ \omega(t_\alpha) = 1\ (\alpha = 1, 2, \ldots, n)\} = \prod_{\alpha=1}^{n} p(t_\alpha - t_{\alpha-1}). \tag{18}$$

Take two copies $\Lambda_1$ and $\Lambda_2$ of $\Lambda$, endowed with copies $\pi_1$ and $\pi_2$ of $\pi$, and (for any $t \in \mathrm{R}$) define a continuous function $\psi_t: \Lambda_1 \times \Lambda_2 \to \Omega_t$ by

$$\begin{aligned} \psi_t(\omega_1, \omega_2)(u) &= \omega_1(t-u), \qquad && (u<t), \\ &= 1, && (u=t), \\ &= \omega_2(u-t), && (u>t). \end{aligned}$$

This function maps the product measure $\pi_1 \times \pi_2$ into a regular Borel measure

$$\lambda_t = (\pi_1 \times \pi_2)\psi_t^{-1}$$

on $\Omega_t$, and it is trivial to check, using (18), that (16) is satisfied with $\lambda$ replaced by $\lambda_t$ so long as $t$ is included in the set $\{t_\alpha\}$. In particular, if

$$s, t \in \{t_1, t_2, \ldots, t_n\},$$

then

$$\lambda_s\{\omega;\ \omega(t_\alpha)=1\} = \prod_{\alpha=2}^{n} p(t_\alpha - t_{\alpha-1}) = \lambda_t\{\omega;\ \omega(t_\alpha)=1\},$$

so that, by the uniqueness assertion of the Kakutani-Nelson theorem,

$$\lambda_s = \lambda_t \quad \text{on} \quad \Omega_s \cap \Omega_t. \tag{19}$$

If $B$ is any Borel set in $\Omega$, there is a countable set $A$ with $B \subseteq \Omega_A$, and we can express $B$ as a disjoint union

$$B = \bigcup_{t \in A} B_t, \tag{20}$$

where

$$B_t \subseteq \Omega_t$$

for each $t \in A$. The decomposition (20) is not unique, but if

$$B = \bigcup_{\tau \in A'} B'_\tau, \qquad B'_\tau \subseteq \Omega_\tau,$$

is another such, then

$$\begin{aligned} \sum_{\tau \in A'} \lambda_\tau(B'_\tau) &= \sum_{\tau \in A'} \sum_{t \in A} \lambda_\tau(B'_\tau \cap B_t) \\ &= \sum_{\tau \in A'} \sum_{t \in A} \lambda_t(B'_\tau \cap B_t), \qquad \text{by (19)} \\ &= \sum_{t \in A} \lambda_t(B_t). \end{aligned}$$

Hence the quantity

$$\lambda(B) = \sum_{t \in A} \lambda(B_t) \tag{21}$$

is defined independently of the decomposition chosen, and clearly defines a Borel measure on $\Omega$, which coincides with $\lambda_t$ on $\Omega_t$. Since $\lambda_t$ is regular and totally finite, there is for each $t \in A$ and each integer $v$ a compact $K_{tv} \subseteq B_t$ such that

$$\lambda_t(K_{tv}) > \lambda_t(B_t) - v^{-1} 2^{-j},$$

whence it follows easily that

$$\lambda(B) = \sup\{\lambda(\bigcup_{t \in a} K_{tv});\ v \leqq 1,\ a \subseteq A,\ a \text{ finite}\},$$

and thus that $\lambda$ is regular.

Since $\lambda=\lambda_t$ on $\Omega_t$, we have, for $t_1<t_2<\cdots<t_n$,

$$\begin{aligned}\lambda\{\omega;\ \omega(t_1)=\omega(t_2)=\cdots=\omega(t_n)=1\}&=\lambda_{t_1}\{\omega;\ \omega(t_\alpha)=1\ (\alpha=1,2,\dots,n)\}\\&=\pi_2\{\omega;\ \omega(t_\alpha-t_1)=1\ (\alpha=2,3,\dots,n)\}\\&=\prod_{\alpha=2}^{n}p(t_\alpha-t_{\alpha-1}),\end{aligned}$$

proving (16). Conversely, any regular Borel measure $\lambda$ satisfying (16) must coincide with $\lambda_t$ on $\Omega_t$, and must thus satisfy (21).

**Theorem 2.** *If $p$ is any $p$-function (standard or not) then, for any real numbers $t_\alpha, \xi_\alpha\ (\alpha=1,2,\dots,n)$,*

$$\sum_{\alpha,\beta=1}^{n} p(|t_\alpha-t_\beta|)\,\xi_\alpha\xi_\beta \geqq 0. \tag{22}$$

*Proof.* The expression (20) is equal to

$$\begin{aligned}\sum_{\alpha,\beta=1}^{n}\xi_\alpha\xi_\beta\,\lambda(\Omega_{t_\alpha}\cap\Omega_{t_\beta})&=\int_\Omega\sum_{\alpha,\beta=1}^{n}\xi_\alpha\xi_\beta\,\omega(t_\alpha)\,\omega(t_\beta)\,d\lambda\\&=\int_\Omega\left\{\sum_{\alpha=1}^{n}\xi_\alpha\,\omega(t_\alpha)\right\}^2 d\lambda\geqq 0.\end{aligned}$$

The use of the Kakutani-Nelson theorem seems to be more natural in this problem than the more familiar Daniell-Kolmogorov construction, since the space $\Omega$ is topologically very simple. The latter approach in effect confines attention to the Baire sets in $\Omega$, but can be made to yield the same results at the cost of some circumlocution.

### 3. Extension of the Measure $\lambda$

Because $\Omega$ is not a Borel set, it makes no sense to ask, for instance, whether $\lambda$ is totally finite, or totally $\sigma$-finite. (Of course, like all Borel measures, $\lambda$ is $\sigma$-finite in the sense that every Borel set is a countable union of sets of finite measure.) In the most important cases, however, $\lambda$ has a canonical extension to a larger class of subsets of $\Omega$, including $\Omega$ itself.

Call a set $E\subseteq\Omega$ a *weakly Borel set* if it is a Borel set in the compact space $\bar{\Omega}$. Since $\Omega$ is open in $\bar{\Omega}$ it is weakly Borel, and it is easy to check that the collection of weakly Borel sets is the $\sigma$-algebra generated by the open sets in $\Omega$. If $E$ is weakly Borel and $B$ Borel, then $E\cap B$ is Borel.

Now suppose that there is a Borel set $\Gamma\subseteq\Omega$ with the property that

$$\lambda(\Omega_t-\Gamma)=0 \qquad \text{for all } t.$$

Then

$$\lambda(\Omega_A-\Gamma)=0 \tag{23}$$

for all countable $A\subseteq R$, so that (14) implies that, for any Borel set $B$,

$$\lambda(B\cap\Gamma)=\lambda(B). \tag{24}$$

For any weakly Borel set $E$,

$$\bar{\lambda}(E)=\lambda(E\cap\Gamma) \tag{25}$$

is well-defined since $E\cap\Gamma$ is Borel, and (24) shows that $\bar{\lambda}$ is an extension of $\lambda$ to a measure on the weakly Borel sets.

Suppose if possible that $\tilde{\lambda}$ is another extension of $\lambda$ to the weakly Borel sets. Then

$$\tilde{\lambda}(E)\geqq\tilde{\lambda}(E\cap\Gamma)=\lambda(E\cap\Gamma)=\bar{\lambda}(E),$$

so that

$$\tilde{\lambda}\geqq\bar{\lambda}.$$

Thus $\bar{\lambda}$ is uniquely characterised as the minimal extension of $\lambda$. In particular, $\bar{\lambda}$ does not depend on the choice of $\Gamma$. Thus, if a Borel set $\Gamma$ can be found to satisfy (23), then $\lambda$ has a unique minimal extension $\bar{\lambda}$ to the weakly Borel sets (which will be denoted simply by $\lambda$ if no confusion can arise).

By (14), there is a countable $A$ with $\Gamma\subseteq\Omega_A$, and then

$$\Omega=(\Omega-\Gamma)\cup\bigcup_{t\in A}\Omega_t$$

exhibits $\Omega$ as a countable union of sets of finite measure;

$$\bar{\lambda}(\Omega-\Gamma)=0, \qquad \bar{\lambda}(\Omega_t)=1.$$

Hence, if $\Gamma$ exists, the minimal extension of $\lambda$ is necessarily totally $\sigma$-finite. (It will appear below that, if no set $\Gamma$ exists to satisfy (23), then $\lambda$ has no totally $\sigma$-finite extension.)

**Theorem 3.** *If $p$ is standard, the measure $\lambda$ defined by Theorem* 1 *has a unique minimal extension to the weakly Borel sets, which is totally $\sigma$-finite, and has total mass*

$$\lambda(\Omega)=\{\lim_{t\to\infty}p(t)\}^{-1}. \tag{26}$$

*Proof.* Let $Q$ be any countable dense subset of $R$, and write

$$\Gamma=\Omega_Q.$$

Then $\Gamma$ is Borel, and for $q\in Q\cap(t,\infty)$,

$$\begin{aligned}\lambda(\Omega_t-\Gamma)&\leqq\lambda(\Omega_t-\Omega_q)\\ &=\lambda(\Omega_t)-\lambda(\Omega_t\cap\Omega_q)\\ &=1-p(q-t)\to 0\end{aligned}$$

as $q\to t$ through $Q\cap(t,\infty)$. Thus $\Gamma$ satisfies (23), and from the preceding discussion (25) defines the unique minimal extension to the weakly Borel sets, which is $\sigma$-finite.

To compute the total mass of $\bar{\lambda}$, take $Q$ to be the set of dyadic rationals, i.e.

$$Q=\bigcup_{m=1}^{\infty}M(2^{-m}),$$

where

$$M(h)=\{\ldots,-2h,-h,0,h,2h,\ldots\}.$$

Then by (25),

$$\bar{\lambda}(\Omega)=\lambda(\Gamma)=\lim \lambda(\Omega_{M(h)}),$$

where the limit is taken as $m\to\infty$ and $h=2^{-m}$. Now

$$\begin{aligned}
\lambda\left(\bigcup_{r=-n}^{\infty}\Omega_{rh}\right) &= \sum_{r=-n}^{\infty}\lambda\left(\Omega_{rh}-\bigcup_{s=-n}^{r-1}\Omega_{sh}\right)\\
&= \sum_{r=-n}^{\infty}\lambda_{rh}\left(\Omega_{rh}-\bigcup_{s=-n}^{r-1}\Omega_{sh}\right)\\
&= \sum_{r=-n}^{\infty}\pi_1\{\omega;\ \omega(\alpha h)=0\,(\alpha=1,2,\dots,n+r)\}\\
&= \sum_{k=0}^{\infty}\pi\{\omega;\ \omega(\alpha h)=0\,(\alpha=1,2,\dots,k)\}.
\end{aligned}$$

Letting $n\to\infty$,

$$\lambda(\Omega_{M(h)})=\sum_{k=0}^{\infty}\pi\{\omega;\ \omega(\alpha h)=0\,(\alpha=1,2,\dots,k)\}. \tag{27}$$

Under $\pi$, the random variables $\omega(\alpha h)\,(\alpha=1,2,\dots)$ are the indicators of a recurrent event with (aperiodic) renewal sequence

$$p(\alpha h)=\pi\{\omega;\ \omega(\alpha h)=1\},\qquad (\alpha=0,1,2,\dots).$$

By a result of Feller ([2], XIII. 10), the right hand side of (27) is equal to

$$\{\lim_{n\to\infty}p(nh)\}^{-1}=\{p(\infty)\}^{-1}, \tag{28}$$

so that for all $h$,

$$\lambda(\Omega_{M(h)})=\{p(\infty)\}^{-1}.$$

Setting $h=2^{-m}$ and letting $m\to\infty$ yields (26), and the proof is complete.

If $p(\infty)>0$, the theorem shows that

$$\mathbf{P}=p(\infty)\,\lambda$$

is a probability measure on $\Omega$, with respect to which the random variables $Z_t\colon \Omega\to\{0,1\}$ defined by

$$Z_t(\omega)=\omega(t) \tag{29}$$

define a stationary stochastic process in the usual sense. This is just the equilibrium regenerative phenomenon defined in [7]. On the other hand, $\lambda$ has infinite total mass whenever $p(\infty)=0$.

Although the standard case is much the most important, some interest attaches to non-standard $p$-functions. The measurable $p$-functions have been characterised in [9], from the result of which we can prove the following.

**Theorem 4.** *If $p$ is measurable but not standard, then either* (i) $p(t)=a\,\bar{p}(t)$, *where* $0<a<1$ *and* $\bar{p}\in\mathscr{P}$, *or* (ii) $p(t)=0$ *for almost all* $t$.

*In case* (i) *the conclusion of Theorem* 3 *is still valid, but in case* (ii) $\lambda$ *has no $\sigma$-finite extension to the weakly Borel sets.*

*Proof.* The fact that (i) and (ii) are the only possibilities is established in [9].

*Case* (i). Let $Q$ be any countable dense subset of $R$, and $\Gamma=\Omega_Q$; we prove that (23) holds and then the rest of the proof goes through without change. Fix $t\in R$ and write $S=\{q-t;\, q\in Q, q>t\}$. Then

$$\begin{aligned}\lambda(\Omega_t-\Gamma)&\leqq\lambda_t\left(\Omega_t-\bigcup_{s\in S}\Omega_{t+s}\right)\\&=\pi\{\omega;\,\omega(s)=0\text{ for all } s\in S\}.\end{aligned}$$

It is shown in [9] that this last expression is equal to

$$\mathbf{E}\{(1-a)^N\},$$

where $N$ is the number of $s\in S$ for which $\bar{Z}_s=1$, if $\bar{Z}$ is a standard regenerative phenomenon with $p$-function $\bar{p}$. Since $S$ is dense, $\mathbf{P}(N=\infty)=1$, so that

$$\lambda(\Omega_t-\Gamma)=0.$$

*Case* (ii). Suppose that $\lambda$ has a $\sigma$-finite extension $\tilde{\lambda}$ to the weakly Borel sets, so that $\Omega$ is a disjoint union

$$\Omega=\bigcup_{k=1}^{\infty}E_k$$

of weakly Borel sets $E_k$ with $\tilde{\lambda}(E_k)<\infty$. For any Borel $B\subseteq E_k$,

$$\lambda(B)=\tilde{\lambda}(B)\leqq\tilde{\lambda}(E_k)$$

so that

$$L_k=\sup\{\lambda(B);\, B\subseteq E_k,\, B\text{ Borel}\}$$

is finite. For any integer $m$, there exists a Borel set $B_{km}\subseteq E_k$ with

$$\lambda(B_{km})>L_k-m^{-1},$$

and therefore

$$B_k=\bigcup_{m=1}^{\infty}B_{km}$$

is Borel and satisfies

$$B_k\subseteq E_k,\qquad \lambda(B_k)=L_k.$$

For any $t$, the set

$$B_k^t=B_k\cup\{(E_k-B_k)\cap\Omega_t\}$$

is also Borel, and $B_k^t\subseteq E_k$, so that, by definition of $L_k$,

$$\begin{aligned}L_k&\geqq\lambda[B_k\cup\{(E_k-B_k)\cap\Omega_t\}]\\&=\lambda(B_k)+\lambda\{(E_k-B_k)\cap\Omega_t\}\\&=L_k+\lambda\{(E_k-B_k)\cap\Omega_t\}.\end{aligned}$$

Thus

$$\lambda\{(E_k-B_k)\cap\Omega_t\}=0,$$

and summing over $k$,

$$\lambda\{(\Omega-\Gamma)\cap\Omega_t\}=0,$$

with

$$\Gamma=\bigcup_{k=1}^{\infty} B_k.$$

Hence $\Gamma$ is a Borel set satisfying (23), and we have shown that the existence of a $\sigma$-finite extension of $\lambda$ implies that of a solution $\Gamma$ of (23).

By hypothesis, the set

$$N=\{t;\ t=0\ \text{or}\ p(t)>0\ \text{or}\ p(-t)>0\}$$

has Lebesgue measure zero. Since $\Gamma$ is Borel, there is a countable $A$ with $\Gamma\subseteq\Omega_A$, and thus the set

$$\begin{aligned} A+N&=\{a+t;\ a\in A,\ t\in N\}\\ &=\bigcup_{a\in A}(a+N)\end{aligned}$$

has measure zero. In particular, there is a real number $x$ with

$$x\notin A+N.$$

Then, for any $a\in A$, $x-a\notin N$, so that $x\neq a$ and

$$p(|x-a|)=0.$$

It follows that

$$\lambda(\Omega_x\cap\Omega_a)=0,$$

and summing over $a\in A$,

$$\lambda(\Omega_x\cap\Omega_A)=0.$$

Since $\Gamma\subseteq\Omega_A$ and $\Gamma$ satisfies (23),

$$1=\lambda(\Omega_x)=\lambda(\Omega_x\cap\Gamma)+\lambda(\Omega_x-\Gamma)=0.$$

The contradiction shows that $\Gamma$ cannot exist, and so that $\lambda$ has no $\sigma$-finite extension to the weakly Borel sets.

It is perhaps illuminating to consider the extreme case in which $p(t)=0$ for all $t>0$, for which $\lambda$ may be described as follows. Define a function $\eta$: $R\to\Omega$ by

$$\begin{aligned}\{\eta(t)\}(s)&=1, &&\text{if } s=t,\\ &=0, &&\text{if } s\neq t.\end{aligned}$$

Then $\lambda$ is the image under $\eta$ of counting measure on $R$;

$$\lambda(B)=\text{number of points in } \eta^{-1}(B).$$

## 4. The Discrete-Time Case

Exactly analogous arguments may be carried through for discrete-time regenerative phenomena (recurrent events), though without the difficulty encountered in the last section, which is caused by the uncountable parameter space. Let $Z$ be the set of (positive and negative) integers, $\Omega^D$ the product space $\{0, 1\}^Z$, and

$\Omega^D = \bar{\Omega}^D - \{o\}$. Then $\Omega^D$ is both locally compact and $\sigma$-compact; it is thus a Borel set in itself, and all Borel measures on $\Omega^D$ are $\sigma$-finite. The argument used in the proof of Theorem 1 thus leads to the following conclusion.

**Theorem 5.** *To every renewal sequence $(u_n)$ there corresponds a unique regular Borel measure $\lambda$ on $\Omega^D$ such that, for integers $t_1 < t_2 < \cdots < t_n$,*

$$\lambda\{\omega;\ \omega(t_\alpha) = 1\ (\alpha = 1, 2, \ldots, n)\} = \prod_{\alpha=2}^{n} u(t_\alpha - t_{\alpha-1}). \tag{30}$$

*If $(u_n)$ has period $d$, then*

$$\lambda(\Omega^D) = d\{\lim_{n\to\infty} u_{nd}\}^{-1}. \tag{31}$$

It is a distinctive property of discrete-time regenerative phenomena that every renewal sequence can be realised in terms of a suitable Markov chain. More precisely, let $(f_n)$ be the first occurence probabilities associated with $(u_n)$ and define

$$g_n = 1 - f_1 - f_2 - \cdots - f_{n-1},$$
$$a_n = f_n/g_n, \qquad (a_n = 0 \text{ if } g_n = 0).$$

Construct a Markov chain on the state space $\{1, 2, \ldots\}$ by the transition probabilities

$$p_{i,i+1} = 1 - a_i, \qquad p_{i1} = a_i \tag{32}$$

all other $p_{ij}$ being zero. Then Chung has remarked that $(u_n)$ is the renewal sequence associated with the state 1:

$$u_n = p_{11}^{(n)} \tag{33}$$

It will be possible to employ Loynes's argument if the chain (32) admits an invariant measure $m_i$,

$$m_j = \sum_i m_i p_{ij} \tag{34}$$

It is easy to check that the only solutions of (34) are of the form

$$m_j = c\, g_j$$

for constants $c$, and then only if

$$f = \sum_{n=1}^{\infty} f_n = 1 \tag{35}$$

Thus, Chung's construction yields a chain with invariant measure if and only if $(u_n)$ is recurrent.

A chain with invariant measure may be constructed by time-reversal of the Chung chain. The matrix $(\bar{p}_{ij})$ defined by

$$\bar{p}_{ij} = (g_j/g_i)\, p_{ji} \tag{36}$$

is substochastic; indeed

$$\sum_{j=1}^{\infty} \bar{p}_{ij} = 1, \qquad (j \neq 1),$$
$$= f, \qquad (j = 1).$$

and it admits the invariant measure $(g_i)$. However, Loynes's argument only works when the chain is honest, and so once again requires recurrence. The usual device of adjoining an absorbing state is unhelpful, since it destroys the invariance of $(g_i)$.

There is however an alternative modification of the Chung chain which admits an invariant measure even when $f<1$. This is defined on the state space $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ by

$$\begin{aligned} p_{i,i+1} &= 1 && (i \leqq 0), \\ &= 1-a_i, && (i \geqq 1), \\ p_{i1} &= a_i, && (i \geqq 1), \end{aligned} \tag{37}$$

with all other $p_{ij}$ zero. Then it is easily checked that (33) still holds, and that the chain has the invariant measure

$$\begin{aligned} m_i &= 1-f && (i \leqq 0), \\ &= g_i && (i \geqq 1). \end{aligned} \tag{38}$$

Using this chain an alternative, though hardly simpler, proof of Theorem 5 may be constructed.

The only possible difficulty with (37) is that it is not irreducible. We therefore pose the question whether, given any renewal sequence, there exists an honest, irreducible Markov chain, with invariant measure, satisfying (33).

## 5. Ergodic Theory

For any $t \in R$, define a continuous function $\theta_t \colon \Omega \to \Omega$ by

$$(\theta_t \omega)(s) = \omega(s+t), \tag{39}$$

and note that $\theta_{t+u} = \theta_t \theta_u$. Define another continuous function $\rho \colon \Omega \to \Omega$ by

$$(\rho\omega)(s) = \omega(-s). \tag{40}$$

If $\lambda$ is the measure associated by Theorem 3 with a standard $p$-function, these functions map $\lambda$ into regular Borel measures

$$\lambda \theta_t^{-1}, \qquad \lambda \rho^{-1}$$

on $\Omega$, and these satisfy (16). By the uniqueness assertions in Theorems 1 and 3,

$$\lambda \theta_t^{-1} = \lambda \rho^{-1} = \lambda; \tag{41}$$

$\lambda$ is invariant under translations and under time-reversal.

**Theorem 6.** *The mappings $\theta_t$ and $\rho$ preserve the measure $\lambda$ defined by Theorem* 3.

In particular, to each $p$ in $\mathscr{P}$ is associated a dynamical system

$$\Sigma_p = (\Omega, \lambda, \theta_t). \tag{42}$$

This at once suggests questions within the purview of ergodic theory. For example, we may define a notion of *ergodic equivalence* in $\mathscr{P}$ by saying that $p_1$ and $p_2$ are ergodically equivalent if $\Sigma_{p_1}$ and $\Sigma_{p_2}$ are weakly isomorphic.

Many of the known metric invariants for (irreducible) Markov shifts, such as different kinds of mixing, are expressible in terms of a single diagonal transition function $p_{ii}$, and it would therefore be tempting to conjecture that the ergodic equivalence class of $p_{ii}$ might itself be a metric invariant of the Markov shift. Were this to be true, it would imply the solidarity theorem that, in an irreducible Markov chain, the $p$-functions $p_{ii}$ and $p_{jj}$ must be ergodically equivalent. This is however false, since (26) shows that the equivalence of $p_1$ and $p_2$ implies that

$$\lim_{t\to\infty} p_1(t) = \lim_{t\to\infty} p_2(t),$$

which is certainly not in general true of $p_{ii}$ and $p_{jj}$.

One might try to rescue the conjecture by weakening the equivalence relation to permit renormalisation of the measure $\lambda$. To show that this is no avail is rather more difficult, and requires the computation of the Kolmogorov-Sinai entropy invariant for $\Sigma_p$. The problem of describing the ergodic equivalence classes in $\mathscr{P}$, and relating ergodic equivalence to what might be called solidarity equivalence, remains open.

## 6. Quasi-Markov Chains

Just as the appropriate context for results about one state of a Markov chain is the theory of regenerative phenomena, so that for the study of a finite number of states (in particular for $p_{ij}$ when $i \neq j$) is the theory of quasi-Markov chains ([8], earlier called "linked systems of regenerative events" [7]). It is therefore of importance to generalise the construction of §§2, 3 to this broader setting.

A quasi-Markov chain is a stochastic process $(Z_t;\ t>0)$ taking values 0, 1, 2, ..., $N$ (where $N$ is a fixed integer) and having finite-dimensional distributions governed by

$$\mathbf{P}_{i_0}\{Z_{t_\alpha} = i_\alpha\ (\alpha = 1, 2, \ldots, n)\} = \prod_{\alpha=1}^{n} p_{i_{\alpha-1} i_\alpha}(t_\alpha - t_{\alpha-1}), \tag{43}$$

for $0 = t_0 < t_1 < \cdots < t_n$ and[3] $i_\alpha \in \{1, 2, \ldots, N\}$. These are therefore determined by the functions $p_{ij}\colon (0, \infty) \to [0, 1]$, which are assembled in an $(N\times N)$-matrix-valued function

$$\mathfrak{p}(t) = (p_{ij}(t);\ i, j = 1, 2, \ldots, N) \tag{44}$$

called the $p$-matrix of the chain. This is said to be standard if

$$\lim_{t\to\infty} \mathfrak{p}(t) = I, \tag{45}$$

(the identity matrix) and the class of standard $p$-matrices of order $N$ is denoted by $\mathfrak{P}_N$.

The class $\mathfrak{P}_N$ is characterised in [7], where it is proved (Theorem 7) that a continuous $(N\times N)$-matrix-valued function $\mathfrak{p}$ belongs to $\mathfrak{P}_N$ if and only if its Laplace transform is of the form

$$\mathfrak{r}(\theta) = \int_0^\infty \mathfrak{p}(t)\, e^{-\theta t}\, dt = [\theta I + A + \int (1 - e^{-\theta x})\, \mu(dx)]^{-1} \tag{46}$$

[3] The subscript on $\mathbf{P}$ allows us, as it were, a choice of starting point, so long as this be not the anomalous state 0.

in $\theta>0$, where $A=(a_{ij})$ is a matrix with

$$a_{ij}\leqq 0 \quad (i\neq j), \qquad \sum_{j=1}^{n} a_{ij}\geqq 0, \tag{47}$$

and $\boldsymbol{\mu}=(\mu_{ij})$ is a matrix of positive measures on $(0,\infty)$, with

$$\int(1-e^{-x})\,\mu_{ii}(dx)<\infty, \qquad \int\mu_{ij}(dx)\leqq -a_{ij} \quad (i\neq j). \tag{48}$$

Moreover, (46) sets up a one-to-one correspondence between $\mathfrak{P}_N$ and the class of pairs $(A,\mu)$ satisfying (47) and (48).

We shall assume throughout that $\mathfrak{p}$ is irreducible in the sense that none of the functions $p_{ij}$ is identically zero. Then, by Theorem 11 of [7], there exist positive numbers $m_i (i=1,2,\dots,N)$ with

$$\sum_{i=1}^{N} m_i a_{ij}\geqq 0, \qquad (j=1,2,\dots,N). \tag{49}$$

In general, these numbers are not uniquely determined, by (49), but in the special case when the chain is *recurrent*, when

$$\sum_{j=1}^{N} a_{ij}=0 \tag{50}$$

for all $i$, they are determined up to constant multiples, and satisfy (49) with equality for all $i$. The next theorem shows that any solution of (49) can be used to construct a stationary quasi-Markov chain, with $\sigma$-finite underlying measure.

**Theorem 7.** *Let* $\mathfrak{p}\in\mathfrak{P}_N$ *be irreducible, and let* $m_i (i=1,2,\dots,N)$ *be positive numbers satisfying* (49). *Define*

$$\bar{\Omega}=\{0,1,2,\dots,N\}^R, \qquad \Omega=\bar{\Omega}-\{o\}, \tag{51}$$

*where* $o$ *is the zero function in* $\bar{\Omega}$. *Then there exists a unique regular Borel measure* $\lambda$ *on* $\Omega$ *such that, for* $n\geqq 1$, $t_1<t_2<\cdots<t_n$ *and* $i_1,i_2,\dots,i_n\in\{1,2,\dots,N\}$,

$$\lambda\{\omega;\ \omega(t_\alpha)=i_\alpha(\alpha=1,2,\dots,n)\}=m_{i_1}\prod_{\alpha=2}^{n} p_{i_{\alpha-1}i_\alpha}(t_\alpha-t_{\alpha-1}). \tag{52}$$

*Moreover,* $\lambda$ *has a unique minimal extension to the weakly Borel sets, which is totally* $\sigma$*-finite.*

*Proof.* This is for the most part a straightforward extension of the proofs of Theorems 1 and 3. Write

$$\Omega_t=\{\omega\in\Omega;\ \omega(t)\neq 0\},$$

so that the $\Omega_t (t\in R)$ form as before a covering of $\Omega$ by open compact sets. Because of (43) there exists, for each $i\neq 0$, a regular probability measure $\pi_i$ on $\Lambda_2=\{0,1,2,\dots,N\}^{(0,\infty)}$ with

$$\pi_i\{\omega;\ \omega(t_\alpha)=i_\alpha(\alpha=1,2,\dots,n)\}=\prod_{\alpha=1}^{n} p_{i_{\alpha-1}i_\alpha}(t_\alpha-t_{\alpha-1}) \tag{53}$$

for $0=t_0<t_1<\cdots<t_n$ and $i_\alpha \neq 0$, $i_0=i$. By Theorem 10 of [7] the matrix $\mathfrak{p}^*$ defined by

$$p_{ij}^*(t)=(m_j/m_i)\,p_{ji}(t)$$

belongs to $\mathfrak{P}_N$, and hence there exists a similar measure $\pi_i^*$ on $\Lambda_1$ (another copy of $\Lambda_2$) satisfying (53) with $\pi_i^*$ replacing $\pi_i$, $\mathfrak{p}^*$ replacing $\mathfrak{p}$. If $\psi_{it}\colon \Lambda_1\times\Lambda_2\to\Omega_t$ is the continuous function defined by

$$\begin{aligned}\psi_{it}(\omega_1,\omega_2)(u)&=\omega_1(t-u), &&(u<t),\\ &=i, &&(u=t),\\ &=\omega_2(u-t), &&(u>t),\end{aligned}$$

then

$$\lambda_t=\sum_{i=1}^{N} m_i(\pi_i^*\times\pi_i)\,\psi_{it}^{-1} \tag{54}$$

defines a finite regular Borel measure on $\Omega_t$. It is easy to check that $\lambda_t$ satisfies (52) when $t\in\{t_\alpha\}$ and thus as before

$$\lambda_s=\lambda_t \quad \text{on} \quad \Omega_s\cap\Omega_t$$

and there is a unique regular Borel measure $\lambda$ on $\Omega$ with

$$\lambda=\lambda_t \quad \text{on} \quad \Omega_t,$$

which satisfies (52).

If $Q$ is any countable dense subset of $R$, and

$$\Gamma=\bigcup_{q\in Q}\Omega_q,$$

then $\Gamma$ is a Borel set, and for $q\in Q\cap(t,\infty)$,

$$\begin{aligned}\lambda(\Omega_t-\Gamma)&\leqq\lambda(\Omega_t-\Omega_q)\\ &=\sum_{i=1}^{N} m_i-\sum_{i=1}^{N}\sum_{j=1}^{N} m_i\,p_{ij}(q-t)\to 0\end{aligned}$$

as $q\to t$, since $\mathfrak{p}$ is standard. Hence

$$\lambda(\Omega_t-\Gamma)=0,$$

and the argument of § 3 shows that

$$\bar{\lambda}(E)=\lambda(E\cap\Gamma)$$

defines the unique minimal extension of $\lambda$ to the weakly Borel sets, which is totally $\sigma$-finite. The proof is therefore complete.

As in Theorem 6, the measure $\lambda$ is preserved by the shift $\theta_t$. The effect of the time-reversal mapping $\rho$ is to map $\lambda$ into the measure similarly associated with the dual $p$-matrix $\mathfrak{p}^*$ given by

$$p_{ij}^*(t)=(m_j/m_i)\,p_{ji}(t). \tag{55}$$

The computation of the total mass of $\lambda$ requires a lemma on the asymptotic behaviour of $p_{ij}$, which is implicit in [7].

**Theorem 8.** *If* $\mathfrak{p}$ *in* $\mathfrak{P}_N$ *is irreducible, then the limits*

$$p_j = \lim_{t\to\infty} p_{ij}(t) \tag{56}$$

*exist and are independent of i. Either all or none of the* $p_j$ *are positive and, for all j,*

$$p_j = \sum_{i=1}^{N} p_i\, a_{ij}. \tag{57}$$

*Proof.* The existence of the limits

$$p_{ij} = \lim_{t\to\infty} p_{ij}(t)$$

is guaranteed by Theorem 5 of [7]. Since

$$p_{ij}(t+a+b) \geqq p_{iI}(a)\, p_{IJ}(t)\, p_{Jj}(b),$$

the $p_{ij}$ are either all positive or all zero. Since in the latter case (56) and (57) are trivial, attention may be confined to the case

$$p_{ij} > 0 \qquad (\text{all } i, j).$$

If

$$\mathfrak{p} = (p_{ij})$$

then

$$\mathfrak{p} = \lim_{\theta\to 0} \theta\, \mathfrak{r}(\theta).$$

Because of (46),

$$\theta\, \mathfrak{r}(\theta)\, \{\theta I + A + \textstyle\int (1 - e^{-\theta x})\, \boldsymbol{\mu}(d\,x)\} = \theta I,$$

and letting $\theta \to 0$,

$$\mathfrak{p} A = 0. \tag{58}$$

Likewise

$$A\, \mathfrak{p} = 0. \tag{59}$$

If $\mathbf{1}$ is the column vector whose elements are all equal to 1, then

$$\mathfrak{p}(A\,\mathbf{1}) = \mathbf{0},$$

and since $\mathfrak{p} > 0$ and $A\,\mathbf{1} \geqq \mathbf{0}$ we must have

$$A\,\mathbf{1} = \mathbf{0}.$$

Hence from (47), the matrix

$$P = I - c\,A$$

is, for sufficiently small $c$, a stochastic matrix, and (58) shows that each row of $\mathfrak{p}$ is a strictly positive left eigenvector for $P$. Thus $P$ defines a finite Markov chain with invariant measure, and thus with no transient states. If $P$ is not irreducible, there is thus a proper subset $M$ of $\{1, 2, \ldots, N\}$ with

$$a_{ij} = 0 \qquad (i \in M, j \notin M \text{ or } i \notin M, j \in M).$$

From (48),

$$\mu_{ij} = 0 \qquad (i \in M, j \notin M \text{ or } i \notin M, j \in M).$$

so that, from (46),

$$r_{ij}(\theta)=0 \qquad (i\in M, j\notin M \text{ or } i\notin M, j\in M).$$

which contradicts the assumption of irreducibility of $\mathfrak{p}$. Thus $P$ is irreducible, and thus the only solutions of $P\mathbf{x}=\mathbf{x}$ are the multiples of $\mathbf{1}$. It therefore follows from (59) that each column of $\mathfrak{p}$ is constant, so that (56) is proved. Finally, (57) follows from (58).

It is to be noted that (57) determines uniquely the ratios $p_1:p_2:\ldots:p_N$, which therefore depend only on $A$ and not on $\boldsymbol{\mu}$. However, the absolute values of the $p_j$ do depend on $\boldsymbol{\mu}$.

**Theorem 9.** *The measure $\lambda$ defined in Theorem 7 is totally finite if and only if the limits $p_j$ in (56) are non-zero. In this case the total mass of $\lambda$ is determined by the equations*

$$m_j=\lambda(\Omega)\,p_j. \tag{60}$$

*Proof.* The function $\zeta_j:\Omega\to\{0,1\}$ defined by

$$\begin{aligned}\zeta_j(\omega)&=1 \qquad \text{if } \omega(0)=j,\\ &=0 \qquad \text{if } \omega(0)\neq j,\end{aligned}$$

belongs to $L_1(\Omega,\lambda)$; indeed

$$\int_\Omega \zeta_j\,d\lambda=m_j.$$

Applying the Birkhoff ergodic theorem to the measure-preserving transformation $\theta_1$, we see that

$$\bar\zeta_j(\omega)=\lim_{k\to\infty}k^{-1}\sum_{r=1}^{k}\zeta_j(\theta_r\,\omega)$$

exists for almost all $\omega$, and $\bar\zeta_j\in L_1(\Omega,\lambda)$. Hence

$$v_j(B)=\int_B \bar\zeta_j\,d\lambda$$

defines a totally finite Borel measure, which is regular since $v_j\leq\lambda$. For $t_1<t_2<\cdots<t_n$ and $i_1,i_2,\ldots,i_n\neq 0$, take an integer $m>t_n$, and use the dominated convergence theorem to show that

$$\begin{aligned}v_j\{\omega;\omega(t_\alpha)=i_\alpha(\alpha=1,2,\ldots,n)\}&=\int\prod_{\alpha=1}^{n}\zeta_{i_\alpha}(\theta_{t_\alpha}\omega)\,\bar\zeta_j(\omega)\,\lambda(d\omega)\\ &=\lim_{k\to\infty}k^{-1}\sum_{r=m}^{k}\int\prod_{\alpha=1}^{n}\zeta_{i_\alpha}(\theta_{t_\alpha}\omega)\,\zeta_j(\theta_r\,\omega)\,\lambda(d\omega)\\ &=\lim_{k\to\infty}k^{-1}\sum_{r=m}^{k}m_{i_1}\prod_{\alpha=2}^{n}p_{i_{\alpha-1}i_\alpha}(t_\alpha-t_{\alpha-1})\,p_{i_nj}(r-t_n)\\ &=m_{i_1}\prod_{\alpha=2}^{n}p_{i_{\alpha-1}i_\alpha}(t_\alpha-t_{\alpha-1})\,p_j,\end{aligned}$$

using (56). Comparing this with (52) and using the uniqueness assertion of Theorem 7, we have
$$\nu_j = p_j \lambda.$$
Since $\nu_j$ is totally finite, $\lambda$ can only have infinite total mass if $p_j = 0$. Conversely, if $\lambda$ is totally finite, the ergodic theorem implies that
$$\int \bar{\zeta}_j \, d\lambda = \int \zeta_j \, d\lambda = m_j,$$
so that $p_j \lambda(\Omega) = \nu_j(\Omega) = m_j$, and the proof is complete.

## 7. Postscript

This paper started from the positive-definite character of $p$-functions, proved in full generality in Theorem 2. Now the statement that a real function $p$ is positive-definite may be expressed in determinantal form, since a necessary and sufficient conditions is that, for all $n \geqq 1$ and all
$$0 < t_1 < t_2 < \cdots < t_n, \tag{61}$$
we have the inequality
$$\begin{vmatrix} 1 & p(t_1) & p(t_2) & \dots & p(t_n) \\ p(t_1) & 1 & p(t_2 - t_1) & \dots & p(t_n - t_1) \\ p(t_2) & p(t_2 - t_1) & 1 & \dots & p(t_n - t_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p(t_n) & p(t_n - t_1) & p(t_n - t_2) & \dots & 1 \end{vmatrix} \geqq 0. \tag{62}$$

It may therefore be of interest to note that the defining inequalities for $p$-functions written out in § 3 of [6] may also be thrown into determinantal form. More precisely, a function $p$ is a $p$-function if and only if, for all (61), $p$ satisfies the inequalities
$$\begin{vmatrix} 1 & p(t_1) & p(t_2) & \dots & p(t_n) \\ 1 & 1 & p(t_2 - t_1) & \dots & p(t_n - t_1) \\ 1 & 0 & 1 & \dots & p(t_n - t_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & \dots & 1 \end{vmatrix} \geqq 0 \tag{63}$$
and
$$\begin{vmatrix} p(t_1) & p(t_2) & p(t_3) & \dots & p(t_n) \\ 1 & p(t_2 - t_1) & p(t_3 - t_1) & \dots & p(t_n - t_1) \\ 0 & 1 & p(t_3 - t_2) & \dots & p(t_n - t_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & p(t_n - t_{n-1}) \end{vmatrix} (-1)^{n-1} \geqq 0. \tag{64}$$

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