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Asymptotics for Interacting Particle Systems on Z^d

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Summary. Two of the simplest interacting particle systems are the coalescing random walks and the voter model. We are interested here in the asymptotic density and growth of these systems as $t \to \infty$. More specifically, let $(\xi_t^{Z^d})$ be a system of coalescing random walks with initial state Z^d, and (ζ_t^O) a voter model with a single individual originating at O. We analyse $p_t = P(O \in \xi_t^{Z^d}) = \hat{P}(\zeta_t^O \neq \emptyset)$, and show that $p_t \sim \frac{1}{\pi} \frac{\log t}{t}$ as $t \to \infty$ for $d=2$, and $p_t \sim (\gamma_d t)^{-1}$ as $t \to \infty$ for $d \geqq 3$ for some γ_d. As a consequence, conditioned on non-extinction of ζ_t^O, $p_t |\zeta_t^O|$ approaches an exponential distribution. Results of a recent paper by Sawyer are applied.

1. Introduction

Two of the simplest interacting particle systems on the d-dimensional integer lattice Z^d are the *coalescing random walks* $(\xi_t^{Z^d})$ and the *voter model* (ζ_t^O). The process $(\xi_t^{Z^d})$ consists of particles, one starting from each site $x \in Z^d$. These particles undergo independent continuous time rate one simple random walks, except that whenever a particle jumps to a site which is already occupied by another, then the two particles coalesce into one. The state space is $S = \{$all subsets of $Z^d\}$, where $x \in \xi_t$ if there is a particle at x at time t. The process (ζ_t^O) is a continuous time Markov chain on the denumerable space $S_0 = \{$finite subsets of $Z^d\}$, with $\zeta_0^O = \{O\}$; it executs the jumps

$$\begin{aligned} &A \to A \cup \{x\} \quad (x \notin A) \quad \text{at rate } \frac{1}{2d} |\{y \in A\colon \|y-x\|=1\}| \\ &A \to A - \{x\} \quad (x \in A) \quad \text{at rate } \frac{1}{2d} |\{y \in A^c\colon \|y-x\|=1\}|, \end{aligned} \tag{1}$$

$A \in S_0$, $x \in Z^d$. (We put $|B|$ = cardinality of $B \in S_0$, $A^c = Z^d - A$, $\| \; \|$ = Euclidean norm.) Here ζ_t^O may be thought of as the sites occupied by particles at time t, so

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that (ζ_t^O) represents the evolution of a finite configuration of particles on Z^d. The process $(\xi_t^{Z^d})$ belongs to the family of coalescing random walks $\{(\xi_t^A);\ A\in S\}$ (the superscript A denoting the initial state), a Markov family of S-valued processes. Similarly, (ζ_t^O) is one of the voter models in $\{(\zeta_t^A);\ A\in S\}$, another such family. These are among the most widely studied interacting particle systems; for references and a recent survey, see [9]. Both $\{(\xi_t^A)\}$ and $\{(\zeta_t^A)\}$ are *additive* in the sense of Harris [10]. Consequently the entire family $\{(\xi_t^A)\}$ can be constructed on a single canonical probability space and the family $\{(\zeta_t^A)\}$ on another probability space, with the aid of their respective *percolation substructures*, in such a way that additivity holds. Namely

$$\xi_t^{A\cup B}=\xi_t^A\cup\xi_t^B \qquad A,B\in S,\ t\geqq 0, \tag{2}$$

$$\zeta_t^{A\cup B}=\zeta_t^A\cup\zeta_t^B \qquad A,B\in S,\ t\geqq 0. \tag{3}$$

Moreover, the substructures for these two systems are *dual:* for P governing $\{(\xi_t^A)\}$ and $\hat{P}$ governing $\{(\zeta_t^A)\}$, a *duality equation* asserts that

$$P(\xi_t^A\cap B\neq\emptyset)=\hat{P}(\zeta_t^B\cap A\neq\emptyset) \qquad A,B\in S,\ t\geqq 0. \tag{4}$$

(See [9, 10] or [11] for more details concerning (2)-(4).) Setting $A=Z^d$ and $B=\{O\}$ in (4), we obtain

$$P(O\in\xi_t^{Z^d})=\hat{P}(\zeta_t^O\neq\emptyset); \tag{5}$$

thus the "particle density" of $\xi_t^{Z^d}$ equals the "survival probability" of ζ_t^O. Since we will be focussing on these two processes, from now on we will abbreviate $\xi_t=\xi_t^{Z^d}$, $\zeta_t=\zeta_t^O$. It follows easily from (1) that

$$n_t=|\zeta_t| \tag{6}$$

is a martingale. In fact, if τ_i is the time of the i'th jump of the process, then (n_{τ_i}) is a simple symmetric random walk with absorption at 0. The holding times $\tau_i-\tau_{i-1}$, however, are determined by the "boundary size" of ζ_{τ_i}, i.e., the length of the borderline curve between sites in ζ_{τ_i} and its complement. Since the geometry of ζ_{τ_i} is not at all obvious, detailed analysis of n_t is considerably more complicated. Nevertheless, since the jump is always at least 2, we do know that $|\zeta_t|$ is eventually absorved at 0 with probability one. Thus, if p_t denotes the common value in (5), i.e.,

$$p_t=P(O\in\xi_t)=\hat{P}(\zeta_t\neq\emptyset), \tag{7}$$

it follows that

$$p_t\downarrow 0 \qquad \text{as } t\to\infty. \tag{8}$$

Our principal objective in this paper is to determine the exact asymptotics for p_t in every dimension d.

By way of motivation, let us first consider the one-dimensional case, where the analysis is quite easy. If $d=1$, then $(|\zeta_t^O|)$ *is* a rate-2 simple random walk on Z with absorption at 0, since (ζ_t^O) remains a "block" with boundary size 2 until it is trapped at $\emptyset$. The reflection principle and local central limit theorem together imply that

$$p_t \sim \frac{1}{\sqrt{\pi t}} \quad \text{as } t \to \infty,$$

and

$$\lim_{t\to\infty} P(n_t \leqq \alpha E[n_t | n_t > 0] \,|\, n_t > 0) = 1 - e^{-\frac{\pi}{4}\alpha^2}, \tag{9}$$

$\alpha \in [0, \infty)$. Using the joint percolation substructure for $\{(\xi_s^A)\}$ and $\{(\zeta_s^A)\}$, $0 \leqq s \leqq t$ (cf. [9] or [10]), one sees that n_t also has an interpretation for the coalescing random walks. Namely (in any dimension d),

$$\begin{aligned} n_t = {} & \text{the number of walks in } (\xi.) \text{ which are at the origin} \\ & \text{at time } t \text{ as a result of coalescence.} \end{aligned} \tag{10}$$

Thus (9) is a weak convergence result for the number of collisions up to time t experienced by the particle at 0, given that 0 is occupied at time t. Alternatively, if we think of the masses of particles as added together upon coalescing, then (9) describes the distribution of masses.

In dimension $d \geqq 2$, matters are not so simple. Here the evolution of the boundary of (ζ_t) seems unmanageable; one must therefore resort to indirect reasoning to investigate p_t. In [4], the fact that the boundary of ζ_t must have dimension at least $d-1$ was used to establish the crude estimate:

$$p_t = O(t^{-d/(d+1)}) \quad \text{as } t \to \infty.$$

To date, this is apparently the only available upper bound. A different approach to the study of (ζ_t) has relied on comparison with a certain "multi-type voter process" $(\bar{\zeta}_t)$, about which more has been proved. $(\bar{\zeta}_t) = ((\bar{\zeta}_t(x);\ x \in Z^d))$ has state space $(Z^d)^{Z^d}$, and may be defined in terms of the percolation substructure $\hat{\mathscr{P}}$ for $\{(\zeta_t^A)\}$ by

$$\bar{\zeta}_t(x) = y \quad \text{if there is a path up from } (y, 0) \text{ to } (x, t) \text{ in } \hat{\mathscr{P}} \tag{11}$$

(cf. [9] or [10]). $(\bar{\zeta}_t)$ is well-defined because there is always exactly one y which works in (11). Introduce

$$N_t = |\{x \in Z^d\colon \bar{\zeta}_t(x) = \bar{\zeta}_t(O)\}|.$$

That is, let N_t be the "patch size" of particles in the multitype process which are the same type as the one at the origin at time t. Sudbury [17] observed that if R_t is the range of a (continuous time rate 1) simple random walk on Z^d up to time t, then

$$\hat{E}[N_t] = E[R_{2t}]. \tag{12}$$

Subsequently, Kelly [12] noted that

$$\hat{P}(N_t = j) = j\hat{P}(n_t = j) \quad j = 0, 1, \ldots, \tag{13}$$

so that

$$p_t = \hat{P}(n_t > 0) = \hat{E}[N_t^{-1}].$$

Asymptotics for $E[R_t]$ were obtained in a celebrated paper by Dvoretsky and Erdös [8]; their results, together with (12), yield

$$\begin{aligned}\hat{E}[N_t] &\sim 2\pi \frac{t}{\log t} \quad \text{as } t\to\infty \quad d=2,\\ &\sim 2\gamma_d t \qquad \text{as } t\to\infty \quad d\geqq 3,\end{aligned} \tag{14}$$

where γ_d is the probability that d-dimensional simple random walk never returns to its initial position. Kelly [12] exhibited lower bounds for p_t by noting that

$$p_t = \hat{E}[N_t^{-1}] \geqq (\hat{E}[N_t])^{-1} \tag{15}$$

and then appealing to (14).

More recently, Sawyer has extended this line of analysis in a remarkable paper [14]. His results deal with a generalization of the multi-type voter model which is known to mathematical geneticists as the "stepping stone model". He is able to compute the asymptotics of all moments for the size of the patch containing the origin in this model. It follows that the patch size, suitably normalized, converges weakly to a Γ-distributed limit. Specialized to $(\bar{\zeta}_t)$, the result is as follows.

Sawyer's Theorem. *For $d\geqq 2$,*

$$\lim_{t\to\infty} \hat{E}\left[\left(\frac{N_t}{E[N_t]}\right)^k\right] = \frac{(k+1)!}{2^k} \qquad k=1,2,\ldots. \tag{16}$$

Thus for any $\alpha\in[0,\infty)$,

$$\lim_{t\to\infty} \hat{P}(N_t \leqq \alpha \hat{E}[N_t]) = \int_0^\alpha 4u e^{-2u}\,du,$$

where $\hat{E}[N_t]$ satisfies (14).

Sawyer's Theorem comes tantalizingly close to determining the asymptotics for p_t. From (13) we see that

$$\hat{E}[n_t^k] = \hat{E}[N_t^{k-1}] \qquad k\geqq 1, \tag{17}$$

so we now asymptotics for all the moments of n_t. Unfortunately, this is not quite enough to handle $p_t = P(n_t>0) = E[N_t^{-1}]$. There is a problem of "tightness near 0"; comparatively small values of N_t may have a drastic influence on p_t without affecting (17) in the limit.

The essential objective in this paper is to bridge the "technical gap" in the approach just outlined. In so doing, we derive exact asymptotics for p_t, as well as a counterpart to Sawyer's weak convergence theorem. Our main result is

Theorem 1'. *Let (ξ_t) be the d-dimensional coalescing random walks starting from Z^d, (ζ_t) the voter model on Z^d starting with a single particle at O. If p_t is given by* (7), *and n_t by* (6) *or* (10), *then*

$$\begin{aligned}p_t &\sim \frac{1}{\pi}\frac{\log t}{t} \quad \text{as } t\to\infty \quad d=2\\ &\sim (\gamma_d t)^{-1} \quad \text{as } t\to\infty \quad d\geqq 3\end{aligned} \tag{18}$$

(γ_d *is as in* (14)). *For any* $d \geqq 2$, $\alpha \in [0, \infty)$,

$$\lim_{t\to\infty} P(n_t \leqq \alpha p_t^{-1} | n_t > 0) = 1 - e^{-\alpha}. \tag{19}$$

We are unable to prove Theorem 1 directly from moment considerations. Rather, our approach is to obtain *good* upper bounds for p_t:

Theorem 1.

$$\begin{aligned} p_t &= O\left(\frac{\log t}{t}\right) \quad \text{as } t\to\infty \quad d=2, \\ &= O(t^{-1}) \quad \text{as } t\to\infty \quad d\geqq 3. \end{aligned} \tag{20}$$

We then apply Sawyer's Theorem in order to determine the correct asymptotic constant. Once this constant is known, the accompanying weak convergence result follows easily.

Section 2 contains the proof that Theorem 1 in conjunction with Sawyer's Theorem yields Theorem 1'. In Sect. 3 we prove Theorem 1. It is instructive to note that the proof deals almost entirely with the infinite system of coalescing random walks. The conventional wisdom of duality theory is that the finite dual system $\{(\zeta_t^A);\ A \in S_0\}$ is simpler to analyse than the infinite system $\{(\xi_t^A);\ A \in S\}$. But the finite voter model in more than one dimension turns out to be sufficiently complicated that the ultimate solution of our problem involves considerable interplay between $\{(\xi_t^A)\}$, $\{(\zeta_t^A)\}$ and $(\bar{\zeta}_t)$. Finally, Sect. 4 contains some additional remarks and open problems.

2. Theorem 1 Implies Theorem 1'

In this section we show, with the aid of Sawyer's Theorem, that (20) implies (18) and (19). The proof is not difficult; for the sake of clarity we make use of two lemmas. Throughout the remainder of the paper the dimension d will be thought of as fixed, and usually suppressed in the notation. It will also be convenient to introduce the notation

$$\begin{aligned} f_t &= \frac{t}{\log t} \quad d=2, \\ &= t \quad d\geqq 3, \\ g_t &= f_t p_t, \end{aligned}$$

and

$$\begin{aligned} K &= \pi \quad d=2, \\ &= \gamma_d \quad d\geqq 3 \ (\gamma_d \text{ is as in (14)}). \end{aligned}$$

We now demonstrate the two lemmas.

Lemma 1. *For any* $\varepsilon > 0$,

$$\lim_{t\to\infty} K f_t \hat{P}(n_t > \varepsilon K f_t) = e^{-\varepsilon}.$$

Proof. By (13), the left side equals

$$\tfrac{1}{2}\hat{E}\left[\left(\frac{N_t}{2Kf_t}\right)^{-1}, \frac{N_t}{2Kf_t} > \frac{\varepsilon}{2}\right]. \tag{21}$$

For N a random variable with density $4ue^{-2u}$, Sawyer's weak convergence result, together with bounded convergence, implies that as $t \to \infty$, (21) converges to

$$\tfrac{1}{2} E\left[N^{-1}, N > \frac{\varepsilon}{2}\right] = \tfrac{1}{2} \int_{\frac{\varepsilon}{2}}^{\infty} \frac{1}{u} 4ue^{-2u}\, du$$
$$= e^{-\varepsilon}. \quad \square$$

Lemma 2. *If* $g_t = O(1)$ *as* $t \to \infty$, *then*

$$\lim_{\varepsilon \to 0} \limsup_{t \to \infty} f_t \hat{P}(n_t \in (0, \varepsilon f_t]) = 0.$$

Proof. For any $s \leqq t$,

$$\hat{P}(n_t \in (0, \varepsilon f_t]) = \hat{P}(n_s \in (0, \varepsilon f_t], n_t > 0) + \hat{P}(n_s > \varepsilon f_t, n_t > 0) - \hat{P}(n_t > \varepsilon f_t).$$

Taking $s = (1 - \sqrt{\varepsilon})t$, and dropping $n_t > 0$ from the middle term on the right, we have

$$f_t \hat{P}(n_t \in (0, \varepsilon f_t]) \leqq f_t \hat{P}(n_{(1-\sqrt{\varepsilon})t} \in (0, \varepsilon f_t], \zeta_t \neq \emptyset) + f_t |\hat{P}(n_{(1-\sqrt{\varepsilon})t} > \varepsilon f_t) - \hat{P}(n_t > \varepsilon f_t)|.$$

By Lemma 1, the second term on the right is

$$O\left(\left|\frac{1}{1-\sqrt{\varepsilon}} e^{-\frac{\varepsilon}{K(1-\sqrt{\varepsilon})}} - e^{-\frac{\varepsilon}{K}}\right|\right) \quad \text{as } t \to \infty,$$

and hence tends to 0 as $\varepsilon \to 0$, uniformly for large t. To bound the first term we use the Markov property, (3), and translation invariance:

$$\begin{aligned} f_t \hat{P}(n_{(1-\sqrt{\varepsilon})t} \in (0, \varepsilon f_t], \zeta_t \neq \emptyset) &= f_t \sum_{A: |A| \in (0, \varepsilon f_t]} \hat{P}(\zeta_{(1-\sqrt{\varepsilon})t} = A)\, \hat{P}(\zeta^A_{\sqrt{\varepsilon}t} \neq \emptyset) \\ &= f_t \sum_{A: |A| \in (0, \varepsilon f_t]} \hat{P}(\zeta_{(1-\sqrt{\varepsilon})t} = A)\, \hat{P}\Big(\bigcup_{x \in A} \zeta^x_{\sqrt{\varepsilon}t} \neq \emptyset\Big) \\ &\leqq f_t p_{(1-\sqrt{\varepsilon})t} \varepsilon f_t p_{\sqrt{\varepsilon}t} \\ &\leqq \frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}} g_{(1-\sqrt{\varepsilon})t}\, g_{\sqrt{\varepsilon}t}. \end{aligned}$$

By hypothesis this last term is $\dfrac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}[O(1)]^2$, and therefore also tends to 0 uniformly for large t as $\varepsilon \to 0$. $\square$

Lemma 2 establishes the "tightness at 0" necessary to apply Sawyer's Theorem to our problem.

Proposition. *If* $g_t = O(1)$ *as* $t \to \infty$, *then* (18) *and* (19) *hold. Hence* (18) *and* (19) *follow from* (20).

Proof. By Lemma 1,

$$\liminf_{t\to\infty} K g_t \geqq \lim_{\varepsilon\to 0}\lim_{t\to\infty} K f_t \hat{P}(n_t > \varepsilon K f_t) = 1.$$

Also, for any $\varepsilon > 0$,

$$\begin{aligned}\limsup_{t\to\infty} K g_t &\leqq \lim_{t\to\infty} K f_t \hat{P}(n_t > \varepsilon K f_t) + \limsup_{t\to\infty} K f_t \hat{P}(n_t \in (0, \varepsilon K f_t]) \\ &= e^{-\varepsilon} + \delta_\varepsilon,\end{aligned}$$

where $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$ by Lemma 2. Thus $\limsup_{t\to\infty} K g_t = 1$. We conclude that

$$p_t \sim (K f_t)^{-1} \quad \text{as } t \to \infty,$$

i.e., (18) holds. Now, using (16)-(18), observe that

$$\begin{aligned}\hat{E}\left[\left(\frac{n_t}{K f_t}\right)^k \middle| n_t > 0\right] &= \hat{E}\left[\left(\frac{N_t}{2 K f_t}\right)^{k-1}\right] 2^{k-1} \\ &\to 2^{k-1} \hat{E}[N^{k-1}] = 2^{k-1} \frac{k!}{2^{k-1}} = k! \quad \text{as } t \to \infty.\end{aligned}$$

Since $\hat{E}[n_t | n_t > 0] = p_t^{-1} \sim K f_t$, and since $k!$ is the k'th moment of the standard exponential distribution, (19) follows by the method of moments. □

3. Proof of Theorem 1

Our objective in this section is to prove

$$\begin{aligned} p_t &= O\left(\frac{\log t}{t}\right) \quad \text{as } t \to \infty \quad d = 2, \\ &= O(t^{-1}) \quad \text{as } t \to \infty \quad d \geqq 3. \end{aligned} \tag{20}$$

The proof constitues the major effort of the paper. We will work directly with the infinite systems of coalescing random walks $\{(\xi_t^A);\ A \in S\}$; since the proof is somewhat long, we pause to introduce some needed terminology.

Extensive use will be made of the graphical representation of $\{(\xi_t^A)\}$, by means of which the entire family is defined on a single probability space; thus a certain familiarity with [9] or [10] will be assumed. $\{(\xi_t^A)\}$ is induced by its percolation substructure $\mathscr{P}$:

$$\xi_t^A = \{x \in Z^d:\ N_t^A(x) > 0\} \qquad A \in S,\ t \geqq 0,$$
where $N_t^A(x)$ is the number of paths up from $(A, 0)$ to (x, t) in $\mathscr{P}$. (22)

More generally, if $N_{t,u}^A(B)$ denotes the number of paths up from (A, t) to (B, u) $(t \leqq u)$, then we can define

$$\xi_{t,u}^A = \{x \in Z^d:\ N_{t,u}^A(x) > 0\} \qquad A \in S,\ 0 \leqq t \leqq u < \infty.$$

Note that

$$\xi_{t,u}^{\xi_t^A} = \xi_u^A \qquad 0 \leqq t \leqq u < \infty. \tag{23}$$

Also, the additivity property (2) extends to

$$\xi_{t,u}^{A \cup B} = \xi_{t,u}^A \cup \xi_{t,u}^B \qquad A, B \in S, \ 0 \leqq t \leqq u < \infty. \tag{24}$$

For brevity's sake, we write $\xi_{t,u} = \xi_{t,u}^{Z^d}$, $\xi_t = \xi_t^{Z^d}$.

We let Λ denote the *box of side* $2R$ *centered at* O, i.e.,

$$\Lambda = \{(x_1, \ldots, x_d) \in Z^d \colon |x_i| \leqq R\},$$

where R is a positive integer. Also, let V be the unique collection of sites in Z^d such that the translations $\Lambda + v$, $v \in V$, partition Z^d. Any choice of R gives rise to a corresponding Λ and V. In what follows, think of R, Λ and V as fixed.

We are now prepared to begin estimation of p_t. As the motivation may tend to become buried in the construction we are to employ, we first present a brief outline. We wish to show that p_t is decreasing at at least a fixed rate: $\log t/t$ if $d = 2$ and $1/t$ if $d \geqq 3$. Our basic approach will be to exploit the fact that the larger p_t is, the more rapidly it must decay: the more coalescing random walks present at time t, the more frequently they coalesce with one another, and the more rapidly their density decreases. Thus, the process has, in some sense, a self-correcting mechanism which limits the number of distinct particles. The upper estimates obtained in this manner for p_t are (perhaps surprisingly) strong enough to yield (20).

The methodology may be considered in analogy with a vector field (or flow) diagram in the context of differential equations. The function p_t has a negative derivative at each point (t, p_t) with $p_t > 0$, and becomes increasingly negative as p_t increases. If it is also true that $g_t = f_t p_t$ (as defined in Sect. 2) has a negative derivative at large enough values of g_t, then g_t will be bounded, which is sufficient for (20). The actual problem is, however, somewhat more complicated in that it does not seem possible to compute p_t' itself. For a given density of particles at some fixed time in the coalescing random walk system, it is not clear what the actual spatial distribution of particles is; the rate at which particles coalesce is however certainly dependent on this distribution. (For example, if the particles are all spread far apart, then this rate will be zero.) Yet, one may circumvent this difficulty by choosing to observe the coalescing random walk system over a fixed spatial region for a time interval of appropriate length.

What we do is to compute the decrease of the particle density over a box Λ with side $2R$. If the box is chosen large enough, it will at time t contain a minimal expected number of particles (two is sufficient for our purposes). In Lemma 3, it is shown that, no matter what the relative positions of the particles in the box, then after additional time s, a minimal number of these particles will have coalesced. Repeating this procedure over disjoint time intervals, we obtain the estimate (27), which bounds g_t in terms of this minimal expectation. This minimal rate at which particles coalesce may be computed without difficulty (Lemma 5). Therefore, appropriate choice of the spatial and time scales R and s (Lemma 4) ensures the desired bound given by (20).

We start off by reformulating the problem slightly, and define

$$e_t(B)=E[|\xi_t\cap B|] \qquad B\in S_0,\ t\geqq 0.$$

Note that by translation invariance

$$e_t(B)=|B|\,p_t.$$

Thus, a little algebra shows that whenever $0\leqq t\leqq u<\infty$,

$$p_u=p_t\left[1-\frac{e_t(\Lambda)-e_u(\Lambda)}{e_t(\Lambda)}\right]. \tag{25}$$

On the other hand, (23), (24) and translation invariance yield

$$\begin{aligned} e_u(\Lambda)&=\sum_{x\in\Lambda} P(x\in\xi_{t,u}^{\xi_t})\\ &\leqq\sum_{x\in\Lambda}\sum_{v\in V} P(x\in\xi_{t,u}^{\xi_t\cap\Lambda+v})\\ &=\sum_{x\in\Lambda}\sum_{v\in V} P(x-v\in\xi_{t,u}^{\xi_t\cap\Lambda})\\ &=E[|\xi_{t,u}^{\xi_t\cap\Lambda}|]. \end{aligned}$$

Substituting this into (25), we obtain

$$p_u\leqq p_t\left[1-\frac{\Delta_{t,u}(\Lambda)}{e_t(\Lambda)}\right], \qquad \text{where } \Delta_{t,u}(\Lambda)=E[|\xi_t\cap\Lambda|-|\xi_{t,u}^{\xi_t\cap\Lambda}|]. \tag{26}$$

Inequality (26) states that p_t decreases at at least some fixed rate which involves the quantity $\Delta_{t,u}(\Lambda)$. The following lemma enables us to make this expression more explicit, and states that particles executing coalescing random walks coalesce at at least a minimal rate independent of their initial configuration.

Lemma 3. *Let P_x be the probability law governing continuous time rate 2 d-dimensional simple random walk starting from x. If τ is the hitting time for the origin, denote $H_s(x)=P_x(\tau<s)$. Let $\{(\xi_t^A)\}$ be the (rate 1) coalescing random walks on Z^d. Then for any $B\in S_0$, $B\neq\emptyset$,*

$$|B|-E[|\xi_s^B|]\geqq(|B|-1)\min_{x,\,y\in B} H_s(y-x).$$

Proof. The left side is the expected number of times particles of $(\xi_\cdot^B)$ coalesce through time s. The right side should be thought of as a lower bound for the expected number of particles which coalesce with some fixed particle through time s, where all interaction among other particles is suppressed. Formally, we note that in the percolation substructure $\mathscr{P}$ for $\{(\xi_t^A)\}$,

$$N_t^B(Z^d)=|B| \qquad B\in S_0,\ t\geqq 0.$$

($N_t^B(Z^d)$ is as in (22).) So for any $\bar{x}\in B$,

$$\begin{aligned} |B|-|\xi_s^B|&=N_s^B(Z^d)-|\{z\in Z^d\colon N_s^B(z)>0\}|\\ &=N_s^B(\xi_s^{\bar{x}})+N_s^B(Z^d-\xi_s^{\bar{x}})-(1+|\{z\in Z^d-\xi_s^{\bar{x}}\colon N_s^B(z)>0\}|)\\ &\geqq N_s^B(\xi_s^{\bar{x}})-1. \end{aligned}$$

Taking expectations, we see that

$$|B|-E[|\xi_s^B|] \geqq \sum_{\substack{y\in B\\ y\neq \bar{x}}} P(\xi_s^y=\xi_s^{\bar{x}}).$$

Since the distance between ξ_t^y and $\xi_t^{\bar{x}}$ is a rate 2 simple random walk, the claim follows. □

As a consequence of the Markov property and Lemma 3, we see that

$$\begin{aligned}\Delta_{t,u}(\Lambda)&=\sum_{B\subset\Lambda} P(\xi_t\cap\Lambda=B)\,E[|B|-|\xi_{t,u}^B|]\\ &\geqq \sum_{\substack{B\subset\Lambda\\ B\neq\emptyset}} P(\xi_t\cap\Lambda=B)(|B|-1)\min_{x,y\in B} H_{u-t}(y-x)\\ &\geqq (e_t(\Lambda)-1)\min_{x,y\in\Lambda} H_{u-t}(y-x).\end{aligned}$$

We also let $u=t+s$, and set

$$h_s=\min_{x,y\in\Lambda} H_s(y-x).$$

Substitution into (26) then yields

$$p_{t+s}\leqq p_t[1-(1-[e_t(\Lambda)]^{-1})h_s] \qquad s,t\geqq 0.$$

Thus, we now have a bound for the rate of decrease of p_t after additional time s. To make the bound completely explicit, we still have to choose Λ and s as functions of t. Given t, we choose $\Lambda=\Lambda_t$ to have side $2R$ with

$$R=R_t=\lceil (2p_t^{-1})^{\frac{1}{d}}\rceil$$

($\lceil k\rceil=$least integer$\geqq k$), in order that

$$e_t(\Lambda)=R^d p_t\geqq 2.$$

With Λ so chosen we obtain the bound

$$p_{t+s}\leqq p_t[1-\tfrac{1}{2}h_s] \qquad s,t\geqq 0.$$

To make this inequality somewhat easier to handle, iterate it a total of $\lfloor ts^{-1}\rfloor$ times ($\lfloor k\rfloor=$greatest integer$\leqq k$). Using the fact that p_t is decreasing we conclude that

$$\begin{aligned}p_{2t}&=p_t[1-\tfrac{1}{2}h_s]^{\lfloor t/s\rfloor}\\ &\leqq p_t\exp\{-\tfrac{1}{2}h_s\lfloor t/s\rfloor\} \qquad s,t\geqq 0.\end{aligned}$$

Since $f_{2t}/f_t\leqq 2$, this last inequality may be rewritten as

$$g_{2t}\leqq g_t\exp\{\log 2-\tfrac{1}{2}h_s\lfloor t/s\rfloor\} \qquad s,t\geqq 0. \tag{27}$$

Our goal in this section is to demonstrate (20), namely, to show that g_t is uniformly bounded on $[0,\infty)$. Inequality (27) is our main tool. As a consequence of the next two lemmas, it will follow that the argument of the exponential term above is negative for large values of g_t. Therefore by (27), g_t cannot become too large, but must remain bounded. We now proceed to make this reasoning precise.

Lemma 4. *For an appropriate choice of* $s = s_t$,

$$\liminf_{t\to\infty} g_t^{-1} h_{s_t} \lfloor t/s_t \rfloor > 0. \tag{28}$$

Assuming (28) for a moment, let us show that (20) follows from (27) and (28). By (28) there is an $\varepsilon > 0$ and a $t_0 < \infty$ such that

$$h_{s_t}\lfloor t/s_t \rfloor > \varepsilon g_t \quad \text{for all } t \geqq t_0. \tag{29}$$

We claim that

$$g_t \leqq \Big(\max_{r\in[0,2t_0]} g_r\Big) \vee \frac{4\log 2}{\varepsilon} \quad \text{for all } t \geqq 0 \tag{30}$$

($a \vee b = \max\{a, b\}$). Suppose that (30) is violated at $t = t_1$, and let

$$t_* = \min\{2^{-n} t_1 \colon (30) \text{ is violated at } 2^{-n} t_1;\ n = 0, 1, \ldots\}. \tag{31}$$

g_{t_*} violates (30), so $t_* > 2t_0$, and hence $t_*/2 > t_0$. With ε as in (29), if $g_{t_*/2} > \frac{2\log 2}{\varepsilon}$, then (29) and (27) imply that

$$g_{t_*} < g_{t_*/2},$$

which contradicts (31). If, on the other hand, $g_{t_*/2} \leqq \frac{2\log 2}{\varepsilon}$, then since $f_{2t}/f_t \leqq 2$ and p_t decreases,

$$g_{t_1} \leqq g_{t_*} \leqq \frac{4\log 2}{\varepsilon}.$$

Thus (30) cannot be violated for any t, i.e., (20) holds.

To prove Lemma 4, thereby completing the proof of Theorem 1, we will need some elementary estimates for rate 2 continuous time simple random walk on Z^d. Let (X_t) denote such a process, P_x its law starting from x, $P_t(x, y) = P_x(X_t = y)$, $G_t(x) = \int_0^t P_s(x, 0)\, ds$, and define $H_t(x)$ as in Lemma 3. Then we have the following lemma.

Lemma 5. *If* $x \in Z^d$ *with* $\|x\| = r$, *then there is a constant* $C_d > 0$ *such that*

$$\begin{aligned} H_{r^2}(x) &\geqq C_2/\log r & d&=2 \\ &\geqq C_d r^{2-d} & d&\geqq 3. \end{aligned}$$

Proof. Use the inequality

$$H_t(x) \geqq G_t(x)/G_t(0),$$

together with the familiar asymptotics for the Green's function: as $r = \|x\| \to \infty$,

$$\begin{aligned} G_{r^2}(x) &\sim \alpha_2 & d&=2, \\ &\sim \alpha_d r^{2-d} & d&\geqq 3; \\ G_{r^2}(0) &\sim \beta_2 \log r & d&=2, \\ &\sim \sup_t G_t(0) = \beta_d & d&\geqq 3; \end{aligned}$$

α_d and β_d are positive (finite) constants. The local central limit theorem yields these asymptotics (cf. the (discrete-time) computations in the proof of Proposition 26.1 of [16].) □

We now proceed to verify (28) for a suitable choice of s_t.

Proof of Lemma 4. Choose

$$s_t = d\lceil (2p_t^{-1})^{\frac{1}{d}}\rceil^2.$$

Since the distance between any two sites in A_t is at most $D_t = \sqrt{d}R_t = \sqrt{d}\lceil (2p_t^{-1})^{\frac{1}{d}}\rceil$, Lemma 5 implies that

$$\begin{aligned} h_{s_t} = \min_{x,y\in A_t} H_{s_t}(y-x) &\geqq \min_{x,y\in A_t} H_{\|x-y\|^2}(y-x) \\ &\geqq C_2/\log D_t \qquad d=2 \\ &\geqq C_d D_t^{2-d} \qquad d\geqq 3. \end{aligned}$$

Now (8), (14) and (15) imply that as $t\to\infty$

$$p_t^{-1}\to\infty \quad \text{and} \quad t p_t^{\frac{2}{d}}\to\infty,$$

so

$$\lceil (2p_t^{-1})^{\frac{1}{d}}\rceil \sim (2p_t^{-1})^{\frac{1}{d}}$$

and

$$\lfloor t/s_t\rfloor \sim t/s_t.$$

It follows that for $d=2$,

$$\begin{aligned} \liminf_{t\to\infty} g_t^{-1} h_{s_t}\lfloor t/s_t\rfloor &\geqq \liminf_{t\to\infty} \frac{\log t}{t} p_t^{-1}\cdot \frac{C_2}{\log(2p_t^{-\frac{1}{2}})}\cdot\frac{t}{4p_t^{-1}} \\ &\geqq \liminf_{t\to\infty} \frac{C_2}{4}\left[\frac{\log t}{\log 2 + \frac{1}{2}\log p_t^{-1}}\right] \\ &\geqq C_2/2 > 0, \end{aligned}$$

where we have used the fact that $\log t/\log p_t^{-1} \geqq 1$ for large t since $t p_t\to\infty$. Similarly, for $d\geqq 3$,

$$\begin{aligned} \liminf_{t\to\infty} g_t^{-1} h_{s_t}\lfloor t/s_t\rfloor &\geqq \liminf_{t\to\infty} t^{-1} p_t^{-1}\cdot C_d[\sqrt{d}(2p_t^{-1})^{\frac{1}{d}}]^{2-d}\frac{t}{d(2p_t^{-1})^{2/d}} \\ &= \frac{C_d}{2d^{d/2}} > 0. \end{aligned}$$

This completes the proof of Lemma 4, and hence of Theorem 1. □

4. Additional Remarks

We discuss briefly three directions in which Theorem 1′ may be extended.

(i) *Domains of attraction.* If the initial state Z^d is replaced by other infinite states $A\in S$, or more generally measures μ on S, what happens to (18)? The answer, for a large class of "nice" states, is that *nothing* new happens, i.e., Theorem 1′ is quite stable under changes in the initial distribution. This phenomenon was studied in the one-dimensional setting in [3]. A simple illustration is the following result.

Theorem 2. *Let* $(\xi_t^{\mu_\theta})$ *be the coalescing random walks on* Z^d *starting from Bernoulli product measure with density* $\theta \in [0,1]$. *Let* $p_t^\theta = P(O \in \xi_t^{\mu_\theta})$. *If* $\theta \in (0,1)$, *then* p_t^θ *has the same asymptotics* (18) *as* p_t.

Proof. A special case of the duality equation for product measures (see e.g. (1.11) in [9]) asserts that

$$P(O \notin \xi_t^{\mu_\theta}) = \hat{E}[(1-\theta)^{|\zeta_t|}].$$

Manipulating this we get

$$0 \leqq 1 - \frac{p_t^\theta}{p_t} = \sum_{k=1}^{\infty} P(|\zeta_t| = k \mid \zeta_t \neq \emptyset)(1-\theta)^k.$$

Bound the rightmost expression by

$$P(|\zeta_t| \in (0, \varepsilon f_t] \mid \zeta_t \neq \emptyset) + (1-\theta)^{\varepsilon f_t},$$

where f_t is defined as in Section 2. If we let $t \to \infty$, then the second term goes to 0; also as $\varepsilon \to 0$, the first term goes to 0 uniformly in t by Theorems 1 and 1′ and Lemma 2. Consequently,

$$\lim_{t\to\infty} \frac{p_t^\theta}{p_t} = 1,$$

which proves the theorem. □

(ii) *Annihilating random walks.* A second family of interacting particle systems closely related to $\{(\xi_t^A)\}$ is the annihilating random walks $\{(\eta_t^A)\}$, where particles annihilate one another rather than coalesce upon collision. The systems (η_t^A) tend to be more difficult to analyse than (ξ_t^A), but fortunately the asymptotic density starting from Z^d is determined by combining Theorem 1′ with a recent result due to Arratia [2].

Theorem 3. *Let* (η_t) *be d-dimensional annihilating random walks starting from* Z^d, *and let* $\tilde{p}_t = P(O \in \eta_t)$. *Then* $\tilde{p}_t$ *satisfies* (18) *with* $\frac{1}{\pi}$ *replaced by* $\frac{1}{2\pi}$, γ_d *replaced by* $2\gamma_d$.

Proof. A special case of Arratia's theorem states that

$$\lim_{t\to\infty} \frac{\tilde{p}_t}{p_t} = \tfrac{1}{2}.$$

(iii) *Shape of the voter model.* Theorem 1′ raises some intriguing and challenging open problems concerning the voter model. The process $(\zeta_t \mid \zeta_t \neq \emptyset)$ may be thought of as the limiting critical case of a one-parameter family of supercritical models for tumour growth introduced by Williams and Bjerknes [18]. The supercriticial models, conditioned on nonextinction, have an asymptotic shape, which grows linearly in "radius" [5, 6]. By analogy, one can ask shape questions about $(\zeta_t|\zeta_t \neq \emptyset)$. The limit law (19) suggests that the normalized shape itself might converge in distribution in some sense. Any results along these lines would be of interest. As one would expect, the situation is altogether different from the supercritical case. For example, whereas the supercritical processes on Z^d have boundary of asymptotic dimension $d-1$, Theorem 1′ shows that $(\zeta_t|\zeta_t \neq \emptyset)$ has boundary of asymptotic dimension d for $d \geqq 3$, "nearly d" for $d=2$.

(iv) *Coalescing Brownian motions.* Closely related systems of coalescing Brownian motions have been studied by Arratia [1] in dimension one, and by Smoluchowski [15], Chandrasekhar [7] and Lang and Nguyen [13] in dimension three.

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