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Probability Theory and Related Fields

On Harmonic Renewal Measures

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Summary. Let μ be a probability and $\nu_h = \sum_{n=1}^{\infty} \frac{1}{n} \mu^{*n}$ the corresponding harmonic renewal measure. Complementing earlier results where μ is concentrated on a halfline we investigate the behaviour of $\nu_h([x, x+1])$ and the harmonic renewal function $G(x) = \nu_h((-\infty, x])$ as $x \to \infty$ if $m_1 = \int x\, \mu(dx) > 0$. We also consider the case $m_1 = 0$.

1. Introduction and Results

For a given probability measure μ on the Borel subsets $\mathfrak{B}$ of the real line let

$$\nu_h = \sum_{n=1}^{\infty} \frac{1}{n} \mu^{*n} \tag{1}$$

denote the corresponding harmonic renewal measure. We assume throughout the paper that μ has finite second moment, i.e.

$$m_2 = \int x^2\, \mu(dx) < \infty, \tag{2}$$

and we only consider non-lattice distributions: beyond this we assume that

$$\begin{array}{l} \mu^{*n} \text{ has a non-vanishing absolutely} \\ \text{continuous component for some } n \in \mathbb{N} \end{array} \tag{3}$$

(see also 3.1 and 3.3 below).

Harmonic renewal measures are of interest in the theory of random walks. Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables with distribution μ on some probability space $(\Omega, \mathfrak{A}, P)$, put $S_0 \equiv 0$, $S_n = \sum_{i=1}^{n} X_i$ for all $n \in \mathbb{N}$. Besides the obvious interpretation of ν_h as a harmonically discounted occupation measure of the random walk $(S_n)_{n \in \mathbb{N}_0}$,

$$\nu_h(A) = E\left[\sum_{n=1}^{\infty} \frac{1}{n} I_A(S_n)\right] \quad \text{for all } A \in \mathfrak{B},$$

we have the following important connection with the Wiener-Hopf factors of μ: let $N = \inf\{n \in \mathbb{N} : S_n > 0\}$, assume $m_1 = \int x\,\mu(dx) \geqq 0$ for the moment, then $P(N < \infty) = 1$. Let μ^+ denote the distribution of S_N, the first positive sum. Then the harmonic renewal measure associated with μ^+ is the restriction of ν_h to $(0, \infty)$ (see [7], § 5 for the setting (Spitzer's identity and fluctuation theory), details, and diverse applications).

In this paper we study asymptotic properties of harmonic renewal measures as for example the behaviour of $\nu_h([x, x+1])$ as $x \to \pm\infty$. In the special case of μ being concentrated on $[0, \infty)$ – we refer to this as the one-sided case in the sequel – asymptotics of harmonic renewal measures have been investigated in some detail by Greenwood, Omey and Teugels in [7] (see also [8]). In [1] Embrechts, Maejima and Omey considered a generalization of (1) where the coefficients $\frac{1}{n}$ are replaced by some regularly varying sequence, again μ is assumed to be concentrated on $[0, \infty)$. In [7] and [8] as well as in [1] the one-sidedness assumption is essential for the methods since these rely on Abelian and Tauberian theorems for Laplace transforms. We work with generalized functions and Gelfand transforms instead, the role of Abelian and Tauberian theorems is taken over by the Wiener-Lévy-Gelfand Theorem for certain convolution algebras of complex measures (see also 3.5 and 3.6 below).

The extension given here of harmonic renewal theory from the one-sided to the two-sided case is analogous for that for ordinary renewal theory. For a brief treatment of this, and references, see e.g. [5], § XI.9.

Let λ denote the Lebesgue measure, let λ_h be the λ-continuous measure with density $x \to 1 \wedge \frac{1}{|x|}$, $x \in \mathbb{R}$, and let λ_h^+ denote the restriction of λ_h to $[0, \infty)$. We call a monotone decreasing function $\tau: [0, \infty) \to [0, \infty)$ dominatedly varying (τ is a DVF for short) if

$$\tau(x) = O(\tau(2x)) \quad \text{as } x \to \infty. \tag{4}$$

Our first result deals with the case of non-zero mean which we may take to be positive. Restricted to a compact interval $\nu_h - \lambda_h^+$ is a finite signed measure, $|\nu_h - \lambda_h^+|$ denotes the corresponding total variation measure.

Theorem 1. *Let μ, ν_h and λ_h^+ be as above, assume* (2), (3) *and* $m_1 > 0$, *let τ be a DVF. Then*

$$\mu(x, \infty)) = o(\tau(x))$$

implies

$$|\nu_h - \lambda_h^+|([x, x+1]) = o(\tau(x)),$$

and

$$\mu((-\infty, -x]) = o(\tau(x))$$

implies

$$\nu_h([-x, -x+1]) = o(\tau(x))$$

as $x \to \infty$. The same holds if o is replaced by O throughout.

Because of (2) the conditions on the μ-tails are automatically satisfied if $\tau(x) = (1+x^2)^{-1}$, so $\nu_h - \lambda_h^+$ is a finite measure and the harmonic renewal function G, $G(x) = \nu_h((-\infty, x])$ is (finite and) asymptotically equal to $\log x$ as $x \to \infty$. We also get asymptotic equality of $\nu_h(x+A)$ and $\lambda_h^+(x+A)$ as $x \to \infty$ for all bounded Borel sets A with $\lambda(A) > 0$ (see also 3.1 below). Furthermore the theorem shows that for bounded $A \in \mathfrak{B}$ the differences $|\nu_h(x+A) - \lambda_h^+(x+A)|$ are bounded by a multiple of $\mu([x, \infty))$ if μ has a dominatedly varying right tail, i.e. if $\mu([x, \infty)) = O(\mu([2x, \infty)))$ as $x \to \infty$; similarly, if the left tail of μ is dominatedly varying a multiple of it bounds $\nu_h(x+A)$ as $x \to -\infty$.

The next theorem deals with the harmonic renewal function, in it γ denotes Euler's constant.

Theorem 2. *Let μ satisfy (2), (3) and $m_1 > 0$, assume that μ has dominatedly varying right tail; $G(x) = \nu_h((-\infty, x])$ for all $x \in \mathbb{R}$. Then*

$$G(x) = \log x - \log m_1 + \gamma + \frac{1}{m_1} \int_x^\infty \mu((t, \infty))\, dt + O(\mu((x, \infty)))$$

as $x \to \infty$; in particular

$$\lim_{x \to \infty} \frac{G(x) - \log x + \log m_1 - \gamma}{\int_x^\infty \mu((t, \infty))\, dt} = \frac{1}{m_1}.$$

The technique we use in the proof of this theorem was suggested by the methods used in [9], it might be of interest in its own right.

We turn to the zero-mean case. Let $\mathfrak{M}$ denote the space of all complex (finite) measures. In order of to make the results (and later on, the proofs) more transparent we introduce

$$\Sigma: \{\mu \in \mathfrak{M}: F_\mu\ \lambda\text{-integrable}\} \to \mathfrak{M}$$

by

$$(\Sigma\mu)(A) = \int_A F_\mu\, d\lambda \qquad \text{for all } A \in \mathfrak{B},$$

where for all $\mu \in \mathfrak{M}\, F_\mu: \mathbb{R} \to \mathbb{C}$ is defined by

$$F_\mu(x) = \mu(\mathbb{R})\, I_{[0, \infty)}(x) - \mu((-\infty, x]) \qquad \text{for all } x \in \mathbb{R}.$$

Theorem 3. *Let μ, ν_h be as above, assume (2), (3) and $m_1 = 0$, let τ be a DVF. Then*

$$\Sigma\mu((x, \infty)) = o(\tau(x))$$

implies

$$|\nu_h - \lambda_h|([x, x+1]) = o(\tau(x)),$$

and

$$\Sigma\mu((-\infty, -x)) = o(\tau(x))$$

implies

$$|\nu_h - \lambda_h|([-x, -x+1]) = o(\tau(x))$$

as $x \to \infty$. The same holds if o is replaced by O throughout.

Because of (2) the conditions on the integrated tails are satisfied if $\tau(x)=(1+x)^{-1}$ which gives asymptotic equality of $\nu_h(x+A)$ and $\lambda_h(x+A)$ as $|x|\to\infty$ for all bounded $A\in\mathfrak{B}$ with $\lambda(A)>0$. Similarly the other remarks following Theorem 1 may be transferred to the present situation. As an analogue of Theorem 2 in the zero-mean case we have the following result.

Theorem 4. *Let μ satisfy* (3), $\int |x|^3\,\mu(dx)<\infty$ *and* $m_1=0$, *assume that the tails of* $\Sigma\,\mu$ *are of dominated variation;* $H(x)=\nu_h([-x,x])$ *for all* $x\geqq 0$. *Then*

$$H(x)=2\log x-\log m_2+2\gamma+\frac{1}{m_2}\int_x^\infty \Sigma\,\mu((t,\infty))\,dt$$
$$-\frac{1}{m_2}\int_{-\infty}^{-x} \Sigma\,\mu((-\infty,t])\,dt+O(\Sigma\,\mu([-x,x]^c))$$

as $x\to\infty$; *in particular*

$$\lim_{x\to\infty}\frac{H(x)-2\log x-\log m_2+2\gamma}{\int_x^\infty \Sigma\,\mu((t,\infty))\,dt-\int_{-\infty}^{-x}\Sigma\,\mu((-\infty,t])\,dt}=\frac{1}{m_2}.$$

From the point of view of random walk theory Theorem 2 gives an expansion of $\sum_{n=1}^{\infty}\frac{1}{n}P(S_n\leqq x)$ down to the order of $P(X_1\geqq x)$; this should be compared with [7], Theorem 3 and Example 3. Similarly Theorem 4 expands $\sum_{n=1}^{\infty}\frac{1}{n}P(|S_n|\leqq x)$ down to the order of $\int_x^\infty P(|X_1|\geqq t)\,dt$.

Proofs are given in Sect. 2. In Sect. 2.1, we regard ν_h as a tempered distribution and give its Fourier transform. Section 2.2 lists some facts from Banach algebra theory, the proofs of the theorems are then given in Sects. 2.3–2.6 respectively. Sect. 3 contains some concluding remarks.

2. Proofs

2.1. Let $\mathscr{S}$ denote the space of rapidly decreasing functions, let $\mathscr{S}'$ be the space of tempered distributions ([15], §7 contains all notions and facts from the theory of generalized functions we need in this section). Locally λ-integrable functions g and positive measures μ which are finite on compact sets define tempered distributions via

$$f\to\int f g\,d\lambda,\quad f\to\int f\,d\mu\quad (f\in\mathscr{S})$$

if $\int_{-x}^{x}|g|\,d\lambda$ and $|\mu|((-x,x))$ grow at most polynomially as $x\to\infty$. Using concentration function estimates as given e.g. in [3] we see that ν_h has these properties and may thus be interpreted as a tempered distribution; we do not always distinguish between such measures (functions) and the associated tempered distributions, $\hat{}$ denotes Fourier transformation in all these cases.

Let $\log: G \to \mathbb{C}$, $G = \mathbb{C} - \{z \in \mathbb{C}: \operatorname{Im} z = 0, \operatorname{Re} z \leqq 0\}$ denote the principal branch of the logarithm (which is analytic on G). Expanding μ about 0 and using (2) we see that $\int_{-\alpha}^{\alpha} |\psi(\theta)| \, d\theta < \infty$ for some $\alpha > 0$, where

$$\psi(\theta) = -\log(1 - \hat{\mu}(\theta)), \qquad \theta \neq 0.$$

We have $|\hat{\mu}(\theta)| < 1$ for all $\theta \neq 0$ since μ is non-lattice, so the integral is finite for all $\alpha > 0$. Further it is known that (3) implies Cramér's condition, so the integral is bounded by a multiple of α and ψ is a tempered distribution.

Proposition 1. *ψ is the Fourier transform of v_h.*

The proof follows standard patterns from the theory of generalized functions and is therefore omitted.

Proposition 1 enables us to compute the Fourier transforms of measures which are sufficiently near to λ_h^+ and λ_h respectively for our later calculations.

Example 1. Let μ be the exponential distribution with parameter 1. Then μ^{*n} has density f_n,

$$f_n(x) = \frac{x^{n-1}}{(n-1)!} e^{-x} \quad \text{if } x > 0, \quad f_n(x) = 0 \quad \text{otherwise},$$

so v_h has density f_h,

$$f_h(x) = \frac{1}{x}(1 - e^{-x}) \quad \text{if } x > 0, \quad f_h(x) = 0 \quad \text{otherwise}.$$

We have $\hat{\mu}(\theta) = \dfrac{1}{1 - i\theta}$, so

$$\hat{v}_h(\theta) = -\log \frac{i\theta}{i\theta - 1}, \qquad \theta \neq 0.$$

Example 2. Applying the transition $x \to -x$ in the situation of Example 1 we see that the measure with density 0 on $[0, \infty)$ and $-\dfrac{1}{x}(1 - e^{x})$ on $(-\infty, 0)$ has Fourier transform $-\log \dfrac{i\theta}{1 + i\theta}$, so if we define v_1 by

$$v_1(A) = \int_A \frac{1}{|x|}(1 - e^{-|x|}) \, dx \qquad \text{for all } A \in \mathfrak{B}$$

we obtain

$$\hat{v}_1(\theta) = -\log \frac{\theta^2}{1 + \theta^2}, \qquad \theta \neq 0.$$

Proposition 1 displays v_1 as the harmonic renewal measure of the bilateral exponential distribution with parameter 1.

2.2 Let $\mathbb{B}$ be a commutative Banach algebra with unit, let $\mathfrak{I}$ denote the set of its maximal ideals. To any $I \in \mathfrak{I}$ there corresponds uniquely a multiplicative functional $\psi_I: \mathbb{B} \to \mathbb{C}$ of norm one such that $I = \psi_I^{-1}(\{0\})$; the Gelfand transform $\tilde{x}: \mathfrak{I} \to \mathbb{C}$ of some $x \in \mathbb{B}$ is defined by $\tilde{x}(I) = \psi_I(x)$. The Wiener-Lévy-Gelfand Theorem states that if $x \in \mathbb{B}$, $U \subset \mathbb{C}$, $\Psi: U \to \mathbb{C}$ are such that

$$\tilde{x}(\mathfrak{I}) \subset U, \qquad U \text{ open}, \qquad \Psi \text{ analytic in } U,$$

then there exists a $y \in \mathbb{B}$ with $\tilde{y} = \Psi \circ \tilde{x}$ ([6], §6). $\mathfrak{M}$, endowed with the total variation norm $\|\,.\,\|_{TV}$ and with convolution as multiplication, is such a Banach algebra; $\tilde{}$ and $\mathfrak{I}$ refer to this space from now on. The important fact on $\mathfrak{I}$ which we need below is the following: $I \in \mathfrak{I}$ either contains $\mathfrak{M}^a$, the set of all absolutely continuous complex measures, or it is of the form $I = I(\theta_0) = \{\mu \in \mathfrak{M}: \hat{\mu}(\theta_0) = 0\}$ for some $\theta_0 \in \mathbb{R}$ ([6], §30). Identifying $\{I(\theta): \theta \in \mathbb{R}\}$ and $\mathbb{R}$ and using

$$\tilde{\mu}(I(\theta)) = \hat{\mu}(\theta) \qquad \text{for all } \theta \in \mathbb{R}$$

we see that we may regard the Gelfand transform as a continuation of the Fourier transform. Its great advantage lies in the fact, expressed in the Wiener-Lévy-Gelfand Theorem, that analytic functions of the transform of a measure are again transforms of measures. This carries over to certain subalgebras of $\mathfrak{M}$ as we now explain.

Let τ be a DVF and put

$$\mathfrak{M}(\tau) = \{\mu \in \mathfrak{M}: |\mu|([x, x+1]) = O(\tau(x)) \text{ as } x \to \infty\},$$
$$\mathfrak{M}^0(\tau) = \{\mu \in \mathfrak{M}: |\mu|([x, x+1]) = o(\tau(x)) \text{ as } x \to \infty\}.$$

Rogozin and Sgibnev [14] proved that these spaces, endowed with a suitable norm, are Banach algebras again, and that any maximal ideal in $\mathfrak{M}(\tau)$ ($\mathfrak{M}^0(\tau)$) is the intersection of some maximal ideal in $\mathfrak{M}$ with $\mathfrak{M}(\tau)$ ($\mathfrak{M}^0(\tau)$). Thus the range of the Gelfand transform with respect to $\mathfrak{M}(\tau)$ ($\mathfrak{M}^0(\tau)$) of some $\mu \in \mathfrak{M}(\tau)$ ($\mathfrak{M}^0(\tau)$) is contained in $\tilde{\mu}(\mathfrak{I})$, and in order to apply the Wiener-Lévy-Gelfand Theorem with $\mathbb{B} = \mathfrak{M}(\tau)$ or $\mathbb{B} = \mathfrak{M}^0(\tau)$ it is enough to consider the range of $\tilde{\mu}$, the Gelfand transform of μ with respect to $\mathfrak{M}$.

2.3. *Proof of Theorem 1.* It is easily checked that

$$\widehat{\Sigma\mu}(\theta) = \frac{\hat{\mu}(\theta) - 1}{i\theta} \quad \text{if } \theta \neq 0, \qquad \widehat{\Sigma\mu}(0) = m_1,$$

and evidently $\mu([x, \infty)) = o(\tau(x))$ implies $\Sigma\mu \in \mathfrak{M}^0(\tau)$.

We put $\mu_1 = \Sigma\mu + \delta_0 - \mu$, then $\mu_1 \in \mathfrak{M}^0(\tau)$.

We show next that the range of $\tilde{\mu}_1$ is contained in G (G is defined in 2.1). Let $I \in \mathfrak{I}$ be of the form $I = I(\theta_0)$. Then $\psi_I(\mu) = \hat{\mu}(\theta_0)$ for all $\mu \in \mathfrak{M}$, hence $\tilde{\mu}_1(I) = \hat{\mu}_1(\theta_0)$ which is $m_1 (\in G)$ if $\theta_0 = \theta$; if $\theta_0 \neq 0$ we have

$$\hat{\mu}_1(\theta_0) = (1 - \hat{\mu}(\theta_0))\left(1 - \frac{1}{i\theta_0}\right).$$

The real parts of both factors are larger than zero, so the product must be in G.

Any other maximal ideal $I \in \mathfrak{J}$ has the property $\psi_I|_{\mathfrak{M}^a} \equiv 0$, so $\tilde{\mu}_1(I) = 1 - \tilde{\mu}(I)$ in this case. If ψ_I vanishes on $\mathfrak{M}^a$ we have

$$\psi_I(\mu^{*n}) = \psi_I((\mu^{*n})_{\text{sing}}) \quad \text{for all } n \in \mathbb{N},$$

where $(.)_{\text{sing}}$ denotes the singular part of a measure. We also have

$$|\psi_I((\mu^{*n})_{\text{sing}})| \leqq \|(\mu^{*n})_{\text{sing}}\|_{TV} \quad \text{for all } n \in \mathbb{N}$$

since ψ_I is of norm 1, so we obtain from (3) for a suitable $n \in \mathbb{N}$

$$|\tilde{\mu}(I)|^n = |\psi_I(\mu^{*n})| < 1.$$

This gives $|\tilde{\mu}(I)| < 1$, so $\tilde{\mu}_1(I) \in G$ also holds for these ideals. Using the Wiener-Lévy-Gelfand Theorem now we see that $\log \circ \hat{\mu}_1 = \hat{\nu}$ for some $\nu \in \mathfrak{M}^0(\tau)$. Since

$$\log \hat{\mu}_1(\theta) = \log(1 - \hat{\mu}(\theta)) - \log\left(1 - \frac{1}{1 - i\theta}\right)$$

for all $\theta \neq 0$, Proposition 1 identifies ν as the difference of the harmonic renewal measures corresponding to the exponential distribution with parameter 1 and μ respectively. Since we can find to any DVF τ a $k \in \mathbb{N}$ such that $x^{-k} = O(\tau(x))$ ([5], p. 289), this together with the computation in Example 1 gives the assertion of the theorem in case of $x \to \infty$. The modifications necessary to obtain the $x \to -\infty$ result should be obvious; replacing $\mathfrak{M}^0(\tau)$ by $\mathfrak{M}(\tau)$ throughout we obtain the corresponding O-results.

2.4. *Proof of Theorem 2.* If $\mathfrak{M}_0$ is some subset of $\mathfrak{M}$ let $\hat{\mathfrak{M}}_0$ denote the set of all Fourier transforms $\hat{\mu}$, $\mu \in \mathfrak{M}_0$. Define $\tau\colon [0,\infty) \to [0,\infty)$ by $\tau(x) = \mu((x,\infty))$ for all $x \geqq 0$. Let μ_1 be defined as in 2.3, F_{μ_1} is λ-integrable because of $m_2 < \infty$.

Lemma 1.

$$\Sigma \mu_1 * (\mu_1 - m_1 \delta_0) \in \mathfrak{M}(\tau).$$

Proof. Let $n \in \mathbb{N}$, $0 < x = x_0 < x_1 < \ldots < x_n = x+1$ be given. Then for any $k \in \{1, \ldots, n\}$ we obtain on using partial integration

$$\begin{aligned}(\mu_1 - m_1 \delta_0) * \Sigma \mu_1((x_{k-1}, x_k]) = &\int_{(-\infty, \frac{x}{2}]} \mu_1((x_{k-1} - y, x_k - y])\, \Sigma \mu_1(dy) \\ &+ \int_{(\frac{x}{2}, \infty)} (\Sigma \mu_1)((x_{k-1} - y, x_k - y])\, \mu_1(dy) - \Sigma \mu_1\left(\left(x_{k-1} - \frac{x}{2}, x_k - \frac{x}{2}\right]\right) \mu_1\left(\left(\frac{x}{2}, \infty\right)\right).\end{aligned}$$

Taking absolute values, summing over k and then taking suprema with respect to partitions of $(x, x+1]$ we get

$$\begin{aligned}|\Sigma \mu_1 * (\mu_1 - m_1 \delta_0)|((x, x+1]) \leqq &\int_{(-\infty, \frac{x}{2}]} |\mu_1|((x-y, x-y+1])\, |\Sigma \mu_1|(dy) \\ &+ \int_{(\frac{x}{2}, \infty)} |\Sigma \mu_1|((x-y, x-y+1])\, |\mu_1|(dy) + |\Sigma \mu_1|\left(\left(\frac{x}{2}, \frac{x}{2} + 1\right]\right) |\mu_1|\left(\left(\frac{x}{2}, \infty\right)\right).\end{aligned}$$

The first term is of order $o(\tau(x))$ since $|\Sigma\mu_1|(\mathbb{R})<\infty$ and

$$\sup_{y\leqq x/2}|\mu_1|((x-y,x-y+1])=O(\sup_{y\geqq x/2}\tau(y))=O(\tau(x)).$$

On $y\in\left(\frac{x}{2}+k,\frac{x}{2}+k+1\right]$ the integrand in the second term is bounded by $|\Sigma\mu_1|\left(\left[\frac{x}{2}-k-1,\frac{x}{2}-k+1\right]\right)$. This gives the bound

$$2|\Sigma\mu_1|(\mathbb{R})\sup_{y\geqq x/2}|\mu_1|([y,y+1])$$

which is $O(\tau(x))$ again.

Turning to the third term we note that both factors are of order $O(\tau_1(x/2))$, where $\tau_1(x)=\int_x^\infty \tau(t)\,dt$ for all $x\geqq 0$. We have $\tau(x)\leqq m_2 x^{-2}$ which gives

$$\begin{aligned}\tau_1(x)&\leqq\int_x^{(m_2/\tau(x))^{1/2}}\tau(x)\,dt+m_2\int_{(m_2/\tau(x))^{1/2}}^\infty t^{-2}\,dt\\&\leqq 2(m_2\tau(x))^{1/2},\end{aligned}$$

so $\tau_1(x/2)^2=O(\tau(x))$ which settles the last term. ///

The proof of the following simple lemma is left to the reader.

Lemma 2. *Let μ', $\mu''\in\mathfrak{M}$ be such that $F_{\mu'}$ is λ-integrable and $\mu'(\mathbb{R})=0$. Then $F_{\mu'*\mu''}$ is also λ-integrable.*

Now let ν_2 denote the harmonic renewal measure of the exponential distribution with parameter 1, put $\nu_3=\nu_2-\nu_h$. From Sect. 2.3 we know that $\nu_3\in\mathfrak{M}$ and that $\hat\nu_3=\Psi\circ\hat\mu_1$ where $\Psi: G\to\mathbb{C}$, $\Psi(z)=\log z$. With a suitable $\Phi: G\to\mathbb{C}$, analytic on G again, we have

$$\Psi(z)=\Psi(m_1)+(z-m_1)\,\Psi'(m_1)+(z-m_1)^2\,\Phi(z)\qquad\text{for all } z\in G.\tag{5}$$

Using Lemma 2 we see that F_{ν_3} is λ-integrable, hence

$$\widehat{\Sigma\nu_3}(\theta)=\frac{1}{m_1}\widehat{\Sigma\mu_1}(\theta)+\widehat{\Sigma\mu_1}(\theta)\,(\hat\mu_1(\theta)-m_1)\,\Phi(\hat\mu_1(\theta)).\tag{6}$$

The Wiener-Lévy-Gelfand Theorem gives $\Phi\circ\hat\mu_1\in\hat{\mathfrak{M}}(\tau)$, this together with Lemma 1 and (6) implies

$$\Sigma\nu_3-\frac{1}{m_1}\Sigma\mu_1\in\mathfrak{M}(\tau).\tag{7}$$

On $(0,\infty)$ a density of this measure is f, where

$$f(x)=(\nu_2-\nu_h)([x,\infty))-\frac{1}{m_1}\int_x^\infty\mu((t,\infty))\,dt+\frac{1}{m_1}\mu((x,\infty)),$$

and (7) means $\int_x^{x+1}|f(t)|\,dt=O(\tau(x))$.

We have

$$|f(x)| \leqq \int_x^{x+1} |f(t)|\,dt + \sup_{x \leqq t \leqq x+1} |f(t)-f(x)|$$
$$\leqq \int_x^{x+1} |f(t)|\,dt + |v_2 - v_h|([x, x+1]) + \frac{1}{m_1}|\mu_1|([x, x+1]).$$

Using Theorem 1 on the middle term we obtain $O(\tau(x))$-behaviour for the density itself, so

$$(v_2 - v_h)([x, \infty)) - \frac{1}{m_1}\Sigma\,\mu((x, \infty)) = O(\tau(x)).$$

We have to relate this quantity with the harmonic renewal function. The total mass of $v_2 - v_h$ is $\hat{v}_3(0)$, hence

$$\int_0^x (1-e^{-t})\,t^{-1}\,dt - G(x) = \log m_1 - (v_2 - v_h)((x, \infty)).$$

Using

$$\gamma = \int_0^1 (1-e^{-t})\,t^{-1}\,dt - \int_1^\infty e^{-t}\,t^{-1}\,dt$$

(see e.g. [16], p. 246) and

$$\int_x^\infty e^{-t}\,t^{-1}\,dt = o(\tau(x))$$

we obtain

$$\int_0^x (1-e^{-t})\,t^{-1}\,dt = \log x + \gamma + o(\tau(x))$$

and the first assertion of the theorem follows. The second is a simple consequence of the first since $\mu((x, \infty)) = o(\Sigma\,\mu((x, \infty)))$ which follows from

$$x\,\mu((x, \infty)) = O(x\,\mu((2x, \infty))),$$
$$x\,\mu((2x, \infty)) \leqq \int_x^{2x} \mu((t, \infty))\,dt \leqq \Sigma\,\mu((x, \infty)).$$

2.5. *Proof of Theorem 3.* Because of $m_2 < \infty$ we may apply Σ to $\Sigma\,\mu$ again, and because of $m_1 = 0$ we have

$$\widehat{\Sigma\Sigma\mu}(\theta) = \frac{\hat{\mu}(\theta) - 1}{\theta^2} \quad \text{if } \theta \neq 0, \qquad \widehat{\Sigma\Sigma\mu}(0) = -m_2(<0);$$

$\Sigma\,\mu((x, \infty)) = o(\tau(x))$ implies $\Sigma\,\Sigma\,\mu \in \mathfrak{M}^0(\tau)$.

Define $\mu_2 = \delta_0 - \mu - \Sigma\,\Sigma\,\mu (\in \mathfrak{M}^0(\tau))$. Then $\hat{\mu}_2(0) = m_2 \in G$, while if $\theta \neq 0$ we have

$$\hat{\mu}_2(\theta) = (1 - \hat{\mu}(\theta))\left(1 + \frac{1}{\theta^2}\right)$$

which evidently is also in G. So $\tilde{\mu}_2(I) \in G$ for all maximal ideals I which are of the form $I = I(\theta_0)$ for some $\theta_0 \in \mathbb{R}$, for all other maximal ideals the same argument as in 2.3 applies. Also as in the proof of Theorem 1 we see that $\hat{\nu}_4 = \log \circ \hat{\mu}_2$ for some $\nu_4 \in \mathfrak{M}^0(\tau)$, using Proposition 1 and Example 2 it follows that $-\nu_4$ deviates from $\nu_h - \lambda_h$ only by a negligible term.

2.6. *Proof of Theorem 4.* We proceed as in the proof of Theorem 2. Put $\tau(x) = \Sigma\, \mu((x, \infty))$ for all $x \geqq 0$, let μ_2 and ν_4 be as in 2.5, F_{μ_2} is λ-integrable because of $\int |x|^3 \mu(dx) < \infty$. We have

$$\widehat{\Sigma \nu_4}(\theta) = \frac{1}{m_2} \widehat{\Sigma \mu_2}(\theta) + \widehat{\Sigma \mu_2}(\theta)(\hat{\mu}_2(\theta) - m_2)\, \Phi_1(\hat{\mu}_2(\theta))$$

with a suitable Φ_1, analytic on G. The same computations as in the proof of Lemma 1 give

$$\Sigma\, \mu_2 * (\mu_2 - m_2\, \delta_0) \in \mathfrak{M}(\tau),$$

so we obtain as in 2.4

$$\int_x^{x+1} |f(t)|\, dt = O(\tau(x))$$

with $f(x) = \nu_4((x, \infty)) - \dfrac{1}{m_2} \mu_2((x, \infty))$. Since

$$f(x) = \nu_4((x, \infty)) - \frac{1}{m_2} \Sigma\Sigma\, \mu((x, \infty)) + O(\tau(x))$$

and

$$\left| \int_x^{x+1} f(t)\, dt - f(x) \right| \leqq |\nu_4|([x, x+1]) + \frac{1}{m_2} |\mu_2|([x, x+1])$$

Theorem 3 yields

$$\nu_4((x, \infty)) - \frac{1}{m_2} \Sigma\Sigma\, \mu((x, \infty)) = O(\Sigma\, \mu((x, \infty))).$$

By symmetry then also

$$\nu_4((-\infty, -x)) - \frac{1}{m_2} \Sigma\Sigma\, \mu((-\infty, -x)) = O(\Sigma\, \mu((-\infty, -x))).$$

We have $\nu_4 = \nu_1 - \nu_h$, where ν_1 denotes the absolutely continuous measure with density $(1 - e^{-|x|})\, |x|^{-1}$, $x \in \mathbb{R}$.

This gives

$$\begin{aligned} H(x) &= 2 \int_0^x (1 - e^{-t})\, t^{-1}\, dt - \nu_4(\mathbb{R}) + \nu_4((x, \infty)) + \nu_4((-\infty, -x)) \\ &= 2 \log x + 2\gamma - \log m_2 + \frac{1}{m_2} \Sigma\Sigma\, \mu((x, \infty)) \\ &\quad + \frac{1}{m_2} \Sigma\Sigma\, \mu((-\infty, -x)) + O(\Sigma\, \mu((-\infty, -x))) + O(\Sigma\, \mu((x, \infty))), \end{aligned}$$

the first assertion of the theorem. The second follows from $\Sigma\,\mu((-\infty, -x)) = o(\Sigma\Sigma\,\mu((-\infty, -x)))$, $\Sigma\,\mu((x, \infty)) = o(\Sigma\Sigma\,\mu((x, \infty)))$ which can be proved in the same way as the corresponding statement in the proof of Theorem 2.

3. Concluding Remarks

3.1. If (3) does not hold then v_h is concentrated on a Lebesgue null set which implies

$$|v_h - \lambda_h|([x, x+1]) \geqq \lambda_h([x, x+1]) \geqq \frac{1}{x+1} \quad \text{for all } x \geqq 0,$$

so in order to obtain (non-trivial) strong uniformity results on the asymptotic behaviour of harmonic renewal measures condition (3) is in some sense necessary (see also [1], 4.(d)).

3.2. Our results relate the asymptotic behaviour of harmonic renewal measures v_h to corresponding properties of the underlying distribution μ. The method may also be used to obtain properties of μ from similar ones of v_h, some arguments even simplify since the logarithm may be replaced by the exponential function.

3.3. Using suitable Banach algebras of absolutely convergent sequences of complex numbers we obtain analogues of our results for lattice distributions. The lattice case is easier to deal with, we can for example dispense with the use of generalized functions; also the results of [9] then become applicable. We will deal with harmonic renewal sequences in a separate paper.

3.4. A problem interesting in its own right is to find the weakest possible assumptions for the estimates and limit relations above, e.g. to what extent can the class of DVFs be enlarged whilst keeping the implications in Theorem 1 in force? Within the framework of our method this leads to an analogous question concerning the Banach algebra results, we refer the reader to the work of Rogozin and Sgibnev [14] for related information.

3.5. Recently Maejima and Omey [12] extended the results of [1] to the two-sided case. Their proof is based on a result of Kalma [11] (see [12], p. 127) which gives an interesting simultaneous comparison of the asymptotic behaviour as $t \to \infty$ of $\mu^{*m}((t, t+h])$, $m \in \mathbb{N}$, with that of $v((t, t+h])$, where $v = \sum_{n=0}^{\infty} \mu^{*n}$ denotes the ordinary renewal mesure. Due to the appearance of v this has – in contrast to our method – no direct analogue in the case of zero mean.

3.6. As the above results should have shown the use of generalized functions and Banach algebra theory is an appropriate method for extending results from the one-sided to the general case in the context of harmonic renewal theory. A somewhat similar situation arises in distribution theory where results relating asymptotic properties of infinitely divisible distributions and their Lévy measures are of interest, see [2] and [10], Sect. 4.2. As a third example we mention the papers of Essén [4] and Rogozin [13] dealing with (ordinary) renewal theory where these tools are also used (at least implicity). Especially the Essén paper has been of great influence to the present work.

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