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Classical Dirichlet Forms On Topological Vector Spaces – the Construction of the Associated Diffusion Process*

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Summary. Given a (minimal) classical Dirichlet form on $L^2(E; \mu)$ we construct the associated diffusion process. Here E is a locally convex topological vector space and μ is a (not necessarily quasi-invariant) probability measure on E. The construction is carried out under certain assumptions on E and μ which can be easily verified in many examples. In particular, we explicitly apply our results to (time-zero and space-time) quantum fields (with or without cut-off).

Contents

0. Introduction

In this paper we continue our study of classical Dirichlet forms on topological vector spaces started in [A/Rö 1, 2] (to which we also refer the reader for the background literature). Our main object here is to construct an associated diffusion process. The exposition is self-contained and includes a review of all cases studied so far (cf. [A/H-K 2–4], [K]). The new results, in particular, imply the existence of the diffusion process in the case of two (and three) dimensional quantum fields (with or without cut-off, cf. below). Let us briefly recall our framework.

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* *Dedication:* Raphael Høegh-Krohn (1938–1988) was an initiator of the theory of Dirichlet forms over infinite dimensional spaces. He has been a continuous source of inspiration. We deeply mourn his departure and dedicate to him this work, as a small sign of our great gratitude

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We study forms of the type

$$\mathscr{E}(u,v)=\sum_{k}\int_{E}\frac{\partial u}{\partial k}\frac{\partial v}{\partial k}\,\mathrm{d}\mu \tag{0.1}$$

on $L^2(E;\mu)$ where E is a locally convex Hausdorff topological vector space which is in addition Souslinean and μ is a finite positive measure on its Borel sets $\mathscr{B}(E)$. $\frac{\partial}{\partial k}$ means partial derivative in the direction given by some $k\in E\setminus\{0\}$. The domain $D(\mathscr{E})$ of (0.1) is $C_0^\infty(\mathbb{R}^d)$ if $E=\mathbb{R}^d$ and $\widetilde{\mathscr{F}C_b^\infty}$, i.e. the set of all (classes of) bounded smooth functions u depending on only finitely many coordinates if E is infinite dimensional. The sum in (0.1) is over an (at most) countable set K_0 of $k\in E$ where K_0 is such that $\mathscr{E}(u,u)<+\infty$ for all $u\in\widetilde{\mathscr{F}C_b^\infty}$ if E is infinite dimensional. K_0 can e.g. be taken to be an orthonormal basis of some Hilbert space H, which is densely and continuously imbedded in E and which plays the role of a tangent space. The resulting form (0.1) is then independent of the special orthonormal basis K_0 (cf. 1.9–1.12 below for details). In addition, one always has to assume that $\mathscr{E}$ is well-defined on $L^2(E;\mu)$, i.e. that $\frac{\partial u}{\partial k}=\frac{\partial v}{\partial k}$ μ-a.e. if $u, v\in\mathscr{F}C_b^\infty$ with $u=v$ μ-a.e. (This is the case if e.g. $\operatorname{supp}\mu=E$).

We emphasize that as in [A/Rö1, 2] we do not assume μ to be k-quasi-invariant for $k\in K_0$ (cf. 3.2 below) which was always done in the earlier literature. One advantage of this more general setting is that e.g., the case where E is replaced by some Borel subset can immediately be reduced to our situation. Furthermore, it can be very hard to prove quasi-invariance in examples, e.g., it is not known whether 3-dimensional quantum fields have this property.

If the form (0.1) is closable it is quite easy to show that its closure is a Dirichlet form in the sense of [F], [S] (cf. 1.6 below) called a *classical Dirichlet form* in [A/Rö1]. In this case one can expect that as in the case $E=\mathbb{R}^d$ there is a probabilistic counterpart to such a Dirichlet form, i.e. there exists an associated diffusion process. To explain this more precisely, we need some preparations.

Suppose that (0.1) is closable and let $(\mathscr{E}^0, D(\mathscr{E}^0))$ denote its closure (cf. 1.1 below). Let A be the associated non-positive definite self-adjoint operator on $L^2(E;\mu)$, i.e. $D(\sqrt{-A})=D(\mathscr{E}^0)$ and $\mathscr{E}^0(u,v)=\langle\sqrt{-A}u,\sqrt{-A}v\rangle_{L^2(E;\mu)}$, u, $v\in D(\mathscr{E}^0)$. Let $T_t=e^{tA}$, $t\geqq 0$. Then $(T_t)_{(t\geqq 0)}$ is a symmetric contraction semigroup such that each T_t is *Markovian*, i.e. $0\leqq T_t u\leqq 1$ μ-a.e. if $0\leqq u\leqq 1$ μ-a.e. (the latter is implied by the fact that normal contractions operate on $(\mathscr{E}^0, D(\mathscr{E}^0))$, cf. 1.8 below and [F]). We say that a diffusion process $(\Omega, \mathscr{F}, (X_t)_{(t\geqq 0)}, (P_z)_{(z\in E)})$ with state space E is associated with $(\mathscr{E}^0, D(\mathscr{E}^0))$ if for any bounded $\mathscr{B}(E)$-measurable function u on E and each $t\geqq 0$

$$T_t u(z)=\int_{\Omega} u(X_t)\,\mathrm{d}P_z \qquad \text{for } \mu\text{-a.e. } z\in E \tag{0.2}$$

(cf. 2.1). Clearly, μ is then an invariant measure for $(X_t)_{(t\geqq 0)}$. Obviously, for (0.2) to hold for some process $(X_t)_{(t\geqq 0)}$ it is necessary that each T_t is Markovian. But for a rigorous existence proof for $(X_t)_{(t\geqq 0)}$ much more than this is needed. In the case where the state space E is locally compact (i.e. $\dim E<+\infty$ in our situation) the existence of a diffusion process satisfying (0.2) follows by a general construction due to M. Fukushima (see [F1] and [F, Sect. 6], [S] for detailed expositions). Unfortunately, this construction does not carry over to non-locally compact state spaces since e.g. it makes essential use of the one-point compactification and of the Riesz-Markov representation theorem for non-negative linear functionals on $C_0(E)$ (i.e. the continuous functions on E with compact support). Hence if E is infinite dimensional i.e. non-locally compact (note that $C_0(E)=\{0\}$ in this case) we cannot apply the results of [F, Sect. 6]. However, under certain conditions there is a method to reduce this existence problem for the associated diffusion process to the "locally compact case" via a suitable compactification (cf. [F0], [F2], [A/HK 2-4] and [K]).

In this paper we show that this compactification method works for the classical Dirichlet form $(\mathscr{E}^0, D(\mathscr{E}^0))$ under some general conditions on E, μ which are easy to check in many examples including all measures occuring in two (and three, cf. 3.11 (ii)) dimensional quantum field theory.

Using [F, Sect. 6] it is fairly easy to prove the existence of a diffusion process $(X_t)_{(t\geqq 0)}$ satisfying (0.2) on some compactification $\bar{E}$ of E. The main problem then is to prove that E is an invariant set for $(X_t)_{(t\geqq 0)}$ and that $(X_t)_{(t\geqq 0)}$ has continuous sample paths w.r.t. the topology on E (which is stronger than that on $\bar{E}$). This procedure has already been described in [A/HK 2-4] and has been carried out in detail in [K] in the case where μ is quasi-invariant and E is a Banach space (see also [F0]). Our method in this paper is an adaptation of that in [K] without assuming quasi-invariance and particularly suitable for quantum fields.

In the special case where $E=\mathscr{S}'(\mathbb{R}^2)$ (i.e. the space of tempered distributions on $\mathbb{R}^2$) and μ is a cut-off space time $P(\Phi)_2$-quantum field (cf. 1.13 (iii) below) the existence of a diffusion process satisfying (0.2) has been claimed in [Bo/Ch/Mi, Sect. 3]. But the corresponding proof in that paper is incomplete since it only refers to [F, Sect. 6] which is not applicable as indicated above. Therefore, our results in this paper also give a justification of the corresponding parts in [Bo/Ch/Mi]. At this point we also want to mention that if μ is some measure in quantum field theory, one might think that the set of "polynomials in the fields" (i.e. the set $\tilde{\mathscr{P}}$ defined in 1.10 (iii) below) is a more convenient domain for the form $\mathscr{E}$ defined by (0.1). But it turns out that this domain is so suitable for proving that the closure of $\mathscr{E}$ is a Dirichlet form i.e. that normal contractions operate on $\mathscr{E}$ (see 1.10 (iii) below for details).

Though we are mainly interested in applications to quantum field theory we present our results in a general framework to illustrate their significance within the theory of infinite dimensional diffusions. We expect the methods to be applicable in many other situations, in particular also for non-symmetric Dirichlet forms.

Let us shortly summarize the contents of the single sections of this paper.

We recall that above we have always assumed that the form (0.1) is closable. This property is absolutely crucial for the whole theory of Dirichlet forms. In Section 1, we therefore summarize the main results of [A/Rö1] where a necessary and sufficient condition for the closability of (the components of) the form (0.1) was proved. In addition we also describe the framework we work in and present in detail the main examples, i.e. abstract Wiener spaces and the quantum fields already mentioned (cf. 1.13 below).

In Sect. 2 we describe in general the compactification method to gain the existence of the diffusion process $(X_t)_{(t\geqq 0)}$. This method is known to experts in the field. We have included this section for the convenience of the reader and in order to isolate the conditions (cf. (2.8)–(2.11)) that an arbitrary (abstract) Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(E; \mu)$ must fulfill so that all relevant techniques work. The final existence result is summarized in Theorem 2.7.

In Sect. 3 we apply the results of Sect. 1, 2 to the classical Dirichlet form $(\mathscr{E}^0, D(\mathscr{E}^0))$. In part 3. a) we start with the case where E is a Banach space and recall the set of conditions (cf. (3.1)–(3.5) below) given in [K] that ensure the existence of an associated diffusion process if μ is quasi-invariant. Since condition (3.4) is in general hard to check in applications, we give a sufficient criterion for (3.4) to hold (cf. proposition 3.4). In part 3. b) we consider the special case where E is a Hilbert space and show that (3.4) is then always fulfilled. It turns out that in this case Theorem 2.7 is applicable to prove the existence of the diffusion process without assuming quasi-invariance. The corresponding proof is given in detail in the appendix. In part 3. c) we show that the case where E is the dual of a nuclear space can be reduced to the Hilbert space case if e.g., μ has finite (second) moments. Then in part 3. d) we apply the "conuclear case" to quantum fields. We also illustrate the significance of the tangent space H mentioned above which determines the Dirichlet form and hence the process (cf. 3.11 (i) below).

Our existence results in Sect. 3 immediately generalize to the cases where the classical Dirichlet form $(\mathscr{E}^0, D(\mathscr{E}^0))$ is replaced by a Dirichlet form of "diffusion type" (cf. [K] and 1.12 (iv) and 3.0 below). It would be interesting to try an alternative construction of processes given a Dirichlet form on $L^2(E; \mu)$ using the Ray-Knight compactification (cf. [Get]) and then trying to apply recent results by [St] to prove that E is an invariant set for the process. But so far, Theorem 2.7 seems to us more suitable for this problem, in particular w.r.t. regularity of sample paths.

Our further interest in Dirichlet forms on topological vector spaces E is two-fold: firstly we want to study the associated potential theory on the infinite dimensional space E. On the one hand many results from the finite dimensional case obtained in [F] carry over to this situation; on the other hand since we now have an associated diffusion process we can apply the "probabilistic potential theory" developed by Dynkin in [Dy 1, 2] for fine Markov processes on standard Borel spaces. Secondly we are interested in the significance of the diffusion processes $(X_t)_{(t\geqq 0)}$ constructed here for the stochastic quantization of quantum fields (cf. [J-L/Mi], [Mi], [Pa/Wu], [Bo/Ch/Mi], [Dö] and also [A/Rö 2]). Crucial for this are ergodic properties of $(X_t)_{(t\geqq 0)}$. The results we have obtained in this direction will be published in a forthcoming paper.

1. Framework and Summary of the Solution of the Closability Problem

In this section we describe the framework we work in and give a summary of the main results of [A/Rö 1]. For proofs we refer to Sects. 1–3 in [A/Rö 2].

Let $(H, <, >)$ be a real Hilbert space and $\| \| := \sqrt{<, >}$. In the sequel we say that a pair $(\mathscr{E}, D(\mathscr{E}))$ is a form on H if $D(\mathscr{E})$ is a linear subspace of H and $\mathscr{E}$: $D(\mathscr{E}) \times D(\mathscr{E}) \to \mathbb{R}$ is a non-negative symmetric bilinear form. Given a form $(\mathscr{E}, D(\mathscr{E}))$ on H and $\alpha > 0$, we set $\mathscr{E}_\alpha := \mathscr{E} + \alpha <, >$, $D(\mathscr{E}_\alpha) := D(\mathscr{E})$. $(\mathscr{E}, D(\mathscr{E}))$ is called closed if the pre-Hilbert space $(D(\mathscr{E}), \mathscr{E}_1)$ is complete and closable if it has a closed extension, i.e. there exists a closed form $(\tilde{\mathscr{E}}, D(\tilde{\mathscr{E}}))$ on H such that $D(\mathscr{E}) \subset D(\tilde{\mathscr{E}})$ and $\mathscr{E} = \tilde{\mathscr{E}}$ on $D(\mathscr{E})$. Clearly, $(\mathscr{E}, D(\mathscr{E}))$ is closable if and only if the following condition is satisfied:

$$(1.1) \quad \begin{cases} \text{If } u_n \in D(\mathscr{E}),\, n \in \mathbb{N}, \text{ such that } u_n \xrightarrow[n \to \infty]{} 0 \text{ in } H \text{ and } (u_n)_{(n \in \mathbb{N})} \text{ is} \\ \mathscr{E}\text{-Cauchy (i.e. } \mathscr{E}(u_n - u_m, u_n - u_m) \xrightarrow[n, m \to \infty]{} 0), \text{ then } \lim\limits_{n \to \infty} \mathscr{E}(u_n, u_n) = 0. \end{cases}$$

1.1 Remark. (i) If a form $(\mathscr{E}, D(\mathscr{E}))$ is given by an operator T on H with domain $D(T)$, i.e.

$$D(\mathscr{E}) = D(T) \quad \text{and} \quad \mathscr{E}(u, v) = \langle Tu, Tv \rangle \qquad \text{for } u, v \in D(\mathscr{E}),$$

$(\mathscr{E}, D(\mathscr{E}))$ is closed or closable if and only if the linear operator T on H with domain $D(T)$ is closed or closable. Furthermore, we recall that an operator T is closable if and only if its adjoint T^* is densely defined.

(ii) Let $\overline{D(\mathscr{E})}$ be the abstract completion of $D(\mathscr{E})$ with respect to $\mathscr{E}_1$. Let i be the natural continuous map from $\overline{D(\mathscr{E})}$ to H. $(\mathscr{E}, D(\mathscr{E}))$ is closable if and only if the map i is one to one.

(iii) If $(\mathscr{E}, D(\mathscr{E}))$ is closable it has a smallest closed extension $(\bar{E}, D(\bar{E}))$, called its closure (cf. [F, Sect. 1.1]).

From now on we will study the case $H := L^2(E; \mu)$ where E is a Hausdorff locally convex topological vector space over $\mathbb{R}$ equipped with its Borel σ-field $\mathscr{B}(E)$ and μ is a (probability) measure on $(E, \mathscr{B}(E))$. Of course now $<, >$ is the usual inner product on $L^2(E; \mu)$. For notational convenience we will denote the μ-class corresponding to a $\mathscr{B}(E)$-measurable function u also by u if no confusion is possible. To have a nice measure theory on $(E, \mathscr{B}(E))$ we assume that E is a Souslin space (in the sense of Bourbaki, i.e. E is the continuous image of a Polish space, cf. [Sch]). Given a form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(E; \mu)$ we call E its "state space" and we sometimes briefly say $(\mathscr{E}, D(\mathscr{E}))$ is a "form on E" (instead of "form on $L^2(E; \mu)$").

In the sequel we denote the Borel σ-field associated with a topological space X by $\mathscr{B}(X)$. Given two measurable spaces $(X_i, \mathscr{B}_i)$, $(i = 1, 2)$, a $\mathscr{B}_1/\mathscr{B}_2$-measurable map T: $X_1 \to X_2$ and a measure μ on $(X_1, \mathscr{B}_1)$ we denote the image measure under T on $(X_2, \mathscr{B}_2)$ by $T(\mu)$.

Let E' be the topological dual of E, $k \in E \setminus \{0\}$ and let $l \in E'$ such that $l(k) = 1$. Define

$$\pi_k(z) := z - l(z)\, k, \qquad z \in E.$$

Let $E_0 := \pi_k(E)$, then E_0 as a closed subspace of E is also a Souslin space. For each $z \in E$, $z = x + sk$, where $x \in E_0$, $s \in \mathbb{R}$, are uniquely determined. Since E, E_0 are Souslinean we can disintegrate μ with respect to π_k: $E \to E_0$ (cf. [D/M, III70] or [R, Proposition 1]), i.e., there exists a kernel $\rho_k\colon E_0\times\mathscr{B}(\mathbb{R})\to[0,1]$ such that for all $u\colon E\to\mathbb{R}$, u bounded, $\mathscr{B}(E)$-measurable

$$\int_E u(z)\,\mu(\mathrm{d}z)=\int_{E_0}\int_{\mathbb{R}} u(x+sk)\,\rho_k(x,\mathrm{d}s)\,\nu_k(\mathrm{d}x) \tag{1.2}$$

where $\nu_k:=\pi_k(\mu)$ and $\rho_k(\cdot,\,\mathrm{d}s)$ is ν_k-a.e. uniquely determined. It is now easy to verify that $(L^2(\mathbb{R};\,\rho_k(x,\,\mathrm{d}s)),\,<,>_x)_{(x\in E_0)}$ is a measurable field of Hilbert spaces over $(E_0,\mathscr{B}(E_0),\nu_k)$ (where, of course $<,>_x$ is the usual L^2-inner product with respect to the measure $\rho_k(x,\mathrm{d}s)$) and that

$$L^2(E;\mu)=\int^{\oplus} L^2(\mathbb{R};\rho_k(x,\mathrm{d}s))\,\nu_k(\mathrm{d}x) \tag{1.3}$$

(cf. [Di, Chap. II, Sect. 1] and [A/Rö1, Sect. 1]). Here for $u\in L^2(E;\,\mu)$ the corresponding field $(u_x)_{(x\in E_0)}$ of vectors in $\int^{\oplus} L^2(\mathbb{R};\,\rho_k(x,\,\mathrm{d}s))\,\nu_k(\mathrm{d}x)$ is given by $u_x:=u(x+\cdot k)$, $x\in E_0$. Let $n\in\mathbb{N}\cup\{\infty\}$. Define the linear space

$$\mathscr{F}C_b^n:=\{u\colon E\to\mathbb{R}\,|\,\text{there exist } l_1,\dots,l_m\in E' \text{ and } f\in C_b^n(\mathbb{R}^m) \text{ such that } u(z)=f(l_1(z),\dots,l_m(z)),\, z\in E\}$$

where $C_b^n(\mathbb{R}^m)$ is the set of all n-times continuously differentiable functions on $\mathbb{R}^m$ such that all partial derivatives of up to order n are bounded. Let $\widetilde{\mathscr{F}C_b^n}$ denote the associated set of classes in $L^2(E;\,\mu)$. Note that if $\operatorname{supp}\mu\neq E$, two different elements in $\mathscr{F}C_b^n$ might belong to the same class in $\widetilde{\mathscr{F}C_b^n}$. Define for $u\in\mathscr{F}C_b^1$ the following Gâteaux-type derivative (in direction k) by

$$\frac{\tilde\partial}{\partial k}u(z):=\frac{\mathrm{d}}{\mathrm{d}s}u(z+sk)\Big|_{s=0},\qquad z\in E. \tag{1.4}$$

If μ has the property

$$\frac{\tilde\partial}{\partial k}u=\frac{\tilde\partial}{\partial k}v\quad \mu\text{-a.e.}\ \text{ if } u,v\in\mathscr{F}C_b^\infty \text{ with } u=v\quad \mu\text{-a.e.}, \tag{1.5}$$

then $\frac{\tilde\partial}{\partial k}$ "respects μ-classes" and therefore defines a linear operator on $L^2(E;\,\mu)$ with domain $\widetilde{\mathscr{F}C_b^\infty}$ which we also denote by $\frac{\tilde\partial}{\partial k}$. In this case we define the corresponding form by

$$\widetilde{\mathscr{E}}_k(u,v)=\int\frac{\tilde\partial}{\partial k}u\,\frac{\tilde\partial}{\partial k}v\,\mathrm{d}\mu,\qquad u,v\in D(\widetilde{\mathscr{E}}_k):=\widetilde{\mathscr{F}C_b^\infty}. \tag{1.6}$$

1.2 Remark. Since E is Souslinean, $\mathscr{B}(E)$ is generated by all $l\in E'$ (cf. [Bad, Exposé n°8, N°7, Corollaire]). Hence if $u\in L^2(E;\,\mu)$ such that $\int\exp(il)\,u\,\mathrm{d}\mu=0$ for all $l\in E'$, it follows that $u=0$. Consequently, since $\cos l$, $\sin l\in\mathscr{F}C_b^\infty$ for $l\in E'$,

$\widetilde{\mathscr{F}C_b^\infty}$ is dense in $L^2(E;\mu)$ (cf. [A/H-K4, Sect. 2]). In particular, if μ satisfies (1.5), then $(\widetilde{\mathscr{E}_k}, \widetilde{\mathscr{F}C_b^\infty})$ is densely defined on $L^2(E;\mu)$.

Let $\mathrm{d}s$ denote Lebesgue measure on $\mathbb{R}$. Given a $\mathscr{B}(\mathbb{R})$-measurable function $\rho\colon \mathbb{R}\to\mathbb{R}_+$ consider the following condition which was introduced in [Ha]:

(H) $\rho=0$ $\mathrm{d}s$-a.e. on $\mathbb{R}\setminus R(\rho)$

$$\text{where } R(\rho):=\left\{t\in\mathbb{R}\,\middle|\,\int_{t-\varepsilon}^{t+\varepsilon}\rho^{-1}\,\mathrm{d}s<+\infty \text{ for some } \varepsilon>0\right\}.$$

(H) is a rather weak assumption. For instance, clearly every $\mathscr{B}(\mathbb{R})$-measurable function $\rho\colon\mathbb{R}\to\mathbb{R}^+$ having the property that for $\mathrm{d}s$-a.e. $s\in\{\rho>0\}$, $\operatorname{ess\,inf}\{\rho(s')\,|\,s-\varepsilon\leqq s'\leqq s+\varepsilon\}>0$ for some $\varepsilon>0$, satisfies (H). In particular, (H) holds for any lower semicontinuous nonnegative function on $\mathbb{R}$. On the other hand, if $C\subset\mathbb{R}$, C closed, with empty interior and strict positive, finite Lebesgue measure, then (H) does not hold for $\rho\colon\mathbb{R}\to\mathbb{R}_+$ defined by

$$\rho(s):=\begin{cases}1, & \text{if } s\in C,\\ 2^{-n}\dfrac{s-a_n}{(b_n-a_n)^2}, & \text{if } s\in\,]a_n,b_n[,\end{cases}$$

where $a_n, b_n\in\mathbb{R}$, $n\in\mathbb{N}$, $a_n<b_n$ such that $\mathbb{R}\setminus C=\bigcup_{n=1}^{\infty}\,]a_n,b_n[$, $]a_n,b_n[\,\cap\,]a_m,b_m[\,=\emptyset$ if $n\neq m$. Note that $\rho\in L^1(\mathbb{R},\mathrm{d}s)$ and even $\rho>0$ on $\mathbb{R}$, but $\mathbb{R}\setminus R(\rho)=C$.

Now we are prepared to state the main result of [A/Rö 1].

1.3 Theorem. *(i) Assume that for ν_k-a.e. $x\in E_0$, $\rho_k(x,\mathrm{d}s)=\rho_k(x,s)\,\mathrm{d}s$ for some $\mathscr{B}(\mathbb{R})$-measurable function $\rho_k(x,\cdot)\colon\mathbb{R}\to\mathbb{R}_+$ satisfying (H). Then the form*

$$\begin{aligned}(1.7)\quad D(\mathscr{E}_k):=\Big\{&u=(u_x)_{(x\in E_0)}\in\int^{\oplus}L^2(\mathbb{R};\rho_k(x,s)\,\mathrm{d}s)\,\nu_k(\mathrm{d}x)\,\big|\,\textit{for } \nu_k\textit{-a.e. } x\in E_0,\\ &u_x \textit{ has an absolutely continuous } (\mathrm{d}s\text{-}) \textit{ version } \widetilde{u_x} \textit{ on } R(\rho_k(x,\cdot))\\ &\textit{and } \frac{\partial}{\partial k}u:=\left(\frac{\mathrm{d}\widetilde{u_x}}{\mathrm{d}s}\right)_{(x\in E_0)}\in\int^{\oplus}L^2(\mathbb{R};\rho_k(x,s)\,\mathrm{d}s)\,\nu_k(\mathrm{d}x)\Big\},\end{aligned}$$

$$(1.8)\qquad \mathscr{E}_k(u,v):=\int\frac{\partial}{\partial k}u\,\frac{\partial}{\partial k}v\,\mathrm{d}\mu,\qquad u,v\in D(\mathscr{E}_k),$$

is closed, or equivalently the operator $\frac{\partial}{\partial k}$ (defined in (1.7)) with domain $D(\mathscr{E}_k)$ is closed. Furthermore, (1.5) *is satisfied and $\frac{\partial}{\partial k}$ is an extension of $\frac{\tilde{\partial}}{\partial k}$. In particular, the form $(\widetilde{\mathscr{E}_k}, \widetilde{\mathscr{F}C_b^\infty})$ is closable.*

(ii) If μ satisfies (1.5) *and the form $(\widetilde{\mathscr{E}_k}, \widetilde{\mathscr{F}C_b^\infty})$ is closable, then for ν_k-a.e. $x\in E_0$, $\rho_k(x,\mathrm{d}s)=\rho_k(x,s)\,\mathrm{d}s$ for some $\mathscr{B}(\mathbb{R})$-measurable function $\rho_k(x,\cdot)\colon\mathbb{R}\to\mathbb{R}^+$ satisfying (H). In particular, $(\mathscr{E}_k, D(\mathscr{E}_k))$ defined by* (1.7), (1.8) *is a closed extension of $(\widetilde{\mathscr{E}_k}, \widetilde{\mathscr{F}C_b^\infty})$.*

1.4. Remark. 1.3 in particular contains the case where $E=\mathbb{R}$ which was completely solved by Rullkötter/Spönemann extending a result by Hamza (see [Ru/Sp], [Sp], [Ha] and also [F, Sect. 2.1]). In fact the proof of 1.3 is based on a "reduction" to the one-dimensional case by disintegrating μ (cf. [A/Rö 1] for details). A slightly modified, but complete exposition of the proof for $E=\mathbb{R}$ is included in Sect. 2 resp. the appendix of [A/Rö 1].

1.5 Remark. (i) Under the assumptions of 1.3 (i) or 1.3 (ii) we can conclude that for each $n\in\mathbb{N}$, $\widetilde{\mathscr{E}_k}$ with domain $\widetilde{\mathscr{F}C_b^n}$ is also well-defined as a form on $L^2(E;\mu)$ and that $(\mathscr{E}_k, D(\mathscr{E}_k))$ is a closed extension of $(\widetilde{\mathscr{E}_k}, \widetilde{\mathscr{F}C_b^n})$ which is therefore closable.

(ii) Our assumptions in 1.3 (ii) do not involve l (or E_0). Hence, 1.3 is independent of the choice of l (or E_0) for a given $k\in E$.

1.6 Definition. Let $k\in E$. k is called *(μ)-admissible* if $k=0$ or for ν_k-a.e. $x\in E_0$, $\rho_k(x, ds)=\rho_k(x,s)\,ds$ for some $\mathscr{B}(\mathbb{R})$-measurable function $\rho_k(x,\cdot)$: $\mathbb{R}\to\mathbb{R}_+$ satisfying (H) or equivalently (cf. 1.3) if (1.5) is satisfied and $(\widetilde{\mathscr{E}_k}, \widetilde{\mathscr{F}C_b^n})$ is closable for some (all) $n\in\mathbb{N}\cup\{+\infty\}$.

1.7 Corollary. *Let K_0 be a finite or countable set of admissible elements in E. Let*

$$D(\mathscr{E}):=\{u\in \bigcap_{k\in K_0} D(\mathscr{E}_k) \mid \sum_{k\in K_0} \mathscr{E}_k(u,u)<+\infty\} \tag{1.9}$$

$$\mathscr{E}(u,v):=\sum_{k\in K_0} \mathscr{E}_k(u,v), \qquad u,v\in D(\mathscr{E}),$$

and let $(\tilde{\mathscr{E}}, D(\tilde{\mathscr{E}}))$ be defined correspondingly with $(\widetilde{\mathscr{E}_k}, \widetilde{\mathscr{F}C_b^\infty})$ replacing $(\mathscr{E}_k, D(\mathscr{E}_k))$. Then $(\mathscr{E}, D(\mathscr{E}))$ is a closed extension of $(\tilde{\mathscr{E}}, D(\tilde{\mathscr{E}}))$.

Now let us recall the definition of a Dirichlet form on $L^2(E;\mu)$.

1.8 Definition. (i) A form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(E;\mu)$ is a *Dirichlet form* if it is closed, $D(\mathscr{E})$ is dense in $L^2(E;\mu)$ and every normal contraction operates on $(\mathscr{E}, D(\mathscr{E}))$, i.e. given T: $\mathbb{R}\to\mathbb{R}$ such that $T(0)=0$ and $|T(x)-T(y)|\leqq|x-y|$ for all $x, y\in\mathbb{R}$ then for every $u\in D(\mathscr{E})$, $T\circ u\in D(\mathscr{E})$ and $\mathscr{E}(T\circ u, T\circ u)\leqq\mathscr{E}(u,u)$.

(ii) The unique negative definite self adjoint operator A on $L^2(E;\mu)$ satisfying $D(\sqrt{-A})=D(\mathscr{E})$ and $\mathscr{E}(u,v)=\langle\sqrt{-A}u, \sqrt{-A}v\rangle_{L^2(E;\mu)}$, $u,v\in D(\mathscr{E})$, is called the *generator* of the Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$.

The fact that every normal contraction operates on $(\mathscr{E}, D(\mathscr{E}))$ is equivalent to the following property of A (cf. [Bou/Hir, Théorème 1.1]):
$\langle Au, (u-1)^+\rangle_{L^2(E;\mu)}\leqq 0$ for each $u\in D(A)$.

1.9 Theorem. *Let K_0 be a finite or countable set of admissible elements in E such that*

$$\sum_{k\in K_0} |l(k)|^2<+\infty \qquad \text{for all } l\in E'. \tag{1.10}$$

Let $(\mathscr{E}, D(\mathscr{E}))$, $(\tilde{\mathscr{E}}, D(\tilde{\mathscr{E}}))$ be defined as in 1.7. *Then $D(\tilde{\mathscr{E}})=\widetilde{\mathscr{F}C_b^\infty}$ and both $(\mathscr{E}, D(\mathscr{E}))$ and the closure of $(\tilde{\mathscr{E}}, \widetilde{\mathscr{F}C_b^\infty})$ are Dirichlet forms.*

1.10 Remark. (i) In accordance with the finite dimensional case we call the Dirichlet forms in 1.9 *classical Dirichlet forms* on E.

(ii) Note that (1.10) is only needed in Theorem 1.9 to assure that the forms $(\mathscr{E}, D(\mathscr{E}))$, $(\tilde{\mathscr{E}}, D(\tilde{\mathscr{E}}))$ are densely defined on $L^2(E;\mu)$.

(iii) In practice it often occurs that a closed densely defined form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(E;\mu)$ is given, and its action is particularly simple and explicit on some core D i.e. a subspace of $D(\mathscr{E})$ which is dense in $D(\mathscr{E})$ w.r.t. $\mathscr{E}_1$ (cf. the latter case in 1.9 for example). To prove that $(\mathscr{E}, D(\mathscr{E}))$ is a Dirichlet form (i.e. every normal contraction operates on it) it is then sufficient to check that for any $\varepsilon>0$ there exists a function $\Phi_\varepsilon\colon \mathbb{R}\to[-\varepsilon, 1+\varepsilon]$ such that $\Phi_\varepsilon(t)=t$ for all $t\in[0,1]$, $0\leqq \Phi_\varepsilon(t')-\Phi_\varepsilon(t)\leqq t'-t$ if $t<t'$, $\Phi_\varepsilon\circ u\in D$ whenever $u\in D$, and $\mathscr{E}(\Phi_\varepsilon\circ u, \Phi_\varepsilon\circ u)\leqq \mathscr{E}(u,u)$ (cf. [F, Theorem 2.1.1]). This is, of course, satisfied for classical Dirichlet forms $\mathscr{E}$ with core $D=\widetilde{\mathscr{F}C_b^\infty}$. In examples from quantum field theory one might also think of taking the classes $\tilde{\mathscr{P}}$ induced by

$$\mathscr{P}:=\{u\colon E\to\mathbb{R}\,|\,u(z)=p(l_1(z),\dots,l_m(z)),\, z\in E,\, l_i\in E',\, 1\leqq i\leqq m,\ p \text{ a polynomial in } m \text{ variables}\}$$

as a core D. But clearly then $\Phi_\varepsilon\circ u\notin\tilde{\mathscr{P}}$ if $u\in\tilde{\mathscr{P}}$. Therefore, such a core is not suitable for classical Dirichlet forms. However, in some cases one can prove that $\Phi_\varepsilon\circ u$ belongs to the closure of $\tilde{\mathscr{P}}$ w.r.t. $\mathscr{E}_1$ if $u\in\tilde{\mathscr{P}}$, which is, of course, sufficient. But more sophisticated methods are needed to prove this. Details on this can be found in [Po/Rö] (see also [A/Hi/P/Rö/St]).

Given an admissible k in E and $u\in D(\mathscr{E}_k)$ we have defined $\frac{\partial u}{\partial k}\in L^2(E;\mu)$. $\frac{\partial u}{\partial k}$ can be considered as a *μ-stochastic partial derivative of u* (w.r.t. k). Of course, one can also study the concept of a "total" μ-stochastic derivative in the sense of Gâteaux. To this end we need to introduce a suitable Hilbert space H that will play the role of a tangent space to E at each pont (cf. [K]). ((1.10) above can be considered as a first step in this direction). Then we are able to define "coordinate free" classical Dirichlet forms similar to those introduced in [K] in the case where E is a separable Banach space and μ is quasi-invariant (cf. 3.2 below).

Suppose that there exists a real separable Hilbert space $(H, <,>_H)$ densely and continuously imbedded in E. Identifying H with its dual we obtain that E' is densely imbedded in H; in this sense

$$E'\subset H\subset E, \tag{1.11}$$

and $<,>_H$ restricted to $E'\times H$ coincides with the dualisation between E' and E. Suppose furthermore, that we can find a dense linear subspace K of $(H, <,>_H)$ consisting of admissible elements in E.

Let $(\mathscr{E}_k, D(\mathscr{E}_k))$, $k\in K$, be defined as in 1.3. Define the linear space

$$S:=\Big\{u\in\bigcap_{k\in K} D(\mathscr{E}_k)\,|\,\text{there exists a } \mathscr{B}(E)/\mathscr{B}(H)-\text{measurable function } \nabla u\colon E\to H \text{ such that for each } k\in K,\ \langle\nabla u(z), k\rangle_H=\frac{\partial u}{\partial k}(z) \text{ for } \mu\text{-a.e. } z\in E \text{ and } \int_E\langle\nabla u,\nabla u\rangle_H\,d\mu<+\infty\Big\}. \tag{1.12}$$

Clearly, ∇u is (μ-a.e.) unique and if $K_0 \subset K$ is an orthonormal basis of H and $(\mathscr{E}, D(\mathscr{E}))$ is defined by (1.9) then $S \subset D(\mathscr{E})$ and for $u, v \in S$

$$\mathscr{E}(u,v) = \int \langle \nabla u, \nabla v \rangle_H \, \mathrm{d}\mu. \tag{1.13}$$

The following theorem corresponds to Theorem 1 in [K].

1.11 Theorem. *Let H, K, S be as above and $K_0 \subset K$ an orthonormal basis of H. Let $(\mathscr{E}, D(\mathscr{E}))$, $(\tilde{\mathscr{E}}, D(\tilde{\mathscr{E}}))$ be defined as in* 1.7 *(depending on K_0). Then condition* (1.10) *holds and hence $D(\tilde{\mathscr{E}}) = \widetilde{\mathscr{F}C_b^\infty}$. Furthermore, $(\mathscr{E}, S)$ is a closed extension of $(\tilde{\mathscr{E}}, \widetilde{\mathscr{F}C_b^\infty})$ and $(\mathscr{E}, S)$ is a Dirichlet form.*

1.12 Remark. (i) It is easy to check that for any $u \in \widetilde{\mathscr{F}C_b^\infty} \subset S$ and $h \in H$

$$\langle \nabla u(z), h \rangle_H = \frac{\mathrm{d}}{\mathrm{d}s} u(z+sh) \Big|_{s=0} \qquad \text{for } \mu\text{-a.e. } z \in E.$$

(ii) It follows by 1.11 that the closure of $(\tilde{\mathscr{E}}, \widetilde{\mathscr{F}C_b^\infty})$ only depends on H and K and not on the special orthonormal basis K_0 in H. We denote this closure by $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$. It will be in the centre of our consideration in Sect. 3 below and we will study several examples. Therefore, we also express its μ-dependence in the notation, but suppress the K-dependence since in many applications $K = E'$ or $K = H$. We have by 1.11

$$\mathscr{E}^0_{\mu,H}(u,v) = \int_E \langle \nabla u, \nabla v \rangle_H \, \mathrm{d}\mu = \sum_{k \in K_0} \int \frac{\partial u}{\partial k} \frac{\partial v}{\partial k} \, \mathrm{d}\mu \qquad \text{for all } u, v \in D(\mathscr{E}^0_{\mu,H}) \tag{1.13}$$

(where ∇ is as in (1.12), $\frac{\partial u}{\partial k}$ as in (1.7) and K_0 is an orthonormal basis of H). Under some additional assumptions on μ one shows ([A/K]) that $(\mathscr{E}, S)$ is maximal among the Dirichlet forms extending $(\tilde{\mathscr{E}}, \widetilde{\mathscr{F}C_b^\infty})$ in some sense. Conditions for its equality with $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$ are given in [A/K].

(iii) Note that a priori the action of $\mathscr{E}^0_{\mu,H}$ on $D(\mathscr{E}^0_{\mu,H})$, i.e. the closure of $\widetilde{\mathscr{F}C_b^\infty}$ is only defined by a limiting procedure. But (1.13) gives the (explicit) action on elements in $D(\mathscr{E}^0_{\mu,H})$ directly. This fact is very important for subsequent sections of this paper.

(iv) Let $\mathscr{L}^\infty(H)$ denote the set of bounded linear operators on H and let $A\colon E \to \mathscr{L}^\infty(H)$ be strongly measurable such that for some $c > 0$, $A(z) - c\,\mathrm{Id}_H$ is a positive definite symmetric operator for each $z \in E$ (cf. [K]). Then it follows by 1.11 and Fatou's lemma that the form

$$\mathscr{E}_A(u,v) := \int \langle A(z) \nabla u(z), \nabla v(z) \rangle_H \, \mu(\mathrm{d}z)$$
$$D(\mathscr{E}_A) := \widetilde{\mathscr{F}C_b^\infty}$$

is closable on $L^2(E;\mu)$. We call its closure *Dirichlet form of diffusion type (with coefficient A).*

1.13 Example. (i) (Abstract Wiener spaces; cf. [G], [Ma] and also [Ku] and the references therein). Assume that E is a separable Banach space and μ a Gaussian (mean zero) measure on $(E, \mathscr{B}(E))$ such that $\operatorname{supp}\mu = E$. Then there exists a unique separable Hilbert space $(H, <, >_H)$ which is densely and continuously imbedded in E such that

$$(1.14) \qquad \int_E l_1(z)\, l_2(z)\, \mu(\mathrm{d}z) = \langle l_1, l_2\rangle_H \quad \text{for all } l_1, l_2 \in E'$$

(cf. e.g. [Wa, Theorem 1.1] for details). Note here that $H \hookrightarrow E$ densely and continuously implies $E' \hookrightarrow H$ densely and continuously, if we identify H with H' and E' is equipped with the norm topology. We identify E' with its image in H so that (1.14) makes sense. We then also have that

$$l(k) = \langle l, k\rangle_H \quad \text{for all } l \in E', \quad k \in H.$$

Now it follows immediately by [A/Rö 1, 5.4] that each $k \in H$ is (μ-)admissible.

(ii) Let $E := \mathscr{S}'(\mathbb{R}^d)$, $d \in \mathbb{N}$, i.e. the space of real tempered distributions on $\mathbb{R}^d$, $\mathscr{S}(\mathbb{R}^d)$ the associated test function space. $\mathscr{S}'(\mathbb{R}^d)$ equipped with the $\sigma(\mathscr{S}', \mathscr{S})$-topology is a Souslin space and $E' = \mathscr{S}'' = \mathscr{S}$. Suppose that μ is a Gaussian (mean zero) measure on $(\mathscr{S}', \mathscr{B}(\mathscr{S}'))$ such that $\langle l, l\rangle_{L^2(\mu)} = 0$ implies $l = 0$ for all $l \in E' = \mathscr{S}$. Let λ^d denote Lebesgue measure on $\mathbb{R}^d$. Identifying $f \in L^2(\mathbb{R}^d; \lambda^d)$ with the map

$$\mathscr{S} \ni l \mapsto \int_{\mathbb{R}^d} l(x) f(x)\, \lambda^d(\mathrm{d}x)$$

$L^2(\mathbb{R}^d; \lambda^d)$ becomes naturally a subspace of $\mathscr{S}'$ and in this sense

$$(1.15) \qquad \mathscr{S} \subset L^2(\mathbb{R}^d; \lambda^d) \subset \mathscr{S}'.$$

Suppose that the inclusion $\mathscr{S} \subset \mathscr{S}'$ is continuous if $\mathscr{S}$ is equipped with the topology induced by $<, >_{L^2(\mu)}$. Then it follows again by [A/Rö, 5.4] that each $k \in \mathscr{S}$ is (μ-)admissible. All assumptions on μ made here hold if e.g. the covariance of μ is given by

$$(1.16) \quad \langle l_1, l_2\rangle_{L^2(\mu)} = \iint (-\Delta + m^2)^{-\alpha}(x-y)\, l_1(x)\, l_2(y)\, \lambda^d(\mathrm{d}x)\, \lambda^d(\mathrm{d}y); \quad l_1, l_2 \in \mathscr{S} = E',$$

where α, $m > 0$ and $(-\Delta + m^2)^{-\alpha}(x)$ denotes the Green function of the operator $-\Delta + m^2$ on $\mathbb{R}^d$. In the case $\alpha = 1$ resp. $\alpha = \frac{1}{2}$, μ is just the "(space-time) free field" resp. "time-zero free field" of mass m in Euclidean quantum field theory (cf. [N], [Si], [Gl/J] and [Rö 2, 3]). The case where $\mathscr{S}'$ is replaced by $\mathscr{D}'$ can be treated similarly.

(iii) (Space-time quantum fields) Let $E := \mathscr{S}'(\mathbb{R}^2)$ and let μ_0^* be the space-time free field defined in (ii). Clearly for $l_1, \ldots, l_j \in \mathscr{S}(\mathbb{R}^2)$, $\prod_{i=1}^{j} l_i \in L^2(\mathscr{S}'(\mathbb{R}^2); \mu_0^*)$. Define for $n \in \mathbb{N}$, $P^{(n)} := P^{(\leqq n)} \ominus P^{(\leqq n-1)}$ with $P^{(\leqq n)}$ being the closed linear span of the monomials $\prod_{i=1}^{j} l_i$, $j \leqq n$ in $L^2(\mathscr{S}'(\mathbb{R}^2); \mu_0^*)$. Now if $\varepsilon \in]0, 1]$, $h \in L^{1+\varepsilon}(\mathbb{R}^2; \lambda^2)$ and $n \in \mathbb{N}$, define $:z^n:(h)$ to be the unique element in $P^{(n)}$ such that

$$\int_{\mathscr{S}'} :z^n:(h) : \prod_{i=1}^{n} l_i : \mathrm{d}\mu_0^* = n! \int_{\mathbb{R}^2} \prod_{i=1}^{n} \left(\int_{\mathbb{R}^2} (-\Delta + m^2)^{-1} (x - y_i)\, l(y_i)\, \lambda^2(\mathrm{d}y_i) \right) h(x)\, \lambda^2(\mathrm{d}x),$$

where $: \prod_{i=1}^{n} l_i :$ is a certain polynomial in $l_1, \ldots, l_n$ (see [Si, p. 12] for the precise definition of the "Wick product" $: \prod_{i=1}^{n} l_i :$ and [Si, Sect. V.1] for existence of $:z^n:(h)$). Clearly, for $h_1, h_2 \in L^{1+\varepsilon}(\mathbb{R}^2; \lambda^2)$, $\alpha, \beta \in \mathbb{R}$,

$$:z^n:(\alpha h_1 + \beta h_2) = \alpha :z^n:(h_1) + \beta :z^n:(h_2) \qquad \mu_0^*\text{-a.e.}$$

Let $u: \mathbb{R} \to \mathbb{R}$ be of the form

$$u(s) = \sum_{n=0}^{2N} a_n s^n \quad \text{with} \quad a_{2N} > 0, \qquad N \in \mathbb{N} \tag{1.17}$$

or

$$u(s) = \int_{\mathbb{R}} e^{\alpha s}\, \nu(\mathrm{d}\alpha) \tag{1.18}$$

with ν any bounded measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with support in $]-\sqrt{4\pi}, \sqrt{4\pi}[$. We call (1.17) the "$P(\Phi)_2$-case" and (1.18) the "exponential case". If u is as in (1.17) or (1.18) and $l \in E' = \mathscr{S}(\mathbb{R}^2)$. Define for $\Lambda \subset \mathbb{R}^2$, Λ bounded

$$:U_\Lambda:(z) := \sum_{n=0}^{2N} a_n :z^n:(1_\Lambda), \qquad z \in E \tag{1.19}$$

resp.

$$:U_\Lambda:(z) := \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} :z^n:(1_\Lambda)\, \nu(\mathrm{d}\alpha), \qquad z \in E, \tag{1.20}$$

where 1_Λ means indicator function of Λ. It is easy to see that the sum in (1.20) converges in $L^2(E; \mu_0^*)$. We have that $\exp(-:U_\Lambda:) \in L^p(E; \mu_0^*)$ for all $p \in [1, \infty[$ and even $\exp(-:U_\Lambda:) \in L^\infty(E; \mu_0^*)$ in the exponential case. Hence the following probability measures (called *space-time cut off quantum field*) are well-defined

$$\mu_\Lambda^* := \frac{\exp(-:U_\Lambda:)}{\int \exp(-:U_\Lambda:)\, \mathrm{d}\mu_0^*} \cdot \mu_0^*. \tag{1.21}$$

It has been proven that the weak limit

$$\lim_{\Lambda \nearrow \mathbb{R}^2} \mu_\Lambda^* =: \mu^*$$

exists as a probability measure on $E = \mathscr{S}'(\mathbb{R}^2)$ (cf. [Gl/J/Sp] for the polynomial case and [A/H-K 1], [Z] for the exponential case; see also [A/H-K 6], [Gl/J], [Si] and the references therein). Now it follows by [A/Rö 2, 3.5] that each $k \in \mathscr{S}(\mathbb{R}^2)$ with compact support (as an element in $\mathscr{S}'$, cf. (1.15)) is μ^*-admissible. Note that [A/Rö 2, Theorem 3.5] is applicable since μ^* is a Guerra/Rosen/Simon-Gibbs state (see e.g. [Rö 1, 6.4–6.6] or [Fr/Si] resp. [Z] in the exponential case, see also [Gu/Ro/Si 1, 2] and [Fö], [Pr]). Note also that in [A/Rö 2] it was proven that in fact μ^* provides us with an example of a "coordinate free" Dirichlet form as considered in Theorem 1.11 with $H = L^2(\mathbb{R}^2; \lambda^2)$ and $K = \{k \in \mathscr{S}(\mathbb{R}^2) | \operatorname{supp} k \text{ compact}\}$ and that μ^* may be replaced by any Guerra/Rosen/Simon-Gibbs state.

(iv) (Time-zero quantum fields) Let $E := \mathscr{S}'(\mathbb{R})$ and let μ_0 be the time zero free field defined in (ii). We know that $\mathscr{S}(\mathbb{R}) \subset L^2(\mathbb{R}; \lambda^1) \subset \mathscr{S}'(\mathbb{R})$ and that each $k \in \mathscr{S}(\mathbb{R})$ is μ_0-admissible. Let $H = L^2(\mathbb{R}; \lambda^1)$, $k = \mathscr{S}(\mathbb{R})$ and $(\mathscr{E}^0_{\mu_0, H}, D(\mathscr{E}^0_{\mu_0, H}))$ be the (minimal) classical Dirichlet form on E defined in 1.12. Let H_0 be the associated self-adjoint operator with domain $D(H_0)$ on $L^2(E; \mu_0)$. For $h \in L^{1+\varepsilon}(\mathbb{R}; \lambda^1)$, $\varepsilon \in]0, 1]$ and $n \in \mathbb{N}$, define $:z^n:(h)$ in exactly the same way as in (iii) but with $\left(-\frac{d^2}{dx^2} + m^2\right)^{-\frac{1}{2}}(x)$, $x \in \mathbb{R}$, replacing $(-\Delta + m^2)^{-1}(x)$, $x \in \mathbb{R}^2$, and μ_0 replacing μ_0^*. Let for $\Lambda \subset \mathbb{R}$, Λ bounded, $:V_\Lambda:$ be the element in $L^2(E; \mu_0)$ now defined analogously to (1.19), (1.20) respectively (cf. [A/Rö 1] and [A/H-K 3] for more details). Fix $\Lambda \subset \mathbb{R}$, Λ bounded, and consider V_Λ as a multiplication operator on $L^2(E; \mu_0)$ with (maximal) domain $D(V_\Lambda)$. Let H_Λ be the operator on $L^2(E; \mu)$ with domain $D(H_\Lambda) = D(H_0) \cap D(V_\Lambda)$ defined by

$$H_\Lambda := H_0 + V_\Lambda.$$

It is known that H_Λ is essentially self-adjoint on $D(H_\Lambda)$, H_Λ is lower bounded in the $P(\Phi)_2$-case and positive in the exponential case and that the infimum of its spectrum is a simple isolated eigenvalue E_Λ (cf. [A/H-K 3] and the references therein for details). Let Ω_Λ be the eigenvector in $L^2(E; \mu_0)$ to E_Λ with norm 1. One has that $\Omega_\Lambda > 0$ μ_0-a.e. and we may assume that $\int \Omega_\Lambda^2 \, d\mu_0 = 1$. The probability measure

$$\mu_\Lambda := \Omega_\Lambda^2 \cdot \mu_0,$$

is called the *space cut-off* $P(\Phi)_2$, *resp. exponential quantum field.*

In the $P(\Phi)_2$-case from now on we only consider the case of weak coupling (i.e., coefficients a_n of u in (1.17) are "sufficiently small"). It has been proven that the weak limit

$$\lim_{\Lambda \nearrow \mathbb{R}} \mu_\Lambda =: \mu$$

exists as a probability measure on $E=\mathscr{S}'(\mathbb{R})$ for $\Lambda=[-L, L]$ as $L\to\infty$ (cf. [Gl/J/S 1, 2] resp. [A/H-K 1], [Z]). μ is called the *weakly coupled* $P(\Phi)_2$ *resp. exponential time zero quantum field* (*Høegh-Krohn model*). It follows from [A/H-K 3, Theorem 5.2] and [A/Rö 1, 4.6] that each $k\in\mathscr{S}(\mathbb{R})$ is a μ-admissible element in $\mathscr{S}'(\mathbb{R})$.

1.14 Remark. (i) We have the same admissibility results for all "space-time cut-off" resp. "space cut-off" quantum fields defined in 1.13 (iii) resp. (iv) (cf. [A/Rö 1, 2]).

(ii) A weakly coupled $P(\Phi)_2$ resp. exponential time zero field can also be reconstructed from the corresponding space-time quantum field μ^*. Indeed, it is known that for each $l\in\mathscr{S}(\mathbb{R})$

$$\int_{\mathscr{S}'(\mathbb{R})} \exp(i\,l(z))\,\mu(\mathrm{d}z)=\lim_{n\to\infty}\int_{\mathscr{S}'(\mathbb{R}^2)} \exp(i\,l_n^*(z^*))\,\mu^*(\mathrm{d}z^*)$$

where l_n^* is a sequence in $\mathscr{S}(\mathbb{R}^2)$ converging to $\delta_0\otimes l$ in the Sobolev space $H^{-1}(\mathbb{R}^2)$ (cf. [Gl/J/S 2], [Fr 2, 3] and [A/H-K 1]). Here δ_0 is the Dirac measure on $\mathbb{R}^1$ with mass in 0. This explains the notion "time-zero quantum fields".

(iii) Replacing u in (1.18) and (1.20) by a "trigonometric function" one can define the "trigonometric case" (cf. e.g. [A/H-K 5]). As above one can obtain similar admissibility results.

(iv) The case of space-time quantum fields with more general underlying Gaussian fields (i.e., superposition of free fields) studied e.g., by Haba (cf. [H] and also [Fr/I/L/Si]) and the case of three space-time dimensions will be studied in forthcoming papers.

2. General Construction of an Associated Diffusion Process

In this section we will not use the linear structure of E. Therefore we only assume that E is a Souslin topological space and μ a probability measure on $(E, \mathscr{B}(E))$. Since we are not using any additional structure on E in this section we may assume without loss of generality that $\operatorname{supp}\mu=E$. (Recall that $\operatorname{supp}\mu$ can be defined in the usual way since E as a Souslin space is Lindelöf, see [Sch, Proposition 3, p. 104]). In particular, there is a one to one correspondence between continuous functions on E and their μ-classes in $L^2(E;\mu)$.

Let $(\mathscr{E}, D(\mathscr{E}))$ be an arbitrary Dirichlet form on $L^2(E;\mu)$. Let A be its generator and let $T_t:=e^{tA}$, $t\geqq 0$. Then $(T_t)_{(t\geqq 0)}$ is a semigroup of symmetric contractions on $L^2(E;\mu)$ such that each T_t is *Markovian*, i.e. $0\leqq T_t u\leqq 1$ μ-a.e. if $0\leqq u\leqq 1$ μ-a.e. The latter is implied by the fact that normal contractions operate on $(\mathscr{E}, D(\mathscr{E}))$ (cf. 1.8 and [F] for details). For simplicity we assume that $1\in D(\mathscr{E})$.

2.1 Definition. A Markov process $(\Omega, \mathscr{F}, (X_t)_{(t\geqq 0)}, (P_z)_{(z\in E)})$ with state space E (cf. e.g., [Dy 1], [Ba]) is said to be *associated to* $(\mathscr{E}, D(\mathscr{E}))$ if for any $u\colon E\to\mathbb{R}$, $\mathscr{B}(E)$-measurable, bounded, and every $t\geqq 0$,

$$(T_t u)(z)=\int_\Omega u(X_t)\,\mathrm{d}P_z \qquad \text{for } \mu\text{-a.e. } z\in E. \tag{2.1}$$

Of course, some Markov process associated with $(\mathscr{E}, D(\mathscr{E}))$ in the sense of 2.1 will not be very useful unless we know some regularity properties of the sample paths. The aim of this section is to explain a general method to construct a Markov process associated to $(\mathscr{E}, D(\mathscr{E}))$ which is a diffusion, i.e., a Hunt process having continuous sample paths P_z-almost surely for each $z \in E$ (cf. e.g., [F, Sect. 4.1]). The conditions we need are listed under (2.8)–(2.11) below. We will make essential use of the capacity given by $(\mathscr{E}, D(\mathscr{E}))$. Therefore, we recall its definition and some of its properties.

For $U \subset E$, U open, define

$$\text{(2.2)} \qquad \operatorname{Cap}(U) := \inf\{\mathscr{E}_1(u,u) \mid u \in D(\mathscr{E}), u \geqq 1 \text{ on } U \ \mu\text{-a.e.}\}$$

(where as before $\mathscr{E}_1(u,u) = \mathscr{E}(u,u) + \int u^2 \, d\mu$) and for an arbitrary subset $A \subset E$ define

$$\text{(2.3)} \qquad \operatorname{Cap}(A) := \inf\{\operatorname{Cap}(U) \mid A \subset U \subset E, U \text{ open}\}.$$

We note that $\operatorname{Cap}(A) \leqq 1$ for any $A \subset E$ since the constant function 1 is in $D(\mathscr{E})$. It follows as in [F, Sect. 3.1] that the set function Cap is a Choquet capacity, i.e., it has the following two properties

(2.4) If $A_n \subset E$, $n \in \mathbb{N}$, are increasing then $\operatorname{Cap}(\bigcup_{n \in \mathbb{N}} A_n) = \sup_{n \in \mathbb{N}} \operatorname{Cap}(A_n)$.

(2.5) If $K_n \subset E$, $n \in \mathbb{N}$, are compact and decreasing then

$$\operatorname{Cap}(\bigcap_{n \in \mathbb{N}} K_n) = \inf_{n \in \mathbb{N}} \operatorname{Cap}(K_n).$$

As in the locally compact case we have the following "capacitability result" (which surprisingly has not been used so far in the theory of Dirichlet forms on non-locally compact spaces).

2.2 Proposition. *Let $A \subset E$, A Borel, then*

$$\operatorname{Cap}(A) = \sup\{\operatorname{Cap}(K) \mid K \subset A, K \text{ compact}\}.$$

Proof. By definition Cap is right-continuous in the sense of [B, Chap, IX, Sect. 6, Def. 9]. Hence the assertion follows from [B, Chap. IX, Sect. 6. Théorème 6 and Proposition 10]. □

2.3 Remark. There is another important property of Cap, namely its subadditivity, i.e.,

$$\text{(2.6)} \qquad \operatorname{Cap}(\bigcup_{n \in \mathbb{N}} A_n) \leqq \sum_{n=1}^{\infty} \operatorname{Cap}(A_n) \quad \text{for } A_n \subset E, \quad n \in \mathbb{N}.$$

Its consequence

$$\text{(2.7)} \qquad \sum_{n=1}^{\infty} \operatorname{Cap}(A_n) < +\infty \Rightarrow \operatorname{Cap}(\limsup_{n \to \infty} A_n) = 0,$$

i.e., the capacitary version of the "first Borel-Cantelli lemma" was used in an essential way in [F 2] to prove "quasi-everywhere statements" on Wiener space.

Now we are prepared to formulate the four conditions we need to construct the desired diffusion process.

(2.8) There exist $K_n \subset E$, $n \in \mathbb{N}$, K_n compact, such that $\lim_{n\to\infty} \operatorname{Cap}(E \backslash K_n) = 0$.

(2.9) There exists a countable set D of bounded continuous functions on E separating the points of E which is dense in $D(\mathscr{E})$ w.r.t. $\mathscr{E}_1$.

(2.10) $\mathscr{E}(u, v) = 0$ if $u, v \in D(\mathscr{E})$, continuous, such that $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$ (where $\operatorname{supp} u$ = closure of $\{u \neq 0\}$ w.r.t. the topology on E).

(2.11) There exist $f_n: E \to \mathbb{R}$, $n \in \mathbb{N}$, generating the topology of E and $(E, \mathscr{B}(E))$ is a standard Borel space (cf. [P, Def. 2.2, p. 133]).

2.4 Remark. (i) Since every Souslin space is separable (cf. [Sch, Proposition 0, p. 96]), (2.11) is e.g., fulfilled if E is completely metrizable.

(ii) We will use the following fact about standard Borel spaces in the proof of the main theorem of this subsection below: If $(X_1, \mathscr{B}_1)$ is standard Borel and $(X_2, \mathscr{B}_2)$ a measurable space with $\mathscr{B}_2$ countably generated then for any one to one $\mathscr{B}_1/\mathscr{B}_2$-measurable map φ we have that $\varphi(X_1) \in \mathscr{B}_2$ and its inverse $\varphi^{-1}: \varphi(X_1) \to X_1$ is $\mathscr{B}_2 \cap \varphi(X_1)/\mathscr{B}_1$-measurable (cf. [P, Theorem 2.4, p. 135]).

Condition (2.8) is crucial and not easy to check in applications. (2.10) and the first half of (2.11) are needed to prove the continuity of the sample paths. (2.9) is a regularity condition which is always fulfilled in the cases we are interested in. This follows from proposition 2.6 below. First we need a lemma which we learnt from J. Brasche (private communication).

2.5 Lemma. *Let $(\mathscr{E}, D(\mathscr{E}))$ be any Dirichlet form on a separable Hilbert space H. Then the Hilbert space $D(\mathscr{E})$ with inner product $\mathscr{E}_1$ is separable.*

Proof. Let A be the generator of $(\mathscr{E}, D(\mathscr{E}))$. Then an application of the spectral theorem to the operator $\sqrt{-A+1}$ on H (cf. e.g., [Re/Si, Theorem VIII.4]) proves the assertion. □

2.6 Proposition. *Let E be as in Sect. 1 and let $(\mathscr{E}, D(\mathscr{E}))$ be a Dirichlet form on $L^2(E; \mu)$ such that $\widetilde{\mathscr{F}C_b^\infty}$ is dense in $D(\mathscr{E})$ w.r.t. $\mathscr{E}_1$. Then (2.9) is satisfied for some $D \subset \widetilde{\mathscr{F}C_b^\infty}$.*

Proof. Since E is Souslinean it follows by [Sch, Corollary, p. 108] that $L^2(E; \mu)$ is separable. Hence by 2.5 and by assumption we can find a countable dense set D in $\widetilde{\mathscr{F}C_b^\infty}$ which is dense in $D(\mathscr{E})$ w.r.t. $\mathscr{E}_1$. By the Hahn-Banach theorem E' separates the points of E, hence since E is Souslinean by [Sch, Proposition, p. 105] there exists a countable subset of E' separating the points of E. Hence (enlarging D if necessary) we may assume that D also separates the points of E. □

By (2.8)–(2.11) we can use a certain compactification method already described in [A/H-K 2–4] and in particular in [K] (see also [F0]) to reduce the construction of the diffusion process to the case where the state space is a locally compact separable metric space. The latter case was entirely solved by M. Fukushima (see [F, Chap. 6]).

2.7 Theorem. *Assume that (2.8)–(2.11) hold. Then there exists a diffusion process with state space E associated with $(\mathscr{E}, D(\mathscr{E}))$.*

Proof. We may assume that D (in (2.9)) is a $\mathbb{Q}$-algebra containing the constants. Let $\bar{D}$ be its closure with respect to uniform norm on E. By Gelfand's representation theorem applied to the algebra $\bar{D}$ and (2.9) there exists a compact separable metric space $\bar{E}$ (namely the maximal ideal space of $\bar{D}$ equipped with the *weak topology) such that E is densely and continuously imbedded in $\bar{E}$ and the restriction to E gives rise to an isomorphism from $C(\bar{E})$(=set of all real continuous functions on $\bar{E}$) to $\bar{D}$ (cf. e.g. [Ga, Chap. 1, Sect. 8]). By 2.4 (ii) the continuous imbedding from E to $\bar{E}$ is in fact bimeasurable from $(E, \mathscr{B}(E))$ to its image in $\bar{E}$ equipped with the trace σ-field induced by $\mathscr{B}(\bar{E})$. Hence we may consider E as a subset of $\bar{E}$ equipped with a stronger topology but such that its Borel σ-field is just the trace of $\mathscr{B}(\bar{E})$. In particular, we may view μ as a measure on $\mathscr{B}(\bar{E})$ defining $\mu(A)=0$ for $A\in\mathscr{B}(\bar{E})$, $A\subset\bar{E}\setminus E$, and we can identify $L^2(E;\mu)$ with $L^2(\bar{E};\mu)$. The Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ becomes a Dirichlet form on $L^2(\bar{E};\mu)$ which is regular (i.e. $D(\mathscr{E})\cap C(\bar{E})$ is dense both in $D(\mathscr{E})$ w.r.t. $\mathscr{E}_1$ and in $C(\bar{E})$ w.r.t. uniform norm on $\bar{E}$) since $D\subset D(\mathscr{E})$. Furthermore, if $u, v\in D(\mathscr{E})\cap C(\bar{E})$ such that $\overline{\{u\neq 0\}}\cap\overline{\{v\neq 0\}}=\emptyset$ (where the closure $\overline{}$ is w.r.t. the metric topology on $\bar{E}$), then $\mathscr{E}(u,v)=0$ by (2.10). By [Rö0, Theorem 4.2] it follows that $(\mathscr{E}, D(\mathscr{E}))$ as a form on $L^2(\bar{E};\mu)$ has the local property in the sense of [F]. Consequently, by [F, Chap. 6] there exists a diffusion process $(\Omega, \mathscr{F}, (X_t)_{(t\geqq 0)}, (P_z)_{(z\in\bar{E})})$ on $\bar{E}$ such that for every $u\colon \bar{E}\to\mathbb{R}$, $\mathscr{B}(\bar{E})$-measurable, bounded, and every $t\geqq 0$,

$$(T_t u)(z)=\int_\Omega u(X_t)\,\mathrm{d}P_z \qquad \text{for } \mu\text{-a.e. } z\in\bar{E}.$$

Therefore, the proof is complete by the following:

Claim. There exists a measurable subset X of E such that $\mu(X)=1$ and

(2.12) X is an invariant set of the diffusion process $(\Omega, \mathscr{F}, (X_t)_{(t\geqq 0)}, (P_z)_{(z\in\bar{E})})$ and

(2.13) $P_z(\{\omega\colon t\mapsto f_n(X_t(\omega))$ is continuous on $[0,\infty[$ for all $n\in\mathbb{N}\})=1$ for all $z\in X$.

(Note that because of (2.11), (2.13) implies the continuity of the sample paths w.r.t. the stronger topology on E, P_z-a.s. for any $z\in X$. The notion "invariant set" in (2.12) just means that the process will not leave the set X if it starts in X; cf. [F. Sect. 4.1] for the precise definition).

Let $\overline{\mathrm{Cap}}$ denote the capacity associated with $(\bar{E}, D(\bar{E}))$ on $L^2(\bar{E};\mu)$, i.e., $\overline{\mathrm{Cap}}$ is analogously defined as Cap but with $\bar{E}$ replacing E. We note that since the topology on E is stronger than the metric topology on $\bar{E}$ we obviously have that $\overline{\mathrm{Cap}}\geqq\mathrm{Cap}$, but $\overline{\mathrm{Cap}}(U)=\mathrm{Cap}(E\cap U)$ for any $U\subset\bar{E}$, U open (in $\bar{E}$). In particular, by (2.8)

$$\overline{\mathrm{Cap}}\Big(\bar{E}\setminus\bigcup_{n\in\mathbb{N}}K_n\Big)\leqq\lim_{n\in\mathbb{N}}\overline{\mathrm{Cap}}(\bar{E}\setminus K_n)=\lim_{n\in\mathbb{N}}\mathrm{Cap}(E\setminus K_n)=0,$$

hence

$$\overline{\mathrm{Cap}}(\bar{E}\setminus E)=0. \tag{2.14}$$

Furthermore, for any $n \in \mathbb{N}$, $f_n \upharpoonright K$ is continuous with respect to the metric topology inherited from $\bar{E}$ for any compact subset K of E. Hence by (2.8) each f_n is quasi-continuous on $\bar{E}$ (where "quasi" is w.r.t. $\overline{\mathrm{Cap}}$; cf. [F, Sect. 3.1] for the relevant definitions). Now the claim follows by [F, Theorem 4.3.2 (i)], (2.14) and [F, Theorem 4.3.1]. □

2.8 Remark. In fact one can show as in [K] that the two capacities Cap and $\overline{\mathrm{Cap}}$ (cf. the preceding proof) coincide on E.

3. Applications

In this section we first consider the case where E is a Banach space and describe a general setting (due to S. Kusuoka, cf. [K]) in which (2.8)–(2.11) are always satisfied. Then we particularly study the case where E is a Hilbert space and subsequently where E is the dual space of a nuclear (countably Hilbert) space. Finally, we apply the latter to quantum fields.

3.0 Remark. We emphasize that in the subsequent discussion the classical Dirichlet form $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$ may always be replaced by a Dirichlet form of diffusion type as introduced in 1.12 (iv) if $\int \|A(z)\|_{\mathscr{L}^\infty(H)}\,\mu(\mathrm{d}z) < +\infty$ (cf. [K]).

a) The Banach Space Case

Consider the following setting introduced in [K]:

(3.1) E is a separable real Banach space with norm $\| \;\|_E$ and dual E' and dualisation ${}_{E'}\langle \cdot,\cdot \rangle_E$.

(3.2) There exists a separable real Hilbert space $(H, \langle\,,\,\rangle_H)$ such that $H \subset E$ densely and the inclusion map is compact. (Note that identifying H with its dual we have that E' is densely and continuously imbedded in H, i.e. $E' \subset H \subset E$ as in (1.11)).

(3.3) μ is a probability measure on $(E, \mathscr{B}(E))$.

(3.4) There exists a sequence $(P_n)_{(n\in\mathbb{N})}$ of projections on H having finite dimensional range contained in E' (i.e. each P_n is of the form $\sum_{j=1}^{m} \langle e_j, \cdot\rangle_H\, e_j$, $m \in \mathbb{N}$, $\{e_j \mid 1 \leqq j \leqq m\} \subset E'$, an orthonormal system in $(H, \langle\,,\,\rangle_H)$) such that:

(i) $P_1(H) \subset P_2(H) \subset \ldots \subset P_n(H) \subset \ldots$ and P_n converges strongly on H to Id_H (= identity on H) as $n \to \infty$.

(ii) $\|z - \tilde{P}_n z\|_E \to 0$, $n \to \infty$, in probability with respect to μ where $\tilde{P}_n$ is the natural continuous extension of P_n to E (i.e. $\tilde{P}_n$ is of the form $\sum_{j=1}^{m} {}_{E'}\langle e_j, \cdot \rangle_E\, e_j$).

(iii) There exists a constant $c > 0$ such that for any $z \in E$

$$c\,\|z\|_E \leqq \sup\left\{ |{}_{E'}\langle l, z\rangle_E| \;\middle|\; l \in \bigcup_{n=1}^{\infty} P_n(H),\ \|l\|_{E'} = 1 \right\}.$$

(3.5) There exists a dense linear subspace K of $(H, <, >_H)$ consisting of μ-admissible elements in E.

3.1 Proposition. *Assume* (3.1)–(3.3), (3.5) *and let* $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$ *be the corresponding (minimal) classical Dirichlet form as defined in* 1.12 *(with K as in* (3.5)*). Then conditions* (2.9)–(2.11) *are satisfied.*

Proof. (2.9) is satisfied by 2.6 and (2.10) holds since E is a normal topological space and the special form (1.13) of $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$. (2.11) is obvious since E is a complete separable metric space. □

To prove the existence of the diffusion process associated with $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$ by Theorem 2.7 it remains to show (2.8). This was done in [K] under the assumption that μ is k-quasi-invariant for any $k \in K$. We recall:

3.2 Definition. For $z_0 \in E$, let $\tau_{z_0}: E \to E$, $\tau_{z_0}(z) := z + z_0$. Let $k \in E$. μ is called k-quasi invariant if $\tau_{sk}(\mu)$ is absolutely continuous with respect to μ for all $s \in \mathbb{R}$.

Hence we have the following theorem (cf. [K, Theorem 2]) which includes the cases studied in [A/H-K 4]:

3.3 Theorem. *Assume* (3.1)–(3.5) *and let* $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$ *be as in* 3.1. *Assume in addition,*

(3.6) $\qquad$ *μ is k-quasi-invariant for any $k \in K$.*

Then conditions (2.8)–(2.11) *are satisfied. Hence, there exists a diffusion process associated with* $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$.

In applications the technical condition (3.4) is hard to check (except when $(E, \| \;\|_E)$ is a Hilbert space, cf. subsection b) below). Therefore, the following is useful.

3.4 Proposition. *Assume* (3.1)–(3.3) *and that in addition, E is reflexive and that there exists an orthonormal basis* $(e_j)_{(j \in \mathbb{N})}$ *of H such that its linear span is dense in* $(E', \| \;\|_{E'})$ *(cf.* (3.2)*) and* $\sum_{j=1}^{\infty} \|e_j\|_E < +\infty$. *If, furthermore, there exists a constant* $c > 0$ *such that for all* $l \in E'$

(3.7) $$\int |_{E'}\langle l, z\rangle_E| \, \mu(\mathrm{d}z) \leqq c \|l\|_H$$

then (3.4) *is satisfied.*

Proof. Define for $n \in \mathbb{N}$, $P_n h := \sum_{j=1}^{n} \langle e_j, h\rangle_H e_j$, $h \in H$. Then

(3.8) $\qquad (\tilde{P}_n)_{(n \in \mathbb{N})}$ converges to Id_E in $L^1(E; \mu; E)$.

Indeed, by (3.7)

$$\sum_{j=1}^{\infty} \int \|_{E'}<e_j, z>_E e_j\|_E \,\mu(\mathrm{d}z) \leqq c \sum_{j=1}^{\infty} \|e_j\|_E$$

which is finite by assumption. Hence $(\tilde{P}_n)_{(n\in\mathbb{N})}$ converges in $L^1(E;\mu;E)$ to some limit F and (selecting a subsequence if necessary) for any $l\in E'$

$$\lim_{n\to\infty} {}_{E'}<l, \tilde{P}_n z>_E = {}_{E'}<l, F(z)>_E \quad \text{for } \mu\text{-a.e. } z\in E. \tag{3.9}$$

On the other hand, for all $l\in E'$, $z\in E$, we have that

$${}_{E'}<l, \tilde{P}_n z>_E = \left\langle l, \sum_{j=1}^{n} {}_{E'}<e_j, z>_E e_j \right\rangle_H = {}_{E'}<\sum_{j=1}^{n} \langle l, e_j\rangle_H e_j, z>_E.$$

But $\sum_{j=1}^{n} \langle l, e_j\rangle_H e_j \to l$, $n\to\infty$, in H, hence by (3.6) (selecting a subsequence if necessary)

$$\lim_{n\to\infty} {}_{E'}<l, \tilde{P}_n z>_E = l(z) \quad \text{for } \mu\text{-a.e. } z\in E. \tag{3.10}$$

By the separability of E' and the Hahn-Banach theorem (3.8) now follows from (3.9) and (3.10); hence (3.4) (i), (ii) hold. (3.4) (iii) is implied by the reflexivity of E. □

b) The Hilbert space case

The proof of the following proposition is standard, we include it for completeness.

3.5 Proposition. *Assume that E is a real separable Hilbert space with inner product* $<,>_E$ *and that* (3.2), (3.3) *hold. Then* (3.4) *holds.*

Proof. For $z\in E$, the map $h\mapsto\langle h,z\rangle_E$ is continuous on H, hence $\langle h,z\rangle_E = \langle h, Az\rangle_H$ for some $Az\in H$. It is easy to check that $A: E\to H$ is linear, injective and continuous and that its restriction to H is non-negative, self-adjoint and compact. By the Riesz-Schauder theorem there exists an orthonormal basis $(e_n)_{(n\in\mathbb{N})}$ of H consisting of eigenvectors of A. Let $\lambda_n\in]0,\infty]$, $n\in\mathbb{N}$, be the corresponding eigenvalues. It is easy to see that the linear span of $\{e_n | n\in\mathbb{N}\}$ is dense in $(E', \| \ \|_{E'})$ and that $\left(\frac{e_n}{\sqrt{l_n}}\right)_{(n\in\mathbb{N})}$ is an orthonormal basis of $(E, <,>_E)$. Hence if

$$P_n h := \sum_{j=1}^{n} \langle e_j, h\rangle_H e_j \quad \text{for } h\in H,$$

then $(P_n)_{(n\in\mathbb{N})}$ satisfies (i)–(iii) in (3.4) (where in (3.4) (ii) we even have convergence for each $z\in E$). □

As a consequence we have:

3.6 Theorem. *Assume that E is a real separable Hilbert space and that* (3.2), (3.3) *and* (3.5) *hold. Let* $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$ *be the corresponding (minimal) classical Dirichlet form as defined in* 1.12 *(with K as in* (3.5)*). Then* (2.8)–(2.11) *are satisfied. Hence, there exists a diffusion process associated with* $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$.

Note that 3.6 is not a special case of Theorem 3.3 since we have dropped the "quasi-invariance condition" (3.6). As we have already emphasized in the introduction, the reason why one should consider this more general situation is that without this assumption the whole theory is "localizable" in the sense that one can replace the state space E by some Borel subset. Furthermore, "quasi-invariance" seems to be very difficult to show for measures μ in 3-dimensional quantum field theory. We give a complete proof of 3.6 in the Appendix based on a modification of the method in [K] and avoiding assumption (3.6).

c) The Conuclear Case

Suppose now that E is the dual space of a countably Hilbert space Φ, i.e. (cf. e.g. [Ge/V, Chap. I] or [Hi, A.3]) Φ is topologised by countably many norms $\| \ \|_n$, $n \geqq 0$, such that it is complete (as a uniform space), where the norms are assumed to come from inner products $<,>_n$ and to be compatible, i.e., if a sequence in E converges to zero w.r.t. $\| \ \|_n$ and is Cauchy w.r.t. $\| \ \|_m$ it also converges to zero w.r.t. $\| \ \|_m$ (for all $n, m \geqq 0$). Of course, we can assume that $\| \ \|_n$, $n \in \mathbb{N}$, are arranged in increasing order:

$$\| \ \|_0 \leqq \| \ \|_1 \leqq \dots \leqq \| \ \|_n \leqq \dots .$$

Consequently, if Φ_n is the completion of Φ w.r.t. $\| \ \|_n$ we have the continuous and dense imbeddings

$$\Phi_0 \supset \Phi_1 \supset \dots \supset \Phi_n \supset \dots$$

and that $\Phi = \bigcap_{n \geqq 0} \Phi_n$. Correspondingly, we have the continuous and dense imbeddings for the dual spaces

$$\Phi'_0 \subset \Phi'_1 \subset \dots \subset \Phi'_n \subset \dots$$

and since every continuous linear functional on Φ is continuous w.r.t. some norm $\| \ \|_n$ (see e.g. [Ge/V, Chap. I, Section 1.2]) it follows that

$$E = \Phi' = \bigcup_{n \geqq 0} \Phi'_n. \tag{3.11}$$

If $E = \Phi'$ is equipped with the strong topology and Φ'' as well, then $E' = \Phi'' = \Phi$ as topological vector spaces i.e. Φ is reflexive (cf. [Ge/V, Chap. I, Section 3.1]).

Suppose now in addition, that Φ is nuclear, i.e. that for any $m \in \mathbb{N}$ there exists $n > m$ such that the injection map $T_m^n \colon \Phi_n \to \Phi_m$ is nuclear i.e. has the form

$$T_m^n z = \sum_{j=1}^{\infty} \lambda_j \langle z, e_j \rangle_n \tilde{e}_j, \qquad z \in \Phi_n,$$

where $(e_j)_{(j \in \mathbb{N})}$, $(\tilde{e}_j)_{(j \in \mathbb{N})}$ are orthonormal systems in Φ_n, Φ_m respectively, and $\lambda_i \in]0, \infty[$ such that $\sum_{j=1}^{\infty} \lambda_j < +\infty$. Then Φ is separable and if follows by [Sch, Corollary 1, p. 115] that E is Lusinean (i.e. the continuous, one to one image of a polish space), hence Souslinean.

In accordance with our earlier notation from now on we only use E, E' instead of Φ' resp. Φ. We have the following:

3.7 Proposition. *Let μ be a probability measure on $(E, \mathscr{B}(E))$ and $p \in [1, \infty[$ such that for any $l \in E'$*

$$\int_E |{}_{E'}\langle l, z \rangle_E|^p \, \mu(dz) < +\infty.$$

Then the following holds.

(i) *There exists $n_0 \in \mathbb{N}$ and a constant $c > 0$ such that for all $l \in E'$*

$$|\int |{}_{E'}\langle l, z \rangle_E|^p \, \mu(dz)|^{\frac{1}{p}} \leqq c \, \|l\|_{n_0}.$$

(ii) *There exists $n \in \mathbb{N} (n > n_0)$ such that $\mu(\Phi_n') = 1$.*

Proof. (i): Define for $M \in \mathbb{N}$

$$q_M(l) := (\int |\inf\{|{}_{E'}\langle l, z \rangle_E|, M\}|^p \, \mu(dz))^{\frac{1}{p}}, \qquad l \in E'.$$

Since E' is a Fréchet space (hence a Baire space) it follows (cf. the proof of the uniform boundedness principle) that there exists $c \in]0, \infty[$ and $n_0 \in \mathbb{N}$ such that for all $M \in \mathbb{N}$, $l \in E'$, $q_M(l) \leqq c \, \|l\|_{n_0}$. Letting M tend to ∞ we obtain (i).

(ii) now immediately follows by (a version of) Minlos' theorem (cf. e.g. [Hi, Theorem 3.1]). □

3.8 Remark. Part (i) of 3.7 is a special case of a general theorem due to Dobrushin and Minlos; cf. [Do/Min, Proposition 5].

Now we are prepared to formulate the main result of this subsection.

3.9 Theorem. *Suppose that:*

(i) *μ is a probability measure on $(E, \mathscr{B}(E))$ such that*

$$\int |{}_{E'}\langle l, z \rangle_E| \, \mu(dz) < +\infty \qquad \text{for each } l \in E'. \tag{3.12}$$

(ii) *$(H, \langle\, , \rangle_H)$ is a separable real Hilbert space such that $H \subset E$ continuously and densely; hence if H is identified with its dual,*

$$E' \subset H \subset E \qquad \text{densely and continuously.} \tag{3.13}$$

(iii) *There exists a dense linear subspace K of* $(H, <, >_H)$ *consisting of* μ-*admissible elements in E.*

Let $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$ *be the corresponding (minimal) classical Dirichlet form as defined in* 1.12 (*with K as in* (iii)). *Then there exists a diffusion process associated with* $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$.

3.10 Remark. (i) Clearly, each Φ'_n, $n \in \mathbb{N}$, satisfies (ii) in 3.9.

(ii) We note that below we will actually prove the following stronger statement: there exists $n \in \mathbb{N}$ such that Φ'_n is an invariant set for the process with $\mu(\Phi'_n) = 1$ and the sample paths are continuous with respect to $\| \ \|_{\Phi'_n}$.

Proof of 3.9. Recall that $\Phi = E'$ and $\Phi' = E$. Since $E' \subset H$ continuously and densely by (3.13), there exists $n_1 \in \mathbb{N}$ such that $\Phi_{n_1} \subset H$ continuously and densely. By (3.12) and 3.7 (ii) we have that $\mu(\Phi'_n) = 1$ for some $n > n_1$. Since H is identified with its dual, it follows that

$$H \subset \Phi'_n \quad \text{continuously and densely.} \tag{3.14}$$

Since the inclusion maps $\Phi_n \subset \Phi_{n_1}$ and hence $\Phi_n \subset H$ are nuclear, the imbedding (3.14) is nuclear, therefore compact. Since $\Phi'_n \subset E$ continuously, it follows by 2.4 (ii) that $\mathscr{B}(\Phi'_n)$ is just the trace of $\mathscr{B}(E)$ on Φ'_n. Hence, we can identify $L^2(E; \mu)$ with $L^2(\Phi'_n; \mu)$ and therefore we can consider $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$ as a Dirichlet form on $L^2(\Phi'_n; \mu)$. Since $E' = \Phi \subset \Phi_n$ densely, it is easy to see that the $L^2(\Phi'_n; \mu)$-classes given by

$$\{u\colon \Phi'_n \mapsto \mathbb{R} \mid \text{there exist } l_1, \ldots, l_m \in \Phi_n, f \in C_b^\infty(\mathbb{R}^m) \text{ such that}$$
$$u(z) = f(l_1(z), \ldots, l_m(z)), z \in \Phi'_n\}$$

are contained in $D(\mathscr{E}^0_{\mu,H})$. Hence $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$ is the (minimal) classical Dirichlet form on $L^2(\Phi'_n; \mu)$ given by H and $K \subset H \subset \Phi'_n$, K as in assumption (iii). Now we can apply Theorem 3.6 with $E = \Phi'_n$ to complete the proof (cf. [F, Theorem 4.1.3]). □

d) Application to quantum fields

In this subsection let E, μ be as in 1.13 (ii)–(iv), i.e., μ is a (generalized) free field, a space-time quantum field or a time-zero quantum field and $E = \mathscr{S}'(\mathbb{R}^d)$ with $d = 1, 2$ in cases (iv), (iii) resp. Clearly, $\mathscr{S}(\mathbb{R}^d)$ is a nuclear countably Hilbert space, i.e. $\mathscr{S}(\mathbb{R}^d) = \Phi$ where Φ is as in subsection c) with (cf. e.g. [Hi, A.3, Example 4])

$$\Phi_n := \text{completion of } \mathscr{S}(\mathbb{R}^d) \text{ w.r.t. } \|l\|_n := \sum_{|\underline{m}| \le n} \int_{\mathbb{R}^d} (1 + |x|^2)^n |l^{(\underline{m})}(x)|^2 \, \lambda^d(\mathrm{d}x).$$

Here $\underline{m} = (m_1, \ldots, m_d) \in (\mathbb{Z}_+)^d$ and $l^{(\underline{m})} = \left(\frac{\partial^{m_1}}{\partial x_1^{m_1}} \cdots \frac{\partial^{m_d}}{\partial x_d^{m_d}}\right) l$. Note that $\Phi_0 = L^2(\mathbb{R}^d; \lambda^d)$. It is well known that for all μ under consideration here, (i) in Theorem 3.9 is fulfilled. This is obvious in case 1.13 (ii); for cases 1.13 (iii) resp. 1.13

(iv) see [Gl/J/Sp 1, 2], [A/H-K 1] and [Gl/J], [Si]. As the Hilbert space H in 3.9 (ii) one can take e.g., $H := L^2(\mathbb{R}^d; \lambda^d)$, then by 1.13 (ii)–(iv) also assumption 3.9 (iii) is fulfilled. Therefore, by 3.9 there exists a diffusion process associated with the corresponding (minimal) Dirichlet form $(\mathscr{E}^0_{\mu,H}, D(\mathscr{E}^0_{\mu,H}))$. We note that there are, of course, other Hilbert spaces H satisfying 3.9 (ii). But one always has to check whether 3.9 (iii) is satisfied. This means on the basis of the admissibility results in 1.13 (ii)–(iv) that there must be "sufficiently many" elements h in $H \subset \mathscr{S}'(\mathbb{R}^d)$ which are of the form

$$h(l) = \int f(x)\, l(x)\, \lambda^d(\mathrm{d}x), \qquad l \in \mathscr{S}(\mathbb{R}^d)$$

for some $f \in \mathscr{S}(\mathbb{R}^d)$. Observe, that this is not automatically fulfilled by 3.9 (ii) since the imbedding (3.13) is, in general, completely different from the imbedding (1.15). But by partial integration it is easy to check that for each $n \in \mathbb{N}$, $H := \Phi'_n$ satisfies both 3.9 (ii) and (iii).

3.11 Remark. (i) We emphasize that the choice of the Hilbert space H which takes the role of the "tangent space" is crucial. Together with μ it determines the Dirichlet form and hence the diffusion process. For example, if μ_0 is the time-zero free field and $H = H^{\frac{1}{2}}(\mathbb{R}^d)$. i.e., the Sobolev space of order $\frac{1}{2}$, the corresponding diffusion process is the ordinary Ornstein-Uhlenbeck process associated with the abstract Wiener space $(\mu_0, H^{\frac{1}{2}}, B)$. Here $B \subset \mathscr{S}'(\mathbb{R}^d)$ is some Banach space supporting μ_0 such that $H^{\frac{1}{2}} \subset B$ continuously and densely (cf. [G], [Ma], [Wa]). In general, B is the completion of $H^{\frac{1}{2}}$ with respect to some μ_0-measurable norm on $H^{\frac{1}{2}}$ (cf. [G], [Ku] for details). In [Rö 2] it was proven that B can be taken to be a (subspace of a) "scaled" Sobolev space of negative order. On the other hand if one takes $H = L^2(\mathbb{R}^d; \lambda^d)$, the corresponding diffusion process is a different kind of Ornstein-Uhlenbeck process, namely the one associated with the free space-time quantum field μ_0^* on $\mathscr{S}'(\mathbb{R}^{d+1})$ (cf. [Rö 3] and [A/H-K 2, Sect. 4]).

(ii) Theorem 3.9 also applies to cut-off quantum fields (cf. 1.14 (i)) and, once the closability question is settled, to quantum fields over 3-dimensional space-time (cf. 1.14 (iv)).

Appendix

Proof of 3.6. Recall that by 3.5 condition (3.4) holds; hence by 3.1 and 2.7 we only have to show (2.8). By 3.5 we know that (3.4) holds.

Let A, e_n, P_n, $n \in \mathbb{N}$, be as in the proof of 3.5 and $\tilde{P}_n$ as in (3.4) (ii). Let $L^\infty(H; E)$ be the set of bounded linear operators from H to E.

Step 1. (cf. [K, Propositions 3.1, 3.2])
Since H is compactly imbedded in E and by (3.4), taking a subsequence if necessary we have that

(A.1)
(i) $\|\mathrm{Id}_H - P_n\|_{L^\infty(H;E)} \leqq 2^{-n}$
(ii) $\mu\{z \in E \mid \|z - \tilde{P}_n z\|_E > 2^{-n}\} \leqq 2^{-n}$.

We define $q\colon E \to \mathbb{R}_+$ by

$$q(z) = \left(\sum_{n=0}^{\infty} 2^n \| \widetilde{P_{n+1}} z - \widetilde{P_n} z \|_E^2 \right)^{\frac{1}{2}}, \qquad z \in E,$$

where $\widetilde{P_0} := 0$. Then $E_0 := \{q < +\infty\}$ is a normed linear subspace of E with norm q having the following properties:

(A.2) (i) $\mu(E_0) = 1$

(ii) (E_0, q) is a Hilbert space compactly imbedded in $(E, \| \ \|_E)$

(iii) $(H, \| \ \|_H)$ is continuously imbedded in (E_0, q).

Indeed (A.2) (i) immediately follows by (A.1) (ii) and the Borel-Cantelli Lemma. (A.2) (iii) holds since by (A.1) (i) for any $h \in H$

$$q(h) = \left(\sum_{n=0}^{\infty} 2^n \| (\mathrm{Id}_H - P_n)(P_{n+1} h) \|_E^2 \right)^{\frac{1}{2}} \leqq \left(\sum_{n=0}^{\infty} 2^{-n} \| P_{n+1} h \|_H^2 \right)^{\frac{1}{2}} \leqq \sqrt{2} \, \| h \|_H.$$

Furthermore, for $h \in H$ we have that $P_n^2 h = P_n h \in E'$, consequently for $z \in E$

$${}_{E'}\langle P_n^2 h, z \rangle_E = {}_{E'}\langle \sum_{j=1}^{n} \langle P_n h, e_j \rangle_H e_j, z \rangle_E = \langle P_n h, \widetilde{P_n} z \rangle_H = {}_{E'}\langle P_n h, \widetilde{P_n} z \rangle_E.$$

Hence we have, due to the fact that the linear span of $\{e_n | n \in \mathbb{N}\}$ is dense in $(E', \| \ \|_{E'})$, that for any $m \in \mathbb{N}$

$$\begin{aligned} \| z \|_E &= \sup_{n > m} \{ | {}_{E'}\langle P_n^2 h, z \rangle_E | \,|\, n \in \mathbb{N}, h \in H \text{ with } \| P_n h \|_{E'} = 1 \} \\ &\leqq \sup_{n > m} \| \widetilde{P_n} z \|_E. \end{aligned}$$

Consequently, if $z \in E_0$ and $m \in \mathbb{N}$,

$$\begin{aligned} \| z - \widetilde{P_m} z \|_E &\leqq \sup_{n > m} \| \widetilde{P_n} z - \widetilde{P_n} \widetilde{P_m} z \|_E = \sup_{n > m} \| \widetilde{P_n} z - \widetilde{P_m} z \|_E \\ &\leqq \sup_{n > m} \sum_{j=m}^{n-1} \| \widetilde{P_{j+1}} z - \widetilde{P_j} z \|_E \leqq \left(\sum_{j=m}^{\infty} 2^{-j} \right)^{\frac{1}{2}} q(z). \end{aligned}$$

Consequently, (E_0, q) is compactly imbedded in $(E, \| \ \|_E)$. The completeness of (E_0, q) is now easy to see by Fatou's lemma. Thus (A.2) (ii) is proven.

Step 2. Let $K_N := \{q \leqq N\}$, $N \in \mathbb{N}$. Then each K_N is a bounded weakly compact subset of E_0, hence by (A.2) (ii), K_N is compact in E. If we can show that

(A.3) $$\lim_{N \to \infty} \mathrm{Cap}(E \setminus K_N) = 0,$$

(2.8) is fulfilled and the proof of the theorem is completed.

To show (A.3) fix $N \in \mathbb{N}$ and $\Phi: \mathbb{R} \to [0,1]$ smooth and increasing such that $\Phi(t)=0$ if $t \leqq (N+\frac{1}{4})^2$, $\Phi(t)=1$ if $t \geqq (N+1)^2$ and $\Phi'(t) \leqq \frac{2}{(N+1)}$ for all $t \in \mathbb{R}$. Let

$$u(z) := \Phi(q(z)^2), \qquad z \in E,$$

where we set $\Phi(+\infty) := 1$. Then $u \in D(\mathscr{E}^0_{\mu,H})$.

Indeed, if $q_m(z) := \left(\sum_{n=0}^{m} 2^n \| \widetilde{P_{n+1}}\, z - \widetilde{P_n}\, z \|_E^2 \right)^{\frac{1}{2}}$, $m \in \mathbb{N}$, $z \in E$, and $u_m := \Phi(q_m(z)^2)$, then $u_m \in \widetilde{\mathscr{F}C_b^\infty}$, $m \in \mathbb{N}$, $(u_m)_{(m \in \mathbb{N})}$ converges in $L^2(E;\mu)$ to u and by the following arguments we see that $\mathscr{E}^0_{\mu,H}(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$: it is clear that for any $v \in \mathscr{F}C_b^\infty$ and $i \in \mathbb{N}$, $\langle \nabla u, e_i \rangle_H = \frac{\tilde{\partial}}{\partial e_i} u$ (cf. 1.12 (i) and (1.4)). Therefore, by the chain rule and Minkowski's inequality we have on E for all $n, m \in \mathbb{N}$

$$\text{(A.4)} \quad \|\nabla(u_n - u_m)\|_H = \left(\sum_{i=1}^{\infty} \left(\frac{\tilde{\partial}}{\partial e_i} (u_n - u_m) \right)^2 \right)^{\frac{1}{2}}$$

$$\leqq |\Phi'(q_n^2) - \Phi'(q_m^2)| \left(\sum_{i=1}^{\infty} \left(\frac{\tilde{\partial}}{\partial e_i} q_n^2 \right)^2 \right)^{\frac{1}{2}} + |\Phi'(q_m^2)| \left(\sum_{i=1}^{\infty} \left(\frac{\tilde{\partial}}{\partial e_i} (q_n^2 - q_m^2) \right)^2 \right)^{\frac{1}{2}}.$$

Furthermore, for all $z \in E$

$$\text{(A.5)} \quad \sum_{i=1}^{\infty} \left(\frac{\tilde{\partial}}{\partial e_i} q_m^2(z) \right)$$

$$= \sum_{i=1}^{\infty} 4 \sum_{n,j=1}^{m} 2^{n+j} \langle \widetilde{P_{n+1}}\, z - \widetilde{P_n}\, z, P_{n+1} e_i - P_n e_i \rangle_E \langle \widetilde{P_{j+1}}\, z - \widetilde{P_j}\, z, P_{j+1} e_i - P_j e_i \rangle_E$$

$$= \sum_{i=1}^{\infty} 4 \sum_{n=1}^{m} 2^{2n} \langle \widetilde{P_{n+1}}\, z - \widetilde{P_n}\, z, P_{n+1} e_i - P_n e_i \rangle_E^2$$

(since for each $i \in \mathbb{N}$, $P_{n+1} e_i - P_n e_i \neq 0 \neq P_{j+1} e_i - P_j e_i$ implies $n=j$)

$$= 4 \sum_{n=1}^{m} 2^{2n} \sum_{i=1}^{\infty} \langle (\widetilde{P_{n+1}} - \widetilde{P_n})\, z, A(P_{n+1} - P_n) e_i \rangle_H^2$$

$$= 4 \sum_{n=1}^{m} 2^{2n} \| (P_{n+1} - P_n) A (\widetilde{P_{n+1}} - \widetilde{P_n})\, z \|_H^2$$

$$\leqq 4 \sum_{n=1}^{m} 2^{2n} \| (P_{n+1} - P_n) A (\widetilde{P_{n+1}} - \widetilde{P_n})\, z \|_E \, \| (\widetilde{P_{n+1}} - \widetilde{P_n})\, z \|_E$$

$$\leqq 4 \|A\|_{\mathscr{L}^\infty(E,H)}\, q_m^2(z).$$

Correspondingly, for all $n, m \in \mathbb{N}$, $n \geqq m$, and each $z \in E$

$$\text{(A.6)} \qquad \sum_{i=1}^{\infty} \left(\frac{\tilde{\partial}}{\partial e_i} (q_n^2(z) - q_m^2(z)) \right)^2 \leqq 4 \|A\|_{\mathscr{L}^\infty(E,H)} (q_n^2(z) - q_m^2(z)).$$

(A.5), (A.6) imply that $(\nabla u_n(z))_{(n\in\mathbb{N})}$ is convergent in H for μ-a.e. $z\in E$ since $\mu(E_0) = 1$. But by the chain rule and (A.5) for all $n\in\mathbb{N}$

$$\text{(A.7)} \qquad \|\nabla u_m\|_H \leqq \Phi'(q_m^2)\cdot 2\cdot \|A\|^{\frac{1}{2}}_{\mathscr{L}^\infty(E,H)}\cdot q_m \leqq 4\,\|A\|^{\frac{1}{2}}_{\mathscr{L}^\infty(E,H)}.$$

Consequently, by the dominated convergence theorem $\mathscr{E}^0_{\mu,H}(u_n-u_m, u_n-u_m)\to 0$, $n, m\to\infty$, and thus $u\in D(\mathscr{E}^\infty_{\mu,H})$.

Furthermore, $u=1$ on $E\setminus K_{N+1}$, hence

$$\text{(A.8)} \qquad \operatorname{Cap}(E\setminus K_{N+1}) \leqq \mathscr{E}^0_{\mu,H,1}(u,u) = \mathscr{E}^0_{\mu,H}(u,u) + \int u^2\, d\mu.$$

But also $u=0$ on a neighbourhood of K_N, therefore, $\nabla u=0$ on K_N. Hence by (A.7)

$$\text{(A.9)} \qquad \mathscr{E}^0_{\mu,H,1}(u,u) \leqq (16\,\|A\|_{\mathscr{L}^\infty E,H)}+1)\,\mu(E\setminus K_N),$$

since $|u|\leqq 1$. Now (A.3) follows from (A.8), (A.9) since $\mu(E_0)=1$. □

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Note added in proof

It follows as in [A/H-K3] (Sect. 3) that the diffusion process X_t defined in $3d$ (for quantum fields) satisfies a stochastic differential equation of the type $dX_t = \beta(X_t)\,dt + dW_t$ in the weak sense, where the drift coefficient β is the "osmotic velocity" given by the measure μ (cfr. [A/H-K3]). Details will be discussed in a forthcoming paper.