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Probability Theory and Related Fields

Wiener-Hopf Factorisation of Brownian Motion

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Summary. We study how Brownian motion behaves under time change by a fluctuating additive functional A_t, in particular letting τ be the first passage time of A_t to zero we compute $\mathbf{P}_{-x}[B_\tau \in dy]$ explicitly in certain cases. The calculation is not an easy one, our method uses the Désiré André relation for the overshoot of a Lévy process and depends on some elliptic function identities. This paper only considers the one boundary case where A_t is increasing (resp. decreasing) on the positive (resp. negative) half line.

The description 'Wiener-Hopf' is sometimes used in probability to denote the decomposition of a random walk into its ascending and descending ladder point processes. In the context of a Lévy process X_t this has to do with decomposing X_t as the sum of its maximum $\bar{X}_t$ and the reflection process $\bar{X}_t - X_t$. Originally these laws were computed by exploiting the Wiener-Hopf technique for solving an integral equation but Feller comments [10] p. 389 'the connections are not so close as is usually made to appear'. Nowadays the preference is to formulate such questions in terms of the Spitzer-Rogozin-Fristedt theory [3]. This describes how $\bar{X}_t$ can be time-changed to get a new Lévy process Y_t and even gives an expression, called Spitzer's formula, for the exponent of Y_t in terms of the law of X_t. This general area is known as fluctuation theory and it is reputed to have considerable practical importance.

In this paper we consider the single boundary Wiener-Hopf factorisation problem for real Brownian motion B_t. Suppose $L(a,t)$ is a bicontinuous version of B_t local time normalised so that for every bounded Borel function f we have

$$\int_0^t f(B_s)\,ds = \int f(a)\,L(a,t)\,da,$$

the occupation density formula. Let $m = m^+ - m^-$ be a signed measure on R with m^+ a positive measure and m^- a positive Radon measure, supported on the positive and negative axes respectively. We write $A_t = \int L(a,t)\,m(da)$ and we define $\tau=\inf\{t>0\colon A_t=0\}$. Then the problem (proposed by Williams et al. [15, 16, 22], but see also [19]) is the explicit computation of the kernel

$$\Pi(x, dy)=\mathbf{P}_{-x}[B_\tau\in dy;\ \tau<+\infty] \qquad (x, y>0).$$

There is an obvious way of doing this, which is to write $X_t=A_{\eta_t}$ with η_t the right continuous version $L^{-1}(0,\cdot)_t$ so that X_t is a Lévy process. We see below that the problem of finding the kernel $\Pi(x, dy)$ is closely related to computing the law of the overshoot of zero by X_t, or equivalently computing the overshoot of zero for $\bar{X}_t$. This gives the connection with fluctuation theory and would be the end of the story were it not for the computational difficulties. The fact is that explicit computation of the overshoot law for a Lévy process is often extremely difficult and using $\bar{X}_t$ doesn't make things any easier, in particular an approach using Spitzer's formula seems to be out of the question. On the other hand there exist examples where one can compute $\Pi(x, dy)$ directly but where the corresponding overshoot law is not known explicitly. This note was partly motivated by a desire to understand why.

Our method goes back to basics. Instead of trying to compute the overshoot directly we will remark that it satisfies an integral equation, and then we use this to derive a system of Eqs. (●, †) containing $\Pi(x, dy)$. One of these equations is of Wiener-Hopf type and it can be solved directly if the original problem has scaling properties (this is our canonical case). The purpose of this note is to develop another method of solution which exploits the behaviour of (●, †) under certain substitutions and which leads to several explicit formulae for the kernel $\Pi(x, dy)$. Previous work on the explicit calculation of $\Pi(x, dy)$ is mainly due to Baker [1, 2]. His methods involved doing some complicated contour integrations, but the snag was that these were somewhat unmotivated. In our work we have tried to be more systematic, and in particular we have sought to clarify the probabilistic structure of the problem before going on to investigate the analytical methods needed to solve it. But in view of the considerable technical difficulties encountered we have not attempted to tackle the general case. Our aim has been to identify the simplest examples and to deal with them comprehensively.

The main results of this paper are to be found in sections three and four below though the material in the second section is maybe not as familiar as it should be. In the first section where we derive our basic system of Eqs. (●, †) for $\Pi(x, dy)$ the approach is fairly straightforward. But though the equations are well-known the methods for solving them are not. So we begin by showing how to solve (●, †) in the 'canonical case' where $m^+(da)=da$ and $m^-(da)=\delta^2\, da$, essentially adapting Ray's method [21] to compute the overshoot for an asymmetric stable process [4, 7]. Our proof is expressed probabilistically since this seems to give some extra insight and is in any case important for other applications.

Section three contains the essential novelty of this research and is where we do most of the work. There we give details of a substitution method, the idea being to show how one can transform our convolution equation (●) by scale change into a form which is suitable for the application of Ray's technique.

The relevant change of variable is computed by inverting the integral transform (†) induced from m^+, its calculation being simplest when (†) defines a Fourier transform or a Fourier series. Our procedure only works when the function concerned satisfies a certain identify, explained at 3.1 below, while the condition $\mathbf{P}[\tau < +\infty]$ is also crucial when we come to justifying the application of Ray's technique.

The final section puts this machinery to work on a variety of examples, mainly gleaned from [2]. In each case we find that $\Pi(x, dy)$ is a conformal image of the canonical answer obtained in section two and, while we determine our substitution analytically, there is nevertheless a clear geometric relationship with the original process B_t. Some of our computations depend on properties of Jacobi's elliptic functions. These no longer form part of every mathematician's toolkit so we have included an appendix summarising their relevant properties. We also found it necessary to do a couple of elliptic contour integrals.

Some of the results obtained here are surprising. For example the geometric link between our original Brownian motion B_t and the conformal function used to obtain $\Pi(x, dy)$ was quite unexpected. It strongly suggests that in the context of this problem we should think of B_t as a real Brownian motion when below zero, and as a purely imaginary process when B_t is positive. See [18] for more on this particular theme. But the main advantage of the geometric analogy is that it allows us to guess the answer before doing the calculation. To put this in perspective just recall the computational difficulties encountered in [16], or indeed when trying to find the overshoot law for a Lévy process [4]. We should also mention that we have no theoretical explanation for what's going on here, just rules of thumb which come from examining the explicit formulae. It seems that a better understanding depends on being able to find other explicitly computable examples. We are looking in particular for a method which dispenses with the restrictive condition 3.1.

The method developed here, together with some of the results, had already been announced in [17].

§ 1. General Results

Throughout this paper we take B_t to be a real Brownian motion with $B_0 = -x < 0$ and we define the additive functional $A_t = \int L(a, t)\, m(da)$ as explained in the introduction, $L(a, t)$ is the B_t local time. Write $T = \inf\{t > 0 \colon B_t = 0\}$ so that by the strong Markov property A_T is independent of $B_t \circ \theta_T$ and let η_t be a right continuous inverse of the local time at zero. Then the following result is well-known.

Lemma 1.1. *Let f be the unique bounded solution of the distributional equation*

$$\frac{d}{dm} f'(x) + 2 i z f(x) = 0$$

normalised by $f(0) = 1$, the derivative f' being discontinuous at zero. Then $X_t = A_{\eta_t}$ is a Lévy process, started at the point $A_T < 0$, with fourier exponent given by

$$\psi(z) = \tfrac{1}{2}[f'(0+) - f'(0-)] = \tfrac{1}{2} \Delta f'(0).$$

Proof. By the generalised Ito formula, if z is real, the process $f(B_t)\exp\{izA_t - \psi(z)L(0,t)\}$ is a local martingale in the B_t filtration. But it is uniformly bounded up to time η_t so doing the time-change $t\to\eta_t$ we still have a martingale. Since $B_{\eta_t}\equiv 0$ then $\mathbf{E}[e^{izX_t}]=e^{\psi(z)t}$ as required.

The Lévy process X_t is of bounded variation so its Lévy measure ν satisfies $\int(|x|\wedge 1)\nu(dx)<+\infty$ and we can write its Fourier exponent as $\psi(z)=\int(e^{izx}-1)\nu(dx)$. Moreover we find that

$$f'(0+)=2\int_0^\infty (e^{izx}-1)\,\nu(dx).$$

This remark will be used below and in fact we mainly consider the case where ν is absolutely continuous.

Definition. Write $m(da)\sim|a|^\gamma\,da$, $a\downarrow 0$ if

$$\lim_{a\downarrow 0}|a|^{-\gamma}\frac{dm}{da}$$

exists and is non-zero.

The next result is well-known, but we have no explicit reference.

Lemma 1.2. *If $m^+(da)\sim da$, $a\downarrow 0$ then $\nu(da)\sim a^{-3/2}\,da$, $a\downarrow 0$.*

Proof. We start by computing estimates for

$$\int_0^\infty (1-e^{-zx})\,\nu(dx)=z\int_0^\infty e^{-zx}\,\nu(x,+\infty)\,dx \qquad (z>0)$$

exploiting the fact that we can explicitly compute the extreme cases. Let us choose positive constants $c_1^2<c_2^2$ and ε such that $dm^+/da\in[c_1^2,c_2^2]$ when $0<a<\varepsilon$.

(a) Suppose we deal with the truncated measure $c_1^2\,1_{[0,\varepsilon]}\,da$. Here, arguing as in 1.1, we must solve $f''=2zf$ subject to the boundary conditions $f(0)=1$ and $f'(\varepsilon)=0$ and then compute $f'(0+)$. This yields the under-estimate $c_1\sqrt{2z}\tanh c_1\sqrt{2z}\,\varepsilon$.

(b) Next we augment the measure to $c_2^2\,1_{[0,\varepsilon]}\,da+\bar{\delta}_\varepsilon$ where $\bar{\delta}_\varepsilon$ denotes an infinite point mass located are the point ε. Solving the same equation subject to the boundary conditions $f(0)=1$ and $f(\varepsilon)=0$ gives our over-estimate $c_2\sqrt{2z}\coth c_2\sqrt{2z}\,\varepsilon$.

As $z\uparrow\infty$ each of these expressions behaves like $c_i\sqrt{2z}$ so a Tauberian theorem [10] p. 443 gives us the required asymptotic behaviour of ν near zero.

The equalisation time $\tau=\inf\{t>0\colon A_t=0\}$ is a B_t stopping time while the time of subsequent return to zero, namely $\tau+T\circ\theta_\tau$, corresponds in the η_t time scale to $U=\inf\{t\colon X_t>0\}$. The random variable X_U is called *the overshoot of level zero* for the process X_t. Finding its law when X_0 is fixed is a celebrated problem [3]. However in the case at hand we have a randomised starting point $A_T<0$ and this seems to make things easier. Nevertheless we will not attempt to compute the law of the overshoot itself, instead we derive a system of equations whose solution will give us the kernel $\Pi(x, dy)$ directly.

The first step is to derive an integral equation for the distribution of X_U. For this we need to establish some notation. Let us write $\kappa(z)$ to denote the Laplace exponent of X_t defined at least on the imaginary axis by the equation $\mathbf{E}[e^{-z(X_t-X_0)}]=e^{-\kappa(z)t}$. Also we introduce the function $\Phi_\lambda(z,x)=\mathbf{E}_{-x}[\mathbf{E}_{A_T}[e^{-\lambda U}e^{-zX_U}]]$, its complementary function $\Psi_\lambda(z,x)=\mathbf{E}_{-x}\left[\mathbf{E}_{A_T}\left[\int_0^U e^{-\lambda t}e^{-zX_t}\,dt\right]\right]$, and we write our initial condition as $\Upsilon(z,x)=\mathbf{E}_{-x}[e^{-zA_T}]$. Note that all these expressions exist, at least for z imaginary, by a majorisation argument. To derive an equation linking them recall that if f is a bounded continously differentiable function then by Ito's formula the process

$$e^{-\lambda t}f(X_t)-\int_0^t e^{-\lambda s}\,ds\int[f(X_s+y)-f(X_s)]\,\nu(dy)+\lambda\int_0^t e^{-\lambda s}f(X_s)\,ds \qquad (\lambda>0)$$

is a uniformly bounded (purely discontinuous) martingale. Taking $f(x)=e^{-zx}$, stopping at the time $t=U$, and taking the expectation, first in X_t and then in A_T, we get

$$\Phi_\lambda(z,x)=\Upsilon(z,x)-[\lambda+\kappa(z)]\,\Psi_\lambda(z,x) \qquad (\bullet)$$

an equation which holds (at least) when z is purely imaginary. This is known as the equation of Désiré André. It seems it can only rarely be solved to give $\mathbf{E}[e^{-zX_U}]$ explicitly.

We now connect this to the problem of computing $\Pi(x, dy)$. Notice that by definition of τ we have $X_U=A_T\circ\theta_\tau$, so using the strong Markov property of B_t at time τ gives

$$\begin{aligned}\Phi_\lambda(z,x)&=\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}e^{-zA_T\circ\theta_\tau};\ \tau<+\infty]\\&=\mathbf{E}_{-x}[\Upsilon(z,-B_\tau)\,\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau];\ \tau<+\infty]\end{aligned}$$

and hence from the definition of Π we get

$$\Phi_\lambda(z,x)=\int_0^\infty \Pi(x,dy)\,\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau=y]\,\Upsilon(z,-y). \qquad (\dagger)$$

Everything done so far is well-known. What is new is our claim that, under certain conditions, one can solve the system of Eqs. ($\bullet$, $\dagger$) for $\Pi(x, dy)$. Our starting point is the following observation which was implicit in [21].

Remark 1.4. The above system has the following structure.

(1) The unknown function Φ_λ (resp. Ψ_λ) is bounded analytic on the right (resp. left) half plane.

(2) For $x>0$ the given function $\Upsilon(\cdot, x)$ is bounded analytic on the left half plane and can be computed by solving a differential equation.

(3) We can write $\kappa(z)=\kappa_+(z)+\kappa_-(z)$ where

$$\kappa_+(z)=\tfrac{1}{2}\Upsilon_x(z,0-); \quad \kappa_-(-z)=-\tfrac{1}{2}\Upsilon_x(-z,0+) \quad (z>0).$$

This means the input data for (●, †), namely Υ and κ, are completely determined once we know the function $\Upsilon(z,x)$.

This resembles the situation in the theory of Wiener-Hopf integral equations (see [6, 20] for example) but there are two complications. The first is that there need not be any open strip of validity for (●), since the equation is only guaranteed to hold on the imaginary axis. The second, and apparently more fundamental, difficulty is that the given data (namely Υ and κ) are not analytic in the entire plane. Such behaviour appears to be new, or at least not well documented, in the context of the Wiener-Hopf method. In any case the standard technique of using factorisation and the Liouville theorem (see [6]) does not work here, nor will the method of [14].

Solving the Désiré André equation (●) on its own would appear to be a very difficult problem, in fact the only non-trivial solution we were able to find in the literature was [21] which deals with the case where X_t is a symmetric stable process. But in the present investigation we shall concentrate on solving (●, †) for $\Pi(x,dy)$. This problem seems to be more tractable than finding the overshoot law, and while the reason for this is rather difficult to pin down we think it is connected with having a randomised starting point $A_T=X_0<0$. In fact we essentially work with $(A_T, A_T\circ\theta_\tau)$ and so, interpreting this as a double transform of the overshoot law, one can make an analogy with computing the law of $\int_0^t 1_{(B_s<0)}\,ds$ by using a suitable double integral transform as in [13] p. 57.

§ 2. Solving a Canonical System

In this section we show how to solve the system

$$\Phi_\lambda(z,x)=\Upsilon(z,x)-[\lambda+\varkappa(z)]\,\Psi_\lambda(z,x), \qquad (●)$$

$$\Phi_\lambda(z,x)=\int_0^\infty \Pi(x,dy)\,\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau=y]\,\Upsilon(z,-y) \qquad (†)$$

for $\Pi(x,dy)$ in the particular case $m^+(da)=da$, $m^-(da)=\delta^2\,da$. The method we use is essentially due to Daniel Ray [21].

Suppose for the moment we concentrate on the Eq. (●) which we know holds at least on the imaginary axis. Recall that $\Upsilon(z,x)$ and $\kappa(z)$ are the given functions here, while all we know about $\Phi_\lambda(z,x)$ and $\Psi_\lambda(z,x)$ to begin with is that they have the analyticity properties prescribed at 1.4. From this information we want to find calculate $\Phi_0(z,x)$. Ray's method is to use (●) to extend the analytic function $z\to\Phi_\lambda(z,x)$ from the right half plane to the complex plane cut along the negative real line. But since $\Upsilon(z,x)$, $\Psi_\lambda(z,x)$ and $\kappa_-(z)$ are already defined on the left half plane this just involves extending the (known) function

$\kappa_+(z)$, something we can do by inspection. We will see in the examples of section four that the function κ_+ cannot be extended analytically to the entire plane, it is either meromorphic or else it has a branch point at zero.

In the present situation $\kappa_+(z)=\frac{1}{2}\sqrt{2z}$, so it has a branch point at zero and we can assume that $\kappa_+(z)$ is defined on the complex plane cut along the negative real axis. However we can exploit the discontinuity of κ_+ along the branch cut to get a solution of (●) as follows. The idea is to multiply (●) by an integrating factor $I(z)$ which also has a branch cut along the negative real axis, where it satisfies the cancellation condition $I(r\,e^{i\pi})\,\kappa(r\,e^{i\pi})=-I(r\,e^{-i\pi})\,\kappa(r\,e^{-i\pi})$. Now suppose we apply the Cauchy theorem to the function $z\to I(z)\,\Phi_\lambda(z,x)$, integrating along the contour $\{z=\mathrm{R}e^{i\theta}\colon -\pi<\theta<\pi\}\cup\{z=\varepsilon\,e^{i\theta}\colon -\pi<\theta<\pi\}\cup\{-y\colon \varepsilon<y<\infty\}$, coming in and going out again on either side of the branch cut to avoid the singularity at zero. If $I(z)$ can be chosen so that as $R\uparrow\infty$ and $\varepsilon\downarrow 0$ the integrals along the circles vanish then we can make the unknown function Ψ_λ cancel as $\lambda\downarrow 0$ thereby leaving an expression for Φ_0 in terms of the integrating factor $I(z)$ and the given function Υ. This is the method of [21].

Our solution follows Ray's idea but with two minor modifications. The first is that we will use martingale stopping instead of Cauchy's theorem, since this formulation is useful and suggestive for our examples. Also we find it convenient to work entirely on the right half plane. Hence we replace z by $z^2/2$ in our system of Eqs. (●, †) and again this change should be thought of as being more than just cosmetic (see [18]).

Since we assume that $m^+(da)=da$, $m^-(da)=\delta^2\,da$ the given data in (●, †) become

$$\Upsilon(-z^2/2)=e^{-z\delta x};\qquad \kappa_+(z^2/2)=\tfrac{1}{2}z;\qquad \kappa_-(-z^2/2)=\tfrac{1}{2}\delta z\qquad (z>0)$$

and our system of equations can be written as

$$\Phi_\lambda(z^2/2,x)=\Upsilon(z^2/2,x)-[\lambda+\kappa_-(z^2/2)+\tfrac{1}{2}z]\,\Psi_\lambda(z^2/2,x),$$

$$\Phi_\lambda(z^2/2,x)=\int_0^\infty \mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau=y]\,e^{-zy}\,\Pi(x,dy).$$

However we cannot go ahead and solve this for $\Phi_0(z,x)$ as it stands, we will need to deploy some extra information in the role of a boundary condition for the system. This point is not emphasised in [21] but will be extremely important for us here. When applying Ray's method one needs to know (as indicated above) that $\lambda\,\Psi_\lambda(-w^2/2,x)\to 0$ when w is real and $\lambda\downarrow 0$. We will see how in our context this depends on knowing that $U<+\infty$ (or equivalently $\tau<+\infty$). Of course one can quote the relevant Lévy process criterion [3] for this but it is easier and more convenient to provide the following direct proof.

Lemma 2.1. *If $m^+(da)=da$, $m^-(da)=\delta^2\,da$ then $U<+\infty$ almost surely.*

Proof. Suppose that $0<\lambda<z$. If we define, for $\gamma_+=\sqrt{2(z-\lambda)}$ and $\gamma_-=\sqrt{2(z\,\delta^2+\lambda)}$ the function

$$f(x)=\begin{cases}\cos\gamma_+x+\dfrac{\gamma_-}{\gamma_+}\sin\gamma_+x & (x>0)\\ e^{\gamma_-x} & (x<0)\end{cases}$$

then the process $f(B_t)\,e^{-\lambda t}e^{zA_t}$ is a uniformly bounded martingale up to time τ. By martingale stopping we get

$$\mathbf{E}_{-x}[f(B_\tau)\,e^{-\lambda\tau}]=f(-x).$$

Letting $\lambda\downarrow 0$ gives $\mathbf{E}_{-x}[f(B_\tau);\ \tau<+\infty]=f(-x)$ so by taking $z\downarrow 0$ we obtain $1=\mathbf{P}[\tau<+\infty]=\mathbf{P}[U<+\infty]$ as required.

The first step in solving for Π is to introduce the asymmetry parameter $\rho=\frac{2}{\pi}\cot^{-1}\delta\in(0,1)$ (in the standard theory of integral equations [20] with analytic coefficients the parameter $1-\rho$ corresponds to the index and is an integer). The idea is that ρ is determined modulo an integer by cancellation conditions while uniqueness comes from the need to comply with integrability conditions at zero and infinity.

Theorem 2.2. *If $m^+(da)=da$, $m^-(da)=\delta^2 da$ then the explicit solution of* ($\bullet$, $\dagger$) *is given by*

$$\Pi(x,dy)=\frac{1}{\pi}\sin\frac{\pi\rho}{2}\left(\frac{\delta x}{y}\right)^{\rho}\frac{dy^2}{y^2+(\delta x)^2}$$

with $\rho=\frac{2}{\pi}\cot^{-1}\delta$.

Proof. It is more convenient not to use the explicit form of Υ in the proof. Consider the function $z\to\Phi_\lambda(z^2/2,x)$, defined on the right half plane by

$$\Phi_\lambda(z^2/2,x)=\int_0^\infty e^{-zy}\,\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau=y]\,\Pi(x,dy).$$

This is an analytic function bounded by one on the right half plane, while by analytic continuation from the lines $\operatorname{Arg} z=\pm\frac{\pi}{4}$ using ($\bullet$) we find that on the imaginary axis $z=iw$ we have

$$\Phi_\lambda(-w^2/2,x)=\Upsilon(-w^2/2,x)-[\lambda+\tfrac{1}{2}\delta|w|+\tfrac{1}{2}iw]\,\Psi_\lambda(-w^2/2,x).$$

Let us define $U_\lambda(z,x)=z^{1-\rho}\Phi_\lambda(z^2/2,x)$ taking the branch of $z^{1-\rho}$ which is real on the positive real axis. Then if Z_t is a complex Brownian motion the process $U_\lambda(Z_t,x)$ is a conformal local martingale. However if $Z_0>0$ and ξ is the hitting time of the imaginary axis we know the hitting distribution is

$$\mathbf{P}[Z_\xi\in i\,dy]=\frac{1}{\pi}\frac{Z_0\,dy}{y^2+Z_0^2}.$$

Now $t\to|U_\lambda(Z_{t\wedge\xi},x)|$ is a submartingale and is therefore uniformly integrable by the estimate $\mathbf{E}[U_\lambda(Z_\xi,x)]\leq\mathbf{E}[|Z_\xi|^{1-\rho}]<+\infty$ since $\rho\in(0,1)$. Therefore applying the Doob stopping theorem at time ξ to the martingale $U_\lambda(z\,Z_t/\delta x,x)$ with

$z>0$ and $Z_0=\delta x$ and using the above relation to substitute for Φ_λ on the imaginary axis we get

$$U_\lambda(z,x)=\frac{1}{\pi}\int_0^\infty\left(\frac{zy}{\delta x}\right)^{1-\rho} e^{i\frac{\pi}{2}(1-\rho)}\frac{\delta x\,dy}{y^2+(\delta x)^2}$$
$$\cdot\left[\Upsilon\left(-\left(\frac{zy}{2\delta x}\right)^2,x\right)-\left[\lambda+\frac{1}{2}\frac{z|y|}{x}+i\frac{1}{2}\frac{zy}{\delta x}\right]\Psi_\lambda\left(-\left(\frac{zy}{2\delta x}\right)^2,x\right)\right]$$
$$+\frac{1}{\pi}\int_{-\infty}^0\left(\frac{z|y|}{\delta x}\right)^{1-\rho} e^{-i\frac{\pi}{2}(1-\rho)}\frac{\delta x\,dy}{y^2+(\delta x)^2}$$
$$\cdot\left[\Upsilon\left(-\left(\frac{zy}{2\delta x}\right)^2,x\right)-\left[\lambda+\frac{1}{2}\frac{z|y|}{x}-i\frac{1}{2}\frac{zy}{\delta x}\right]\Psi_\lambda\left(-\left(\frac{zy}{2\delta x}\right)^2,x\right)\right].$$

Adding the integrals and using the definition of ρ we find there is a cancellation in the coefficient of Ψ_λ and so

$$U_\lambda(z,x)=\frac{2}{\pi}\sin\frac{\rho\pi}{2}\int_0^\infty\left(\frac{zy}{\delta x}\right)^{1-\rho}\Upsilon\left(-\left(\frac{zy}{2\delta x}\right)^2,x\right)\frac{\delta x\,dy}{y^2+(\delta x)^2}$$
$$-\lambda\frac{2}{\pi}\sin\frac{\rho\pi}{2}\int_0^\infty\left(\frac{zy}{\delta x}\right)^{1-\rho}\Psi_\lambda\left(-\left(\frac{zy}{2\delta x}\right)^2,x\right)\frac{\delta x\,dy}{y^2+(\delta x)^2}.$$

However for $w>0$ we have the uniform estimate

$$\lambda\Psi_\lambda(-w^2/2,x)=\mathbf{E}_{-x}\left[\mathbf{E}_{A_T}\left[\lambda\int_0^U e^{-\lambda t}e^{w^2X_t/2}\,dt\right]\right]$$
$$\leqq\mathbf{E}_{-x}\left[\mathbf{E}_{A_T}\left[\lambda\int_0^U e^{-\lambda t}\,dt\right]\right]=\mathbf{E}_{-x}[\mathbf{E}_{A_T}[1-e^{-\lambda U}]].$$

Using the previous lemma we can apply the dominated convergence theorem as $\lambda\downarrow 0$ to see that $\lambda\Psi_\lambda(-w^2/2,x)\to 0$. And since $0<\rho<1$ we can let $\lambda\downarrow 0$ in the above formula to get, again by the dominated convergence theorem, the solution

$$U_0(z,x)=\frac{2}{\pi}\sin\frac{\rho\pi}{2}\int_0^\infty\left(\frac{zy}{\delta x}\right)^{1-\rho}\Upsilon\left(-\left(\frac{zy}{2\delta x}\right)^2,x\right)\frac{\delta x\,dy}{y^2+(\delta x)^2}.$$

But we know from (†), again because $\tau<+\infty$, that

$$U_0(z,x)=z^{1-\rho}\Phi_0(z^2/2,x)=z^{1-\rho}\int_0^\infty\Pi(x,dy)\,e^{-zy}.$$

Comparing this with the previous expression, substituting $\Upsilon(-z^2/2,x)=e^{-zx}$ gives us what we want.

Remark. The argument used here is easier than the one used by Ray [21] since we know to begin with that our function $z \to \Phi_\lambda(z^2/2, x)$ is uniformly bounded on the right half plane. This means we can dispense with quoting the Phragmen-Lindelöf theorem [23].

The reasoning in the above proof does not depend on the explicit form of $\Upsilon(z,x)$ for $x<0$. In fact if we examine the argument more closely we find it is important to have $m^+(da)=da$ and $\kappa_-(-w^2/2)=\frac{1}{2}\delta|w|$, but we only need minimal assumptions on the functions Υ and Ψ_λ. It transpires that the following generalisation provides just what we need in order to deal with the examples of section four.

Suppose we consider the modified system

$$\Phi_\lambda(z^2/2,x)=\Upsilon(z^2/2,x)-\lambda\Psi_\lambda(z^2/2,x)-[\tfrac{1}{2}z+\kappa_-(z^2/2)]\,\bar{\Psi}_\lambda(z^2/2,x), \qquad (\star)$$

$$\Phi_\lambda(z^2/2,x)=\int_0^\infty \mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau=y]\,e^{-zy}\,\Pi(x,dy) \qquad (\bowtie)$$

where the first relation holds at least on the imaginary axis and $\kappa_-(-w^2/2)=\frac{1}{2}\delta|w|$. Then the reasoning used in the above proof yields the following result.

Corollary 2.3. *Assume that* $w\to\Upsilon(-w^2/2,x)$ *and* $w\to\bar{\Psi}_\lambda(-w^2/2,x)$ *are bounded even functions of the real variable* w, *while* $\lambda\Psi_\lambda(-w^2/2,x)\to 0$ *as* $\lambda\downarrow 0$. *Then the solution of* $(\star,\bowtie)$ *satisfies*

$$\Phi_{0+}(z^2/2,x)=\frac{2}{\pi}\sin\frac{\rho\pi}{2}\int_0^\infty\left(\frac{y}{\delta x}\right)^{1-\rho}\Upsilon\left(-\left(\frac{zy}{2\delta x}\right)^2,x\right)\frac{\delta x\,dy}{y^2+(\delta x)^2}$$

when $z>0$.

We shall refer to this later on as *the canonical case.* It will be needed in connection with the change of variable idea to be developed in the next section.

The result stated in our theorem has already been derived in [2] by using much different methods. See also [22]. Moreover, it is not hard to see that the above approach will also work when we have

$$m^+(da)=|a|^\gamma\,da; \qquad m^-(da)=\delta^{2+\gamma}|a|^\gamma\,da \qquad (\gamma>-1)$$

in which case the underlying Lévy process X_t is asymmetric stable of order $\alpha=1/(2+\gamma)$. However we have restricted ourselves to the case $\alpha=\frac{1}{2}$ for two reasons; firstly in order to simplify the exposition, and also because this more general situation does not lend itself so easily to realising explicit answers for $\Pi(x,dy)$.

§ 3. Change of Variable

The essence of our solution to the Wiener-Hopf problem for Brownian motion is the substitution procedure described in detail below. The starting point for this investigation was to note that the solution obtained in [16] (by a series of complicated calculations but see also [2]) could be written as the conformal image of the answer at 2.2 under the analytic function $z \to \sin\frac{\pi z}{2}$. We wanted to relate this observation to the structure of the corresponding integral equations in the hope of obtaining further solutions by the same method. What we found was that under suitable conditions there is a scale change which puts the system (●, †) into our canonical form (⋆, ⋈). The substitution depends, as we shall see, on the properties of the function $\Upsilon(z, x)$ and in particular on comparing its behaviour on either side of zero, the boundary point.

We begin with the following observation, which is both crucial for our argument and interesting in its own right. Suppose f is an odd function, defined and analytic on a symmetric interval I containing the origin. Then we want a condition under which we have the identity

$$\int_I h(y_1)\, dy_1\, d_{y_2}\left(\frac{f'(0)}{f(y_2-y_1)}\right) = \int_{\bar{I}} \bar{h}(u)\, du\, d_v\left(\frac{1}{v-u}\right) \tag{1}$$

where $\bar{h}\circ f = h$, $\bar{I} = f(I)$, $u = f(y_1)$, $v = f(y_2)$, and where h is an arbitrary even function, the integrals being wrt y_1 and u respectively. Obviously this condition restricts the function f quite severely.

To write (1) in a more convenient form let us introduce the function G defined by

$$\frac{G'}{G} = \frac{f'(0)}{f}$$

so our identity can be written as

$$\int_I h(y_1)\, d_{y_1} d_{y_2} \log|G(y_2-y_1)| = \int_{\bar{I}} \bar{h}(u)\, d_u d_v \log|v-u|.$$

Then remembering that h is even we obtain the following simple test.

Lemma 3.1. *Suppose G is an analytic function in a neighbourhood of zero such that for some function f*

$$\frac{G(y_2+y_1)}{G(y_2-y_1)} = \frac{f(y_2)+f(y_1)}{f(y_2)-f(y_1)}$$

for pairs (y_1, y_2). Then f complies with the requirement (1).

Proof. Putting $y_2 = 0$ we find that G is odd, so f is odd as well. Next we show our condition implies $G'/G = f'(0)/f$. But this follows if we take logarithms and differentiate in y_1 at $y_1 = 0$. It is now clear, by reversing the argument given before, that f satisfies (1).

Later on (in the final section) we will see several examples of such functions f, each one giving rise to a solution of the system (●, †). But for now we want to examine (●, †) in more detail. Our dilemma is that the general case is very difficult to deal with. For example, in the cases we can compute explicitly the function $z \to \mathbf{E}_y[e^{-zA_T}]$ defines an integral transform *on the negative real axis* but we have no idea of how to prove this when $m^+(da)$ is arbitrary. So we will specialise, looking only at those cases where $m^+(da)$ is Lebesgue measure, or Lebesgue measure on an interval $[0,l]$, or where $m^+(da)$ is given by Lebesgue measure plus an infinite point mass. In this section the measure $m^-(da)$ is required to satisfy a regularity condition at zero, and we assume that the gap diffusion on $(-\infty, 0]$ with speed measure m^- is persistent.

In the case $m^+(da)=da$ we have already seen that $\mathbf{E}_y[e^{-zA_T}]=e^{-\sqrt{2z}y}$ and so when we run z to the negative real axis we end up dealing with a Fourier transform. We claim this is the simplest case and we will examine it in detail in subsection I. The cases $m^+(da)=da\,1_{[0,l]}$ and $m^+(da)=da+\bar{\delta}_l$, where $\bar{\delta}_l$ is an infinite point mass at l, are also not too difficult since we get respectively a Fourier cosine series and a Fourier sine series. These are discussed in subsections II and III.

I. $m^+(da)=da;\ m^-(da)\sim da,\ a\uparrow 0.$

The system $(\bullet, \dagger)$ can now be written as

$$\begin{aligned}\Phi_\lambda(z^2/2, x) &= \Upsilon(z^2/2, x)-[\lambda+\tfrac{1}{2}z+\kappa_-(z^2/2)]\,\Psi_\lambda(z^2/2)\\ \Phi_\lambda(z^2/2, x) &= \int_0^\infty e^{-zy}\,\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau=y]\,\Pi(x,dy).\end{aligned} \qquad (\natural)$$

As already noted in the proof of Theorem 2.1, the function $z\to\Phi_\lambda(z^2/2, x)$ is defined everywhere on the right hand half plane. Our observation is that $(\natural)$ is well-defined on the imaginary axis, namely we have

$$\begin{aligned}&\int_0^\infty e^{-iwy}\,\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau=y]\,\Pi(x,dy)\\ &\qquad=\Upsilon(-w^2/2, x)-[\lambda+\tfrac{1}{2}iw+\kappa_-(-w^2/2)]\,\Psi_\lambda(-w^2/2, x).\end{aligned}$$

Given this, it seems sensible to do the inverse Fourier transform of $w\to\kappa_-(-w^2/2)$. This should give us some idea of the special functions which arise when solving for $\Pi(x,dy)$. Note however that we must use generalised Fourier transforms [11] here because the function $w\to\kappa_-(-w^2/2)$ is the Fourier transform of a tempered distribution.

Lemma 3.2. *Let μ be the distribution defined by the relation*

$$\kappa_-(-w^2/2)=-\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iwy}\,\mu(dy).$$

Then there is a function f which satisfies the following conditions.

(a) If ϕ is a test function whose support avoids zero then

$$\int \phi(y)\,\mu(dy)=-\int \phi(y)\,f^{-2}(y)\,f'(y)\,dy.$$

(b) f is odd, $f(\infty)=\infty$, and f increases no faster then exponentially.
(c) f is non-negative, increasing, and infinitely differentiable on R^+.
(d) We have $f(y)\sim y$, $y\downarrow 0$.

In the terminology of the theory of distributions [11] $1/f$ is said to be a regularisation of μ.

Proof. We can actually give a formula for f in terms of the Lévy measure ν of $X_t=A_{\eta_t}$. Let us define

$$\frac{1}{f(y)}=\pi\int_0^\infty du\,\frac{y}{\sqrt{2\pi u^3}}\,e^{-y^2/2u}\,\nu(-\infty,-u]$$

where we recall ([13] p. 217) that ν is absolutely continuous. By Lemma 1.2 $\nu(du)\sim|u|^{-3/2}\,du$ as $u\uparrow 0$ so the integral converges when $y\neq 0$. Property (c) and the first part of (b) are now obvious. To get the second part of (b) we use the dominated convergence theorem, while for the third assertion we use the monotone dependence of f on ν and then, since $\nu(du)\sim|u|^{-3/2}\,du$, $u\uparrow 0$, we can compare it with the case where $m^-(da)=1_{(-p,0)}\,da$. That the growth is at most exponential now follows from the computation of f in the first example of section four. Next, substituting $a=u\,y^{-2}$ we obtain

$$\frac{1}{f(y)}=\pi\int_0^\infty da\,\frac{1}{\sqrt{2\pi a^3}}\,e^{-1/2a}\,\nu(-\infty,-a\,y^2]\qquad(y>0)$$

and since $\lim_{y\downarrow 0} y\,\nu(-\infty,-a\,y^2]\neq 0$ property (d) follows by applying the dominated convergence theorem. To check (a) note that

$$\int_{-\infty}^\infty \sin w\,y\,\frac{dy}{f(y)}=\pi\int_0^\infty \nu(-\infty,-u]\,du\int_{-\infty}^\infty \sin w\,y\,\frac{y}{\sqrt{2\pi u^3}}\,e^{-y^2/2u}\,dy$$

where we justify the change in order of integration by noting that $y(2\pi u^3)^{-1/2}\,e^{-y^2/2u}$ is decreasing (which allows us to write the second integral as a sum of positive terms). But now, this is the imaginary part of the following Fourier transform (using (d) to interpret the real part as a Cauchy principal value integral)

$$-\int_{-\infty}^\infty e^{-iwy}\frac{dy}{f(y)}=-\pi\int_0^\infty \nu(-\infty,-u]\,du\int_{-\infty}^\infty e^{-iwy}\frac{y}{\sqrt{2\pi u^3}}\,e^{-y^2/2u}\,dy.$$

Integrating by parts this can be rewritten as

$$i\,w\,\pi\int_0^\infty \nu(-\infty,-u]\,e^{-w^2u/2}\,du=\frac{2\pi i}{w}\int_{-\infty}^0 (1-e^{w^2u/2})\,\nu(du)=\frac{2\pi i}{w}\,\kappa_-(-w^2/2).$$

By definition of the distributional derivative this gives

$$\int_{-\infty}^{\infty} e^{-iwy} d_y\left(\frac{1}{f(y)}\right) = -2\pi\kappa_-(-w^2/2)$$

which by the uniqueness result for Fourier transforms of distributions is precisely what we mean by (a).

Recall now the explicit evaluation [11] p. 359

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iwy}\frac{dy}{y^2} = \frac{-iw}{2\pi}\int_{-\infty}^{\infty} e^{-iwy}\frac{dy}{y} = \tfrac{1}{2}|w|$$

where we take the Cauchy principal value of the integral. In view of the results of the previous section this suggests the following tactic. First invert the Fourier transform in (#), then do the change of variable $v=f(y)$ in the resulting convolution equation, and finally solve the scale-changed equation by (the standard procedure of) taking its Fourier transform.

Now as a general method this is inadequate. The objection comes at the last step where if the Fourier transform is to be successful we need to again have a convolution equation, and change of variable will usually destroy the convolution from the original expression. But look at what we want here: we begin with the convolution of two measures $\mu*\eta$, where μ is even (but unknown) and $f(\eta)(dy)=y^{-2}dy$, and we ask that the image measure $f(\mu*\eta)$ be representable in the form $\bar{\mu}*f(\eta)$ (so we want our scale change to *preserve the convolution structure*). It turns out that there are non-trivial examples where this is possible. We will see below how condition 3.1 provides just what we need.

So let us look more carefully at what we propose. Suppose $\tilde{\Pi}(x,dv)$ is the kernel obtained from $\Pi(x,dy)$ under the substitution $v=f(y)$. Then we write

$$\tilde{\Phi}_\lambda(z^2/2,x) = \int_0^{\infty} e^{-zv}\,\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau = f^{-1}(v)]\,\tilde{\Pi}(x,dv) \qquad (z>0)$$

which by analytic continuation defines the function $\tilde{\Phi}_\lambda(z^2/2,x)$ on the right half plane. Doing the same with the functions Ψ_λ, Υ we obtain $\tilde{\Psi}_\lambda$, $\tilde{\Upsilon}$, and moreover we observe that these functions will still have the analyticity properties listed at Remark 1.4.

All this is reasonably straightforward. However inversion and change of variable for the product

$$w \to [\tfrac{1}{2}iw + \kappa_-(-w^2/2)]\,\Psi_\lambda(-w^2/2,x)$$

is somewhat more difficult. The term $w\to\frac{1}{2}iw$ inverts to give minus one half the distributional derivative of the Dirac mass at zero, this being a tempered distribution, while the inversion of $w\to\kappa_-(-w^2/2)$ is described at 3.2. So let us look at the inversion of $w \to \Psi_\lambda(-w^2/2, x)$ as a Fourier transform. Recall from the definition (in section one) that

$$\Psi_\lambda(z, x) = \mathbf{E}_{-x}\left[\int_0^U e^{-\lambda s} e^{-zX_s} ds\right] = \int_{-\infty}^{0} e^{-zk} \int_0^\infty e^{-\lambda t} \mathbf{P}_{-x}[X_t \in dk, t < U]\, dt.$$

Hence $\Psi_\lambda(-w^2/2, x)$ can be written as

$$\mathbf{E}_{-x}\left[\int_0^U e^{-\lambda_s} \mathbf{E}[e^{iw\Xi(-X_s)}]\, ds\right] = \int_0^\infty \mathbf{E}[e^{iw\Xi_k}] \int_0^\infty e^{-\lambda t} \mathbf{P}_{-x}[X_t \in -dk, t < U]\, dt$$

where Ξ_{σ^2} is an independent normal random variable with variance σ^2, which means that the Fourier transform inverts to give us

$$h(y)\, dy = dy \int_0^\infty \frac{1}{\sqrt{2\pi k}} e^{-y^2/2k} \int_0^\infty e^{-\lambda t} \mathbf{P}_{-x}[X_t \in -dk, t < U]\, dt,$$

in particular the law is absolutely continuous with an even density which is infinitely differentiable and of rapid decrease (it is easy to see that $X_U = 0$ occurs with probability zero here). Consequently we have no difficulty in defining the convolution and if we invert the product

$$[\tfrac{1}{2} i w + \kappa_-(-w^2/2)]\, \Psi_\lambda(-w^2/2, x)$$

then assuming f satisfies the conditions at 3.1 we find an expression of the form

$$-\tfrac{1}{2} h'(y_2)\, dy_2 - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(y_1)\, dy_1\, d_{y_2}\left(\frac{1}{f(y_2 - y_1)}\right)$$

$$= -\tfrac{1}{2} \bar{h}'(v)\, dv - \frac{c_0}{2\pi} \int_{-\infty}^{\infty} \bar{h}(u)\, du\, d_v\left(\frac{1}{v-u}\right)$$

where $c_0^{-1} = f'(0)$. Because the rhs is again a convolution it follows that its Fourier transform is given by

$$[\tfrac{1}{2} i w + \tfrac{1}{2} c_0 |w|]\, \bar{\Psi}_\lambda(-w^2/2, x),$$

denoting by $\bar{\Psi}_\lambda(-w^2/2, x)$ the Fourier transform of the even function $\bar{h}$ (notice how this exists and is bounded since $\bar{h} = h \circ f^{-1}$ is also infinitely differentiable and of rapid decrease). The outcome is that under the scale change by f our Eq. (♮) transforms to

$$\tilde{\Phi}_\lambda(-w^2/2, x) = \tilde{\Upsilon}(-w^2/2, x) - \lambda \tilde{\Psi}_\lambda(-w^2/2, x) - [\tfrac{1}{2} i w + \tfrac{1}{2} c_0 |w|]\, \bar{\Psi}_\lambda(-w^2/2, x).$$

This resembles the situation at 2.3, so we now summarise the net effect of our change of variable as follows.

Theorem 3.3. *Suppose that f (constructed at 3.2) satisfies the hypotheses of 3.1. Then substituting $v=f(y)$ puts the system (#) into the form*

$$\tilde{\Phi}_\lambda(z^2/2,x)=\tilde{\Upsilon}(z^2/2,x)-\lambda\,\tilde{\Psi}_\lambda(z^2/2,x)-[\tfrac{1}{2}z+\tilde{\kappa}_-(z^2/2)]\,\Psi_\lambda(z^2/2,x), \qquad (\star)$$

$$\tilde{\Phi}_\lambda(z^2/2,x)=\int_0^\infty \mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau=f^{-1}(v)]\,e^{-zv}\,\tilde{\Pi}(x,dy) \qquad (\bowtie)$$

with $\tilde{\kappa}(-w^2/2)=\frac{1}{2}|w|/f'(0)$ when w is real. Moreover if $\tau<+\infty$ then this satisfies the conditions at 2.3 and the solution is given by

$$\begin{aligned}\tilde{\Phi}_0(z^2/2,x)&=\int e^{-zy}\,\tilde{\Pi}(x,dy)\\&=\frac{2}{\pi}\sin\frac{\rho\pi}{2}\int_0^\infty\left(\frac{y}{\delta x}\right)^{1-\rho}\tilde{\Upsilon}\left(-\left(\frac{zy}{2\delta x}\right)^2,x\right)\frac{\delta x\,dy}{y^2+(\delta x)^2}\end{aligned}$$

where $\rho=\dfrac{2}{\pi}\tan^{-1}f'(0)$.

Proof. The only point remaining is to see that $\lambda\tilde{\Psi}_\lambda(-w^2/2,x)\to 0$ as $\lambda\downarrow 0$. But scale-change does not affect the behaviour in λ so this holds provided the same is true for $\lambda\Psi_\lambda(-w^2/2,x)$, which we deduce exactly as in the proof of Theorem 2.2 using the finiteness of τ.

II. $m^+(da)=da\,1_{[0,l]}$; $m^-(da)\sim da$, $a\uparrow 0$

Here the situation is slightly more complicated since we are working with a Fourier cosine series rather than with the Fourier transform. However the relevant inversion formulae can again be found in [11]. Substituting our data for this case we find the system (•, †) now becomes

$$\left.\begin{aligned}\Phi_\lambda(z^2/2,x)&=\Upsilon(z^2/2,x)-[\lambda+\kappa_-(z^2/2)+\tfrac{1}{2}z\tanh z\,l]\,\Psi_\lambda(z^2/2),\\ \Phi_\lambda(z^2/2,x)&=\int_0^l\frac{\cosh(l-y)z}{\cosh lz}\,\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau=y]\,\Pi(x,dy).\end{aligned}\right\} \qquad (\natural)$$

We observe that the second equation above defines a meromorphic function on the right half plane, and that on the imaginary axis it gives us

$$\begin{aligned}&\int_0^l\frac{\cos(l-y)w}{\cos lw}\,\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau=y]\,\Pi(x,dy)\\&\quad=\Upsilon(-w^2/2,x)-[\lambda+\kappa_-(-w^2/2)-\tfrac{1}{2}w\tan w\,l]\,\Psi_\lambda(-w^2/2,x).\end{aligned}$$

Putting $w=n\pi/l$ gives the Fourier cosine coefficients for these and we will use this observation to reduce ($\natural$) to our standard form.

Lemma 3.2′. *Let μ be the distribution supported on $(-l,l)$ which is defined by*

$$-\frac{1}{2\pi}\int_{-l}^{l} e^{-iwy}\mu(dy)=\kappa_-(-w^2/2).$$

Then there is a function f which satisfies the following conditions.

(a) If ϕ is a test function whose support avoids zero then

$$\int \phi(y)\,\mu(dy)=-\int \phi(y) f^{-2}(y) f'(y)\,dy.$$

(b) f is odd, $f(l)=+\infty$, and f increases no faster than a multiple of $\tan\frac{\pi y}{2l}$.

(c) f is non-negative, increasing, and infinitely differentiable on $(0, l)$.

(d) We have $f(y)\sim y$, $y\downarrow 0$.

Proof. This follows the pattern mapped out for Lemma 3.1, the main difference being that the functions are now less familiar while the estimates are more difficult to come by. We begin by noting that a formal argument leads to the expression in terms of the Lévy measure ν of X_t to wit

$$\frac{1}{f(y)}=-\frac{\pi}{2l}\int_0^\infty \frac{d}{dy}\theta_3\left(\frac{y}{2l}\middle| q\right)\nu(-\infty,-u)\,du$$

where θ_3 is one of Jacobi's theta functions and $\log q=\log q(u)=-\pi^2 u/2l^2$. Our first task is to verify that the integral converges. Using the Fourier series of [9] p. 355, $\theta_3(z|q)=1+2\sum_{n=1}^{\infty} q^{n^2}\cos n\pi z$, we see that convergence at infinity presents no problem if $y\neq 0$. To examine the convergence at zero we can use Jacobi's imaginary transformation [9] p. 370

$$\theta_3\left(\frac{z}{\tau};-\frac{1}{\tau}\right)=(-i\tau)^{1/2}\exp\left(\frac{i\pi z^2}{\tau}\right)\theta_3(z;\tau)$$

where $\theta_3(z;\tau)=\theta_3(z|e^{i\pi\tau})$ is the standard alternative notation for theta functions. Moreover the same estimates work for all the derivatives of θ_3 so we find that $1/f$ is odd, infinitely differentiable, and vanishes at $\pm l$. To investigate the singularity at zero we again use Jacobi's imaginary transformation and since we know the limiting behaviour of ν (from Lemma 1.2) this allows us to check that $\lim_{y\downarrow 0} y f(y)\neq 0$. That f is non-negative and increasing on $(0, l)$ follows from the product representation [9], p. 357 (16)

$$\theta_3(z,q)=q_0\prod_{n=1}^{\infty}(1+2q^{2n-1}\cos\pi z+q^{4n-2})$$

with $q_0=\prod_{n=1}^{\infty}(1-q^{2n})$. It remains to verify that $1/f$ has the correct Fourier series but this is just a computation starting from the definition of $\mu(dy)$.

We can now run through the same procedure as before, namely we treat the Eq. (♮) as the Fourier cosine transform of a convolution equation. The idea is to invert, to do the substitution $v=f(y)$, and then to solve by taking the Fourier transform. There are some points of difference with the previous case, such as the fact that

$$\int_{-l}^{l} \frac{\cos(l-y)\,w}{\cos l w}\,\eta(dy)=\tfrac{1}{2} w \tan w\, l$$

identifies η as one half the distributional derivative of the Dirac mass at zero. But otherwise the argument is straightforward and leads to the following result.

Theorem 3.3′. *Let f be defined as in 3.2′, and suppose it satisfies the hypotheses of 3.1. Then the above substitution puts the system (♮) into the canonical form*

$$\tilde{\Phi}_\lambda(z^2/2, x)=\tilde{\Upsilon}(z^2/2, x)-\lambda\,\tilde{\Psi}_\lambda(z^2/2, x)-[\tfrac{1}{2}z+\tilde{\kappa}_-(z^2/2)]\,\tilde{\Psi}_\lambda(z^2/2, x), \qquad (\star)$$

$$\tilde{\Phi}_\lambda(z^2/2, x)=\int_0^\infty \mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau=f^{-1}(v)]\,e^{-zv}\,\tilde{\Pi}(x, dv) \qquad (\bowtie)$$

with $\tilde{\kappa}_-(-w^2/2)=\tfrac{1}{2}|w|/f'(0)$ when w is real, and the system can be solved as at 2.3.

III. $m^+(da)=da\,1_{[0,l]}+\bar{\delta}_l\,;\ m^-(da)\sim da,\ a\uparrow 0$

Here $\bar{\delta}_l$ represents an infinite point mass at l, so we can suppose that our Brownian motion has an absorbing boundary there. Notice that now the measure m^+ is not Radon, something which is quite natural from our probabilistic point of view. Then by substituting our data we find the system (●, †) becomes

$$\begin{aligned}\Phi_\lambda(z^2/2, x)&=\Upsilon(z^2/2, x)-[\lambda+\kappa_-(z^2/2)+\tfrac{1}{2}z\coth z\,l]\,\Psi_\lambda(z^2/2),\\ \Phi_\lambda(z^2/2, x)&=\int_0^l \frac{\sinh(l-y)\,z}{\sinh l z}\,\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau=y]\,\Pi(x, dy).\end{aligned} \qquad (^\flat)$$

Again we observe that the first equation extends $\Phi_\lambda(z^2/2, x)$ as a meromorphic function to the imaginary axis, where ($^\flat$) gives us the relation

$$\int_0^l \frac{\sin(l-y)\,w}{\sin l w}\,\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}\,|\,B_\tau=y]\,\Pi(x, dy)$$
$$=\Upsilon(-w^2/2, x)-[\lambda+\kappa_-(-w^2/2)+\tfrac{1}{2}w\cot w\,l]\,\Psi_\lambda(-w^2/2, x).$$

Putting $w_n=\theta_n\,\pi/l$, where $\theta_n=n+\tfrac{1}{2}$, we have the coefficients for a Fourier expansion in terms of the functions $\cos\frac{\pi y}{l}\theta_n$. These are the eigenfunctions of the

Laplacian on $(-l, l)$ when the boundary conditions are absorbing, and we want to invert the transform as before again looking to reduce ($^\flat$) to the form ($\star$, $\bowtie$). The change of variable is computed from the following.

Lemma 3.2″. *Let μ be the distribution supported on $(-l, l)$ and defined by the equation*

$$-\frac{1}{2\pi}\int_{-l}^{l} e^{-iwy}\,\mu(dy)=\kappa_-(-w^2/2).$$

Then there is a function f which satisfies the following conditions.

(a) If ϕ is a test function whose support avoids zero then

$$\int \phi(y)\,\mu(dy)=-\int \phi(y)\,f^{-2}(y)\,f'(y)\,dy.$$

(b) $1/f$ is odd and its derivative vanishes at l.

(c) f is non-negative, increasing, and infinitely differentiable on $(0, l)$.

(d) We have $f(y)\sim y$, $y\downarrow 0$.

Proof. This is very much like the previous argument, the essential difference being that we use Jacobi's function θ_2 in place of θ_3. A formal argument leads to the expression

$$\frac{1}{f(y)}=-\frac{\pi}{2l}\int_0^\infty \frac{d}{dy}\,\theta_2\left(\frac{y}{2l}\middle|q\right)\nu(-\infty, -u]\,du$$

where ν is the Lévy measure of X_t and $\theta_2(z|q)=2\sum_1^\infty q^{(n+\frac{1}{2})^2}\cos(2n+1)\pi z$ with $\log q=\log q(u)=-\frac{\pi^2 u}{2l^2}$. It is clear from this representation that (b) above is true, and verifying the other properties is a matter of routine, though we need to use Jacobi's imaginary transformation

$$\theta_4\left(\frac{z}{\tau};\,-\frac{1}{\tau}\right)=(-i\tau)^{1/2}\exp\left(\frac{i\pi z^2}{\tau}\right)\theta_2(z;\tau).$$

We omit the details.

We now want to do a substitution and transform this to the canonical setup. But, unlike the previous cases, we have $f(l)<+\infty$ here and this provides an extra complication. Suppose we normalise by $f(l)=1$, invert our Eq. ($\bullet$) according the recipe just indicated, and do the substitution $v=f(y)$. Then we obtain, assuming f satisfies the condition 3.1, an equation of the type

$$\begin{aligned}&\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)}|B_\tau=f^{-1}(v)]\,\tilde{\Pi}(x, dv)\\&=\tilde{\Sigma}(x, dv)-\lambda\,\tilde{h}(v)\,dv+\tfrac{1}{2}\bar{h}'(v)\,dv+\frac{c_0}{2\pi}\int_{-1}^{1} d_v\left(\frac{1}{v-u}\right)\bar{h}(u)\,du\end{aligned}$$

where we take $v\in(-1, 1)$ and we know that $\tilde{\Sigma}$ and $\bar{h}$ are even in the variable v. Extending $\bar{h}$, $\tilde{h}$ and $\tilde{\Pi}$ to the entire line by truncation and extending the convolution in the obvious way, we modify $\tilde{\Sigma}(x, dv)$ to ensure that the equation remains valid for $|v|>1$. Then this extension corresponds to our canonical case and so we get the following. Notice how $\tau<+\infty$ is automatic here.

Theorem 3.3″. *Suppose the function f satisfies condition at 3.1. Then the above substitution puts the system* ($^{\flat}$) *into the canonical form*

$$\tilde{\Phi}_\lambda(z^2/2, x) = \tilde{\Upsilon}(z^2/2, x) - \lambda\, \tilde{\Psi}_\lambda(z^2/2, x) - [\tfrac{1}{2}z + \tilde{\kappa}_-(z^2/2)]\, \tilde{\Psi}_\lambda(z^2/2, x), \qquad (\star)$$

$$\tilde{\Phi}_\lambda(z^2/2, x) = \int_0^\infty \mathbf{E}_{-x}[e^{-\lambda L(0,\tau)} \,|\, B_\tau = f^{-1}(v)]\, e^{-zv}\, \tilde{\Pi}(x, dv) \qquad (\bowtie)$$

with $\tilde{\kappa}_-(-w^2/2) = \frac{1}{2}|w|/f'(0)$ *when* w *is real. In this case we have* $\Pi(x, dy) = \mathbf{P}_x[B_\tau \in dy;\, y < l] = 1_{(v<1)}\tilde{\Pi}(x, dv)$ *with* $v = f(y)$ *and* $f(l) = 1$.

Thus we have a technique for solving ($\bullet$, †) when $m^+(da)$ takes one of the three forms listed above, namely we use the above substitution to transform it to our canonical form ($\star, \bowtie$) and then quote 2.3. It seems there are other situations where the same general idea applies though we will not look at them here. The difficulty is that one cannot expect much in the way of explicit formulae.

Remarks. (1) Differentiability of the function $1/f$ depends on the measures $m^+(da)$ and $m^-(da)$ having the same behaviour close to zero. See [17] for more information on this point.

(2) In principle one can compute f by using the formulae given in the proofs. However, since v is not known to begin with it is often easier to find f by direct inversion of the appropriate transform.

(3) The above formulae for f lead to some interesting identities involving special functions. Even in the simple cases considered below the Lévy measures can have (e.g.) theta function densities.

(4) The condition at 3.1 suffices for the examples we consider below but is nevertheless extremely restrictive. We were unable to formulate a suitable alternative.

Finally we should say something about the importance of our boundary condition, the requirement that $\mathbf{P}[\tau<+\infty]=1$. Recall that to apply our method in any of the three cases studied above we not only have to compute the change of variable $v=f(y)$, but we also must check that f satisfies 3.1 and that $\tau<+\infty$ (this last is automatic if $m^+(da) = da + \delta_l$). The reason we are restricted to the case $\tau<+\infty$ is because the method of solution for the canonical case needs this. In order to deal with other boundary conditions we need other examples with different methods of solution. The guiding principle is that we can use our substitution to transform only between problems with the same boundary conditions.

§ 4. Some Explicit Calculations

In this section we apply the method of Sect. 3 to our four examples. In each case we set up the Eqs. ($\bullet$, †), we invert the appropriate transform to find the function f, do the substitution, and then solve the resulting equation by taking the Fourier transform. In each case we need to verify the boundary condition, which means showing that $\tau < +\infty$ holds almost surely. Remember: the canonical example at 2.3 is the only one we can solve directly, for all the other cases we must detour via our substitution procedure.

We begin by recalling the problem solved in section two, where we had the input data

$$\Upsilon(-z, x) = e^{-\sqrt{2z}\,\delta x}; \qquad \kappa_-(-z) = \tfrac{1}{2}\delta\sqrt{2z} \qquad (z>0)$$

for our system $(\bullet, \dagger)$. There we found the explicit expression

$$\Pi(x, dy) = \frac{1}{\pi}\sin\frac{\pi\rho}{2}\left(\frac{\delta x}{y}\right)^{\rho}\frac{dy^2}{(\delta x)^2+y^2}$$

with $\rho = \frac{2}{\pi}\cot^{-1}\delta$. As already pointed out this answer is not new but it is useful since it provides us with a model for the solutions derived below.

Example 1. Suppose that

$$m_+(da) = da; \qquad m_-(da) = \delta^2\, da\, 1_{[-p,0]}$$

where $p>0$. This is essentially the case considered in [16], but see also [1].

The first thing to check is that $\tau < +\infty$. But note that if

$$f_\theta(y) = \begin{cases} \cosh\delta\theta(p+y) & \text{if } y<0 \\ \cosh\delta\theta p\cos\theta y + \delta\sinh\delta\theta p\sin\theta y & \text{if } y>0 \end{cases}$$

where θ may be any positive number, then the process $f(B_t)\,e^{(\theta^2/2)A_t}$ is a uniformly bounded martingale up to time τ. Then if we apply the Doob stopping theorem at time τ this gives $\mathbf{E}_{-x}[f_\theta(B_\tau);\ \tau<+\infty] = f_\theta(-x)$, recalling from [3] that the real-valued Lévy process $X_t = A_{\eta_t}$ (notation at 1.1) either oscillates or drifts to infinity. So letting $\theta \downarrow 0$ shows that $\mathbf{P}_{-x}[\tau<+\infty]=1$ as required.

In order to solve the $(\bullet, \dagger)$ note that we are in the situation of 3.I with the data (see [13], p. 29)

$$\Upsilon(z, x) = \frac{\cos(p-x)\,\delta\sqrt{2z}}{\cos\delta p\sqrt{2z}}; \qquad \kappa_-(z) = -\tfrac{1}{2}\delta\sqrt{2z}\tan\delta p\sqrt{2z} \qquad (z>0)$$

when $x \in (0, -p)$. Explicitly, this means

$$\Phi_\lambda(z^2/2, x) = \Upsilon(z^2/2, x) - [\lambda - \tfrac{1}{2}\delta z\tan\delta p z + \tfrac{1}{2}z]\,\Psi_\lambda(z^2/2),$$

$$\Phi_\lambda(z^2/2, x) = \int_0^\infty e^{-zy}\,\mathbf{E}_{-x}[e^{-\lambda L(0,\tau)} \mid B_\tau = y]\,\Pi(x, dy)$$

and we can apply the results of 3.2 and 3.3. Using the formula in [8] p. 33 (28)

$$\int_{-\infty}^{\infty} \frac{1}{y} \tanh \delta y \cos(w y)\, dy = -2 \log \left| \tanh \frac{\pi w}{4\delta} \right|$$

we compute our change of variable as

$$f(y) = \frac{2p}{\pi} \sinh \frac{\pi y}{2\delta p}.$$

Next using [8] p. 31 (12) we have

$$\Upsilon(-w^2/2, x) = \frac{1}{\delta p} \int_{-\infty}^{\infty} e^{-iwy} \frac{\cosh \frac{\pi y}{2\delta p} \sin \frac{\pi x}{2p}}{\cosh \frac{\pi y}{\delta p} - \cos \frac{\pi x}{p}}\, dy$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-iwy} \frac{\sin \frac{\pi x}{2p}\, d\left(\sinh \frac{\pi y}{2\delta p}\right)}{\sinh^2 \frac{\pi y}{2\delta p} + \sin^2 \frac{\pi x}{2p}}.$$

If we make the change of variable $v = \frac{2p}{\pi} \sinh \frac{\pi y}{2\delta p}$ then $v'(0) = 1/\delta$ and by direct computation we get $\log \tilde{\Upsilon}(-w^2/2, x) = -|w| \frac{2p}{\pi} \sin \frac{\pi x}{2p}$. Moreover it is easy to check (directly or else see Appendix) that the function $G(x) = \tanh(x/2)$ satisfies the condition of 3.1. This means that we can apply Theorem 3.3 and so reduce the system to canonical form by means of the conformal mapping $z \to \frac{2p}{\pi} \sin \frac{\pi z}{2\delta p}$. Thus we read off the answer from Theorem 2.3 as

$$\Pi(x, dy) = \frac{1}{\pi} \sin \frac{\pi \rho}{2} \left(\frac{\sin \frac{\pi x}{2p}}{\sinh \frac{\pi y}{2\delta p}} \right)^{\rho} \frac{d\left(\sinh^2 \frac{\pi y}{2\delta p}\right)}{\sin^2 \frac{\pi x}{2p} + \sinh^2 \frac{\pi y}{2\delta p}}$$

where $\rho = \frac{2}{\pi} \cot^{-1} \delta$ and this completes the solution.

This result is not new. The case $\delta = 1$ was treated (with difficulty) in [16], while the above expression first appeared in [1] though the methods used there were quite different. See also [18].

It is no accident that the analytic function $z \to \frac{2p}{\pi} \sin \frac{\pi z}{2\delta p}$ is involved in the answer. In fact one can find $\tilde{\Upsilon}$ as follows. Note that the conformal mapping $z \to \frac{2p}{\pi} \sin \frac{\pi z}{2\delta p}$ takes the strip $\{x + iy\colon 0 \leqq x \leqq \delta p\}$ to the left half plane, with the boundary points $\delta p + iy$ going onto the interval $\left(\frac{2\pi}{p}, +\infty\right)$. Now let X_t be a reflecting Brownian motion on $(-\infty, \delta p]$ and denoting by ξ its hitting time of zero we use the definition of A_t to get

$$\Upsilon(-z^2/2, x) = \mathbf{E}_{-x}[e^{z^2 A_T/2}] = \mathbf{E}_{\delta x}[e^{-z^2 \xi/2}] = \mathbf{E}_{\delta x}[e^{iz\beta_\xi}]$$

if β_t is an independent Brownian motion. This identifies $\Upsilon(-z^2/2, x)$ as the Fourier transform in z of the hitting distribution on the imaginary axis for the process $W_t = X_t + i\beta_t$ started at δx. It remains only to compute this law. But W_t is a complex Brownian motion until it hits the line $\delta p + iy$, so by Paul Lévy's theorem $Z_t = \frac{2p}{\pi} \sin \frac{\pi W_t}{2\delta p}$ is a time-changed complex Brownian motion up to when it hits the interval $\left(\frac{2p}{\pi}, +\infty\right)$, from which it is reflected vertically. However we only sample Z_t on the imaginary axis so by the strong Markov property it is immaterial whether the process reflects along $\left(\frac{2p}{\pi}, +\infty\right)$ or not. Thus in the change of variable $z \to \frac{2p}{\pi} \sin \frac{\pi z}{2\delta p}$ we get the hitting distribution for complex Brownian motion itself and this explains the appearance of the conformal image of the Cauchy law when we invert $\Upsilon(-w^2/2, x)$.

Example 2. Here we take

$$m^+(da) = da + \bar{\delta}_l; \qquad m^-(da) = \delta^2 \, da.$$

Since $\bar{\delta}_l$ is an infinite Dirac mass at the point l we see τ is bounded by B_t the hitting time of l, and is hence finite almost surely. Also the input data for (●, †) are

$$\Upsilon(-z^2/2, x) = e^{-\delta z x}; \qquad \Upsilon(-z^2/2, x) = \frac{\sinh z(l-x)}{\sinh z l} \qquad (z > 0)$$

using [13] p. 29 for the second one and so (●, †) becomes

$$\Phi_\lambda(-z^2/2, x) = \Upsilon(-z^2/2, x) - [\lambda + \tfrac{1}{2} z \cot z\, l + \tfrac{1}{2}\delta z]\, \Psi_\lambda(-z^2/2, x),$$

$$\Phi_\lambda(z^2/2, x) = \int_0^l \Pi(x, dy)\, \mathbf{E}_{-x}[e^{-\lambda L(0,\tau)} \mid B_\tau = y]\, \frac{\sinh z(l-y)}{\sinh z l}$$

when z is real. We saw in 3.III how to solve this for $\Pi(x, dy)$; take $z = iw$, $\theta_n = n + \frac{1}{2}$, and then $w = \frac{\pi}{l}\theta_n$ to find that the above determines a Fourier series in $\cos \frac{\pi y}{l}\theta_n$. We invert the first equation wrt this transform.

Starting with Υ we notice that since $\int_0^l \cos^2 \theta_n \frac{\pi y}{l} dy = l/2$, the inversion gives us

$$\frac{2}{l} \Re\left[\sum_{n=0}^{\infty} e^{i\theta_n \pi y/l} e^{-\theta_n \pi \delta x/l}\right] = \frac{2}{\pi} \frac{\sinh \frac{\pi \delta x}{2l}\, d\left(\sin \frac{\pi y}{2l}\right)}{\sinh^2 \frac{\pi \delta x}{2l} + \sin^2 \frac{\pi y}{2l}}.$$

Coming to the inversion of $\kappa_-(-w^2/2) = \frac{1}{2}\delta|w|$ we first look at the

$$\sum_{n=0}^{\infty} \frac{1}{\theta_n} \cos \theta_n \frac{\pi y}{l} = \tfrac{1}{2} \log \cot \frac{\pi y}{4l}$$

using [12], p. 38 to do the sum. This we can differentiate to get the distributional equality

$$\sum_{n=0}^{\infty} \sin \theta_n \frac{\pi y}{l} = \tfrac{1}{4} \operatorname{cosec} \frac{\pi y}{2l}.$$

In order to determine our substitution what we need is the formal sum

$$\frac{2\pi\delta}{l} \sum_{n=0}^{\infty} \frac{\pi \theta_n}{l} \cos \theta_n \frac{\pi y}{l} = \frac{\pi \delta}{2l} d_y\left(\operatorname{cosec} \frac{\pi y}{2l}\right)$$

which gives us $f(y) = \frac{2l}{\pi\delta} \sinh \frac{\pi y}{2l}$. But as one can easily check (see Appendix) the function $G(x) = \tan(x/2)$ verifies 3.1 while $\tau < +\infty$ is automatic so we can now apply 3.2′ and 3.3′ to reduce the system to our canonical form ($\star, \bowtie$). Hence the solution is given by the conformal image of 2.2 using the function $t \to \sin \frac{\pi z}{2l}$ and the final answer is therefore

$$\Pi(x, dy) = \frac{1}{\pi} \sin \frac{\pi \rho}{2} \left(\frac{\sinh \frac{\pi \delta x}{2l}}{\sin \frac{\pi y}{2l}}\right)^{\rho} \frac{d\left(\sin^2 \frac{\pi y}{2l}\right)}{\sinh^2 \frac{\pi \delta x}{2l} + \sin^2 \frac{\pi y}{2l}} \qquad (y < l)$$

where $\rho = \frac{2}{\pi} \cot^{-1} \delta$.

For $\delta = 1$ this example was treated in [1] from the perspective of a two boundary problem but it seems that the connection with absorbing Brownian motion was overlooked. Our result is new.

This completes the proof but again there is a probabilistic explanation as to why the inversion procedure applied to $\Upsilon(-w^2/2, x)$ gives us a conformal image of the Cauchy law. Suppose we take $W_t = X_t + i\bar{\beta}_t$ where X_t is a real Brownian motion and $\bar{\beta}_t$ is an independent Brownian motion having absorbing barriers at $\pm l$. Let ξ represent the hitting time of zero by the process X_t and consider $t \to e^{-\pi\theta_n W_t/l}$ where θ_n is defined as above. Then this is a conformal martingale and since it is purely imaginary on the lines $\Im z = \pm l$ it follows by the Cauchy-Riemann equations (see [18] for more detail) that $M_t = \Re[e^{-\pi\theta_n W_t/l}]$ is a bounded martingale up to time ξ. By the Doob theorem this gives us

$$\int_{-l}^{l} \cos\frac{\theta_n \pi y}{l} \mathbf{P}_{\delta x}[W_\xi \in i\,dy] = e^{-\pi\theta_n \delta x/l}$$

which identifies $\mathbf{P}_{\delta x}[W_\xi \in i\,dy]$ as the distribution we want. To compute this let us consider the conformal mapping $z \to \sinh\frac{\pi z}{2l}$ which takes the strip $\{x+iy: -l<y<l\}$ to the left half plane, the boundaries $x \pm il$ going onto the intervals $(i, +i\infty)$ and $(-i\infty, -i)$ respectively. But, since W_t is a complex Brownian motion until it hits the boundary $x \pm il$, then by Paul Lévy's theorem $Z_t = \sinh\frac{\pi W_t}{2l}$ is a time-changed complex Brownian motion up to when it hits the intervals $(i, i\infty)$, $(-i\infty, -i)$. Thus in the change of variable $z \to \sinh\frac{\pi z}{2l}$ we get the hitting distribution on the imaginary axis for complex Brownian motion killed when it hits these intervals. Hence the law we seek is the conformal image of the truncated Cauchy distribution by this function, precisely what was computed above.

Example 3. In this case we have

$$m^+(da) = da\, 1_{[0,l]}; \qquad m^-(da) = \delta^2\, da\, 1_{[-p,0]} \qquad (l \geqq \delta^2 p).$$

This example is much more difficult than the first two, since the solution requires some familiarity with elliptic functions. The standard references are [5, 9, 24], but see also the Appendix for a summary of the relevant facts.

For the solution our first step is to check $\tau < +\infty$. Let us consider the Lévy process $X_t = A_{\eta_t}$ whose Laplace exponent is given by

$$\kappa(z) = \tfrac{1}{2}\sqrt{2z}\tanh l\sqrt{2z} - \tfrac{1}{2}\sqrt{2z}\,\delta\tan\delta p\sqrt{2z}.$$

It follows that X_t has exponential moments, so if $l > \delta^2 p$ the strong law of large numbers applies to show that $\lim X_t = +\infty$ so we have $\tau < +\infty$ almost surely. If $\delta^2 p = l$ we have the same conclusion from the law of the iterated logarithm.

The data for the system (●, †) are

$$\Upsilon(z, x) = \frac{\cos[(p-x)\,d\sqrt{2z}]}{\cos[p\,\delta\sqrt{2z}]}; \qquad \kappa_-(z) = -\tfrac{1}{2}\delta\sqrt{2z}\tan p\,\delta\sqrt{2z}$$

so we need to solve the equations

$$\Phi_\lambda(z^2/2, x) = \frac{\cos z\,\delta(p-x)}{\cos z\,\delta p} - [\lambda + \tfrac{1}{2} z \tanh z\, l - \tfrac{1}{2}\delta \tan \delta p\, z]\, \Psi_\lambda(z^2/2, x);$$

$$\Phi_\lambda(z^2/2, x) = \int_0^l \Pi(x, dy)\, \mathbf{E}_{-x}[e^{-\lambda L(0,\tau)} \mid B_\tau = y]\, \frac{\cosh z(l-y)}{\cosh z\, l}.$$

This is the situation covered by 3.II, which means we substitute $z \mapsto i w$ with $w > 0$, write $w = \frac{n\pi}{l}$, and look to sum the resulting Fourier series. The inversion of each term now gives Jacobi elliptic functions and the reader may wish to consult the Appendix since we assume a familiarity with their basic properties. First we try inverting the term $\kappa_-(z^2/2) = -\frac{1}{2}\delta \tan \delta p\, z$ which gives us the series

$$\frac{1}{l}\sum_0^\infty \frac{\delta\pi}{l} \cos\frac{n\pi z}{l} \tanh\frac{\delta p \pi n}{l}.$$

To evaluate this we use the formula (proved in the Appendix)

$$\int_{-K}^{K} \cos\frac{\pi n u}{K} \log|\widehat{\mathrm{Sn}}\, u|\, du = -\frac{K}{n} \tanh\frac{n\pi K'}{2K}.$$

Here $\widehat{\mathrm{Sn}}$ is Jacobi's elliptic function with imaginary quarter period $K' = 2p\delta$ and real quarter period $K = l$. Differentiating $\widehat{\mathrm{Sn}}$ by using [9] p. 343, gives us our change of variable

$$f = \frac{\widehat{\mathrm{Sn}}}{\delta\, \widehat{\mathrm{Cn}}\, \widehat{\mathrm{Dn}}}.$$

As in previous examples we now consider the unique conformal extension of f, initially considered to be defined on $(-il, il)$, to a suitable rectangle in the right half plane. Using Jacobi's imaginary transformation the function we want turns out to be

$$z \to \frac{\widehat{\mathrm{sn}}\, z\, \widehat{\mathrm{cn}}\, z}{\delta\, \widehat{\mathrm{dn}}\, z}$$

where (see Appendix) lower case notation means these are functions of complementary modulus, the real and imaginary quarter periods being interchanged.

However by analogy with the previous examples we expect that f should have periods $p\delta, l$. That this is in fact the case one can check directly using the formulae of [9], p. 350. Moreover the above expression for f can be simplified as follows using [9], p. 341. First of all remark that all Jacobi's elliptic functions have the same poles so that f has zeros at the zeros of $\widehat{\mathrm{Sn}}$ and the poles of $\widehat{\mathrm{Cn}}$, namely $2ml + 4ni\delta p$ and $2ml + (4n+2)i\delta p$. On the other hand the poles of f are located at the zeros of $\widehat{\mathrm{Cn}}\,\widehat{\mathrm{Dn}}$ these being $(2m+1)l+4ni\delta p$ and $(2m+1)l+(4n+2)i\delta p$. Next we introduce sn, Jacobi's elliptic function of real quarter period δp and imaginary quarter period l, which we find has zeros at $2m\delta p+2nil$ and poles at $2m\delta+(2n+1)il$. Since all poles and zeros are simple it follows from Jacobi's imaginary transformation that

$$z \to \frac{\widehat{\mathrm{sn}}\, z\, \widehat{\mathrm{cn}}\, z}{\widehat{\mathrm{dn}}\, z} \frac{1}{\mathrm{sn}\, z}$$

is an elliptic function with neither zeros nor poles in the complex plane. Therefore by the Liouville theorem it is constant, and again invoking Jacobi's imaginary transformation we identify f as a multiple of

$$z \to \frac{\mathrm{Sn}\, z}{\mathrm{Cn}\, z}.$$

We will verify in the Appendix that this function satisfies the condition of 3.1.

The final part of the computation is to see that our answer is the conformal image of 2.2 by the function $z \to \mathrm{sn}\, z$. For this we compute the function $\tilde{Y}(-w^2/2, x)$ by finding the image by $\mathrm{sn}\, z$ of the rectangle with corners $\pm il$, $\pm il+p\delta$. But it suffices to determine the image of the boundary and we see that $(-il, il)$ is mapped onto the imaginary axis while the other boundary points go to the positive real axis. In fact we find from [9], p. 351, that if k is the (elliptic) modulus of sn then

$$\mathrm{sn}(p\delta+il)=k^{-1}; \qquad \mathrm{sn}\, p\delta=1$$

(recall that $0<k<1$). Thus we are in a similar situation to before; namely the distribution we seek can be realised as the hitting distribution on the imaginary axis for complex Brownian motion with reflecting boundaries on the other sides of the rectangle whose corners are $\pm il$, $\pm il+p\delta$. But because the image motion under $z \to \mathrm{sn}\, z$ is a time-changed Brownian motion with vertical reflection along the real axis one can use the strong Markov property to see that $Y(-w^2/2, x)$ inverts to give us the conformal image of the Cauchy law. So after doing the substitution we get

$$\log \tilde{Y}(-w^2/2, x) = -|w|\, \mathrm{sn}\, \delta x$$

(if the reader is unhappy with this argument then a direct computation can be found in the Appendix where of course the answer is the same!). So we have everything in place for the solution: $\tau<+\infty$, the function f satisfies 3.1, and $\tilde{Y}$ is the Fourier transform of the conformal image of the Cauchy law by $\mathrm{sn}\, z$. The answer is therefore

$$\Pi(x, dy)=\frac{1}{\pi} \sin \frac{\pi\rho}{2} \left(\frac{X}{Y}\right)^{\rho} \frac{dY^2}{X^2+Y^2}$$

with $X=\mathrm{sn}\, \delta x$, $Y=-i\, \mathrm{sn}\, iy$.

This result was announced in [17] (omitting the restriction $l \geqq \delta^2 p$) and to our knowledge the solution is new. A related problem appears in [2].

Example 4. Consider

$$m^+(da) = da + \bar{\delta}_l; \quad m^-(da) = \delta^2 da\, 1_{[-p,0]}.$$

In this case we must solve

$$\Phi_\lambda(z^2/2, x) = \frac{\cos z\,\delta(p-x)}{\cos z\,\delta p} + [\lambda + \tfrac{1}{2} z \coth z\, l - \tfrac{1}{2}\delta z \tan \delta z\, p]\,\Psi_\lambda(z^2/2, x),$$

$$\Phi_\lambda(z^2/2, x) = \int_0^\infty \Pi(x, dy)\, \mathbf{E}_{-x}[e^{-\lambda L(0,\tau)} \,|\, B_\tau = y]\, \frac{\sinh z(l-y)}{\sinh z\, l}$$

and we are in the setting of 3.III; so we take $z = \pi i \theta_n / l$, where $\theta_n = (n + \frac{1}{2})$, and do the inversion of the resulting series. From the inversion of κ_- we get $\frac{1}{l} \sum_{n=0}^{\infty} \frac{\delta \pi}{l} \cos \frac{\pi y}{l} \theta_n \tanh \theta_n \frac{\pi p \delta}{l}$, so if we integrate in y and use the formula (see Appendix)

$$\frac{1}{2\pi} \int_{-K}^{K} \sin \frac{\pi u}{K} \theta_n \frac{\operatorname{dn} u}{\operatorname{sn} u}\, du = \tfrac{1}{2} \tanh \frac{\pi K'}{K} \theta_n$$

then we find that

$$\frac{\delta}{l} \sum_{n=0}^{\infty} \sin \frac{\pi y}{l} \theta_n \tanh \theta_n \frac{\pi p \delta}{l} = \frac{\delta}{\pi} \frac{\operatorname{Dn} y}{\operatorname{Sn} y}$$

where these are Jacobi elliptic functions with imaginary quarter period $K' = \delta p$, and real quarter period l. Using Jacobi's imaginary transformation on this gives us

$$z \to \frac{\operatorname{sn} z}{\operatorname{dn} z}$$

as the required conformal mapping. To complete our argument we need to verify the following. First we must check that our mapping satisfies 3.1; this is not so trivial and we refer to the Appendix. Next we want to see that $\Upsilon(-z^2/2, x)$ is the Fourier transform of the image by sn/dn of the Cauchy law; this follows either by a conformal mapping argument as before, or else by direct computation starting from the Fourier series (see the Appendix for more details on this). Finally we know that in this case $\tau<+\infty$ is automatic and so we obtain the answer

$$\Pi(x, dy)=\frac{1}{\pi}\sin\frac{\pi\rho}{2}\left(\frac{X}{Y}\right)^{\rho}\frac{dY^2}{X^2+Y^2} \qquad (y<l)$$

with the notation

$$X=\frac{\operatorname{sn}\delta x}{\operatorname{dn}\delta x}; \qquad Y=\frac{\operatorname{Sn} y}{\operatorname{Dn} y}.$$

This result appears to be new though, as for the previous example, it is related to the 'factorisation on a circle' problem considered in [2].

Remarks. (1) These last two examples respectively generalise the first two as can be seen by using the degenerate forms of Jacobi functions when the periods tend to infinity. Thus if $l\uparrow+\infty$ in example 3 then we recover example 1, while taking $p\uparrow+\infty$ in example 4 gives example 2. Doing both we obtain the answer at 2.2.

(2) Given the difficulty of doing these computations it seems sensible to list what corroborative evidence we have. The first example has been computed in [2] and a special case was treated in [16]. Moreover we have checked it by using a conformal martingale construction similar to the eigenvalue method of [2]. The second example is new but a special case appears in [2] and we have again checked the answer by another argument (see [18]). For the two examples involving elliptic functions we currently have no other proof.

(3) The case $m^+(da)=1_{(0,l)}da$, $m^-(da)=\delta^2 da$ was considered in [17] but the answer there is incorrect, the error coming since the condition $\tau<+\infty$ is not satisfied. However we have since managed to calculate $\Pi(x, dy)$ by using the information, suggested by the method of 3.II, that the relevant change of variable is $z\to\tanh\frac{\pi z}{2l}$. In fact the solution is

$$\Pi(x, dy)=\frac{2}{\pi}\sin\frac{\pi\rho}{2}\left(\frac{\sin\frac{\pi y}{2l}}{\sinh\frac{\pi\delta x}{2l}}\right)^{1-\rho}\frac{\tanh\frac{\pi\delta x}{2l}\,d\left(\tan\frac{\pi y}{2l}\right)}{\tan^2\frac{\pi y}{2l}+\tanh^2\frac{\pi\delta x}{2l}}.$$

Details of the calculation are to be found in [18].

(4) The condition at 3.1 is rather intriguing. One would expect it to be well-known but I was unable to find any reference in the literature. However several people have suggested the possibility of characterising all such functions and I managed to do so. It turns out that the only functions which satisfy 3.1 are essentially (i.e. up to rotations) of the form sn or sn/dn, although we can now take the period parallelogram to be skew instead of just rectangular as in our examples and we also allow degenerate behaviour. The proof, which uses only elementary complex variable theory, has been written up separately.

(5) It would be interesting to find a better probabilistic justification for the conformal mappings used here. In the context of [18] their occurence is somehow

'obvious', but here we lack a probabilistic interpretation for the equation at 2.3.

(6) I want to thank David Williams and Neil Baker for introducing me to these problems and for convincing me that explicit computations were indeed possible. In particular they conjectured that the solution of example 3 would involve Jacobi elliptic functions.

(7) The hypothesis of 3.1 was omitted from [17] so the main theorem there is false. I am indebted to Ph. Biane, and to at least one of the referees of a previous version of this paper, for pointing out my error.

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Appendix

Elliptic functions have so many properties that we should mention the ones used in this paper. In fact we are only concerned with the elliptic functions of Jacobi, which we write as sn, cn, and dn, these being biperiodic functions on the complex plane. They all have simple poles and zeros, two of each in any fundamental region, and we only discuss the case where the period parallelogram is a rectangle. It is standard to denote by K (resp. K') the real (resp. imaginary) quarter period of the *family*, the individual periodicities being specified in terms of these [24], p. 504. The usual way of defining sn is as the inverse of the function

$$z \to \int_0^z \frac{1}{\sqrt{(1-w)(1-k^2 w)}}\, dw$$

provided the path from 0 to z avoids branch points of the integrand. The parameter $0<k<1$ is known as the modulus of $\operatorname{sn} z = \operatorname{sn}(z, k)$, and we denote the elliptic function of complementary modulus $\operatorname{sn}(z, \sqrt{1-k^2})$ by upper case notation $\operatorname{Sn} z$. We can define functions Cn and Dn, related by the same rule to cn and dn, and it can be shown [5], p. 394 that K' is the real quarter period for the functions of complementary modulus while K is their imaginary quarter period. It is not entirely surprising then that from [9], p. 344 we get

$$\operatorname{sn}(iz) = i\frac{\operatorname{Sn} z}{\operatorname{Cn} z}; \quad \operatorname{cn}(iz) = \frac{1}{\operatorname{Cn} z}; \quad \operatorname{dn}(iz) = \frac{\operatorname{Dn} z}{\operatorname{Cn} z}.$$

These formulae are known collectively as Jacobi's imaginary transformation for elliptic functions.

Another well known property is the degenerate behaviour of Jacobi elliptic functions when the periods tend to infinity [9], p. 354. This can be summarised as follows where we determine the constants using the expansion [9], p. 344

(11) for small values of z. When the real quarter period K is infinite and the imaginary quarter period K' finite we have

$$\operatorname{sn} z = \frac{2K'}{\pi} \tanh \frac{\pi z}{2K'}; \quad \operatorname{cn} z = \operatorname{sech} \frac{\pi z}{2K'}; \quad \operatorname{dn} z = \operatorname{sech} \frac{\pi z}{2K'}.$$

Conversely when the real quarter period K is finite with imaginary quarter period K' infinite the degenerate forms are

$$\operatorname{sn} z = \frac{2K}{\pi} \sin \frac{\pi z}{2K}; \quad \operatorname{cn} z = \cos \frac{\pi z}{2K}; \quad \operatorname{dn} z = 1.$$

These are extremely useful for checking algebraic identities.

Next we have a look at the formula

$$\int_{-K}^{k} \cos \frac{\pi n x}{K} \log|\operatorname{sn} x|\, dx = -\frac{K}{n} \tanh \frac{n \pi K'}{2K}$$

(the case $n=1$ appears in the Mathematical Tripos of 1902). To prove this consider the contour integral

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{\operatorname{dn} z \operatorname{cn} z}{\operatorname{sn} z} e^{i\pi n z/K} dz$$

where Γ is the rectangular contour with corners $\pm K$, $\pm K + i\infty$. But using [9], p. 350 and Jacobi's imaginary transformation

$$\frac{\operatorname{dn} \operatorname{cn}}{\operatorname{sn}} (i y \pm K) = -(k')^2 i \frac{\operatorname{Sn} y \operatorname{Cn} y}{\operatorname{Dn} y}$$

so if we apply the residue theorem then the integrals along the infinite lines cancel. Using integration by parts and $\operatorname{sn}(\pm K) = \pm 1$ we see the contribution along the real axis gives us

$$\frac{1}{2\pi i} \int_{-K}^{K} \frac{\operatorname{dn} x \operatorname{cn} x}{\operatorname{sn} x} e^{i\pi n x/K} dx = -\frac{n}{2K} \int_{-K}^{K} \cos \frac{n \pi x}{K} \log|\operatorname{sn}|\, x\, dx.$$

It remains to calculate the residues at the poles of the integrand inside Γ. Some of these occur at the zeros of sn, which by [24], p. 504, are the points $2miK'$ and where we know that $\operatorname{cn}(2miK') = \operatorname{dn}(2miK') = (-1)^m$. The residues for these are therefore $e^{-2mn\pi K'/K}$, with half this for the contribution from the origin. The other singularities of the integrand occur at the poles $(2m+1)iK'$ of cn, because the poles of dn cancel with those of sn, and we compute the residues here to be $-e^{-(2m+1)n\pi K'/K}$ by using [9], p. 341. That's all that's needed to verify our claim.

By doing a similar argument one can prove that

$$\int_{-K}^{K} \sin \frac{\pi x}{K} \theta_n \frac{\operatorname{dn} x}{\operatorname{sn} x} dx = \pi \tanh \frac{\pi K'}{K} \theta_n \qquad (\theta_n = n + \tfrac{1}{2}).$$

The contour for this calculation is the same as before and we use the integrand

$$\int_{\Gamma} \frac{\operatorname{dn} z}{\operatorname{sn} z} e^{i\pi\theta_n z/K} dz$$

with the information that

$$\frac{\operatorname{dn}}{\operatorname{sn}}(i y \pm K) = \pm k' \operatorname{Cn} y,$$

which gives us the cancellation of the infinite integrals and calculating the residues as before. Notice how the poles are now just at those zeros of $\operatorname{sn} z$ which lie inside Γ, namely the points $2 m i K'$.

We also promised to do the direct computation of $\tilde{\Upsilon}(-w^2/2, x)$ in the case of examples 3 and 4 above. For example 3 we begin with the kernel

$$\frac{1}{l}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cosh \dfrac{n\pi(p-x)\delta}{l}}{\cosh \dfrac{n\pi\delta p}{l}} \cos \frac{n\pi y}{l}\right] dy.$$

However if we write $\cosh \frac{n\pi\delta(p-x)}{l} \cos \frac{n\pi y}{l}$ as the real part of $\cos \frac{n\pi}{l}(i\delta[p-x]+y)$ then we have the real part of

$$\frac{1}{l}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2q^n}{1+q^{2n}} \cos \frac{n\pi}{l}(i\delta[p-x]+y)\right] dy$$

where $q = e^{-\pi\delta p/l}$. From [12], p. 911, this is the real part of $\frac{1}{\pi} \operatorname{Dn}(i\delta[p-x]+y)$ provided $p < +\infty$. But by [9], p. 350, this is just

$$\frac{1}{\pi} \Re\left[-i \frac{\operatorname{Cn}}{\operatorname{Sn}}(y - i\delta x)\right] dy$$

which we can write, using Jacobi's imaginary transformation and the addition formula for sn, as

$$\begin{aligned}
\frac{1}{\pi} \Re\left[\frac{dy}{\operatorname{sn}(\delta p - i y)}\right] &= \frac{1}{\pi} \Re\left[\frac{(1 - k^2 \operatorname{sn}^2 \delta x \operatorname{sn}^2 i y)\, dy}{\operatorname{sn} \delta x \operatorname{cn} i y \operatorname{dn} i y - \operatorname{cn} \delta x \operatorname{dn} \delta x \operatorname{sn} i y}\right] \\
&= \frac{1}{\pi} \frac{\operatorname{sn} \delta x \operatorname{cn} i y \operatorname{dn} i y [1 - k^2 \operatorname{sn}^2 \delta x \operatorname{sn}^2 i y]\, dy}{\operatorname{sn}^2 \delta x \operatorname{cn}^2 i y \operatorname{dn}^2 i y - \operatorname{cn}^2 \delta x \operatorname{dn}^2 \delta x \operatorname{sn}^2 i y} \\
&= \frac{1}{\pi} \frac{\operatorname{sn} \delta x \operatorname{cn} i y \operatorname{dn} i y\, dy}{\operatorname{sn}^2 \delta x - \operatorname{sn}^2 i y} = -\frac{i}{\pi} \frac{\operatorname{sn} \delta x\, d(\operatorname{sn} i y)}{\operatorname{sn}^2 \delta x - \operatorname{sn}^2 i y}
\end{aligned}$$

as required.

The computation for example 4 is much the same. We have to sum

$$\frac{1}{l}\left[\sum_{n=0}^{\infty}\frac{\sinh\theta_n\dfrac{\pi\delta(p-x)}{l}}{\sinh\theta_n\dfrac{\pi\delta p}{l}}\cos\theta_n\frac{\pi y}{l}\right]dy.$$

However if we write $\sinh\theta_n\dfrac{\pi\delta(p-x)}{l}\cos\theta_n\dfrac{\pi y}{l}$ as the imaginary part of $\sin\theta_n\dfrac{\pi}{l}(i\delta[p-x]+y)$ then we obtain

$$\mathfrak{J}\left[\frac{1}{l}\sum_{n=0}^{\infty}\frac{2q^{n+\frac{1}{2}}}{1-q^{2n+1}}\sin\theta_n\frac{\pi}{l}(i\delta[p-x]+y)\right]dy.$$

But from [12] p. 911 this is the imaginary part of $\dfrac{k'}{\pi}\operatorname{Sn}(i\delta[p-x]+y)$ and using the addition formulae for elliptic functions we obtain

$$\frac{-ik'}{\pi}\frac{\operatorname{Cn}y\operatorname{Dn}y\operatorname{Sn}i\delta(p-x)}{1-(k')^2\operatorname{Sn}^2y\operatorname{Sn}^2i\delta(p-x)}=\frac{-i}{\pi}\left[\frac{\operatorname{Cn}y\operatorname{Dn}y\operatorname{Sn}i\delta x}{\operatorname{Sn}^2i\delta x+\operatorname{Sn}^2y}\right].$$

Using Jacobi's imaginary transformation this comes out to be

$$\frac{1}{\pi}\left[\frac{\operatorname{sn}\delta x\operatorname{cn}\delta x\, d(\operatorname{Sn}y)}{\operatorname{sn}^2\delta x+\operatorname{Sn}^2y\operatorname{cn}^2\delta x}\right]$$

as we claimed above.

Finally we come to the problem of verifying the condition at 3.1. Recall that this is

$$\frac{G(y_1+y_2)}{G(y_1-y_2)}=\frac{f(y_2)+f(y_1)}{f(y_2)-f(y_1)}$$

and we need to verify it for the functions f encountered above, namely $\sin z$, $\sinh z$, $\operatorname{sn}z$ and $\operatorname{sn}z/\operatorname{dn}z$. Because the first two are degenerate cases of the third and fourth respectively we only have to check our condition on these, and replacing $z\to iz$ it suffices by Jacobi's imaginary transformation to check 3.1 only for the functions $\operatorname{Sn}z/\operatorname{Cn}z$ and $\operatorname{sn}z/\operatorname{dn}z$. Recall again our convention that $\widehat{\operatorname{Sn}}\,z$ is the Jacobi elliptic function with the same imaginary period as $\operatorname{Sn}z$ but whose real period is doubled.

Case 1. Suppose $f=\operatorname{Sn}/\operatorname{Cn}$. In this case, by using the alternative expression (see example 4.3)

$$f(z)=\frac{\widehat{\operatorname{Sn}}\,z}{\widehat{\operatorname{Cn}}\,z\,\widehat{\operatorname{Dn}}\,z}$$

we see immediately that $G=\widehat{\operatorname{Sn}}$. Condition 3.1 now follows by using the addition formula for $\widehat{\operatorname{Sn}}$.

Case 2. Consider $f=\mathrm{sn}/\mathrm{dn}$. In this case we know [12] p. 630 that $G=\sqrt{(1-\mathrm{cn})(1+\mathrm{cn})^{-1}}$ but the verifying 3.1 is not so easy. If we use the convention that $\mathrm{sn}\, y_i = s_i$ etc. then we need to check the identity

$$\frac{1-\mathrm{cn}(y_2+y_1)}{1+\mathrm{cn}(y_2+y_1)}\,\frac{1+\mathrm{cn}(y_2-y_1)}{1-\mathrm{cn}(y_2-y_1)}=\left(\frac{s_1\,d_2+s_2\,d_1}{s_1\,d_2-s_2\,d_1}\right)^2.$$

For this expand $\mathrm{cn}(y_2 \pm y_1)$ by using the addition formula [9] p. 344 whereupon the lhs gives us

$$\frac{(1-k^2\,s_1^2\,s_2^2+s_1\,s_2\,d_1\,d_2)^2-c_1^2\,c_2^2}{(1-k^2\,s_1^2\,s_2^2-s_1\,s_2\,d_1\,d_2)^2-c_1^2\,c_2^2}.$$

Expanding, using the identities $d_i^2=1-k^2\,s_i^2$, $c_i^2=1-s_i^2$, we find that the top line comes out to be

$$(s_1^2\,d_2^2+s_2^2\,d_1^2+2\,s_1\,d_1\,s_2\,d_2)\,(1-k^2\,s_1^2\,s_2^2),$$

so by symmetry the result is clear.

Remark. The period parallelogram is rectangular if and only if the elliptic modulus $k\in(0,1)$, and we have restricted ourselves to this case since it suffices for the examples considered above. However, when checking the condition 3.1 we only needed the addition formulae for elliptic functions and these are true in full generality. Thus we have found a pair of one parameter families of functions satisfying 3.1 where the parameter k^2 takes its values in the complex plane cut along $(-\infty, 0)$ and $(1, +\infty)$ (see [5] p. 390 for the full story on this). But we were unable to find Brownian motion examples giving rise to elliptic functions with a skew period parallelogram.

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