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Do Minimal Solutions of Heat Equations Characterize Diffusions?

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Summary. Brownian motion may be characterized as a process which, when composed with minimal parabolic functions, gives martingales. This note explores the extent to which this is true in general. For the diffusion associated with the Kohn Laplacian on the Heisenberg group it is shown to be false.

Introduction

In [13] it was shown that for Brownian motion on a non-compact symmetric space X the minimal solutions u of the heat equation associated with the Laplace-Beltrami operator determine martingales which characterize the process. More explicitly, if a process $(X_t)_{t\geqq 0}$ on X is such that $u(X_t, t)$ is always a martingale of expectation one whenever u is a minimal solution, then $(X_t)_{t\geqq 0}$ on $\mathbb{R}^d$ is a Brownian motion started from the origin if the processes $(\exp\{-\|y\|^2(t/2)+\langle y, X_t\rangle\})_{t\geqq 0}$, $y\in\mathbb{R}^d$, are all martingales of expectation one.

This raises the question as to when this type of result is valid. In this article, while no general answer is given, several examples are discussed for which it holds and one where it does not.

It is shown for a reasonable elliptic operator L defined on an open set U in $\mathbb{R}^d$ that for a minimal solution u of the equation $Lu+u_t=0$ on $U\times\mathbb{R}$, the process $u(X_t, t)$ is a martingale if and only if u has no zeros (here $(X_t)_{t\geqq 0}$ is the associated diffusion killed when it hits the boundary). In the case of Brownian motion on $\mathbb{R}^d\times(0,\infty)$ killed when it hits 0 it is shown that the corresponding minimal solutions without zeros on $\mathbb{R}^d\times(0,\infty)\times\mathbb{R}$ characterize the process in the above sense.

This also suggests a similar characterization of reflecting Brownian motion on $\mathbb{R}^d\times[0,\infty)$. These characterizations also hold for a simple random walk on $\mathbb{Z}^d$.

In the case of the non-compact symmetric space X a minimal solution of the heat equation factors as a product of a non-negative eigenfunction and

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an exponential in t. As is known [5], this is false for the Ornstein-Uhlenbeck process on $\mathbb{R}^d$. However, it is still true that the minimal solutions characterize the process in the same way (Theorem 5.1).

The article concludes with a counterexample. Let $\mathbb{H}^n$ denote the Heisenberg group $\mathbb{C}^n \times \mathbb{R}$ and let L denote the Kohn Laplacian on $\mathbb{H}^n$. The minimal solutions of the equation $Lu+u_t=0$ are determined. As they are of the form $e^{ct} f(g)$, where $Lf=cf$ and $f \geqq 0$ is minimal, this amounts to determining the functions f. It is shown that they depend only on $\mathbb{C}^n$ and are the minimal eigenfunctions for the equation $\Delta f=cf$ on $\mathbb{C}^n$, where Δ is the usual Laplacian. As a result, the resulting martingales cannot characterize the associated diffusion. This has nothing to do with the fact that L is degenerate. It still happens if L is replaced by $L+\varepsilon\,\partial^2/\partial t^2$.

Consequently, it looks as though the fact that in euclidean coordinates $L+\varepsilon^2\,\partial^2/\partial t^2$ is not uniformly elliptic is the reason for the failure of the theorem. However, a recent result of Alexopoulos [1] also suggests that it is due to the nilpotence of the group $\mathbb{H}^n$. It is an open question as to whether the theorem is valid for reasonable uniformly elliptic operators on $\mathbb{R}^d$.

1. Martingales and Minimal Parabolic Functions: A Problem

First some well known facts about solutions of parabolic equations are recalled. Consider the parabolic operator $L^\# = L+\partial/\partial t$ defined on functions $u \in \mathfrak{C}^{2,1}(U)$, U any open set $\subset \mathbb{R}^d \times \mathbb{R}$, by

$$L^\# u = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x,t)\, D_i D_j u + \sum_{i=1}^{n} b_i(x,t)\, D_i u + u_t, \qquad \text{where } D_i = \partial/\partial x_i.$$

For simplicity's sake assume the coefficients to be bounded Lipschitz and the matrix (a_{ij}) to be uniformly elliptic.

Definition 1.1. A *box* B in U is a set of the form $B=D\times(a,b)$, where D is an open ball. The *parabolic boundary* of B, denoted by $\partial^* B$, is $D\times\{b\} \cup \partial D \times [a,b]$. The *lower boundary* of B is $D\times\{a\}$.

It is well known (cf. Freedman [6]) that the first boundary value problem for any box B has a unique solution which is non-negative when the boundary value is $\geqq 0$. In other words to any continuous function ϕ on the parabolic boundary there is a unique continuous extension $u=u_\phi$ to the box for which $L^\# u=0$.

Therefore, to each $y\in B$ there is a probability measure μ_y on the parabolic boundary such that $u_\phi(y)=\int \phi\, d\mu_y$. This "harmonic measure" is the exit law for the box of the continuous Markov process with generator $L^\#$ started at y.

If $y\in U$ let $B(y)$ be the family of boxes in U for which y is in the lower boundary.

Proposition 1.2. *A continuous function f on U is in $\mathscr{C}^{2,1}(U)$ and satisfies $L^{\#}u=0$ if and only if for all $y\in U$ and all boxes B in $B(y)$ that are $\subset U$, $f(y)=\int f\,d\mu_y$, where μ_y is the harmonic measure for B associated with $y\in B$.*

Proof. Fix a box $B=D\times(a,b)$ and let u be the solution of $L^{\#}u=0$ on B with boundary value $\phi=f\,|\,\partial^* B$. Consider $g=f\text{-}u$ on B. Let $y=(x,t)\in B$ and choose an increasing sequence of boxes $B_n=D_n\times(b_n,t)$ with $D=\bigcup D_n$ and $b_n\uparrow b$. Then $g(y)=\int g\,d\mu_{n,y}=\int g\,d\mu_y$ since the exit law for D_n converges weakly to that of D. Hence, $g(y)=0$.

Now assume that $U=0\times(a,c)$, 0 a domain $\subset\mathbb{R}^d$. Consider the diffusion associated with $L^{\#}$ killed when it first leaves U (cf. [12] for the diffusion on $\mathbb{R}^d$). For any $y=(x,t)$ let $E^{(x,t)}$ denote expectation with respect to this diffusion started from $y=(x,t)$. The random position of the diffusing particle at time t will be denoted by (X_t,t). Let $(\mathfrak{F}_s)_{a<s<c}$ be an appropriate filtration on the underlying probability space $(\Omega,\mathfrak{F},P)$.

Corollary 1.3. *Assume that $u\geqq 0$ and that $L^{\#}u=0$ on $O\times(a,c)$. Let $a<b<c$ and define $v(x,t)=u(x,t)$ for $b\leqq t<c$ and $=E^{(x,t)}[u(X_b,b)]$ for $a<t<b$. Then $L^{\#}v=0$.*

Proof. To begin with v is continuous. Further, for any box B in $B(y)$, $y=(x,t)$ that is $\subset O\times(a,c)$, the Strong Markov property implies that $E^{(x,t)}[v(X_T,T)]=v(y)$, where T is the first exit time from B.

Theorem 1.4. *Assume $u\geqq 0$ and that $L^{\#}u=0$ on $O\times(a,c)$. Let u be a minimal solution (i.e., if $0\leqq v\leqq u$ and v is a solution then $v=\lambda u$). Let $a<t_0<c$. Define the process $M=(M_t)_{t_0\leqq t<c}$, where $M_t=u(X_t,t)$. Then, for any $x_0\in O$, M is a $P^{(x_0,t_0)}$-martingale with non-zero expectation if and only if u has no zeros.*

Proof. Assume that u has no zeros. Since u is minimal, it follows from the previous corollary that for any $s<t$, $a<t_0<s<t<c$, $u(X_s,s)=E^{(X_s,s)}u(X_t,t)$. By the Markov property this is $E^{(x_0,t_0)}[u(X_t,t)\,|\,\mathfrak{F}_s]$.

Proposition 1.2 implies that if u vanishes at $y_1=(x_1,t_1)$ then u vanishes on $O\times[t_1,c)$. As a result, $M_t=0$ for $t_1\leqq t<c$.

Now assume that the coefficients a_{ij} and b_i do not depend on t.

Problem 1.5. Let (X_t) be a continuous Markov process on X. Let $u\geqq 0$ be a minimal solution of $L^{\#}u=0$ on $X\times(a,c)$ without zeros and let $a\leqq b<c$. Denote by $M(u)$ the process $(M_t)_{b\leqq t<c}$ where $M_t=u(X_t,t)$. *Is (X_t) equivalent to the diffusion on X with generator L, started from x_0, if M is a martingale with expectation $u(x_0,b)$ for all minimal u?*

When $X=\mathbb{R}^n$, $(a,c)=\mathbb{R}$ and $b=0$ the answer is affirmative for Brownian motion as was pointed out in the introduction. If X is a non-compact symmetric space and L is the Laplace-Beltrami operator the answer is again affirmative [13].

In what follows it will be shown that for Brownian motions on $\mathbb{R}^+$ and for Ornstein-Uhlenbeck processes the answer is affirmative.

However, for invariant diffusions on the Heisenberg group the answer is negative. The reason for this seems to be due to a lack of uniform ellipticity and/or the nilpotence of the group.

All processes will be defined on a probability space $(\Omega,\mathfrak{F},P)$ equipped with an increasing filtration $(\mathfrak{F}_t)_{t\geqq 0}$.

2. Brownian Motion on $\mathbb{R}^d \times \mathbb{R}^+$ Killed at Zero

A list of the minimal solutions of the heat equation $\Delta u + 2u_t = 0$ on $\mathbb{R}^d \times \mathbb{R}^+$ is given in [10] (see Corollary 3.5):

(1) To each $Y \in \mathbb{R}^d \times \mathbb{R}^+$ there corresponds the zero-free function

$$K(Y; u, t) = \{\exp - \|Y\|^2 t/2\} \{\exp\langle u, Y\rangle - \exp\langle u, \bar{Y}\rangle\},$$

where $\bar{Y} = (y, -a)$ if $Y = (y, a)$ with $y \in \mathbb{R}^d$. If $u = (x', x)$ then

$$K(Y; u, t) = \{\exp - \|y\|^2 (t/2) + \langle x', y\rangle\} \{\exp - a^2 t/2\} \, 2 \sinh a x.$$

(2) Furthermore, to each $y \in \mathbb{R}^d$, there corresponds the zero-free function

$$K(y; u, t) = x \exp\{-\|y\|^2 (t/2) + \langle x', y\rangle\}.$$

Remark 2.1. Note that $K(y; u, t)$ is the derivative in a at $a=0$ of $K(Y; u, t) = \{\exp - \|y\|^2 (t/2) + \langle x', y\rangle\} \{\exp - a^2 t/2\} \, 2 \sinh a x$. Therefore if $K(Y; U_t, t)$ is a martingale for all $Y = (y, a)$, where $U_t = (X'_t, X_t)$, the fact $\sinh 2y = 2 \sinh y \cosh y$ implies that $\{\exp - \|y\|^2 (t/2) + \langle X'_t, y\rangle\} \times \{\exp - a^2 t/2\} X_t \cosh a X_t \in L^1$. Since $\sinh a x / a \leqq x \cosh a x$ it follows that $(\partial/\partial a)(K(y, a; U_t, t))|_{a=0}$ is also a martingale. Consequently, the functions in the second list produce martingales whenever those in the first list do so.

The analogue in this context of the key Proposition 3.4 in [13] is the following result.

Proposition 2.1. *Let $u = (x', x)$ denote a generic point of $\mathbb{R}^d \times \mathbb{R}^+$. There is a unique probability μ_t on $\mathbb{R}^d \times \mathbb{R}^+$ such that, for all $y \in \mathbb{R}^d$ and $a > 0$,*

$$\int \exp\langle x', y\rangle \sinh(a x) \, \mu_t(du) = \exp\{\langle x'_0, y\rangle + (\|y\|^2 + a^2) t/2\} \sinh(a x_0). \quad (*)$$

Further, this probability is the law of Brownian motion on $\mathbb{R}^d \times \mathbb{R}^+$ at time t when started from x_0 and killed when it hits $x = 0$.

Proof. To each probability v on $\mathbb{R}^d \times \mathbb{R}^+$ there corresponds a unique signed measure η on $\mathbb{R}^d \times \mathbb{R}$ such that (i) η agrees with v on $\mathbb{R}^d \times \mathbb{R}^+$ and (ii) η is "odd" i.e. $J * \eta = -\eta$ where $J(x', x) = (x', -x)$.

Let $\tau(y, a) = \int \exp\langle x', y\rangle \sinh(a x) \, v(du)$. Then τ can be extended analytically to $\mathbb{C}^n \times \mathbb{C}$. Hence, for all $z \in \mathbb{C}$ and $\zeta \in \mathbb{C}^n$, if v satisfies $(*)$,

$$\int \exp\langle x', \zeta\rangle \sinh(z x) \, v(du) = \exp\{\langle x'_0, \zeta\rangle + (\|\zeta\|^2 + z^2) t/2\} \sinh(a x_0),$$

where $\langle x', \zeta\rangle = \sum_{i=1}^{n} x'_i \zeta_i$ and $\|\zeta\|^2 = \sum_{i=1}^{n} \zeta_i^2$. As a result, the Fourier transform of the signed measure η is $\exp\{i\langle x'_0, y\rangle - (\|y\|^2 + a^2) t/2\} \sin(a x)$.

Since for killed Brownian motion $(B_t)_{t\geqq 0}$, $K(Y; B_t, t)=\{\exp - \|y\|^2 (t/2) + \langle b'_t, y\rangle\}\{\exp - a^2 t/2\}\, 2\sinh a\, b_t$ is a martingale, where $B_t=(b'_t, b_t)\in\mathbb{R}^d\times\mathbb{R}^+$, it follows that the law μ_t of B_t satisfies (*).

As a fairly immediate corollary of this characterization of probabilities one has

Proposition 2.2. *Let $(X_t)_{t\geqq 0}$ be a process on $\mathbb{R}^d\times\mathbb{R}^+$, where $X_t=(x'_t, x_t)$. It is equivalent to Brownian motion $B=(B_t)_{t\geqq 0}$ started from $X_0=(x'_0, x_0)$ and killed when x_t first equals zero if and only if, for all $a>0$ and $y\in\mathbb{R}^d$, the process $M^{a,y}=(M_t^{a,y})_{t\geqq 0}$, where $M_t^{a,y}=\{\exp - \|y\|^2 (t/2)+\langle x'_t, y\rangle\}\{\exp - a^2 t/2\}\, 2\sinh a\, x_t$, is a martingale of expectation $\{\exp\langle x'_0, y\rangle\}\, 2\sinh a\, x_0$.*

Proof. To begin with, it follows from the martingale condition that the distribution of X_t is the same as that of B_t.

To determine the finite-dimensional joint distributions let $0<t(1)<t(2)<\ldots<t(k)=s<t$ and let v be the joint distribution of $(X_0, X_{t(1)}, X_{t(2)}, \ldots, X_s, X_t)$. Denote by $\pi(v, du)$ a conditional distribution of v given v, where $(v, u)\in\{\mathbb{R}^d\times\mathbb{R}^+\}^d\times\{\mathbb{R}^d\times\mathbb{R}^+\}$. The martingale condition implies that for $v=v(\omega)=(X_0, X_{t(1)}, X_{t(2)}, \ldots, X_s)$,

$$\int \pi(v, du)\exp\langle x', y\rangle \sinh(a\,x)=\exp\{(\|y\|^2+a^2)(t-s)/2\}\exp\{\langle X'_s, y\rangle \sinh(a\,X_s)$$

P-a.s. From Proposition 2.1 it follows that P-a.s., for $v=(X_0, X_{t(1)}, X_{t(2)}, \ldots, X_s)$, $\pi(v, du)$ is the law of killed Brownian motion at time $t-s$ started from $X_s(\omega)$. Since $v(dv, du)=\mu(dv)\,\pi(v, du)$ this implies that one may compute the finite-dimensional distributions by induction.

Remark 2.3. The proof of Proposition 2.1 uses the well-known fact that two probabilities v and v' agree on $\mathbb{R}^d\times\mathbb{R}^+$ if and only if, for all $y\in\mathbb{R}^d$ and $a>0$,

$$\int \exp\langle x', i\,y\rangle \sin(a\,x)\, v(du)=\int \exp\langle x', i\,y\rangle \sin(ax)\, v'(du).$$

Using this fact, one may determine the finite-dimensional joint distributions without using regular conditional probabilities (the following argument was pointed out to the author by C.S. Herz). Let $0<t(1)<t(2), \ldots<t(k)=s<t$ and let f_i, $1\leqq i\leqq k$ and $f\in\mathfrak{C}_b^\infty(\mathbb{R}^d\times\mathbb{R}^+)$. Define $\int f\,dv=E[\prod_{1\leqq i\leqq k} f_i(X_{t(i)})\,f(X_t)]$ and define $\int f\,dv'$ to be the corresponding integral for killed Brownian motion B. Now $\int f\,dv$ and $\int f\,dv'$ may be computed by first conditioning on $\mathfrak{F}_s$ and then integrating. Doing this with $f(u)=\exp\langle x', i\,y\rangle \sin(a\,x)$ and using the martingale property reduces the computation of $\int f\,dv$ and of $\int f\,dv'$ to an expectation of functions of $X_{t(i)}$, $1\leqq i\leqq k$. By an obvious induction on k these expectations coincide and so $v=v'$.

3. Reflected Brownian Motion on $\mathbb{R}^d\times[0, +\infty)$

Let v be a fixed vector of length one in $\mathbb{R}^{d+1}$ with $v_{d+1}>0$. There is a unique linear R: $\mathbb{R}^{d+1}\to\mathbb{R}^{d+1}$ which leaves $\mathbb{R}^d=\mathbb{R}^d\times\{0\}\subset\mathbb{R}^{d+1}$ pointwise fixed and maps v to $-v$ transformation (the reflection of $\mathbb{R}^{d+1}$ in $\mathbb{R}^d$ along v).

An obliquely reflected Brownian motion on $\mathbb{R}^d \times [0, \infty)$ may be obtained by starting a Brownian motion on $\mathbb{R}^{d+1}$ at a point x_0 whose $d+1$-st co-ordinate is >0 and mapping it onto $\mathbb{R}^d \times [0, \infty)$ by S, where $S(x)=x$ if the $d+1-$st co-ordinate of x is $\geqq 0$ and $S(x)=R(x)$ otherwise. In the case of killed Brownian motion the solutions of the heat equation that are of interest are those that are "odd" with respect to the normal reflection. By analogy, here one considers those solutions u that are "even" i.e. $u(x,t)=u(R(x)\,t)$ for all $x\in\mathbb{R}^{d+1}$. The important ones are again minimal: namely the functions

$$k(y; x, t)=\{\exp - \|y\|^2 t/2\} \{\exp\langle y, x\rangle + \exp\langle {}^tR(y), x\rangle\}, \qquad \text{for } y\in\mathbb{R}^d \times [0, \infty).$$

A probability v on $\mathbb{R}^d \times (0, +\infty)$ determines a unique measure on $\mathbb{R}^{d+1}$ that is invariant under reflection along v. Consequently, v is determined by the transform $\tilde{v}(y)=\int \{\exp\langle i\,y, x\rangle + \exp\langle i^t R(y), x\rangle\}\, v(dx)$, $y\in\mathbb{R}^{d+1}$.

Given this information it is not hard to copy the proof of Proposition 2.2 to get a proof for

Proposition 3.1. *Let $(X_t)_{t\geqq 0}$ be a process on $\mathbb{R}^d \times [0, +\infty)$. Assume that, for each $t\geqq 0$, $P(\{X_t\in\mathbb{R}^d\})=0$. Then the following are equivalent:*

(1) *the process is equivalent to Brownian motion started from x_0 and reflected obliquely at $\mathbb{R}^d \times \{0\}$ by the vector v;*

(2) *for all $y\in\mathbb{R}^d \times [0, \infty)$, the processes*

$$(\{\exp - \|y\|^2 t/2\} \{\exp\langle y, X_t\rangle + \exp\langle {}^tR(y), X_t\rangle\})_{t\geqq 0}$$

are all martingales of expectation $\{\exp\langle y, x_0\rangle + \exp\langle {}^tR(y), x_0\rangle\}$.

Remark 3.2. The condition that the process at any time t is not on the boundary ensures that the joint distributions are carried by the product of copies of $\mathbb{R}^d \times (0, +\infty)$ and as a result all the measures involved can be determined by their transforms.

4. Remarks on Simple Random Walks

The above simple computations for Brownian motion raise the obvious question as to whether they can be carried out for random walks. In the case of simple random walks on $\mathbb{Z}^d$ it is relatively straightforward to compute the minimal solutions of the associated space-time walk on $\mathbb{Z}^{d+1}$. As in the Brownian motion case they give rise to martingales which characterize the original walk. Details are given in [1] for this and more general random walks. Killing and reflection at $\mathbb{Z}^{d-1} \times \{0\}$ can be handled as in Sect. 3.

5. A Characterization of the Ornstein-Uhlenbeck Process

Consider the generator $Lu(x)=\Delta u(x)-\langle x, \nabla u(x)\rangle$ of the Ornstein-Uhlenbeck process on $\mathbb{R}^d$. Using a scaling in x by $\sqrt{2}$ and the computations in [5] the minimal solutions of the equation $Lu+u_t=0$ on $\mathbb{R}^d \times \mathbb{R}$ are easily computed.

One may also simply calculate by Martin's method starting from the formula for the fundamental solution $G(x,t;y,s)=P_{t-s}(x,y)$ if $s<t$ and $=0$ otherwise, where

$$P_t(x,y)=[1/2\pi(1-e^{-2t})]^{d/2}\exp\{(-1/2(1-e^{-2t}))\,\|e^{-t}x+y\|^2\}.$$

They are the functions $K(y;x,t)=\exp\{-(e^{2t}-1)\,\|y\|^2+\sqrt{2}e^t\langle y,x\rangle\}$, where $y\in\mathbb{R}^d$.

Since they have no zeros, by Theorem 1.5 they give rise to martingales when composed with the space-time process corresponding to the Ornstein-Uhlenbeck process. As a consequence so does every non-negative solution of $Lu+u_t=0$ on $\mathbb{R}^d\times\mathbb{R}$. These martingales characterize the process as stated below.

Theorem 5.1. *Let $X=(X_t)_{t\geqq 0}$ be a process on $\mathbb{R}^d$ and let*

$$M_t^y=K(y;X_t,t)=\exp\{-(e^{2t}-1)\,\|y\|^2+\sqrt{2}e^t\langle y,X_t\rangle\}.$$

The process X is equivalent to an Ornstein-Uhlenbeck process started from $x_0\in\mathbb{R}^d$ if and only if, for all $y\in\mathbb{R}^d$, the process $M^y=(M_t^y)_{t\geqq 0}$ is a martingale with expectation $\exp\sqrt{2}\langle y,x_0\rangle$.

Proof. It remains to show that if the processes M^y are all martingales, then X is as stated.

To begin with, if $y_t=e^tX_t$, the process $(y_t)_{t\geqq 0}$ has independent increments since $E[\exp\{\sqrt{2}\langle y,y_t-y_s\rangle\}\,|\,\mathfrak{F}_s]=\exp\{\|y\|^2(e^{2t}-e^{2s})\}$ if $s<t$. Hence, X is a Markov process.

The value of the expectation of the martingale implies that the Laplace transform of y_t is $E[\exp\langle u,Y_t\rangle]=\exp\{(e^{2t}-1)\,\|u\|^2/2+\langle u,x_0\rangle\}$. It is then a routine calculation to show that the distribution of X_t is $P_t(x_0,x)\,dx$.

Remark 5.2. Replacing the generator L by L^+, where $L^+u(x)=\Delta u(x)+\langle x,\nabla u(x)\rangle$, the space-time process associated with $L^+\,\partial/\partial t$ is obtained from the Ornstein-Uhlenbeck process by conditioning by $h(x,t)=\exp\{(-dt/2)+\|x\|^2\}$. The minimal functions are then of the form $hK(y;\,)$. Given a process for which they give rise to martingales one may recognize it as a diffusion with generator L^+ by "unconditioning" with the aid of the martingale determined by h (i.e. for $y=0$) and observing that with the resulting change of probability one is then looking at a diffusion with generator L.

6. Brownian Motion on the Heisenberg Group

Let $\mathbb{H}^n=\mathbb{C}^n\times\mathbb{R}$ denote the Heisenberg group, where $(z,t)\cdot(w,s)=(z+w,t+s+2\operatorname{Im} z\,w^-)$. It is an easy computation to show that the vector fields

$$X_k=\partial/\partial x_k+2y_k\,\partial/\partial t \quad\text{and}\quad Y_k=\partial/\partial y_k-2x_k\,\partial/\partial t$$

are left invariant and satisfy $[X_k, Y_k] = -4T = -4\partial/\partial t$, for $1 \leqq k \leqq n$. The operator $\Delta_k = \sum_{k=1}^{n} [X_k^2 + Y_k^2]$ is called the Kohn Laplacian. It is a sub-elliptic operator on $\mathbb{H}^n$ and because the vector fields X_k and Y_k, $1 \leqq k \leqq n$, satisfy Hörmander's condition it behaves in many respects as though it were elliptic. In particular, one may carry out the usual arguments of potential theory with it (cf. [3]). This second order operator and the related operators $\Delta_{k,\varepsilon} = \Delta_k + \varepsilon T^2$, $\varepsilon \geqq 0$, are all left invariant on $\mathbb{H}^n$ and it is reasonable to expect that all the standard properties of classical potential theory on $\mathbb{R}^n$ should carry over to them.

It will now be shown that in one respect this is false, namely there are not enough solutions of the associated parabolic equations for the corresponding martingales to characterize the diffusions. What happens is that in each case a non-negative global solution u of the parabolic equation depends only on $z \in \mathbb{C}^n$ and the time variable. In other words, $\Delta_{k,\varepsilon} u + u_t = 0$ and $u \geqq 0$ implies that $u = v \circ \pi$, where $\pi: \mathbb{C}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}^n \times \mathbb{R}$ is the projection $\pi(z, s, t) = (z, t)$ and $\Delta v + 4v_t = 0$.

To begin with assume $\varepsilon = 0$. The sheaf of solutions of the equation $\Delta_k u + u_t = 0$ satisfies Doob's axioms of abstract potential theory (cf. [2]). This means roughly that the potential theory associated with the heat equation on $\mathbb{R}^n$ applies here. In particular, Moser's form of the Harnack Inequality is valid as will now be shown.

Proposition 6.1 (Harnack's Inequality). *Let $h \geqq 0$ be a solution of the equation $\Delta_k u + u_t = 0$. Let compact $K \subset \{(g,t) \mid t > t_0\}$. There is a constant $C(K, (g_0, t_0))$ such that*

$$h(g,t) \leqq C h(g_0, t_0) \qquad \textit{for all } (g,t) \in K.$$

Proof. In view of Satz 1.4.4 of [2], it will suffice to show that $\{(g,t) \mid t \geqq t_0\}$ is the smallest absorbing set containing (g_0, t_0). This is so if for a non-negative hyperharmonic function u on $\mathbb{H}^n \times \mathbb{R}$ (hyperharmonic wrt $\Delta_k + \partial/\partial t$, cf. [2] or [4]) $(g_0, t_0) = \lim (g_n, t_n)$ and $u(g_n, t_n) < \infty$ implies that $u < \infty$ on $\{(g,t) \mid t \geqq t_0\}$. Since u is lower-semicontinuous and $u(g,t) \geqq \int P_\varepsilon(g, dg') u(g', t+\varepsilon)$ this result follows from the fact that the support of the measure $P_\varepsilon(g, dg')$ is $\mathbb{H}^n$ for any $g \in \mathbb{H}^n$ and $\varepsilon > 0$, cf. [7].

By group invariance it is clear that the constant in the Harnack Inequality is independent under translations of the "space" variable g. As a result, it follows as in [9] that if $u \geqq 0$ is a minimal global solution of $\Delta_k u + u_t = 0$, then $u(g,t) = f(g) e^{-\lambda t}$, where f is a minimal solution of $\Delta_k h = \lambda h$ on $\mathbb{H}^n$. Non-negative solutions of this equation will be shown to depend only on $z \in \mathbb{C}^n$.

Proposition 6.2. *Let f be a minimal solution of $\Delta_k h = \lambda h$ on $\mathbb{H}^n$. If $g = (z,t) \in \mathbb{C}^n \times \mathbb{R}$, then $f(g) = \phi(z) = e^{2\sqrt{\lambda}\,\Re e\langle z, b\rangle}$, where $\|b\| = 1$.*

Proof. Let $a = (0, a) \in \mathbb{R} \subset \mathbb{C}^n \times \mathbb{R}$. If $g \in \mathbb{H}^n$, then $a \cdot g = g \cdot a$ and so by Harnack's Inequality $f(a \cdot g) \leqq C(a) f(g)$ for all g. Since $h(g) = f(a \cdot g)$ satisfies $\Delta_k h = \lambda h$, it follows from minimality that $f(a \cdot g) = C(a) f(g)$ for all g and so $f(z,t) = C(t) f(z, 0)$. Since $f(z, a+t) = f(a \cdot (z,t)) = C(a) f(z,t) = C(a) C(t) f(z, 0)$, it follows that $C(t) = e^{At}$. Hence, $f(z,t) = e^{At} \phi(z)$.

If $A \neq 0$, then by averaging over $U(n)$ there is a non-negative solution h of $\Delta_k h = \lambda h$ such that $h(z,t) = e^{At}\psi(\|z\|)$. Since ψ is radial,

$$\Delta_k h(z,t) = e^{At}\{\Delta\psi(\|z\|) + 4A^2\|z\|^2\psi(\|z\|)\}$$

and so

$$\psi''(r) + \frac{(n-1)}{r}\psi'(r) = 4\{\lambda - A^2 r^2\}\psi(r). \tag{$*$}$$

It follows from this that, for $r \geqq r_0$, one may assume that $\xi(z) = \psi(\|z\|)$ is superharmonic and decreasing in r.

If $n=1$ this implies that ψ is a constant for large r, which contradicts $(*)$.

If $n \geqq 2$ then it follows from $(*)$ that for all $r \geqq r_0$

$$\psi(r) = \int_r^\infty 4\{\lambda - A^2 s^2\} s\psi(s)\,ds. \tag{$**$}$$

One sees this by considering, for each $r \geqq r_0$, the superharmonic function $\zeta(z)$ equal to $\psi(r)$ if $\|z\| = s$ and equal to $\psi(s)$ if $r \leqq \|z\| = s$. The formula $(*)$ gives the density for the measure that determines the potential part τ of ζ. Hence, $\tau(r) = \tau(0) = \int_r^\infty 4\{\lambda - A^2 s^2\} s\psi(s)\,ds$. Since the harmonic part of ζ is a constant, it follows that $\zeta = \tau$. As a result of $(**)$.

$$-\psi'(r) = 4\{\lambda - A^2 r^2\} r\psi(r), \quad \text{and so } \psi(r) = B\,e^{4\{\lambda r^2/2 - A^2 r^4/4\}}.$$

Substitution of this value in $(*)$ leads to a contradiction and so $A=0$ unless $\psi = 0$.

The explicit form of ϕ is well-known (cf. [8]).

Theorem 6.3. *Let $u \geqq 0$ be a function on $\mathbb{H}^n \times \mathbb{R}$. Then u is a solution of $\Delta_{k,\varepsilon} u + u_t = 0$ if and only if $u = v \circ \pi$, where π: $\mathbb{C}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}^n \times \mathbb{R}$ is the projection $\pi(z,s,t) = (z,t)$ and $\Delta v + 4v_t = 0$.*

Proof. In case $\varepsilon = 0$ this is an immediate corollary of the above proposition. The only thing that changes in the argument when $\varepsilon \neq 0$ is that in $(*)$ λ is replaced by $\lambda + A^2\varepsilon/4$. The argument applies as before.

Corollary 6.4. *Let $(g_t)_{t \geqq 0}$ be a process valued in $\mathbb{H}^n$. The following conditions are equivalent:*

(1) *for each non-negative solution u of the parabolic equation $\Delta_{k,\varepsilon} u + u_t = 0$ on $\mathbb{H}^n \times \mathbb{R}$, the process $(u(g_t, t))_{t \geqq 0}$ is a martingale of expectation one;*

(2) *the projection $(z_t)_{t \geqq 0}$ of the process $(g_t)_{t \geqq 0}$ onto $\mathbb{C}^n$ is equivalent to Brownian motion started from the origin.*

Remark 6.5. In [11], Margulis showed that for certain random walks on a finitely generated discrete nilpotent group G the positive harmonic functions u factor through the commutator subgroup $[G,G]$. This means that they are of the form $u = v \circ \pi$, where π: $G \to G/[G,G]$ and v is harmonic for the image walk. In [1] Alexopoulos has shown that in this context the analogue of Theorem

6.3 (for $\varepsilon=0$) is valid for the space-time random walk on $G\times\mathbb{Z}$. The measures involved have finite symmetric support. This suggests that Theorem 6.3 should hold for any sub-Laplacian on a nilpotent Lie group.

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