Probab. Th. Rel. Fields 76, 533–555 (1987)

Probability Theory and Related Fields

Generalized Ornstein-Uhlenbeck Process Having a Characteristic Operator with Polynomial Coefficients

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Summary. Let Φ be a weighted Schwartz's space of rapidly decreasing functions, Φ' the dual space and $\mathscr{L}(t)$ a perturbed diffusion operator with polynomial coefficients from Φ into itself. It is proven that $\mathscr{L}(t)$ generates the Kolmogorov evolution operator from Φ into itself via stochastic method. As applications, we construct a unique solution of a Langevin's equation on Φ':

$$d\xi(t) = dW(t) + \mathscr{L}^*(t)\,\xi(t)\,dt,$$

where $W(t)$ is a Φ'-valued Brownian motion and $\mathscr{L}^*(t)$ is the adjoint of $\mathscr{L}(t)$ and show a central limit theorem for interacting multiplicative diffusions.

1. Introduction

Since McKean [18] proved that the empirical distribution $U^{(n)}(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i^{(n)}(t)}$ for an interacting n-particle diffusion process $X^{(n)}(t) = (X_1^{(n)}(t), X_2^{(n)}(t), \ldots, X_n^{(n)}(t))$ converges to a non random measure $u(t) = u(t, dx)$, several authors (Itô [10], Kusuoka and Tamura [17], Shiga and Tanaka [23], Sznitman [24], Tanaka and Hitsuda [25], Tanaka [26]) studied from different point of views the limit behavior of $S_n(t) = \sqrt{n}(U^{(n)}(t) - u(t))$. Further it was obtained by Hitsuda and Mitoma [6] that the limit process of $S_n(t)$ is governed by a Langevin's equation on a distribution space Φ', (dual space of Φ):

$$d\xi(t) = dW(t) + \mathscr{L}^*(t)\,\xi(t)\,dt, \tag{1.1}$$

where the characteristic operator $\mathscr{L}^*(t)$ is the adjoint of a perturbed diffusion operator $\mathscr{L}(t)$ with uniformly bounded coefficients acting from Φ into itself and $W(t)$ is a Φ'-valued Brownian motion.

On the other hand, Dawson [2] studied the fluctuation phenomena for a simple model of interacting diffusions with polynomial coefficients called by

Graham and Shenzle [5] multiplicative processes. Inspired by him to study the fluctuation problem for interacting multiplicative diffusions, we essentially need to consider the Langevin equation where $\mathscr{L}(t)$ is a perturbed diffusion operator with polynomial coefficients for solving the identification problem of the limit processes of $S_n(t)$.

The aim of this paper is to prove that the Langevin equation with such a characteristic operator has a unique solution represented similarly as a finite dimensional Ornstein-Uhlenbeck process (Theorem 1). Then we will show that $S_n(t)$ converges weakly to a generalized Ornstein-Uhlenbeck process studied in Theorem 1 in the case where $X^{(n)}(t)$ is a multiplicative diffusion process with mean-field-like polynomial interacting drift (Theorem 2).

In Sect. 2 we will prove Theorem 1, where it is essential to verify via stochastic method that $\mathscr{L}(t)$ generates the Kolmogorov evolution operator, (defined precisely later), from Φ into itself. This implies that $\mathscr{L}^*(t)$ generates the usual evolution operator from Φ' into itself. In Sect. 3, Theorem 2 will be proved in three steps. In Step 1 we will prove the tightness of $S_n(t)$ in $C([0, \infty); \Phi')$ of continuous mappings from $[0, \infty)$ into Φ'. In the course of the proof it is sufficient to check the Kolmogorov tightness criterion for each real process $(S_n(t))(\phi)$, $\phi \in \Phi$, ([21]). In Step 2, the limit equation of $S_n(t)$ having the form of (1.1) will be derived along the same line as Hitsuda and Mitoma [6]. The uniqueness for the limit equation proved in Theorem 1 will complete the proof in the last Step 3.

2. Generalized Ornstein-Uhlenbeck Process

Before stating results, we will define a suitable space Φ modified from the Schwartz space $\mathscr{S}$ of rapidly decreasing functions and give some notations. Let $\rho(x)$ be the Friedrichs mollifier and Supp $[\rho(x)] \subset [-1, 1]$. Set $g(x) = \int_{\mathbb{R}} e^{-|y|} \rho(x-y)\, dy$, $h(x) = 1/g(x)$ and $\Phi = \{\phi(x) = h(x)\, \varphi(x);\, \varphi \in \mathscr{S}\}$. According to Gelfand-Vilenkin (3.6 in Chap. 1) [3], we will metrize Φ by the countably many semi-norms:

$$\|\phi\|_n = \sup_{\substack{x \in \mathbb{R} \\ 0 \leqq k \leqq n}} (1+x^2)^n |D^k(g(x)\, \phi(x))|, \qquad n = 0, 1, 2, \ldots$$

where $D = \frac{d}{dx}$. Let Φ' be the topological dual space of Φ and $\langle x, \phi \rangle = x(\phi)$, $x \in \Phi'$, $\phi \in \Phi$. Denote the space of continuous mappings from $[0, \infty)$ into Φ' by $C([0, \infty); \Phi')$ whose topology and Borel field were introduced in [21].

We will give precise definitions concerning a Langevin's equation considered in this paper. Let $W(t)$ be a Φ'-valued strongly continuous Gaussian additive process of mean 0 and $W(0) = 0$. For any $t \in [0, \infty)$, let $\mathscr{L}(t)$ be a continuous linear operator from Φ into itself and for any $\phi \in \Phi$, $\mathscr{L}(t)\phi$ continuous from $[0, \infty)$ into Φ. We consider the following integral equation on Φ':

$$\xi(t)=\xi(0)+W(t)+\int_0^t \mathscr{L}^*(s)\,\xi(s)\,ds, \tag{2.1}$$

where $\mathscr{L}^*(t)$ is the adjoint of $\mathscr{L}(t)$, the initial value $\xi(0)$ and $W(t)$ are defined on a complete probability space $(\Omega, \mathscr{F}, P)$ and the integral on the space Φ' denotes the Riemann integral. We say that (2.1) has a unique solution if there exists a Φ'-valued strongly continuous process $\xi(t)$ defined on $(\Omega, \mathscr{F}, P)$ satisfies (2.1) and for any such processes $\xi(t)$ and $\bar{\xi}(t)$, $\xi(t)=\bar{\xi}(t)$ for all $t\in[0,\infty)$ a.s. whenever $\xi(0)=\bar{\xi}(0)$ a.s. We also consider

$$\hat{\xi}(t)=\hat{\xi}(0)+\hat{W}(t)+\int_0^t \mathscr{L}^*(s)\,\hat{\xi}(s)\,ds, \tag{2.2}$$

where the joint distribution of $(\hat{\xi}(0), \hat{W}(t))$ coincides with that of $(\xi(0), W(t))$ and $\hat{\xi}(0)$ and $\hat{W}(t)$ may be defined on the other probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$. When (2.1) and (2.2) have solutions $\xi(t)$ and $\hat{\xi}(t)$, we say that the uniqueness in law for (2.1) holds if the laws P_ξ and $P_{\hat{\xi}}$ of $\xi(t)$ and $\hat{\xi}(t)$ on $C([0,\infty);\Phi')$ coincide.

Let $T(t)$ be a continuous linear operator from Φ into itself and for any $\phi\in\Phi$, $T(t)\phi$ continuous from $[0,\infty)$ into Φ. We call $T(t)$ generates the Kolmogorov evolution operator $U(t,s)$ if $U(t,s)$ is a continuous linear operator from Φ into itself such that

(T.1) for any $\phi\in\Phi$, $U(t,s)\phi$ is continuous from $\{(t,s);\, 0\leqq s\leqq t\}$ into Φ,

(T.2) $U(t,t)=U(s,s)=$ identity operator,

(T.3) $\dfrac{d}{dt}U(t,s)\phi=U(t,s)\,T(t)\phi$, $0\leqq s\leqq t$ on Φ,

(T.4) $\dfrac{d}{ds}U(t,s)\phi=-T(s)\,U(t,s)\phi$, $0\leqq s\leqq t$, $t>0$ on Φ.

Let $T^*(t)$ and $U^*(t,s)$ be the adjoint operator of $T(t)$ and $U(t,s)$ respectively. By the nuclearity of Φ, we get

Remark. If $T(t)$ generates the Kolmogorov evolution operator $U(t,s)$, then $T^*(t)$ generates the usual evolution operator $U^*(t,s)$ on Φ' equipped with the strong topology. Namely $U^*(t,s)$ satisfies the following (1)–(5).

(1) For any $x\in\Phi'$, $U^*(t,s)x$ is continuous from $\{(t,s);\, 0\leqq s\leqq t\}$ into Φ'.

(2) For $0\leqq s\leqq r\leqq t$, $U^*(t,r)\,U^*(r,s)=U^*(t,s)$.

(3) $U^*(s,s)=$ identity operator.

(4) $\dfrac{d}{dt}U^*(t,s)x=T^*(t)\,U^*(t,s)x$, $0\leqq s\leqq t$ on Φ'.

(5) $\dfrac{d}{ds}U^*(t,s)x=-U^*(t,s)\,T^*(s)x$, $0\leqq s\leqq t$, $t>0$ on Φ'.

Following Holley and Stroock [7] and Itô [11], we begin with a generalization of the finite dimensional Ornstein-Uhlenbeck process.

Before proceeding to a proposition, we give a definition of a stochastic integral $\int_0^t U^*(t,s)\,dW(s)$ whenever $U(t,s)$ is the Kolmogorov evolution operator. Since Φ is a nuclear Fréchet space, there is another system of Hilbertian semi-norms, $|||\cdot|||_1 \leqq |||\cdot|||_2 \leqq \dots \leqq |||\cdot|||_n \leqq \dots$, defining the topology equivalent to that introduced before on Φ. Let Φ_n be the completion of Φ by $|||\cdot|||_n$, Φ'_n the dual space of Φ_n, $|||\cdot|||_{-n}$ the dual norm of Φ'_n and $\langle\ ,\ \rangle_n$ the canonical bilinear form on $\Phi'_n \times \Phi_n$. Since $W(t)$ is a Φ'-valued strongly continuous Gaussian process, for any $T>0$ there exists a positive integer n_1 such that $E[\sup_{0\leqq t\leqq T} |||W(t)|||^2_{-n_1}] < \infty$, ([19]). By (T.1), $\sup_{0\leqq s\leqq t\leqq T} |||U(t,s)\phi|||_{n_1} < +\infty$, so that the stochastic integral $\int_0^t \langle dW(s), U(t,s)\phi\rangle_{n_1}$ is well defined, (Kunita [15]). We denote the value by $Y_t(\phi)$. Since for any fixed t, $Y_t(\phi)$ is continuous from Φ into L_0 of all real random variables with the probability convergence topology and for any fixed $\phi\in\Phi$, $Y_t(\phi)$ has a continuous version, combining Itô and Nawata [12] and [20], there exists a Φ'-valued strongly continuous process Y_t such that $\langle Y_t, \phi\rangle = Y_t(\phi)$ almost surely. Define $\int_0^t U^*(t,s)\,dW(s) = Y_t$.

Then we have

Proposition 1. *Suppose that $\mathscr{L}(t)$ generates the Kolmogorov evolution operator $U(t,s)$. Then* (2.1) *has a unique solution*

$$\xi(t) = U^*(t,0)\,\xi(0) + \int_0^t U^*(t,s)\,dW(s).$$

Further the law uniqueness for (2.1) *holds.*

Proof. Using (T.3) we get

$$\int_0^t U^*(t,s)\,dW(s) = W(t) + \int_0^t \mathscr{L}^*(\tau)\left(\int_0^\tau U^*(\tau,s)\,dW(s)\right)d\tau,$$

so that noticing $\int_0^t \mathscr{L}^*(\tau)\,U^*(\tau,0)\,\xi(0)\,d\tau = U^*(t,0)\,\xi(0) - \xi(0)$, we have that $\xi(t)$ satisfies (2.1). The uniqueness will be proved by applying the arguments in the proof of Proposition 7.3 of Komatsu [14] for bilinear form $\langle\ ,\ \rangle$. Since $U^*(t,s)$ is non random, $\xi(t)$ and $\hat{\xi}(t)$ are the same measurable functional of $(\xi(0), W(t))$ and $(\hat{\xi}(0), \hat{W}(t))$, and hence the law uniqueness easily follows from the structure of the Borel field of $C([0,\infty); \Phi')$, ([21]).

Next we will consider the case where $\mathscr{L}(t) = A(t) + J(t)$ and $J(t)$ satisfies the following condition:

(H) There exists a positive integer n_0 such that for any integer $n\geqq 0$ and any $T>0$,

$$\sup_{0\leqq t\leqq T}\ \sup_{\substack{\|\phi\|_{n_0}\leqq 1\\ \phi\in\Phi}} \|J(t)\phi\|_n < \infty.$$

In the subsequent discussions, for simplicity we denote positive constants by C_i or $C_i(\tau_1, \tau_2, \dots)$ with depending parameters $\tau_1, \tau_2, \dots$, for ambiguous cases, $i=1, 2, \dots$ and also positive integers by n_i, $i=2, 3, \dots$.

Proposition 2. *Suppose that for any $t \in [0, \infty)$, $A(t)$ and $J(t)$ are continuous linear operators from Φ into itself, for any $\phi \in \Phi$, $A(t)\phi$ and $J(t)\phi$ are continuous from $[0, \infty)$ into Φ, $A(t)$ generates the Kolmogorov evolution operator and $J(t)$ satisfies the condition* (H). *Then $\mathscr{L}(t)$ generates the Kolmogorov evolution operator $V(t, s)$ and the conclusion of Proposition* 1 *holds if $U(t, s)$ is replaced by $V(t, s)$.*

Proof. It is enough to show that $\mathscr{L}(t)$ generates the Kolmogorov evolution operator. Denote an evolution operator that $A(t)$ generates by $U(t, s)$. Following Theorem 1.19 of Chap. IX in Kato [13], we will consider an integral equation on Φ:

$$y(t, s, \phi) = U(t, s)\phi + \int_s^t U(\tau, s)\, J(\tau)\, y(t, \tau, \phi)\, d\tau. \tag{2.3}$$

By Baire's category theorem ([8], p. 62), for any integer $n \geqq 0$, we get

$$\sup_{0 \leqq s \leqq t \leqq T} \|U(t, s)\phi\|_n \leqq C_1(n, T)\, \|\phi\|_{n_2}, \quad (n_2 > n), \tag{2.4}$$

$$\sup_{0 \leqq t \leqq T} \|A(t)\phi\|_n \leqq C_2(n, T)\, \|\phi\|_{n_3}, \quad (n_3 > n). \tag{2.5}$$

Hence (2.4) and the condition (H) quarantee that (2.3) is uniquely solved by the method of successive approximations. Define $V(t, s)\phi = y(t, s, \phi)$. Then Gronwall's lemma gives

$$\sup_{0 \leqq s \leqq t \leqq T} \|V(t, s)\phi\|_n \leqq C_3(n, T)\, \|\phi\|_{n_4}, \quad (n_4 > n). \tag{2.6}$$

Using (T.3) and (T.4) of $U(t, s)$, (2.4) and (2.5), we get

$$\|(U(t', s') - U(t, s))\phi\|_n \leqq C_4(n, T)\{|t - t'| + |s - s'|\}\, \|\phi\|_{n_5}, \quad (n_5 > n). \tag{2.7}$$

By (T.4) of $U(t, s)$,

$$\int_s^t \left\{\frac{1}{\varepsilon}(U(\tau, s+\varepsilon) - U(\tau, s))\, J(\tau)\, V(t, \tau)\phi\right\} d\tau \to$$
$$-A(s) \int_s^t (U(\tau, s)\, J(\tau)\, V(t, \tau)\phi)\, d\tau \quad \text{as } \varepsilon \to 0,$$

since the n-th seminorm of the integrand is bounded uniformly in $\tau \in [s, t]$. Using (2.6), (2.7) and Gronwall's lemma, we have

$$\|(V(t, s') - V(t, s))\phi\|_n \leqq C_5(n, T)\, |s - s'|\, \|\phi\|_{n_6}, \quad (n_6 > n). \tag{2.8}$$

By (2.4), (2.6), (2.7) and (2.8),

$$-\frac{1}{\varepsilon} \int_s^{s+\varepsilon} (U(\tau, s+\varepsilon)\, J(\tau)\, V(t, \tau)\phi)\, d\tau \to -J(s)\, V(t, s)\phi \quad \text{as } \varepsilon \to 0.$$

Thus, together with (T.4) of $U(t, s)$, we find that $V(t, s)$ satisfies (T.4).

Now, it is evident that $V(t, s)$ satisfies (T.1) and (T.2). Given ε and δ, set

$$O_1(s, \tau) = \left\| \int_t^{t+\varepsilon} \left(\frac{1}{\varepsilon} U(\tau, s) A(\tau) \phi \right) d\tau - U(t, s) A(t) \phi \right\|_n,$$

$$O_2(s, \varepsilon) = \left\| \int_t^{t+\varepsilon} \left(\frac{1}{\varepsilon} U(\tau, s) J(\tau) V(t+\varepsilon, \tau) \phi \right) d\tau - U(t, s) J(t) \phi \right\|_n$$

and

$$R_V(s, \varepsilon, \delta) = \left\| \frac{1}{\varepsilon} (V(t+\varepsilon, s) - V(t, s)) \phi - \frac{1}{\delta} (V(t+\delta, s) - V(t, s)) \phi \right\|_n.$$

Then if $n \geqq n_0$, we have

$$R_V(s, \varepsilon, \delta) \leqq \sup_{0 \leqq s \leqq t} \{O_1(s, \varepsilon) + O_1(s, \delta) + O_2(s, \varepsilon) + O_2(s, \delta)\} + C_6(n, T) \int_s^t R_V(\tau, \varepsilon, \delta)\, d\tau, \qquad 0 \leqq s \leqq t \leqq T.$$

By (2.4), (2.5) and (2.7), $\lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \sup_{0 \leqq s \leqq t} \{O_1(s, \varepsilon) + O_1(s, \delta)\} = 0$. By (2.4), (2.6), (2.7) and (2.8), also $\lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \sup_{0 \leqq s \leqq t} \{O_2(s, \varepsilon) + O_2(s, \delta)\} = 0$. Applying the generalized Gronwall lemma for $R_V(s, \varepsilon, \delta)$ and taking the above two equalities into account, we find that $\left\{ \frac{1}{\varepsilon} (V(t+\varepsilon, s) - V(t, s)) \phi \right\}$ forms a Cauchy sequence and hence $V(t, s)\phi$ is differentiable with respect to t. Moreover the above argument gives that $\left\| \frac{1}{\varepsilon} (V(t+\varepsilon, s) - V(t, s)) \phi \right\|_n$ is bounded uniformly in $0 \leqq s \leqq T$. Therefore similarly to the proof of (T.4) of $V(t, s)$, we have

$$\frac{d}{dt} V(t, s) \phi = U(t, s) \mathscr{L}(t) \phi + \int_s^t U(\tau, s) J(\tau) \frac{dV(t, \tau) \phi}{dt} d\tau.$$

The uniqueness of this equation implies (T.3) of $V(t, s)$.

Before proceeding to Theorem 1, we will introduce some definitions. For any integer $p \geqq -1$, $C_{p,u}^{\infty}$ denotes a set of real functions $f(t, x)$ such that the following three conditions are satisfied.

(i) $f(t, x)$ is infinitely differentiable, $(= C^{\infty})$, with respect to x.

(ii) For any integer $n \geqq 0$ and any $T > 0$, there exists a constant $C(T, n)$ such that

(ii-1) $\qquad \sup_{0 \leqq t \leqq T} |D^n f(t, x)| \leqq C(T, n)(1 + |x|)^{p-n} \quad$ if $p - n > 0$, (ii-2) $$\sup_{0 \leqq t \leqq T} |D^n f(t, x)| \leqq C(T, n) \quad \text{if } p-n \leqq 0.$$

(iii) For any integer $n \geqq 0$ and any $M > 0$

$$\lim_{t \to s} \sup_{|x| \leqq M} |D^n(f(t, x) - f(s, x))| = 0.$$

Suppose that

$$(\mathscr{L}(t)\phi)(x) = \tfrac{1}{2}\alpha(t, x)^2 \phi''(x) + \beta(t, x)\phi'(x) + (J(t)\phi)(x),$$

where $\alpha(t, x) \in C^\infty_{p,u}$, $\beta(t, x) = \bar{\beta}(t, x) + \beta_{2p+1}(t) x^{2p+1}$, $\bar{\beta}(t, x) \in C^\infty_{2p,u}$, $\beta_{2p+1}(t)$ is continuous in $t \in [0, \infty)$, $\beta_{2p+1}(t) < 0$ if $p \geqq 0$ and $\beta_{-1}(t) = 0$.

Theorem 1. *Suppose that for any $t \in [0, \infty)$, $J(t)$ is a continuous linear operator from Φ into itself, for any $\phi \in \Phi$, $J(t)\phi$ is continuous from $[0, \infty)$ into Φ and $J(t)$ satisfies the condition* (H). *Then $\mathscr{L}(t)$ generates the Kolmogorov evolution operator $U(t, s)$ and* (2.1) *has a unique solution*

$$\xi(t) = U^*(t, 0)\,\xi(0) + \int_0^t U^*(t, s)\, dW(s).$$

Further if $\xi(0)$ is a Φ'-valued Gaussian random variable independent of $W(t)$, the law uniqueness for (2.1) *holds and the law is Gaussian.*

Proof. Set $(A(t)\phi)(x) = \frac{1}{2}\alpha(t, x)^2 \phi''(x) + \beta(t, x)\phi'(x)$. Then by Proposition 2, it is enough to check that $A(t)$ generates the Kolmogorov evolution operator. The proof will be devided into two Steps and carried out via stochastic method. In Step 1 we will derive the pointwise Kolmogorov forward and backward equations and in Step 2 verify that these equations hold in an abstract sense.

Step 1. We will consider the following Itô's stochastic differential equation:

$$\begin{aligned} \eta_{s,t}(x) &= x + \int_s^t \alpha(r, \eta_{s,r}(x))\, dB(r) + \int_s^t \beta(r, \eta_{s,r}(x))\, dr \\ \eta_{s,s}(x) &= x. \end{aligned} \tag{2.9}$$

Here $B(t)$ is 1-dimensional Brownian motion.

If $p = 0, -1$, Eq. (2.9) has a unique non-explosive solution $\eta_{s,t}(x)$ because the coefficients are globally Lipshitz continuous.

Suppose that $p \geqq 1$. For each natural number N, we choose globally Lipshitz continuous functions $\alpha^N(t, x)$ and $\beta^N(t, x)$ such that

$$\begin{aligned} &\alpha^N(t, x) = \alpha(t, x) && \text{and} \quad \beta^N(t, x) = \beta(t, x) && \text{if } |x| \leqq N, \\ &\alpha^N(t, x) = \alpha(t, N) && \text{and} \quad \beta^N(t, x) = \beta(t, N) && \text{if } x > N, \\ &\alpha^N(t, x) = \alpha(t, -N) && \text{and} \quad \beta^N(t, x) = \beta(t, -N) && \text{if } x < -N. \end{aligned}$$

Then Eq. (2.9) corresponding to coefficients $\{\alpha^N(t,x), \beta^N(t,x)\}$ has a unique solution $\eta^N_{s,t}(x)$. For any $T>0$, let $\alpha_T>0$ be a constant such that $\sup_{0\leqq t\leqq T}|\alpha(t,x)|^2 \leqq \alpha_T(1+x^{2p})$. Let ζ be a real number such that $0<\zeta<(\min_{0\leqq t\leqq T}|\beta_{2p+1}(t)|)/\alpha_T$. By the Itó formula we get

$$E[e^{\zeta|\eta^N_{s,t}(x)|^2}]-e^{\zeta|x|^2}$$
$$=E\left[\int_s^t e^{\zeta|\eta^N_{s,r}(x)|^2}\{2\zeta\,\eta^N_{s,r}(x)\,\beta^N(r,\eta^N_{s,r}(x))+(2\zeta^2\,\eta^N_{s,r}(x)^2+\zeta)\,\alpha^N(r,\eta^N_{s,r}(x))^2\}\,dr\right].$$

Noticing a manner of choosing ζ and $\beta_{2p+1}(t)<0$, we have a constant $C_7(T)$ independent of x and N such that

$$\{2\zeta\,\eta^N_{s,r}(x)\,\beta^N(r,\eta^N_{s,r}(x))+(2\zeta^2\,\eta^N_{s,r}(x)^2+\zeta)\,\alpha^N(r,\eta^N_{s,r}(x))^2\}\leqq C_7(T).$$

Therefore Gronwall's lemma implies

$$\sup_N \sup_{|x|\leqq M} \sup_{0\leqq s\leqq t\leqq T} E[e^{\zeta|\eta^N_{s,t}(x)|^2}]<\infty,$$

so that Eq. (2.9) has a unique solution $\eta_{s,t}(x)$ and it has no explosions, (Theorem 5.2, p. 229 of Kunita [16], Ikeda and Watanabe [9]).

By Proposition 2 in [6] and a calculation similar to the above, we have

Lemma 1. *For any* $\varepsilon>0$, $T>0$ *and* $M>0$,

$$\sup_{|x|\leqq M} \sup_{0\leqq s\leqq t\leqq T} E[e^{\varepsilon|\eta_{s,t}(x)|}]<\infty. \tag{2.10}$$

Therefore by the strict conservativeness of $\eta_{s,t}(x)$, (p. 232) and Theorem 5.4 of Kunita [16], $\eta_{s,t}(x)$ is infinitely differentiable with respect to x for any $s\leqq t$ and further the proof of Theorem 5.2 of [16] implies that the following differential formulae for $\eta_{s,t}(x)$ hold like in the case of stochastic differential equations with globally Lipschitz smooth coefficients (for example § 8 in Chap. 2 of Gihman and Skorohod [4]):

$$D^k\eta_{s,t}(x)=D^k x+\int_s^t D^k\alpha(r,\eta_{s,r}(x))\,dB(r)+\int_s^t D^k\beta(r,\eta_{s,r}(x))\,dr \tag{2.11}$$

for any integer $k\geqq 0$.

Define $(U(t,s)\phi)(x)=E[\phi(\eta_{s,t}(x))]$. Since the coefficients of (2.9) satisfy the condition of Theorem 1.1 (p. 256) of [16], if we prove the following integrabilities for $D^k\eta_{s,t}(x)$ quaranteeing the uniform integrabilities used in [22], Itô's forward and backward formulae for $\phi(\eta_{s,t}(x))$ lead us to the pointwise Kolmogorov forward and backward equations:

$$\frac{d}{dt}(U(t,s)\phi)(x)=(U(t,s)A(t)\phi)(x), \tag{2.12}$$

$$\frac{d}{ds}(U(t,s)\phi)(x)=-(A(s)U(t,s)\phi)(x). \tag{2.13}$$

Lemma 2. *For any integers $i \geqq 1$ and $j \geqq 1$ and any $T>0$,*

$$\sup_{0 \leqq s \leqq t \leqq T} E[|D^i \eta_{s,t}(x)|^j] \leqq C_8(T)(1+|x|)^{j(i-1)\{(2p-1) \vee 0\}}, \tag{2.14}$$

where $a \vee b = \max\{a, b\}$.

Proof. Also in this case, it is enough to check (2.14) for $p \geqq 0$, (Theorem 1, p. 61 of Gihman and Skorohod [4, 22]). We will show this by a mathematical induction. For brevity, we use that notation $f^{(k)}(t, x) = D^k f(t, x)$. By (2.11) and the Itô formula,

$$E[(D\eta_{s,t}(x))^{2j}] - 1 = E\left[\int_s^t (D\eta_{s,r}(x))^{2j}\{2j\beta^{(1)}(r, \eta_{s,r}(x)) + j(2j-1)(\alpha^{(1)}(r, \eta_{s,r}(x)))^2\}\, dr\right].$$

In fact, by a manner used in the proof of deriving Lemma 1, there exists a constant $C_9 = C_9(j, T)$ such that

$$\sup_{0 \leqq s \leqq r \leqq T} \{2j\beta^{(1)}(r, \eta_{s,r}(x)) + j(2j-1)(\alpha^{(1)}(r, \eta_{s,r}(x)))^2\} \leqq C_9,$$

and therefore Gronwall's lemma gives (2.14) for $i=1$.

Suppose that (2.14) holds for every integer $1 \leqq i \leqq k$. Set

$$\alpha_{k,\eta}(s, r) = D^{k+1}\alpha(r, \eta_{s,r}(x)) - \alpha^{(1)}(r, \eta_{s,r}(x))\, D^{k+1}\eta_{s,r}(x)$$

and

$$\beta_{k,\eta}(s, r) = D^{k+1}\beta(r, \eta_{s,r}(x)) - \beta^{(1)}(r, \eta_{s,r}(x))\, D^{k+1}\eta_{s,r}(x).$$

Again by (2.11) and the Itô formula,

$$\begin{aligned} E[(D^{k+1}\eta_{s,t}(x))^{2j}] &= E\Big[\int_s^t (2j(D^{k+1}\eta_{s,r}(x))^{2j-1}\{\beta^{(1)}(r, \eta_{s,r}(x)) \\ &\quad \times D^{k+1}\eta_{s,r}(x) + \beta_{k,\eta}(s, r)\} + j(2j-1)(D^{k+1}\eta_{s,r}(x))^{2j-2}\{\alpha^{(1)}(r, \eta_{s,r}(x)) \\ &\quad \times D^{k+1}\eta_{s,r}(x) + \alpha_{k,\eta}(s, r)\}^2)\, dr\Big] \\ &\leqq E\Big[\int_s^t \{(D^{k+1}\eta_{s,r}(x))^{2j}(2j\beta^{(1)}(r, \eta_{s,r}(x)) + 2j(2j-1)(\alpha^{(1)}(r, \eta_{s,r}(x)))^2) \\ &\quad + 2j|D^{k+1}\eta_{s,r}(x)|^{2j-1}|\beta_{k,\eta}(s, r)| \\ &\quad + 2j(2j-1)|D^{k+1}\eta_{s,r}(x)|^{2j-2}|\alpha_{k,\eta}(s, r)|^2\}\, dr\Big]. \end{aligned}$$

By a manner used in the proof for $i=1$ and Hölder's inequality, there exists a constant $C_{10} = C_{10}(j, T)$ such that the right hand side of the above inequality is dominated by

$$C_{10}\int_s^t E[(D^{k+1}\eta_{s,r}(x))^{2j} + (\alpha_{k,\eta}(s, r))^{2j} + (\beta_{k,\eta}(s, r))^{2j}]\, dr.$$

Since $\alpha_{k,\eta}(s, r)$ is a finite sum of terms of the type

$$\alpha^{(j)}(r, \eta_{s,r}(x))(D\eta_{s,r}(x))^{j_1}(D^2\eta_{s,r}(x))^{j_2}\dots(D^k\eta_{s,r}(x))^{j_k},$$
$$2 \leqq j \leqq k, \qquad \sum_{n=1}^{k} n j_n = k+1,$$

and $\beta_{k,\eta}(s, r)$ is also written by the same way as the above, we have for $0 \leqq s \leqq r \leqq T$

$$\begin{aligned} &E[(\alpha_{k,\eta}(s,r))^{2j}] \vee E[(\beta_{k,\eta}(s,r))^{2j}] \\ &\quad \leqq C_{11}(j, T) \sup_{2 \leqq n \leqq k} \{E[(\alpha^{(n)}(r, \eta_{s,r}(x)))^{4j}]^{1/2} \vee E[(\beta^{(n)}(r, \eta_{s,r}(x)))^{4j}]^{1/2}\} \\ &\quad (1+|x|)^{2j\left(k+1-\sum_{n=1}^{k} j_n\right)((2p-1)\vee 0)}. \end{aligned}$$

By a manner used in the proof of deriving Lemma 1, we get for any integer $n \geqq 0$,

$$E[(\eta_{s,t}(x))^{2n}] \leqq C_{12}(T)(1+|x|^{2n}), \qquad 0 \leqq s \leqq t \leqq T, \tag{2.15}$$

and hence

$$\begin{aligned} &\sup_{2 \leqq n \leqq k} \{E[(\alpha^{(n)}(r, \eta_{s,r}(x)))^{4j}]^{1/2} \vee E[(\beta^{(n)}(r, \eta_{s,r}(x)))^{4j}]^{1/2}\} \\ &\qquad \leqq C_{13}(j, T)(1+|x|)^{2j((2p-1)\vee 0)}. \end{aligned}$$

Since $\left(k+1-\sum_{n=1}^{k} j_n\right) \leqq k-1$, summing up the above inequalities and using Gronwall's lemma, we have (2.14) for $i=k+1$.

Step 2. We shall prove (T.1)–(T.4) of $U(t, s)$ below. Suppose that for any integer $n \geqq 0$, the following equalities have been proved.

$$\lim_{\substack{t \to t' \\ s \to s'}} \|U(t,s)\phi - U(t',s)\phi\|_n = 0. \tag{2.16}$$

$$\lim_{\substack{t \to t' \\ s \to s'}} \|U(t,s)\phi - U(t,s')\phi\|_n = 0. \tag{2.17}$$

Then (2.16) and (2.17) implies (T.1). (T.2) is immediate from the definition of $U(t, s)$. For any $\phi \in \Phi$,

$$\begin{aligned} &\|A(t)\phi\|_n \leqq C_{14}(n, T)\|\phi\|_{n+(p+2)}, \qquad 0 \leqq t \leqq T, \\ &\lim_{t \to s} \|A(t)\phi - A(s)\phi\|_n = 0, \end{aligned} \tag{2.18}$$

so that together with (2.4) and (2.18), we have $U(\tau, s)A(\tau)\phi$, $\tau \geqq s$ and $A(\tau)U(t, \tau)\phi$, $0 \leqq \tau \leqq t$, are $\|\cdot\|_n$-continuous in τ and hence both are the $\|\cdot\|_n$-Riemann integrable for every integer $n \geqq 0$. Therefore by Kolmogorov's equations and the definition of n-th semi-norm $\|\cdot\|_n$, we have

$$\left(\int_s^t U(\tau,s)A(\tau)\phi\,d\tau\right)(x)=(U(t,s)\phi)(x)-\phi(x),$$
$$\left(-\int_s^t A(\tau)U(t,\tau)\phi\,d\tau\right)(x)=\phi(x)-(U(t,s)\phi)(x), \tag{2.19}$$

which gives (T.3) and (T.4).

To check (2.16) and (2.17), we will begin to prepare a regularity lemma for $\eta_{s,t}(x)$ and the differentials.

Lemma 3. *For any* $T>0$, $M>0$, $0\leqq s\leqq t\leqq T$, $0\leqq s'\leqq t'\leqq T$ *and any integers* $n\geqq 1$ *and* $m\geqq 0$,

$$\sup_{|x|\leqq M} E[|\eta_{s,t}(x)-\eta_{s',t'}(x)|^n]\leqq C_{15}(n,T,M)\{|t-t'|^{\frac{n}{2}}+|s-s'|^{\frac{n}{2}}\}, \tag{2.20}$$

$$\sup_{|x|\leqq M} E[|\eta_{s,t}(x)-x|^n]\leqq C_{16}(n,T,M)\,|t-s|^{\frac{n}{2}}, \tag{2.21}$$

$$\sup_{|x|\leqq M} E[|D^m\eta_{s,t}(x)-D^m\eta_{s',t'}(x)|^n]\leqq C_{17}(n,m,T,M)\{|t-t'|^{\frac{n}{2}}+|s-s'|^{\frac{n}{2}}\}, \tag{2.22}$$

$$\sup_{|x|\leqq M} E[|D^m(\eta_{s,t}(x)-x)|^n]\leqq C_{18}(n,m,T,M)\,|t-s|^{\frac{n}{2}}. \tag{2.23}$$

Proof. For the proofs of (2.20) and (2.22), it is enough to show them only for the case where $0\leqq s<s'<t<t'\leqq T$. Since

$$\eta_{s',t'}(x)=x+\int_{s'}^t \alpha(r,\eta_{s',r}(x))\,dB(r)+\int_{s'}^t \beta(r,\eta_{s',r}(x))\,dr$$
$$+\int_t^{t'} \alpha(r,\eta_{s',r}(x))\,dB(r)+\int_t^{t'} \beta(r,\eta_{s',r}(x))\,dr,$$

$$\eta_{s,t}(x)=\eta_{s,s'}(x)+\int_{s'}^t \alpha(r,\eta_{s,r}(x))\,dB(r)+\int_{s'}^t \beta(r,\eta_{s,r}(x))\,dr,$$

the left hand side of (2.20) is dominated by

$$5^{n-1}\sup_{|x|\leqq M} E\Bigg[|x-\eta_{s,s'}(x)|^n+\left|\int_{s'}^t (\alpha(r,\eta_{s',r}(x))-\alpha(r,\eta_{s,r}(x)))\,dB(r)\right|^n$$
$$+\left|\int_{s'}^t (\beta(r,\eta_{s',r}(x))-\beta(r,\eta_{s,r}(x)))\,dr\right|^n+\left|\int_t^{t'} \alpha(r,\eta_{s',r}(x)\,dB(r)\right|^n$$
$$+\left|\int_t^{t'} \beta(r,\eta_{s',r}(x))\,dr\right|^n\Bigg]. \tag{2.24}$$

Noticing Lemma 1 and the Burkholder inequality, ((3) on p. 193 of Kunita [16]), we get

$$\sup_{|x|\leqq M} E\left[\left|\int_t^{t'} \alpha(r, \eta_{s',r}(x))\,dB(r)\right|^n + \left|\int_t^{t'} \beta(r, \eta_{s',r}(x))\,dr\right|^n\right]$$
$$\leqq C_{19}(n, T, M)\,|t-t'|^{\frac{n}{2}}. \tag{2.25}$$

Quite similarly we have

$$\sup_{|x|\leqq M} E[|x-\eta_{s,s'}(x)|^n] \leqq C_{20}(n, T, M)\,|s-s'|^{\frac{n}{2}}, \tag{2.26}$$

which proves (2.21) simultaneously and have also (2.23) by Lemmas 1 and 2 and (2.11).

Setting $g_k(x, y) = \sum_{j=0}^{k-1} x^{k-1-j} y^j$ for any integer $k \geqq 1$ and noticing that $g_{2p+1}(x, y) \geqq \frac{1}{2}\{x^{2p}+y^{2p}\}$ and $\beta_{2p+1}(t) < 0$ if $p \geqq 0$, we have the following by a manner used in the proofs of the previous lemmas;

$$E[|\eta_{s',r}(x)-\eta_{s',r}(y)|^{2n}] \leqq C_{21}(n, T)\,|x-y|^{2n}, \quad 0 \leqq s' \leqq r \leqq T. \tag{2.27}$$

Since $\eta_{s,r}(x) = \eta_{s',r}(\eta_{s,s'}(x))$ if $r \geqq s'$ and $\eta_{s',r}(y) - \eta_{s',r}(x)$ is independent of $\eta_{s,s'}(x)$, we get by (2.26) and (2.27) that for $|x| \leqq M$,

$$E[|\eta_{s',r}(x)-\eta_{s,r}(x)|^{2n}] \leqq C_{22}(n, T, M)\,|s-s'|^n. \tag{2.28}$$

Thus (2.24), together with (2.25), (2.26), (2.28), Lemma 1 and the Burkholder and Schwarz inequalities, leads us to (2.20).

We will show (2.22) by a mathematical induction. If $m=0$, (2.22) is immediate from (2.20). Suppose that for any integers $k \geqq 0$ and $n \geqq 1$, (2.22) holds for every integer $0 \leqq m \leqq k$. Using the notations in the proof of Lemma 2, we get

$$D^{k+1}\eta_{s,t}(x) - D^{k+1}\eta_{s',t}(x) = D^{k+1}(\eta_{s,s'}(x) - x)$$
$$+ \int_{s'}^{t} \{(\alpha_{k,\eta}(s, r) - \alpha_{k,\eta}(s', r)) + \alpha^{(1)}(r, \eta_{s,r}(x))(D^{k+1}\eta_{s,r}(x) - D^{k+1}\eta_{s',r}(x))$$
$$+ D^{k+1}\eta_{s',r}(x)(\alpha^{(1)}(r, \eta_{s,r}(x)) - \alpha^{(1)}(r, \eta_{s',r}(x)))\}\,dB(r)$$
$$+ \int_{s'}^{t} \{(\beta_{k,\eta}(s, r) - \beta_{k,\eta}(s', r)) + \beta^{(1)}(r, \eta_{s,r}(x))(D^{k+1}\eta_{s,r}(x) - D^{k+1}\eta_{s',r}(x))$$
$$+ D^{k+1}\eta_{s',r}(x)(\beta^{(1)}(r, \eta_{s,r}(x)) - \beta^{(1)}(r, \eta_{s',r}(x)))\}\,dr.$$

Then by the assumption of the induction, Lemmas 1 and 2, (2.20), (2.23) and a manner used in the proof of Lemma 2, we have

$$\sup_{|x|\leqq M} E[|D^{k+1}\eta_{s,t}(x) - D^{k+1}\eta_{s',t}(x)|^{2n}] \leqq C_{23}(n, k+1, T, M)\,|s-s'|^n. \tag{2.29}$$

Of course, by Lemmas 1 and 2, we get

$$\sup_{|x|\leqq M} E[|D^{k+1}\eta_{s',t}(x) - D^{k+1}\eta_{s',t'}(x)|^n] \leqq C_{24}(n, k+1, T, M)\,|t-t'|^{\frac{n}{2}}. \tag{2.30}$$

Therefore (2.29) and (2.30) imply (2.22). This completes the proof.

Now we will return to the proofs of (2.16) and (2.17). First we will discuss the case where $p \geqq 1$. Suppose that $0 \leqq s \leqq t \leqq T$. Since

$$\|U(t,s)\phi\|_n = \sup_{\substack{x\in\mathbb{R}\\ 0\leqq k\leqq n}} (1+x^2)^n |D^k(g(x)(U(t,s)\phi)(x))|, \tag{2.31}$$

changing the order of the differentials and the expectation because of Lemmas 1 and 2, we have by the Leibniz formula that the right hand side of (2.31) is dominated by a finite sum of the terms;

$$\sup_{x\in\mathbb{R}} (1+x^2)^n |D^l g(x) E[h^{(m)}(\eta_{s,t}(x))\, \varphi^{(q)}(\eta_{s,t}(x))(D\eta_{s,t}(x))^{n_1}$$
$$(D^2\eta_{s,t}(x))^{n_2}\dots(D^k\eta_{s,t}(x))^{n_k}]|, \tag{2.32}$$

where $0 \leqq l+k \leqq n$, $0 \leqq m+q \leqq k$, $n_1+2n_2+\dots+kn_k=k$, $h^{(m)}(x)=D^m h(x)$ and $\varphi^{(q)}(x)=D^q\varphi(x)$. By the definitions of $g(x)$ and $h(x)$,

$$\begin{aligned} |D^l g(x)| &\leqq C_{25}(l)\, e^{-|x|}, \\ |h^{(m)}(\eta_{s,t}(x))| &\leqq C_{26}(m)\, e^{|\eta_{s,t}(x)|}. \end{aligned} \tag{2.33}$$

Let $y_{s,t}(x)$ be a unique solution of

$$y_{s,t}(x) = x + \int_s^t \beta_{2p+1}(r)(y_{s,r}(x))^{2p+1}\, dr.$$

Set $K_{s,t}(x)=\eta_{s,t}(x)-y_{s,t}(x)$. Then we have

Lemma 4. *For any $T>0$, there exists a constant $\zeta>0$ such that*

$$\sup_{0\leqq s\leqq t\leqq T} E[e^{\zeta|K_{s,t}(x)|^2}] \leqq C_{27}(T)(1+x^{2p}). \tag{2.34}$$

Proof. Let $\alpha_T>0$ and $\beta_T>0$ be real numbers such that $\sup_{0\leqq t\leqq T}|\alpha(t,x)|^2 \leqq \alpha_T(1+x^{2p})$ and $\sup_{0\leqq t\leqq T}|\bar\beta(t,x)| \leqq \beta_T(1+x^{2p})$. We choose a real number ζ such that $0<\zeta < (\min_{0\leqq t\leqq T}|\beta_{2p+1}(t)|)/4\alpha_T$. Then by a manner similar to that used in the proofs of deriving Lemma 1 and (2.27), we get

$$E[e^{\zeta K_{s,t}(x)^2}] \leqq 1 + E\left[\int_s^t \{F(r)\,\eta_{s,r}(x)^{2p} + (G(r)+I)\, e^{\zeta K_{s,r}(x)^2}\}\, dr\right],$$

where

$$\begin{aligned} F(r) &= \zeta e^{\zeta K_{s,r}(x)^2}\{(\tfrac{1}{2}\beta_{2p+1}(r)+2\zeta\alpha_T)K_{s,r}(x)^2 + 2\beta_T|K_{s,r}(x)| + \alpha_T\}, \\ G(r) &= \zeta K_{s,r}(x)^2\{\tfrac{1}{2}\beta_{2p+1}(r)(\eta_{s,r}(x)^{2p}+y_{s,r}(x)^{2p}) + 2\zeta\alpha_T + 1\}, \\ I &= \zeta(\zeta\beta_T^2+\alpha_T). \end{aligned}$$

Noticing a manner of choosing ζ and $\beta_{2p+1}(t)<0$, we have

$$F(r)\leqq C_{28}(T) \quad \text{and} \quad G(r)\leqq C_{29}(T).$$

Therefore the inequality (2.15) and Gronwall's lemma give us (2.34).

By making use of (2.33), (2.34), Lemma 2 and the Schwarz inequality, we get that (2.32) is dominated by

$$C_{30}(n, T)\sup_{x\in\mathbb{R}}(1+x^2)^{n(p+1)+p}e^{-|x|+|y_{s,t}(x)|}E[|\varphi^{(q)}(\eta_{s,t}(x))|^2]^{\frac{1}{2}}. \tag{2.35}$$

Since

$$y_{s,t}(x)=\frac{x}{\left(1-2p\left(\int_s^t \beta_{2p+1}(r)\,dr\right)x^{2p}\right)^{1/2p}},$$

and

$$\begin{aligned}E[|\varphi^{(q)}(\eta_{s,t}(x))|^2]^{\frac{1}{2}} &= E\left[\frac{(1+\eta_{s,t}(x)^2)^{(4n(p+1)+8p+2)p}}{(1+\eta_{s,t}(x)^2)^{(4n(p+1)+8p+2)p}}|\varphi^{(q)}(\eta_{s,t}(x))|^2\right]^{\frac{1}{2}} \\ &\leqq \|\phi\|_{(2n(p+1)+4p+1)p}E\left[\left(\frac{1}{1+\eta_{s,t}(x)^2}\right)^{(4n(p+1)+8p+2)p}\right]^{\frac{1}{2}},\end{aligned}$$

(2.35) is dominated by

$$C_{31}(n, T)\sup_{x\in\mathbb{R}}(1+x^2)^{(L-2p-1)/2}e^{-\gamma(t,s)|x|}\|\phi\|_{Lp}E\left[\left(\frac{1}{1+\eta_{s,t}(x)^2}\right)^{2Lp}\right]^{\frac{1}{2}}, \tag{2.36}$$

where $\gamma(t, s)=1-(1+\lambda(t, s)x^{2p})^{-1/2p}$, $\lambda(t, s)=-2p\int_s^t \beta_{2p+1}(r)dr$ and $L=2n(p+1)+4p+1$.

For the simplicity we will omit the parameters t, s of $\gamma(t, s)$ and $\lambda(t, s)$ except for ambiguous cases.

Setting $H(x)=1+\lambda x^{2p}+((1+\lambda x^{2p})^{1/2p}K_{s,t}(x)+x)^{2p}$, we get

$$\begin{aligned}&E\left[\left(\frac{1}{1+\eta_{s,t}(x)^2}\right)^{2Lp}\right]^{\frac{1}{2}} \\ &\leqq C_{32}(n, T)\left\{E\left[\left(\frac{1}{H(x)}\right)^{2L}\right]^{\frac{1}{2}}+E\left[\left(\frac{\lambda x^{2p}}{H(x)}\right)^{2L}\right]^{\frac{1}{2}}\right\}.\end{aligned} \tag{2.37}$$

Let $(\Sigma, \mathscr{B}, Q)$ be a probability space where the 1-dimensional Brownian motion $B(t)$ is defined.

Suppose that $|x|\geqq 1$. Setting

$$\Lambda=\left\{\sigma\in\Sigma; \frac{(1+\lambda x^{2p})^{1/2p}}{|x|}|K_{s,t}(x)|<\frac{1}{2}\right\}$$

and noticing that

$$H(x)=1+\lambda x^{2p}+x^{2p}\left(1+\frac{(1+\lambda x^{2p})^{1/2p}}{x}K_{s,t}(x)\right)^{2p},$$

we get

$$E\left[\left(\frac{1}{H(x)}\right)^{2L}\right]=\left(\int_{\Lambda}+\int_{\Sigma\setminus\Lambda}\right)\left(\frac{1}{H(x)}\right)^{2L}dQ$$
$$\leqq\left(\frac{1}{1+x^{2p}/4^{p}}\right)^{2L}+\left(\frac{1}{1+\lambda x^{2p}}\right)^{2L}Q(\Sigma\setminus\Lambda). \tag{2.38}$$

By (2.34), $\sup_{0\leqq s\leqq t\leqq T} E[|K_{s,t}(x)|^{4Lp}]\leqq C_{33}(n, T)(1+x^{2p})$, so that the Čebyšev inequality implies

$$Q(\Sigma\setminus\Lambda)\leqq C_{34}(n, T)\left(\frac{1+\lambda x^{2p}}{x^{2p}}\right)^{2L}(1+x^{2p}),$$

and hence combining this with (2.38), we get

$$E\left[\left(\frac{1}{H(x)}\right)^{2L}\right]^{\frac{1}{2}}\leqq C_{35}(n, T)\left\{\left(\frac{1}{1+x^{2p}}\right)^{L}+\left(\frac{1}{x^{2p}}\right)^{(2L-1)/2}\right\}. \tag{2.39}$$

Quite similarly we have

$$E\left[\left(\frac{\lambda x^{2p}}{H(x)}\right)^{2L}\right]^{\frac{1}{2}}\leqq C_{36}(n, T)\,\lambda^{L}(1+x^{2p}). \tag{2.40}$$

Since $\frac{1}{\gamma(t, x)}\leqq C_{37}(p, T)\frac{1}{\lambda(t, s)}$, $t, s\in[0, T]$, we have

$$(1+x^{2})^{(L-2p-1)/2}e^{-\gamma|x|}\leqq L!\,2^{L/2}\left(\frac{1}{\lambda}\right)^{L}\frac{C_{37}(p, T)^{L}}{(1+x^{2})^{(2p+1)/2}}$$

and hence combining this with (2.37)–(2.39) and (2.40), we have if $|x|\geqq 1$,

$$(1+x^{2})^{n(p+1)+p}e^{-\gamma|x|}E\left[\left(\frac{1}{1+\eta_{s,t}(x)^{2}}\right)^{(4n(p+1)+8p+2)}\right]^{\frac{1}{2}}$$
$$\leqq C_{38}(n, T)\left\{\left(\frac{1}{1+x^{2p}}\right)^{(n(p+1)+3p+1)}+\left(\frac{1}{x^{2p}}\right)^{(n(p+1)+3p+1/2)}\right.$$
$$\left.+\frac{1}{(1+x^{2})^{1/2}}\right\}. \tag{2.41}$$

If $t=s$, for any integer $0\leqq k\leqq n$,

$$(1+x^{2})^{n}|D^{k}(g(x)\,\phi(x))|$$
$$\leqq\|\phi\|_{(2n(p+1)+4p+1)p}\left(\frac{1}{1+x^{2}}\right)^{(2n(p+1)+4p+1)p-n}, \tag{2.42}$$

By (2.41) and (2.42), we obtain that for any $\varepsilon>0$ there exists an $M>0$ such that

$$\sup_{0\leqq s\leqq t\leqq T}\ \sup_{\substack{|x|\geqq M\\ 0\leqq k\leqq n}} (1+x^2)^n\,|D^k(g(x)(U(t,s)\phi)(x))|<\frac{\varepsilon}{3}.$$

Hence we get

$$\begin{aligned}&\|U(t,s)\phi-U(t',s)\phi\|_n\\ &\quad\leqq\frac{2}{3}\varepsilon+\sup_{\substack{|x|\leqq M\\ 0\leqq k\leqq n}}(1+x^2)^n\,|D^k(g(x)((U(t,s)\phi)(x)-(U(t',s)\phi)(x)))|. \qquad (2.43)\end{aligned}$$

By Lemma 3 and the Schwarz inequality, we get that the second term of the right hand side of (2.43) is dominated by $C_{39}(n,T)\,|t-t'|^{1/2}$, which proves (2.16). Now (2.17) will be proved similarly.

For the case where $p=-1, 0$, as we proved in [22] we get

$$(1+x^2)^n\,|D^k(g(x)(U(t,s)\phi)(x))|\leqq C_{40}(n,T)\,\|\phi\|_{n+1}\frac{1}{1+x^2},\qquad t,s\in[0,T],$$

by making use of Lemma 2.3 (p. 212) of Kunita [16]. The rest of the proof will be carried out similarly. This completes the proof.

3. Central Limit Theorem for Interacting Multiplicative Diffusions

Inspired by Graham and Schenzle [5], we will study a central limit theorem for interacting multiplicative diffusions. Before explaining the circumstance, we will introduce some notations. For any integer $p\geqq -1$, let C_p^∞ be a set of real C^∞-functions $f(x)$ such that for any integer $i\geqq 0$ there exists a constant $C(i)$ satisfying $|D^i f(x)|\leqq C(i)(1+|x|)^{p-i}$ if $p-i>0$ and $|D^i f(x)|\leqq C(i)$ if $p-i\leqq 0$. Further $C_p^\infty\times C_p^\infty$ denotes a set of real C^∞-functions $f(x,y)$ such that for any integer $i\geqq 0$ there exists a constant $\bar{C}(i)$ satisfying $|\bar{D}^i f(x,y)|\leqq \bar{C}(i)(1+|x|+|y|)^{p-i}$ if $p-i>0$ and $|\bar{D}^i f(x,y)|\leqq\bar{C}(i)$ if $p-i\leqq 0$, where $\bar{D}^i=\dfrac{\partial^i}{\partial x^{i_1}\partial y^{i_2}}$ and $i_1\geqq 0$, $i_2\geqq 0$ are integers such that, $i_1+i_2=i$.

Now we will consider the following Itô's stochastic differential system.

$$\begin{aligned}X_i^{(n)}(t)&=\sigma_i+\int_0^t a(X_i^{(n)}(r))\,dB_i(r)\\ &\quad+\int_0^t\{b(X_i^{(n)}(r))+c(X_i^{(n)}(r),\bar{X}_n(r))\}\,dr \qquad (3.1)\\ \bar{X}_n(t)&=\frac{1}{n}\sum_{j=1}^n X_j^{(n)}(t),\qquad i=1,2,\dots,n,\end{aligned}$$

where $a(x)\in C_p^\infty$, $b(x)=\bar{b}(x)+b_{2p+1}x^{2p+1}$, $\bar{b}(x)\in C_{2p}^\infty$, $b_{2p+1}<0$ if $p\geqq 0$, $b_{-1}=0$ and $c(x,y)\in C_{2p}^\infty\times C_{2p}^\infty$. Further the coefficients $\{\sigma_i\}$ are independent copies of a real random variable σ which satisfies $E[e^{\varepsilon_0|\sigma|^2}]<\infty$ for some $\varepsilon_0>0$ and is independent of $B(t)$ and $\{B_i(t)\}$ are independent copies of $B(t)$.

Similarly to Step 1 in Sect. 2, we assume $p\geqq 1$ and denote by $a^N(x)$, $b^N(x)$ and $c^N(x, y)$ the truncations of $a(x)$, $b(x)$ and $c(x, y)$. Let $X_i^{(n),N}$ be a unique solution of Eq. (3.1) corresponding to the coefficients

$$\{a^N(x), b^N(x), c^N(x, y)\} \quad \text{and} \quad \bar{X}_{n,N}(r)=\frac{1}{n}\sum_{j=1}^{n} X_j^{(n),N}(r).$$

Let $a_p>0$ and $b_p>0$ be constants such that $|a(x)|^2\leqq a_p(1+x^{2p})$ and $|b(x)| \leqq b_p(1+x^{2p})$. We choose a real number ζ such that $0<\zeta<\min\{\varepsilon_0, |b_{2p+1}|/a_p\}$. Since $X_i^{(n),N}(t)$ and $X_j^{(n),N}(t)$ have the same distribution, by the Itô formula, we have

$$\begin{aligned} E[e^{\zeta|X_i^{(n),N}(t)|^2}] \leqq E[e^{\zeta|\sigma_i|^2}] + E\Bigg[\int_0^t \frac{1}{n}\sum_{i=1}^{n} e^{\zeta(X_i^{(n),N}(r))^2} \\ \cdot \{2\zeta(X_i^{(n),N}(r))\, b(X_i^{(n),N}(r)) \\ +2\zeta c_p |X_i^{(n),N}(r)|(1+|X_i^{(n),N}(r)|^{2p}+|\bar{X}_{n,N}(r)|^{2p}) \\ +(2\zeta^2(X_i^{(n),N}(r))^2+\zeta)\, a_p(1+|X_i^{(n),N}(r)|^{2p})\}\, dr\Bigg], \end{aligned} \tag{3.2}$$

where c_p is a constant such that $|c(x, y)|\leqq c_p(1+x^{2p}+y^{2p})$. By the Hölder inequality, we get for any integer $m\geqq 0$,

$$\frac{1}{n}\sum_{i=1}^{n} |X_i^{(n),N}(r)|^{2m+1}\, |\bar{X}_{n,N}(r)|^{2p} \leqq \left(\frac{1}{n}\sum_{i=1}^{n} |X_i^{(n),N}(r)|^{2m+2p+1}\right),$$

so that the right hand side of (3.2) is dominated by

$$\begin{aligned} E[e^{\zeta|\sigma_i|^2}] + E\Bigg[\int_0^t \sum_{m=0}^{\infty} \frac{\zeta^m}{m!}\Bigg\{\frac{1}{n}\sum_{i=1}^{n} (2\zeta b_{2p+1}(X_i^{(n),N}(r))^{2m+2p+2} \\ +2\zeta(b_p+c_p)\,|X_i^{(n),N}(r)|^{2m+1}(1+|X_i^{(n),N}(r)|^{2p}) \\ +2\zeta c_p\,|X_i^{(n),N}(r)|^{2m+2p+1} \\ +|X_i^{(n),N}(r)|^{2m}(2\zeta^2\,|X_i^{(n),N}(r)|^2+\zeta)\, a_p(1+|X_i^{(n),N}(r)|^{2p})\Bigg\}\, dr\Bigg]. \end{aligned} \tag{3.3}$$

By a manner of choosing ζ and of proving previous lemmas, there exists a constant $C_{41}>0$ such that (3.3) is dominated by

$$E[e^{\zeta|\sigma_i|^2}] + C_{41}\int_0^t \frac{1}{n}\sum_{i=1}^{n} E[e^{\zeta|X_i^{(n),N}(r)|^2}]\, dr.$$

and hence again using the fact that $X_i^{(n),N}(t)$ and $X_j^{(n),N}(t)$ have the same distribution, we get by (3.2),

$$E[e^{\zeta|X_i^{(n),N}(t)|^2}] \leqq E[e^{\zeta|\sigma_i|^2}] + C_{41}\int_0^t E[e^{\zeta|X_i^{(n),N}(r)|^2}]\, dr.$$

Therefore Gronwall's lemma gives

$$\sup_N \sup_{0\leqq t\leqq T} E[e^{\zeta|X_i^{(n),N}(t)|^2}]<\infty,$$

so that similarly to the manner of deriving Lemma 1, (3.1) has a unique non explosive solution $X_i^{(n)}(t)$ and together with the exponential integrability for $p=-1, 0$ proved in [6], we have the following lemma which will be used later.

Lemma 5. *For any $\varepsilon>0$ and $T>0$,*

$$\sup_{0\leqq t\leqq T} E[e^{\varepsilon|X_i^{(n)}(t)|}]\leqq C_{42}(T,\varepsilon)<\infty \tag{3.4}$$

where $C_{42}(T,\varepsilon)$ is independent of n and i.

We will proceed to the discussion of the following non linear stochastic differential equation because the equation is the formal limit of (3.1).

$$\begin{aligned} X_i(t)=\sigma_i&+\int_0^t a(X_i(r))\,dB_i(r)\\ &+\int_0^t \{b(X_i(r))+c(X_i(r), \int_R x u(r,dx))\}\,dr, \end{aligned} \tag{3.5}$$

$u(r,dx)$ is the probability distribution of $X_i(r)$.

Let $Y_0(t)=\sigma_i$ and $Y_n(t)$, $n=1, 2, \ldots$, be defined successively as follows:

$$Y_n(t)=\sigma_i+\int_0^t a(Y_n(r))\,dB_i(r)+\int_0^t \{b(Y_n(r))+c(Y_n(r), E[Y_{n-1}(r)])\}\,dr.$$

For integers $m\geqq(2p(2p-\frac{1}{2})\vee 1)$, by the Hölder inequality we get

$$\begin{aligned} &|Y_n(r)|^{2m-1}|E[Y_{n-1}(r)]|^{2p\vee 0}\\ &\quad\leqq C_{43}\{1+|Y_n(r)|^{2m}|Y_n(r)|^{\left((2p-1)\frac{m}{m-p}\right)\vee 0}+E[|Y_{n-1}(r)|^{2m}]\}, \end{aligned}$$

so that noticing $(2p-1)\dfrac{m}{m-p}\leqq 2p-\dfrac{1}{2}$ and $b_{2p+1}<0$ if $p\geqq 0$ and using the Itô formula we have

$$\begin{aligned} E[|Y_n(t)|^{2m}]\leqq C_{44}\Big\{&1+E[|\sigma_i|^{2m}]+\int_0^t E[|Y_n(r)|^{2m}]\,dr\\ &+\int_0^t E[|Y_{n-1}(r)|^{2m}]\,dr\Big\}. \end{aligned}$$

Hence by the generalized Gronwall lemma, there exists a constant C_{45} independent of n such that

$$\sup_{0\leqq t\leqq T} E[|Y_n(t)|^{2m}]\leqq C_{45}. \tag{3.6}$$

Noticing $\sum_{j=0}^{2p} (Y_n(t))^{2p-j}(Y_{n-1}(t))^j \geqq \frac{1}{2}\{(Y_n(t))^{2p}+(Y_{n-1}(t))^{2p}\}$ and $b_{2p+1}<0$ if $p \geqq 0$ and using the Itô formula, Hölder's inequality, (3.6) and the assumptions of the coefficients $\{a(x), b(x), c(x, y)\}$, we have for $m \geqq (2p \vee 1)$,

$$E[|Y_n(t)-Y_{n-1}(t)|^{2m}] \leqq C_{46} \int_0^t E[|Y_{n-1}(r)-Y_{n-2}(r)|^{2m}]\,dr. \tag{3.7}$$

Therefore by the iteration, (3.5) has a solution $X_i(t)$. Similarly the inequality (3.6) holds if $Y_n(t)$ is replaced by $X_i(t)$ and also the inequality (3.7) holds if $Y_n(t)-Y_{n-1}(t)$ is replaced by $X_i^1(t)-X_i^2(t)$ for two solutions $X_i^1(t)$, $X_i^2(t)$ of the equation (3.5), so that Gronwall's lemma gives the uniqueness of solutions of (3.5).

Now the central limit theorem is as follows.

Theorem 2. *Suppose that* $b_{2p+1}<0$ *if* $p \geqq 0$, $b_{-1}=0$ *and* $E[e^{\varepsilon_0|\sigma|^2}]<\infty$ *for some* $\varepsilon_0>0$. *Then* $S_n(t)$ *converges weakly in* $C([0,\infty)\colon \Phi')$ *to a generalized Ornstein-Uhlenbeck process* $S(t)$ *given by a unique solution of*

$$d\xi(t)=dW(t)+\mathscr{L}^*(t)\,\xi(t)\,dt,$$

where

$$(\mathscr{L}(t)\phi)(x)=\tfrac{1}{2}a(x)^2\,\phi''(x)+\Big(b(x)+c\Big(x, \int_{\mathbb{R}} y\,u(t,dy)\Big)\Big)\phi'(x)$$
$$+x\int_{\mathbb{R}} \phi'(z)\,c_y\Big(z, \int_{\mathbb{R}} x\,u(t,dx)\Big)\,u(t,dz)$$

and

$$c_y(x,y)=\frac{d}{dy}c(x,y).$$

The Φ'-valued Brownian motion $W(t)$ has a covariance functional $E[\langle W(t),\phi_1\rangle\langle W(s),\phi_2\rangle]=\int_0^{t\wedge s}\Big(\int_{\mathbb{R}} a(x)^2\,\phi_1'(x)\,\phi_2'(x)\,u(\tau,dx)\Big)\,d\tau$, $t\wedge s=\min\{t,s\}$ and the initial value $\xi(0)$ is a Φ'-valued Gaussian random variable independent of $W(t)$ and of mean 0 and covariance $E[\langle\xi(0),\phi_1\rangle\langle\xi(0),\phi_2\rangle] = E[\phi_1(\sigma)\,\phi_2(\sigma)]-E[\phi_1(\sigma)]\,E[\phi_2(\sigma)]$.

The proof will be devided into three steps as we mentioned in the introduction.

Step 1. Tightness of $\{S_n(t)\}$ *in* $C([0,\infty)\colon\Phi')$

By Theorem 12.3 of Billingsley [1] and Theorem 3.1 and (R.2) of [21], it is enough to examine the Kolmogorov test for $\langle S_n(t),\phi\rangle$, $\phi\in\Phi$ such that for any $T>0$,

$$\begin{aligned} &E[|\langle S_n(t)-S_n(s),\phi\rangle|^4]\leqq C_{47}(\phi)\,|t-s|^2 \qquad 0\leqq s\leqq t\leqq T,\\ &E[|\langle S_n(0),\phi\rangle|^2]\leqq C_{48}(\phi), \end{aligned} \tag{3.8}$$

where $C_{47}(\phi)$ and $C_{48}(\phi)$ are independent of n.

To prove (3.8), we prepare two lemmas

Lemma 6. *For any integer* $m \geqq (2p(2p-\frac{1}{2}) \vee 1)$ *and any* $T>0$ *there exists a constant* $C_{49}(m, T)$ *independent of* n *and* i *such that*

$$\sup_{0 \leqq t \leqq T} E[|X_i^{(n)}(t) - X_i(t)|^{2m}] \leqq C_{49}(m, T)/n^m. \tag{3.9}$$

Proof. First we remark that Fatou's lemma, together with (3.6), implies

$$E[|X_i(t)|^{2m}] \leqq C_{50}(m, T), \qquad 0 \leqq t \leqq T. \tag{3.10}$$

Since

$$\begin{aligned} 2m\,|X_i^{(n)}(r) - X_i(r)|^{2m-1}\,|c(X_i^{(n)}(r), \bar{X}_n(r)) - c(X_i(r), E[X_i(r)])| \\ \leqq C_{51}\,|X_i^{(n)}(r) - X_i(r)|^{2m-1}\{|X_i^{(n)}(r) - X_i(r)|\,(1 + |X_i(r)|^{(2p-1)\vee 0} \\ + |X_i^{(n)}(r)|^{(2p-1)\vee 0} + |\bar{X}_n(r)|^{(2p-1)\vee 0}) \\ + |\bar{X}_n(r) - E[X_i(r)]|\,(1 + |X_i(r)|^{(2p-1)\vee 0} + E[X_i(r)]^{(2p-1)\vee 0} \\ + |\bar{X}_n(r) - E[X_i(r)]|^{(2p-1)\vee 0})\}, \end{aligned} \tag{3.11}$$

so that noticing

$$\bar{X}_n(r) = \frac{1}{n}\sum_{j=1}^{n} (X_j^{(n)}(r) - X_j(r)) + \frac{1}{n}\sum_{j=1}^{n} (X_j(r) - E[X_j(r)]) + E[X_j(r)]$$

and using (3.10) and the Hölder inequality, we have that for $0 \leqq r \leqq T$, the right hand side of (3.11) is dominated by

$$\begin{aligned} C_{52}\Big\{|X_i^{(n)}(r) - X_i(r)|^{2m}(1 + |X_i(r)|^{(2p-\frac{1}{2})\vee 0} + |X_i^{(n)}(r)|^{(2p-\frac{1}{2})\vee 0}) \\ + \frac{1}{n}\sum_{j=1}^{n} |X_j^{(n)}(r) - X_j(r)|^{2m} + \Big(\frac{1}{n}\sum_{j=1}^{n} (X_j(r) - E[X_j(r)])\Big)^{2m}\Big\}. \end{aligned}$$

Noticing $b_{2p+1} < 0$ if $p \geqq 0$ and using the Itô formula, the above inequalities and the independence of $X_i(t)$, $i = 1, 2, \ldots$, we have

$$\begin{aligned} E[|X_i^{(n)}(t) - X_i(t)|^{2m}] \leqq C_{53}\frac{1}{n^m} + C_{54}\int_0^t \Big\{E[|X_i^{(n)}(r) - X_i(r)|^{2m}] \\ + \frac{1}{n}\sum_{j=1}^{n} E[|X_j^{(n)}(r) - X_j(r)|^{2m}]\Big\}\,dr, \qquad 0 \leqq t \leqq T, \end{aligned}$$

and hence Gronwall's lemma gives (3.9).

By the above lemma, Fatou's lemma can be applied for (3.4). Hence we have

Lemma 7. *For any* $\varepsilon > 0$ *and* $T > 0$,

$$\sup_{0 \leqq t \leqq T} \{E[e^{\varepsilon|X_i^{(n)}(t)|}] \vee E[e^{\varepsilon|X_i(t)|}]\} \leqq C_{55}(T, \varepsilon) < \infty \tag{3.12}$$

where $C_{55}(t, \varepsilon)$ *is independent of* n *and* i.

Once we know Lemmas 6 and 7, we will be able to prove (3.8) similarly to the proof of Theorem 1 of Hitsuda and Mitoma [6].

Step 2. Langevin's Equation of Limit Process

Let $S(t)$ be the limit process of a convergent subsequence $\{S_{n_\nu}(t)\}$ of $\{S_n(t)\}$. The existence is guaranteed by Step 1 and the proof of Proposition 5.1 of [21]. Noticing the form of $(\mathscr{L}(t)\phi)(x)$, we get that for any $T>0$ and any integer $n\geqq 0$,

$$\|\mathscr{L}(t)\phi\|_n \leqq C_{56}(n, T)\|\phi\|_{n+p+2}, \qquad 0\leqq t\leqq T,$$

$$\|\mathscr{L}(t)\phi-\mathscr{L}(s)\phi\|_n \leqq C_{57}(n, T)\|\phi\|_{n+p+1}|t-s|^{1/2}, \qquad t, s\in[0, T].$$

Hence $\mathscr{L}^*(t)S(t)$ is continuous in t on Φ', so that we can define

$$W_S(t)=S(t)-S(0)-\int_0^t \mathscr{L}^*(r)S(r)\,dr.$$

Applying the Itô formula to $\phi(X_i^{(n)}(t))$ and $\phi(X_i(t))$ and using Lemmas 6 and 7, we have

$$\begin{aligned}&\langle S_n(t)-S_n(0), \phi\rangle\\ &\quad=\int_0^t \langle S_n(r), \tfrac{1}{2}a(\cdot)^2\phi''(\cdot)+(b(\cdot)+c(\cdot, \int_{\mathbb{R}} yu(r, dy)))\phi'(\cdot)\rangle\,dr\\ &\qquad+\frac{1}{\sqrt{n}}\sum_{i=1}^n\int_0^t \phi'(X_i(r))\,a(X_i(r))\,dB_i(r)+R_{2,n}(t, \phi)+I_1^{(n)}(t),\end{aligned}$$

where

$$\begin{aligned}I_1^{(n)}(t)=\frac{1}{\sqrt{n}}\sum_{i=1}^n\int_0^t &\phi'(X_i^{(n)}(r))(c(X_i^{(n)}(r), \bar{X}_n(r))\\ &-c(X_i^{(n)}(r), E[X_i(r)]))\,dr\end{aligned}$$

and

$$\lim_{n\to\infty} E[|R_{2,n}(t, \phi)|]=0.$$

Rewrite $I_1^{(n)}(t)$ as the sum of terms

$$\begin{aligned}I_{11}^{(n)}(t)=\int_0^t \frac{1}{\sqrt{n}}\sum_{i=1}^n &(\phi'(X_i^{(n)}(r))-\phi'(X_i(r)))(c(X_i^{(n)}(r), \bar{X}_n(r))\\ &-c(X_i^{(n)}(r), E[X_i(r)]))\,dr,\end{aligned}$$

$$\begin{aligned}I_{12}^{(n)}(t)=\int_0^t \frac{1}{\sqrt{n}}\sum_{i=1}^n &\phi'(X_i(r))(c(X_i^{(n)}(r), \bar{X}_n(r))-c(X_i^{(n)}(r), E[X_i(r)])\\ &-c_y(X_i(r), E[X_i(r)])(\bar{X}_n(r)-E[X_i(r)]))\,dr,\end{aligned}$$

$$I_{13}^{(n)}(t)=\int_0^t \frac{1}{\sqrt{n}} \sum_{i=1}^n (\phi'(X_i(r))\, c_y(X_i(r), E[X_i(r)]) - E[\phi'(X_i(r))\, c_y(X_i(r), E[X_i(r)])])(\bar{X}_n(r)-E[X_i(r)])\, dr,$$

$$I_{14}^{(n)}(t)=\int_0^t \langle S_n(r), E[\phi'(X_i(r))\, c_y(X_i(r), E[X_i(r)])](\cdot)\rangle\, dr.$$

By Lemmas 6 and 7 and the independence of $X_i(t)$, $i=1, 2, \ldots$, we get

$$\lim_{n\to\infty} E[|I_{11}^{(n)}(t)|]=0, \quad \lim_{n\to\infty} E[|I_{12}^{(n)}(t)|]=0 \quad \text{and} \quad \lim_{n\to\infty} E[|I_{13}^{(n)}(t)|]=0.$$

Therefore setting $R_n(t, \phi)=R_{2,n}(t, \phi)+I_{11}^{(n)}(t)+I_{12}^{(n)}(t)+I_{13}^{(n)}(t)$, we obtain

$$\langle S_n(t)-S_n(0), \phi\rangle - \int_0^t \langle S_n(r), \mathscr{L}(r)\phi\rangle\, dr = \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \phi'(X_i(r))\, a(X_i(r))\, dB_i(r)+R_n(t, \phi),$$

where $\lim_{n\to\infty} E[|R_n(t, \phi)|]=0$, which will complete the proof similarly to the proof of Theorem 2 in [6].

Step 3. *Proof of Theorem 2*

By Step 2, any limit process $S(t)$ of $S_n(t)$ satisfies a Langevin's equation of type stated in Theorem 1. It is easy to check the condition (H), so that appealing to Theorem 1 we have that the probability measures induced by any limit process of $S_n(t)$ on $C([0, \infty); \Phi')$ coincide. Thus Step 1, the proof of Proposition 5.1 in [21] and Theorem 2.3 in Billingsley [1] lead us to the completion of the proof of Theorem 2.

Acknowledgment. The author would like to express his hearty thanks to the referees for valuable suggestions.

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Received June 12, 1986; received in revised form December 1, 1986